

# ASSIGNMENT 4

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1. Consider the linear program (**LP**).

$$\begin{aligned} Z &= \max 2x_1 + x_2 \\ \text{s.t.} & \\ 2x_1 + 3x_2 &\leq 3 \\ x_1 + 5x_2 &\leq 1 \\ 2x_1 + x_2 &\leq 4 \\ 4x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0. \end{aligned} \tag{LP}$$

- (a) Draw the feasible region of (**LP**).
- (b) Solve (**LP**) using the simplex method.
- (c) At every step of the simplex method, read the feasible solution from the dictionary, and locate it in your drawing of the feasible region.
- (d) Which inequalities and non-negativity constraints are **tight** at the optimal solution of (**LP**)?
- (e) Write down the dual of (**LP**).
- (f) Find the optimal solution of dual of (**LP**) from the final dictionary of the simplex method of (**LP**).

**Answer:** The solutions to the first three bits are standard. To determine which inequalities and non-negative constraints are tight, look at variables whose optimal

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value are zero, i.e., the set of all non-basic variables and basic variables whose values are zero. The dual of this linear program is

$$\begin{aligned} \min \quad & 3y_1 + y_2 + 4y_3 + 5y_4 \\ \text{s.t.} \quad & \\ & 2y_1 + y_2 + 2y_3 + 4y_4 \geq 2 \\ & 3y_1 + 5y_2 + y_3 + y_4 \geq 1 \\ & y_1, y_2, y_3, y_4 \geq 0. \end{aligned}$$

The optimal solution of the dual can be read from the optimal solution of the primal problem by looking at the coefficients of the slack variables and multiplying them by  $-1$ .

2. Consider the linear program (**LP-2**).

$$\begin{aligned} Z = \max \quad & 3x_1 + 2x_2 + 4x_3 \\ \text{s.t.} \quad & \\ & x_1 + x_2 + 2x_3 \leq 4 \\ & 2x_1 + 3x_3 \leq 5 \\ & 2x_1 + x_2 + 3x_3 \leq 7 \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \tag{LP-2}$$

Solve (**LP-2**) using the simplex method, and identify the tight constraints at the optimal solution.

**Answer:** Standard solution using simplex as above.

3. Consider the linear program (**SP**)

$$\begin{aligned} Z = \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \\ & \sum_{j=1}^n a_j x_j \leq b \\ & x_j \geq 0 \quad \forall j \in \{1, \dots, n\}. \end{aligned} \tag{SP}$$

Assume that  $c_j > 0$  and  $a_j > 0$  for all  $j \in \{1, \dots, n\}$ , and  $b > 0$ . Prove that  $Z = b \max_{j \in \{1, \dots, n\}} \frac{c_j}{a_j}$ .

**Answer:** Let  $s$  be the slack variable in the first dictionary of the simplex. So, the first dictionary looks as:

$$\begin{aligned} Z &= c_1x_1 + \dots + c_nx_n \\ s &= b - a_1x_1 - \dots - a_nx_n \end{aligned}$$

Now we choose  $s$  as the leaving variable. The entering variable is  $x_j$  such that  $j = \arg \max_{j \in \{1, \dots, n\}} \frac{c_j}{a_j}$ . Since coefficient of  $x_j$  is positive, this is a valid entering variable. Variable  $s$  is also a valid leaving variable since  $\frac{b}{a_j} > 0$ . After this pivot operation, the second dictionary will look as:

$$\begin{aligned} Z &= b\frac{c_j}{a_j} + \sum_{i \neq j} [c_i - c_j\frac{a_i}{a_j}]x_i - s\frac{c_j}{a_j} \\ x_j &= \frac{b}{a_j} - \frac{s}{a_j} - \sum_{i \neq j} \frac{a_i}{a_j}x_i \end{aligned}$$

Since for all  $i \neq j$ , we have  $\frac{c_i}{a_i} \leq \frac{c_j}{a_j}$ , we get  $c_i \leq c_j\frac{a_i}{a_j}$ . Hence, all coefficients of non-basic variables in the objective function row of the second dictionary are non-positive. Hence, this is an optimal solution. Thus, the optimal solution value is  $Z = b \max_{j \in \{1, \dots, n\}} \frac{c_j}{a_j}$ .

4. Solve the following linear program using the (two-phase) simplex method.

$$\begin{aligned} \max & -x_1 - 2x_2 \\ \text{s.t.} & \\ & -3x_1 + x_2 \leq -1 \\ & x_1 - x_2 \leq 1 \\ & -2x_1 + 7x_2 \leq 6 \\ & 9x_1 - 4x_2 \leq 6 \\ & -5x_1 + 2x_2 \leq -3 \\ & 7x_1 - 3x_2 \leq 6 \\ & x_1, x_2 \geq 0. \end{aligned}$$

**Answer:** Use the first phase to determine if it has a feasible solution. If it has a feasible solution, then use the second phase to determine if it is unbounded or has an optimal solution.

5. Consider the linear program (**P**).

$$\begin{aligned}
& \max \sum_{j=1}^n c_j x_j \\
& \text{s.t.} \\
& \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \forall i \in \{1, \dots, m\} \\
& x_j \geq 0 \quad \forall j \in \{1, \dots, n\}.
\end{aligned} \tag{P}$$

$$\tag{1}$$

Show that if (P) is unbounded then there exists  $x_k$  ( $k \in \{1, \dots, n\}$ ) such that the following LP (P- $k$ ) is unbounded.

$$\begin{aligned}
& \max x_k \\
& \text{s.t.} \\
& \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \forall i \in \{1, \dots, m\} \\
& x_j \geq 0 \quad \forall j \in \{1, \dots, n\}.
\end{aligned} \tag{P- $k$ }$$

$$\tag{2}$$

**Answer:** If (P) is unbounded, then in some dictionary of the simplex method, the value of some entering variable can be increased arbitrarily. Let that entering variable be  $x_k$ . This means for any real number  $M$ , there exists a feasible solution (P) such that  $x_k > M$ . This implies that (P- $k$ ) is unbounded.