Assignment 5

Debasis Mishra*

April 12, 2011

1. Write the dual of the linear program (**LP**). How will the dual change when we impose $x_2 \geq 0$. Write the complementary slackness conditions.

$$
\begin{aligned}
Z = \max\ & 2x_1 + x_2 \\
\text{s.t.} & \\
2x_1 + 3x_2 &\leq 3 \\
x_1 + 5x_2 &\leq 1 \\
2x_1 + x_2 &= 4 \\
4x_1 + x_2 &= 5 \\
x_1 &\geq 0.
\end{aligned}
\tag{LP}
$$

2. Write the dual of the linear program (**LP-2**).

$$
\begin{aligned}
Z = \min\ & y_1 - y_2 + 3y_3 \\
\text{s.t.} & \\
y_1 + 3y_2 + y_3 &\leq 3 \\
y_1 + 5y_2 - 6y_3 &\geq 1 \\
y_1 + y_2 + y_3 &= 4 \\
y_1, y_2 &\geq 0.
\end{aligned}
\tag{LP-2}
$$

*Planning Unit, Indian Statistical Institute, 7 Shahid Jit Singh Marg, New Delhi 110016, India, E-mail: dmishra@isid.ac.in

3. Consider the linear program (**SP**)

$$
\begin{aligned}
Z = \max \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & \qquad \qquad \textbf{(SP)} \\
\sum_{j=1}^{n} a_j x_j \le b & \\
x_j \ge 0 \qquad \forall\, j \in \{1, \dots, n\}. &
\end{aligned}
$$

Assume that $c_j > 0$ and $a_j > 0$ for all $j \in \{1, \dots, n\}$, and $b > 0$. Write the dual of (**SP**). Prove $Z = b \max_{j \in \{1,\dots,n\}} \frac{c_j}{a_j}$ using linear programming duality.

4. The **uncapacitated facility location** (UFL) problem is defined as follows. A set of potential facility locations $N = \{1, \dots, n\}$ is given. The set of all clients is denoted by $M = \{1, \dots, m\}$. Every client needs to be served by exactly one facility. The cost of opening a facility in location $j \in N$ is given by f_j. The cost of serving client $i \in M$ by facility $j \in N$ is given by c_{ij}. A facility may serve any number of clients (**uncapcitated**) but each client must be served by exactly one facility. The objective is to **serve all clients by minimizing the total cost of opening the facilities and serving the clients**. Note that you have to decide (a) which facilities to open (b) which clients get served by which (opened) facility. Formulate the UFL problem as an integer program.

5. Consider the following integer program (**IP**): $\min \sum_{j=1}^{5} c_j x_j$ subject to $\frac{7}{4}x_1 - \frac{2}{3}x_2 + \frac{5}{2}x_3 - \frac{5}{12}x_4 + \frac{19}{6}x_5 = \frac{8}{3}$ for $x_i \in \mathbb{Z}_+$ for $i \in \{1, 2, 3, 4, 5\}$.

   (a) Show that the problem (**IP-1**) $\min \sum_{j=1}^{5} c_j x_j$ subject to $\frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 = \frac{2}{3} + w$ for $w \in \mathbb{Z}_+$ and $x_j \in \mathbb{Z}_+$ for all $j \in \{1, 2, 3, 4, 5\}$ is a relaxation of (**IP**).

   (b) Show that the problem $\min \sum_{j=1}^{5} c_j x_j$ subject to $\frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 \ge \frac{2}{3}$ for $x_j \in \mathbb{R}_+$ for all $j \in \{1, 2, 3, 4, 5\}$ is a relaxation of (**IP-1).**

6. Consider the constraints of an integer program (**IP**) with variables $x_1, \dots, x_m$ and y. Let $\mathbb{Z}_+$ be the set of non-negative integers.

$$
\begin{aligned}
& x_i \le y \qquad \forall\, i \in \{1, \dots, m\} \\
& x_i \in \mathbb{Z}_+ \qquad \forall\, i \in \{1, \dots, m\} \\
& y \in \{0, 1\}
\end{aligned}
$$

Show that the LP relaxation of (**IP**) gives an optimal solution of (**IP**).