END-TERM EXAMINATION
MATHEMATICAL PROGRAMMING WITH APPLICATIONS TO ECONOMICS
TOTAL SCORE: 50

1. Consider the following system of equations (**F**) with variables $(x_1, \ldots, x_n)$.

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \qquad \forall\, i \in \{1, \ldots, m\}$$
$$\sum_{j=1}^{n} x_j = 1$$
$$x_j \geq 0 \qquad \forall\, j \in \{1, \ldots, n\}.$$

- Write down the Farkas Alternative for the system in (**F**). **(3 marks)**

  **Answer:** The Farkas alternative is

$$y_{m+1} < 0$$
$$\sum_{i=1}^{m} a_{ij}y_i + y_{m+1} \geq 0$$
$$y_1, \ldots, y_{m+1} \text{ free.}$$

- Show that the system (**F**) has a solution if and only if the following system with variables $(y_1, \ldots, y_m)$ has no solution. **(5 marks)**

$$\sum_{i=1}^{m} a_{ij}y_i > 0 \qquad \forall\, j \in \{1, \ldots, n\}$$
$$y_i \quad \text{free} \qquad \forall\, i \in \{1, \ldots, m\}.$$

  **Answer:** If (**F**) has a solution then Farkas alternative has no solution. Hence, for every $y_{m+1} < 0$, we have $\sum_{i=1}^{m} a_{ij}y_i \leq -y_{m+1}$. Taking $y_{m+1}$ arbitrarily close to zero, we get $\sum_{i=1}^{m} a_{ij}y_i \leq 0$. Hence, the given system has no solution. For the converse, if the given system has no solution, assume for contradiction that (F) has no solution. Then, the Farkas alternative has a solution, say $(y_1, \ldots, y_{m+1})$. Since $y_{m+1} < 0$, we construct $y'_i = \frac{y_i}{-y_{m+1}}$ for all $i = 1, \ldots, m$. This satisfies, $\sum_{i=1}^{m} a_{ij}y'_i > 1 > 0$. This is a contradiction.

2. We are given a linear program (**P**) and its dual (**D**). I claim that (**P**) is unbounded. How can you verify my claim if you are only allowed to run the first phase of the two-phase simplex method for (**P**) and (**D**)? **(5 marks)**

**Answer:** We can run the first phase of (**P**) to find out if it is feasible or not. If my claim is correct, then we should conclude here that (**P**) is feasible. To conclude, whether my claim is correct or not we run the first phase of (**D**). If (**D**) is infeasible, then (**P**) has to be unbounded (it cannot be infeasible since we have already ruled that out).

3. A linear program (**P**) is solved using the two-phase simplex method. The original variables of (**P**) are $(x_1, x_2, x_3, x_4)$ with three constraints. A dictionary of the simplex method while solving (**P**) is given below.

$$
\begin{aligned}
x_2 &= 14 - 2x_1 - 4x_3 - 5x_5 - 3x_7 \\
x_4 &= 5 - x_1 - x_3 - 2x_5 - x_7 \\
x_6 &= 1 + 5x_1 + 9x_3 + 21x_5 + 11x_7 \\
\\
z &= 29 - x_1 - 2x_3 - 11x_5 - 6x_7.
\end{aligned}
$$

- Find the optimal solution (objective function and variables) of (**P**). **(2 marks)**
  **Answer:** $z^* = 29$ and $x_1^* = 0, x_2^* = 14, x_3^* = 0, x_4^* = 5$.
- Find the optimal solution (objective function and variables) of the dual of (**P**). **(3 marks)**
  **Answer:** $z^* = 29$ and $y_1^* = 11, y_2^* = 0, y_3^* = 6$.
- Which constraints are tight in the optimal solution of dual of (**P**)? **(3 marks)**
  **Answer:** By CS conditions, constraints corresponding to $x_2$ and $x_4$ must be tight.
- Identify the basic variables in the final dictionary of the dual of (**P**). **(2 marks)**
  **Answer:** Since $y_1$ and $y_3$ have positive value, they are basic. Suppose slack variables are $y_4, y_5, y_6, y_7$ (corresponding to $x_1, x_2, x_3, x_4$). Since $x_2$ and $x_4$ have positive value, the coefficients of $y_5$ and $y_7$ are negative in the final dictionary in the objective function row. This implies that $y_5$ and $y_7$ are non-basic, which means $y_4$ and $y_6$ are basic.

