END-TERM EXAMINATION
MATHEMATICAL PROGRAMMING WITH APPLICATIONS TO ECONOMICS
TOTAL SCORE: 50

1. Consider an undirected graph $G = (N, E)$, where N is the set of vertices and E is the set of edges. A **matching** of G is a subset of edges $S \subseteq E$ such that no two edges in S share an endpoint (i.e., disjoint edges). A maximum matching of G is a matching S of G such that for any other matching T of G we have $|S| \geq |T|$.

   - Formulate the problem of finding a maximum matching of G as an (binary) integer program. Denote this formulation as (**IP**). **4 marks**

     SOLUTION: The decision variable is $x_e \in \{0, 1\}$ for every edge $e \in E$. If $x_e = 1$, it means that edge e is included in the matching, else it is not included. The constraint must ensure that it is a matching. For this, we need to ensure at every node, maximum one edge is selected. Let E_i denote the set of edges incident on node i.

$$\max_x \sum_{e \in E} x_e$$

$$\text{s.t.}$$

$$\sum_{e \in E_i} x_e \leq 1 \qquad \forall\, i \in N$$

$$x_e \in \{0, 1\} \qquad \forall\, e \in E.$$

   - Show that if G is a bipartite graph (G is bipartite if there is a partitioning of N into B and L such that for every edge $\{i, j\} \in E$, we have $i \in B$ and $j \in L$) then the LP relaxation of (**IP**) gives an integral optimal solution. Denote the LP relaxation of (**IP**) as (**LP**). **2 marks**

     SOLUTION: The LP relaxation of this formulation is:

$$\max_x \sum_{e \in E} x_e$$

$$\text{s.t.}$$

$$\sum_{e \in E_i} x_e \leq 1 \qquad \forall\, i \in N$$

$$x_e \geq 0 \qquad \forall\, e \in E.$$

If G is bipartite, then we can partition the set of constraints into B and L. Any variable x_e will appear once in B and once in L by definition (if e is the edge $\{i, j\}$, where $i \in B$ and $j \in L$, x_e will appear once in E_i and in E_j). Since the coefficient variables is 1 everywhere, the constraint matrix is TU. The right hand side of the constraints is integral. Hence, this LP relaxation gives integral solution.

- Show that the dual of (**LP**) has an integral optimal solution for bipartite G and corresponds to the problem of finding the minimum vertex cover of G (Note: a vertex cover of G is a set of vertices $C \subseteq N$ such that every edge in E has at least one end point in C and a minimum vertex cover of G is a vertex cover of G having the minimum number of vertices over all vertex covers of G). **4 marks**

  SOLUTION: The dual of the LP relaxation for bipartite graph G is defined as follows. There is a variable for every node: y_i for every $i \in N$.

$$\min_{y} \sum_{i \in N} y_i$$

$$\text{s.t.}$$

$$\begin{aligned} y_i + y_j &\geq 1 \qquad \forall\, e = \{i, j\} \in E \\ y_i &\geq 0 \qquad \forall\, i \in N. \end{aligned}$$

  Again, the constraint matrix is TU (it is the transpose of the primal constraint matrix, which is TU). The right hand side of the constraint matrix is integral. Hence, the dual has integral optimal solution. Further, in the optimal solution of the dual $y_i \leq 1$ for all $i \in N$. To see this, assume for contradiction $y_i > 1$ for some $i \in N$ in the optimal solution. The optimal solution value can be decreased by setting $y_i = 1$, and this new solution is still feasible. This contradicts the optimality of the earlier solution.

  So, $y_i \in \{0, 1\}$ for all $i \in N$ in any optimal solution. We can think $y_i = 1$ if node i is chosen in a vertex cover. The constraints ensure that for every edge at least one node is chosen with value one - this is exactly the definition of *covering an edge*. The objective function minimizes the vertex cover.

- Use this to show that the cardinality of a minimum vertex cover equals the cardinality of a maximum matching for a bipartite graph. **2 marks**

  SOLUTION: This follows from strong duality.

2. A linear program is solved using two-phase simplex method. The first-phase of the simplex method gives an optimal solution value which is non-zero. Which of the following states can the dual of this linear program have? (a) it can have an optimal

solution (b) it can be infeasible (c) it can be unbounded. Explain your answer for each case. **6 marks**

SOLUTION: This means that the original LP has no feasible solution. By strong duality, the dual cannot have an optimal solution. But the dual can be infeasible or unbounded (**note:** you need to give examples in both cases to show that dual can be infeasible or unbounded).

