End-Term Examination
Mathematical Programming with Applications to Economics
Total Score: 60

1. Suppose $G = (N, E, w)$ is a weighted strongly connected directed graph with $w : E \to \mathbb{R}$. Denote the shortest path from node i to node j in G as $s(i, j)$. Show that G has a potential if and only if $s(i, j) + s(j, i) \geq 0$ for all $i, j \in N$. (**5 marks**)

2. Let $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq \frac{1}{x_1}\}$ and $T = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$. Can S and T be strictly separated? Explain your answer. (**5 marks**)

3. Consider the following dictionary which appears while solving a linear program using the simplex method.

$$x_3 = 3 - 2x_1 + x_4$$
$$x_2 = \frac{1}{2} + x_1 - 2x_4$$

$$z = 1 - 2x_1 - 4x_4.$$

   - Write down the set of basic and non-basic variables in this dictionary. (**2 marks**)
   - Find the optimal solution of this linear program. (**2 mark**)

4. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be two sets of positive integers, each containing n positive integers. Suppose $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_n$. Consider the problem of associating with each $i = 1, \ldots, n$ a distinct index $j(i)$ such that the sum $\sum_{i=1}^{n} a_i b_{j(i)}$ is maximized.

   (a) Formulate this problem as an assignment problem. (**3 marks**)

   (b) Write its dual and complementary slackness conditions. (**3+3 marks**)

   (c) Use linear programming duality theory to show that the optimal solution is to set $j(i) = i$ for all $i = 1, \ldots, n$. (**7 marks**)

5. Consider the linear fractional program.

$$\min_{x \in \mathbb{R}^n} \frac{cx + \gamma}{dx + \delta} \quad \text{s.t.} \quad Ax \leq b \qquad \textbf{(FP)}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c, d \in \mathbb{R}^n$, and $\gamma, \delta \in \mathbb{R}$. Assume that the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is bounded and $dx + \delta > 0$ for all $x \in P$.

Show that the linear fractional program (**FP**) can be solved by solving the following linear program (**LFP**). (**10 marks**)

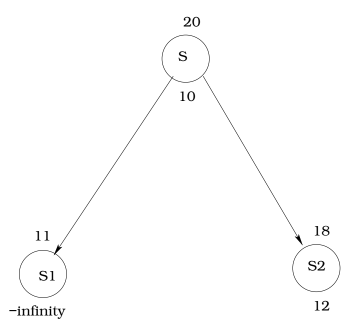
$$\begin{aligned} \min_{y\in\mathbb{R}^n, z\in\mathbb{R}} \ & cy + \gamma z \\ \text{s.t.} \ & \\ & Ay - bz \leq 0 \\ & dy + \delta z = 1 \\ & z \geq 0. \end{aligned} \tag{LFP}$$

More precisely, suppose $\tilde{y} \in \mathbb{R}^n$ and $\tilde{z} \in \mathbb{R}$ are an optimal solution of (**LFP**), then show that (a) $\tilde{z} > 0$ and (b) $\tilde{x} = \frac{\tilde{y}}{\tilde{z}}$ is an optimal solution of (**FP**).

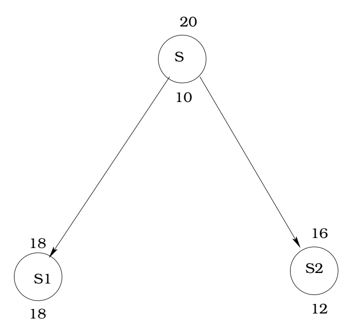
6. Consider an integer program (in standard maximization form) whose feasible region is S. Figures 1(a) and 1(b) give you two instances where the feasible region is partitioned into $S1$ and $S2$ in a branch and bound tree. The figures near each node reflect the lower bound (written below a node) and the upper bound (written above a node). For both Figures 1(a) and 1(b), answer the following.

   (a) Update the lower and upper bounds of the original integer program. (**4 marks**)

   (b) Which of the nodes can be pruned and why? (**6 marks**)



(a)



(b)

Figure 1: How to do pruning

7. Consider the standard linear program (**LP-I**).

$$\begin{aligned} &\max_{x \in \mathbb{R}^n} \sum_{j=1}^{n} c_j x_j \\ &\text{s.t.} \\ &\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \forall\, i \in \{1, \ldots, m\} \\ &\qquad x_j \geq 0 \qquad \forall\, j \in \{1, \ldots, n\}. \end{aligned} \tag{LP-I}$$

Answer the following questions.

- Write the auxiliary linear programming problem of (**LP-I**) to start the first phase of the simplex method. (**5 marks**)
- Show that the original LP (**LP-I**) has a feasible solution if and only if the auxiliary linear program has optimal value zero. (**5 marks**)