Assignment 5

Debasis Mishra[*]

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1. Write the dual of the linear program (**LP**). How will the dual change when we impose $x_2 \geq 0$. Write the complementary slackness conditions.

$$
\begin{aligned}
Z = \max\ & 2x_1 + x_2 \\
\text{s.t.}\ & \\
2x_1 + 3x_2 &\leq 3 \\
x_1 + 5x_2 &\leq 1 \\
2x_1 + x_2 &= 4 \\
4x_1 + x_2 &= 5 \\
x_1 &\geq 0.
\end{aligned} \tag{LP}
$$

Answer: There is a variable for every constraint: y_1, y_2, y_3, y_4. Since first two constraints are $\leq$, y_1 and y_2 are non-negative. Since last two are equality constraints, y_3 and y_4 are free. For the constraints in dual, since $x_1 \geq 0$, the corresponding constraint is $\geq$ and the other constraint (corresponding to x_2) is equality. The dual of the linear program (**LP**) is:

$$
\begin{aligned}
Z = \min\ & 3y_1 + y_2 + 4y_3 + 5y_4 \\
\text{s.t.}\ & \\
2y_1 + y_2 + 2y_3 + 4y_4 &\geq 2 \\
3y_1 + 5y_2 + y_3 + y_4 &= 1 \\
y_1, y_2 &\geq 0
\end{aligned} \tag{DLP}
$$

[*]Planning Unit, Indian Statistical Institute, 7 Shahid Jit Singh Marg, New Delhi 110016, India, E-mail: dmishra@isid.ac.in

If $x_2 \geq 0$, the the second constraint will also be $\geq$. The CS conditions are:

$$\begin{aligned} y_1[3 - 2x_1 - 3x_2] &= 0 \\ y_2[1 - x_1 - 5x_2] &= 0 \\ x_1[2y_1 + y_2 + 2y_3 + 4y_4 - 2] &= 0 \\ x_2[3y_1 + 5y_2 + y_3 + y_4 - 1] &= 0. \end{aligned}$$

2. Write the dual of the linear program (**LP-2**).

$$\begin{aligned} Z = \min\, & y_1 - y_2 + 3y_3 \\ \text{s.t.} & \\ y_1 + 3y_2 + y_3 &\leq 3 \\ y_1 + 5y_2 - 6y_3 &\geq 1 \\ y_1 + y_2 + y_3 &= 4 \\ y_1, y_2 &\geq 0. \end{aligned} \qquad \textbf{(LP-2)}$$

Answer: Since this is a minimization problem, we first write all the constraints in $\geq$ or $=$ form.

$$\begin{aligned} Z = \min\, & y_1 - y_2 + 3y_3 \\ \text{s.t.} & \\ -y_1 - 3y_2 - y_3 &\geq -3 \\ y_1 + 5y_2 - 6y_3 &\geq 1 \\ y_1 + y_2 + y_3 &= 4 \\ y_1, y_2 &\geq 0. \end{aligned} \qquad \textbf{(LP-2')}$$

Then, the dual is written as before. Now, the objective function of the dual will be in maximization form.

$$\begin{aligned} Z = \max\, & -3x_1 + x_2 + 4x_3 \\ \text{s.t.} & \\ -x_1 + x_2 + x_3 &\leq 1 \\ -3x_1 + 5x_2 + x_3 &\leq -1 \\ -x_1 - 6x_2 + x_3 &= 3 \\ x_1, x_2 &\geq 0. \end{aligned} \qquad \textbf{(DLP-2)} \quad (1)$$

3. Consider the linear program (**SP**)

$$
\begin{aligned}
Z = \max \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & \qquad \textbf{(SP)} \\
\sum_{j=1}^{n} a_j x_j \leq b & \\
x_j \geq 0 \qquad \forall\, j \in \{1, \ldots, n\}. &
\end{aligned}
$$

Assume that $c_j > 0$ and $a_j > 0$ for all $j \in \{1, \ldots, n\}$, and $b > 0$. Write the dual of (**SP**). Prove $Z = b \max_{j \in \{1,\ldots,n\}} \frac{c_j}{a_j}$ using linear programming duality.

Answer: The dual of this problem is

$$
\begin{aligned}
Z = \min by & \\
\text{s.t.} & \qquad \textbf{(DSP)} \\
a_j y \geq c_j \qquad \forall\, j \in \{1, \ldots, n\} & \\
y \geq 0. & \qquad (2)
\end{aligned}
$$

The constraint of (**DSP**) can be rewritten as $y \geq \max_{j \in \{1,\ldots,n\}} \frac{c_j}{a_j}$. Since $b > 0$, value of by is minimized by choosing $y = \max_{j \in \{1,\ldots,n\}} \frac{c_j}{a_j}$. Hence, $Z = b \max_{j \in \{1,\ldots,n\}} \frac{c_j}{a_j}$.