3. Consider the following linear program.

$$\begin{aligned} &\max x_1 + x_2 \\ &\text{s.t.} && \textbf{(LP)} \\ &8x_1 + 5x_2 \leq 32 && (1) \\ &8x_1 + 6x_2 \leq 33 && (2) \\ &8x_1 + 7x_2 \leq 35 && (3) \\ &x_1, x_2 \geq 0. \end{aligned}$$

Let x_3, x_4, x_5 be the slack variables corresponding to Equations 1, 2, and 3 respectively. The unique optimal solution of this linear program (after solving using a simplex method) is $(x_1, x_2, x_3, x_4, x_5) = (0, 5, 7, 3, 0)$.

- Identify the basic and non-basic variables in the final dictionary of the simplex method for solving (**LP**). **2 marks**

  SOLUTION: There will be three basic variables and two non-basic variables. The non-basic variables must have zero value. Hence, x_1 and x_5 are the non-basic variables.

- Write down the final dictionary of the simplex method for solving (**LP**). Note that you do not need to solve (**LP**) to find the final dictionary. **3 marks**

  SOLUTION: We need to write the basic variable (x_2, x_3, x_4) and the objective function in terms of non-basic variables (x_1, x_5). This will looks as follows.

$$\begin{aligned} x_2 &= 5 - \frac{8}{7}x_1 - \frac{1}{7}x_5 \\ x_3 &= 7 - \frac{16}{7}x_1 + \frac{5}{7}x_5 \\ x_4 &= 3 - \frac{8}{7}x_1 + \frac{6}{7}x_5 \end{aligned}$$

$$z = 5 - \frac{1}{7}x_1 - \frac{1}{7}x_5.$$

- Write down the dual of (**LP**). **2 marks**

  SOLUTION: The dual of this LP is:

$$\begin{aligned} \min 32y_1 + 33y_2 + 35y_3& \\ \text{s.t.}& \\ 8y_1 + 8y_2 + 8y_3 &\geq 1 \\ 5y_1 + 6y_2 + 7y_3 &\geq 1 \\ y_1, y_2, y_3 &\geq 0. \end{aligned}$$

- Write down the complementary slackness (CS) conditions. **2 marks**

  SOLUTION: The CS conditions are as follows. Consider any feasible solution pair (x, y). This pair is optimal if and only if the following holds.

$$\begin{aligned} y_1(32 - 8x_1 - 5x_2) &= 0 \\ y_2(33 - 8x_1 - 6x_2) &= 0 \\ y_3(35 - 8x_1 - 7x_2) &= 0 \\ x_1(8y_1 + 8y_2 + 8y_3 - 1) &= 0 \\ x_2(5y_1 + 6y_2 + 7y_3 - 1) &= 0. \end{aligned}$$

- Given the optimal solution of (**LP**), which dual variables will have positive value in the optimal solution of dual of (**LP**). **2 marks**

  SOLUTION: Since $x_5 = 0$, that constraint is binding. So, the only dual variable which will have positive value is y_3.

- Find the optimal solution of dual of (**LP**) without solving it explicitly. **3 marks**

  SOLUTION: The optimal solution can be read from the final dictionary of the simplex. It is the negative of coefficients of slack variables, which gives, $y_1 = y_2 = 0$ and $y_3 = \frac{1}{7}$.

4. Write the Farkas Alternatives for the following system of linear inequalities. **4 marks**

$$\begin{aligned} x_1 - 3x_2 + x_3 &\leq -3 \\ x_1 + x_2 - x_3 &\geq 2 \\ x_1 + 2x_2 + 3x_3 &= 5 \\ x_1 - x_2 &= 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution: We rewrite the inequalities in standard form.

