End-Term Examination

Mathematical Programming with Applications to Economics

Total Score: 50

1. Consider the following system of equations (**F**) with variables $(x_1, \ldots, x_n)$.

$$\sum_{j=1}^{n} a_{ij} x_j = 0 \qquad \forall\, i \in \{1, \ldots, m\}$$
$$\sum_{j=1}^{n} x_j = 1$$
$$x_j \geq 0 \qquad \forall\, j \in \{1, \ldots, n\}.$$

- Write down the Farkas Alternative for the system in (**F**). **(3 marks)**
- Show that the system (**F**) has a solution if and only if the following system with variables $(y_1, \ldots, y_m)$ has no solution. **(5 marks)**

$$\sum_{i=1}^{m} a_{ij} y_i > 0 \qquad \forall\, j \in \{1, \ldots, n\}$$
$$y_i \quad \text{free} \qquad \forall\, i \in \{1, \ldots, m\}.$$

2. We are given a linear program (**P**) and its dual (**D**). I claim that (**P**) is unbounded. How can you verify my claim if you are only allowed to run the first phase of the two-phase simplex method for (**P**) and (**D**)? **(5 marks)**

3. A linear program (**P**) is solved using the two-phase simplex method. The original variables of (**P**) are (x_1, x_2, x_3, x_4) with three constraints. A dictionary of the simplex method while solving (**P**) is given below.

$$x_2 = 14 - 2x_1 - 4x_3 - 5x_5 - 3x_7$$
$$x_4 = 5 - x_1 - x_3 - 2x_5 - x_7$$
$$x_6 = 1 + 5x_1 + 9x_3 + 21x_5 + 11x_7$$

$$z = 29 - x_1 - 2x_3 - 11x_5 - 6x_7.$$

- Find the optimal solution (objective function and variables) of (**P**). **(2 marks)**
- Find the optimal solution (objective function and variables) of the dual of (**P**). **(3 marks)**
- Which constraints are tight in the optimal solution of dual of (**P**)? **(3 marks)**

- Identify the basic variables in the final dictionary of the dual of (**P**). **(2 marks)**

4. Consider the following linear program (call it **(CP)**) with variables $(C, x_1, \ldots, x_n)$.

$$\begin{aligned} & \max C \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij}x_j - C \geq 0 \qquad \forall\, i \in \{1, \ldots, m\} \\ & \sum_{j=1}^{n} x_j = 1 \\ & x_j \geq 0 \qquad \forall\, j \in \{1, \ldots, n\}. \end{aligned}$$

   - Write down the dual of (**CP**). **(5 marks)**
   - Write down the complementary slackness conditions. **(3 marks)**

5. The uncapacitated facility location (UFL) problem is defined as follows. A set of potential facility locations $N = \{1, \ldots, n\}$ is given. A set of clients is given, and denoted by $M = \{1, \ldots, m\}$. Every client needs to be served by exactly one facility. The cost of opening a facility in location $j \in N$ is f_j. The cost of serving client $i \in M$ by facility $j \in N$ is c_{ij}. A facility may serve any number of clients (thus, the term "uncapcitated"). But each client must be served by exactly one facility. The objective is to **serve all clients by minimizing the total cost of opening the facilities and serving the clients**. Note that you have to decide (a) which facilities to open (b) which clients get served by which (opened) facility. Formulate the UFL problem as an integer program. **(10 marks)**

6. Suppose an integer program is solved by branch and bound method. For this, its feasible region S is partitioned into S_1 and S_2. The LP relaxation of S_1 gives integral solution, with optimal solution value α. The LP relaxation of S_2 does not give integral solution, but has an optimal solution β. What can you say about the optimal solution of the original integer program. **(4 marks)**

7. Consider the matrices below.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   - Draw the $cone(A)$ and decide if $\{x \in \mathbb{R}^3 : Ax = b, x_1, x_2, x_3 \geq 0\}$ is non-empty. (**3 marks**)
   - Write the Farkas alternatives for $\{x \in \mathbb{R}^3 : Ax = b, x_1, x_2, x_3 \geq 0\}$. (**2 marks**)