4. The **uncapacitated facility location** (UFL) problem is defined as follows. A set of potential facility locations $N = \{1, \ldots, n\}$ is given. The set of all clients is denoted by $M = \{1, \ldots, m\}$. Every client needs to be served by exactly one facility. The cost of opening a facility in location $j \in N$ is given by f_j. The cost of serving client $i \in M$ by facility $j \in N$ is given by c_{ij}. A facility may serve any number of clients (**uncapcitated**) but each client must be served by exactly one facility. The objective is to **serve all clients by minimizing the total cost of opening the facilities and serving the clients**. Note that you have to decide (a) which facilities to open (b) which clients get served by which (opened) facility. Formulate the UFL problem as an integer program.

Answer: There are two sets of variables: y_j is the variable denoting if facility $j \in N$ is opened or not and x_{ij} is the variable denoting if facility $j \in N$ serves client $i \in M$. The first constraint is every client is served.

$$\sum_{j \in N} x_{ij} = 1 \qquad \forall\, i \in M.$$

The second constraint is that a facility can serve only if it is opened.

$$x_{ij} \leq y_j \qquad \forall\, i \in M,\ \forall\, j \in N.$$

The objective function is to minimize total cost.

$$\min \sum_{j \in N} f_j y_j + \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij}.$$

Hence, the IP can be formulated as follows.

$$\min \sum_{j \in N} f_j y_j + \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij}$$

$$\text{s.t.} \tag{\textbf{UFL}}$$

$$\sum_{j \in N} x_{ij} = 1 \qquad \forall\, i \in M \tag{3}$$

$$x_{ij} \leq y_j \qquad \forall\, i \in M,\ \forall\, j \in N \tag{4}$$

$$x_{ij} \in \{0,1\} \qquad \forall\, i \in M,\ \forall\, j \in N \tag{5}$$

$$y_j \in \{0,1\} \qquad \forall\, j \in N. \tag{6}$$

5. Consider the following integer program (**IP**): $\min \sum_{j=1}^{5} c_j x_j$ subject to $\frac{7}{4}x_1 - \frac{2}{3}x_2 + \frac{5}{2}x_3 - \frac{5}{12}x_4 + \frac{19}{6}x_5 = \frac{8}{3}$ for $x_i \in \mathbb{Z}_+$ for $i \in \{1,2,3,4,5\}$.

   (a) Show that the problem (**IP-1**) $\min \sum_{j=1}^{5} c_j x_j$ subject to $\frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 = \frac{2}{3} + w$ for $w \in \mathbb{Z}_+$ and $x_j \in \mathbb{Z}_+$ for all $j \in \{1,2,3,4,5\}$ is a relaxation of (**IP**).

   Answer: Since objective functions of both the problems are same, we only need to show that every feasible solution of (**IP**) is also a feasible solution of (**IP-1**). Let $x_1, \ldots, x_5$ be a feasible solution of (**IP**). Then, we can write

$$\frac{7}{4}x_1 - \frac{2}{3}x_2 + \frac{5}{2}x_3 - \frac{5}{12}x_4 + \frac{19}{6}x_5 = \frac{8}{3}$$

$$\Leftrightarrow (\frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5) + (x_1 - x_2 + 2x_3 - x_4 + 3x_5) = 2 + \frac{2}{3}.$$

   Let $w = 2 - (x_1 - x_2 + 2x_3 - x_4 + 3x_5)$. Let $\bar{w} = \frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5$. We first argue that $w \geq 0$. Assume for contradiction $w < 0$. Since $x_1, \ldots, x_5$ are integers, then $2-(x_1-x_2+2x_3-x_4+3x_5) \leq -1$. Hence $x_1-x_2+2x_3-x_4+3x_5 \geq 3$. But $x_1 - x_2 + 2x_3 - x_4 + 3x_5 = \frac{8}{3} - \bar{w}$. Hence, $\bar{w} \leq -\frac{1}{3}$. This is impossible since $x_1, \ldots, x_5 \geq 0$ implies that $\bar{w} \geq 0$.

   Now, since $x_1, \ldots, x_5$ are integers w is also a non-negative integer. Hence $x_1, \ldots, x_5, w \in \mathbb{Z}_+$ is a feasible solution of (**IP-1**).

(b) Show that the problem $\min \sum_{j=1}^{5} c_j x_j$ subject to $\frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 \geq \frac{2}{3}$ for $x_j \in \mathbb{R}_+$ for all $j \in \{1, 2, 3, 4, 5\}$ is a relaxation of (**IP-1**).

Answer: This follows from the fact that $w \geq 0$ in (**IP-1**).

6. Consider the constraints of an integer program (**IP**) with variables $x_1, \ldots, x_m$ and y. Let $\mathbb{Z}_+$ be the set of non-negative integers.

$$\begin{aligned} x_i &\leq y \qquad \forall\, i \in \{1, \ldots, m\} \\ x_i &\in \mathbb{Z}_+ \qquad \forall\, i \in \{1, \ldots, m\} \\ y &\in \{0, 1\} \end{aligned}$$

Show that the LP relaxation of (**IP**) gives an optimal solution of (**IP**).

Answer: Consider the LP relaxation of this (**IP**). The constraint matrix is clearly TU since in every row, either there are two variables or one variable, and whenever there are two variables, the coefficients of these variables have opposite sign.