$$\begin{aligned} x_1 - 3x_2 + x_3 &\leq -3 \\ -x_1 - x_2 + x_3 &\leq -2 \\ x_1 + 2x_2 + 3x_3 &= 5 \\ x_1 - x_2 &= 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The Farkas alternative will have four variables: y_1, y_2, y_3, y_4. Out of this, $y_1, y_2 \geq 0$ and y_3, y_4 are free. The Farkas alternatives can be written as

$$\begin{aligned} -3y_1 - 2y_2 + 5y_3 + 2y_4 &< 0 \\ y_1 - y_2 + y_3 + y_4 &\geq 0 \\ -3y_1 - y_2 + 2y_3 - y_4 &\geq 0 \\ y_1 + y_2 + 3y_3 &= 0 \\ y_1, y_2 &\geq 0 \end{aligned}$$

5. An integer program is solved using branch and bound technique. Let the feasible region of the integer program be S. First, S is partitioned into S_1 and S_2 ($S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$). The LP relaxation of S_1 gave an integral solution with optimal solution value of objective function equal to 20. The LP relaxation of S_2 gave a fractional solution with optimal solution value of objective function equal to 24. A feasible solution of S_2 has objective function value equal to 22.

   - Find an upper and lower bound of optimal solution value of objective function of S. **2 marks**

     Solution: A lower bound of S is 22 ($\max(20, 22)$) and an upper bound of S is 24 ($\max(20, 24)$).

   - Which of the nodes amongst S_1 and S_2 can be pruned and why? **2 marks**

     Solution: S_1 can be pruned due to optimality and bound. S_2 cannot be pruned.

6. Consider the integer program $\max x_1 + x_2 + x_3$ subject to $\frac{5}{2}x_1 - \frac{1}{3}x_2 + \frac{7}{4}x_3 = \frac{9}{2}$ and x_1, x_2, x_3 are non-negative integers. Show that $\max x_1 + x_2 + x_3$ subject to $\frac{1}{2}x_1 + \frac{2}{3}x_2 + \frac{3}{4}x_3 \geq \frac{1}{2}$ and $x_1, x_2, x_3 \geq 0$ is a relaxation of this integer program. **5 marks**

   Solution: Let $X = \{x_1, x_2, x_3 \text{ non-negative integers} : \frac{5}{2}x_1 - \frac{1}{3}x_2 + \frac{7}{4}x_3 = \frac{9}{2}\}$ and $T = \{x_1, x_2, x_3 \geq 0 : \frac{1}{2}x_1 + \frac{2}{3}x_2 + \frac{3}{4}x_3 \geq \frac{1}{2}\}$. Since the objective function of the two problems is the same, it is enough to show that $X \subseteq T$. For this, choose any $(x_1, x_2, x_3) \in X$. By definition, x_1, x_2, x_3 are non-negative integers. Further, by definition of X, $(0, 0, 0) \notin X$. Hence, $(x_1, x_2, x_3) \neq (0, 0, 0)$. But, this means either x_1 or x_2 or x_3 is greater than or equal to 1. Hence, $\frac{1}{2}x_1 + \frac{2}{3}x_2 + \frac{3}{4}x_3 \geq \frac{1}{2}$, i.e., $(x_1, x_2, x_3) \in T$.

7. Let A be a $m \times n$ matrix. For any $\alpha \in \mathbb{R}$, define

$$K(\alpha) = \{b \in \mathbb{R}^m : b = Ax, \text{ for some } x \in \mathbb{R}^n \text{ with } \|x\| \leq \alpha\}.$$

Show that $K(\alpha)$ is convex for all $\alpha \in \mathbb{R}$. **5 marks**

SOLUTION: Pick $b^1, b^2 \in K(\alpha)$. By definition, there is $x^1, x^2 \in \mathbb{R}^n$ with $\|x^1\| \leq \alpha$ and $\|x^2\| \leq \alpha$ and $b^1 = Ax^1$ and $b^2 = Ax^2$. Choose $\lambda \in (0, 1)$ and let $b^3 = \lambda b^1 + (1-\lambda)b^2 = \lambda Ax^1 + (1-\lambda)Ax^2 = A[\lambda x^1 + (1-\lambda)x^2] = Ax^3$, where $x^3 = \lambda x^1 + (1-\lambda)x^2$. Since $\mathbb{R}^n$ is convex, $x^3 \in \mathbb{R}^n$. Also, it is easy to check that $\|x^3\| = \|\lambda x^1 + (1-\lambda)x^2\| \leq \|\lambda x^1\| + \|(1-\lambda)x^2\| \leq \alpha$, where the last but one inequality comes from the triangle inequality. Hence, $b^3 \in K(\alpha)$.