End-Term Examination

Mathematical Programming with Applications to Economics

Total Score: 50

1. Consider an undirected graph $G = (N, E)$, where N is the set of vertices and E is the set of edges. A **matching** of G is a subset of edges $S \subseteq E$ such that no two edges in S share an endpoint (i.e., disjoint edges). A maximum matching of G is a matching S of G such that for any other matching T of G we have $|S| \geq |T|$.

   - Formulate the problem of finding a maximum matching of G as an (binary) integer program. Denote this formulation as (**IP**). **4 marks**
   - Show that if G is a bipartite graph (G is bipartite if there is a partitioning of N into B and L such that for every edge $\{i, j\} \in E$, we have $i \in B$ and $j \in L$) then the LP relaxation of (**IP**) gives an integral optimal solution. Denote the LP relaxation of (**IP**) as (**LP**). **2 marks**
   - Show that the dual of (**LP**) has an integral optimal solution and corresponds to the problem of finding the minimum vertex cover of G if G is bipartite. (Note: a vertex cover of G is a set of vertices $C \subseteq N$ such that every edge in E has at least one end point in C and a minimum vertex cover of G is a vertex cover of G having the minimum number of vertices over all vertex covers of G). **4 marks**
   - Use this to show that the cardinality of a minimum vertex cover equals the cardinality of a maximum matching if G is bipartite. **2 marks**

2. A linear program is solved using two-phase simplex method. The first-phase of the simplex method gives an optimal solution value which is non-zero. Which of the following states can the dual of this linear program have? (a) it can have an optimal solution (b) it can be infeasible (c) it can be unbounded. Explain your answer for each case. **6 marks**

3. Consider the following linear program.

$$
\begin{aligned}
&\max x_1 + x_2 \\
&\text{s.t.} && (\mathbf{LP}) \\
&8x_1 + 5x_2 \leq 32 && (1) \\
&8x_1 + 6x_2 \leq 33 && (2) \\
&8x_1 + 7x_2 \leq 35 && (3) \\
&x_1, x_2 \geq 0.
\end{aligned}
$$

Let x_3, x_4, x_5 be the slack variables corresponding to Equations 1, 2, and 3 respectively. The unique optimal solution of this linear program (after solving using a simplex method) is $(x_1, x_2, x_3, x_4, x_5) = (0, 5, 7, 3, 0)$.

- Identify the basic and non-basic variables in the final dictionary of the simplex method for solving (**LP**). **2 marks**
- Write down the final dictionary of the simplex method for solving (**LP**). Note that you do not need to solve (**LP**) to find the final dictionary. **3 marks**
- Write down the dual of (**LP**). **2 marks**
- Write down the complementary slackness (CS) conditions. **2 marks**
- Given the optimal solution of (**LP**), which dual variables will have positive value in the optimal solution of dual of (**LP**). **2 marks**
- Find the optimal solution of dual of (**LP**) without solving it explicitly. **3 marks**

4. Write the Farkas Alternatives for the following system of linear inequalities. **4 marks**

$$\begin{aligned} x_1 - 3x_2 + x_3 &\leq -3 \\ x_1 + x_2 - x_3 &\geq 2 \\ x_1 + 2x_2 + 3x_3 &= 5 \\ x_1 - x_2 &= 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

5. An integer program is solved using branch and bound technique. Let the feasible region of the integer program be S. First, S is partitioned into S_1 and S_2 ($S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$). The LP relaxation of S_1 gave an integral solution with optimal solution value of objective function equal to 20. The LP relaxation of S_2 gave a fractional solution with optimal solution value of objective function equal to 24. A feasible solution of S_2 has objective function value equal to 22.

   - Find an upper and lower bound of optimal solution value of objective function of S. **2 marks**
   - Which of the nodes amongst S_1 and S_2 can be pruned and why? **2 marks**

6. Consider the integer program $\max x_1 + x_2 + x_3$ subject to $\frac{5}{2}x_1 - \frac{1}{3}x_2 + \frac{7}{4}x_3 = \frac{9}{2}$ and x_1, x_2, x_3 are non-negative integers. Show that $\max x_1 + x_2 + x_3$ subject to $\frac{1}{2}x_1 + \frac{2}{3}x_2 + \frac{3}{4}x_3 \geq \frac{1}{2}$ and $x_1, x_2, x_3 \geq 0$ is a relaxation of this integer program. **5 marks**

7. Let A be a $m \times n$ matrix. For any $\alpha \in \mathbb{R}$, define $K(\alpha) = \{b \in \mathbb{R}^m : b = Ax, \|x\| \leq \alpha\}$. Show that $K(\alpha)$ is convex for all $\alpha \in \mathbb{R}$. **5 marks**