4. Consider the following linear program (call it **(CP)**) with variables $(C, x_1, \ldots, x_n)$.

$$
\begin{aligned}
&\max C \\
\text{s.t.} \quad &\sum_{j=1}^{n} a_{ij}x_j - C \geq 0 \qquad \forall\, i \in \{1, \ldots, m\} \\
&\sum_{j=1}^{n} x_j = 1 \\
&x_j \geq 0 \qquad \forall\, j \in \{1, \ldots, n\}.
\end{aligned}
$$

- Write down the dual of (**CP**). **(5 marks)**

  **Answer:**

$$V = \min_{R,y} R$$

$$\text{s.t.}$$

$$R - \sum_{i=1}^{m} a_{ij} y_i \geq 0 \qquad \forall\ j$$

$$\sum_{i=1}^{m} y_i = 1$$

$$y_i \geq 0 \qquad \forall\ i.$$

- Write down the complementary slackness conditions. **(3 marks)**

  Suppose $(x_1, \ldots, x_n, C)$ is feasible for (CP) and $(y_1, \ldots, y_m, R)$ is feasible for its dual. Then, they are optimal if and only if

$$[\sum_{j=1}^{n} a_{ij} x_j - C] y_i = 0 \qquad \forall\ i [R - \sum_{i=1}^{m} a_{ij} y_i] x_j \qquad = 0 \qquad \forall\ j.$$

5. The uncapacitated facility location (UFL) problem is defined as follows. A set of potential facility locations $N = \{1, \ldots, n\}$ is given. A set of clients is given, and denoted by $M = \{1, \ldots, m\}$. Every client needs to be served by exactly one facility. The cost of opening a facility in location $j \in N$ is $f_j$. The cost of serving client $i \in M$ by facility $j \in N$ is $c_{ij}$. A facility may serve any number of clients (thus, the term "uncapcitated"). But each client must be served by exactly one facility. The objective is to **serve all clients by minimizing the total cost of opening the facilities and serving the clients**. Note that you have to decide (a) which facilities to open (b) which clients get served by which (opened) facility. Formulate the UFL problem as an integer program. **(10 marks)**

   **Answer:** There are two sets of variables: $y_j$ is the variable denoting if facility $j \in N$ is opened or not and $x_{ij}$ is the variable denoting if facility $j \in N$ serves client $i \in M$. The first constraint is every client is served.

$$\sum_{j \in N} x_{ij} = 1 \qquad \forall\ i \in M.$$

   The second constraint is that a facility can serve only if it is opened.

$$x_{ij} \leq y_j \qquad \forall\ i \in M,\ \forall\ j \in N.$$

The objective function is to minimize total cost.

$$\min \sum_{j \in N} f_j y_j + \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij}.$$

Hence, the IP can be formulated as follows.

$$\min \sum_{j \in N} f_j y_j + \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij}$$

$$\begin{aligned}
&\text{s.t.} && && \textbf{(UFL)}\\
&\sum_{j \in N} x_{ij} = 1 && \forall\, i \in M && (1)\\
&x_{ij} \le y_j && \forall\, i \in M,\ \forall\, j \in N && (2)\\
&x_{ij} \in \{0,1\} && \forall\, i \in M,\ \forall\, j \in N && (3)\\
&y_j \in \{0,1\} && \forall\, j \in N. && (4)
\end{aligned}$$

6. Suppose an integer program is solved by branch and bound method. For this, its feasible region $S$ is partitioned into $S_1$ and $S_2$. The LP relaxation of $S_1$ gives integral solution, with optimal solution value $\alpha$. The LP relaxation of $S_2$ does not give integral solution, but has an optimal solution $\beta$. What can you say about the optimal solution of the original integer program. **(4 marks)**

   **Answer:** If $\alpha \ge \beta$, then $\alpha$ is the optimal solution of IP. Else, $\beta$ is an upper bound and $\alpha$ is a lower bound of the IP.

7. Consider the matrices below.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Draw the *cone*$(A)$ and decide if $\{x \in \mathbb{R}^3 : Ax = b, x_1, x_2, x_3 \ge 0\}$ is non-empty. (**3 marks**)

  **Answer:** The cones generated by columns of $A$ is shown in Figure 1.

  As can be seen from Figure 1, $b \notin cone(A)$. Hence, the given system is empty.
- Write the Farkas alternatives for $\{x \in \mathbb{R}^3 : Ax = b, x_1, x_2, x_3 \ge 0\}$. (**2 marks**)

  **Answer:** The Farkas alternatives is:

$$\begin{aligned}
y_1 &< 0\\
2y_1 + y_2 &\ge 0\\
y_2 &\ge 0\\
-y_1 + 2y_2 &\ge 0\\
y_1, y_2 &\text{ free}
\end{aligned}$$

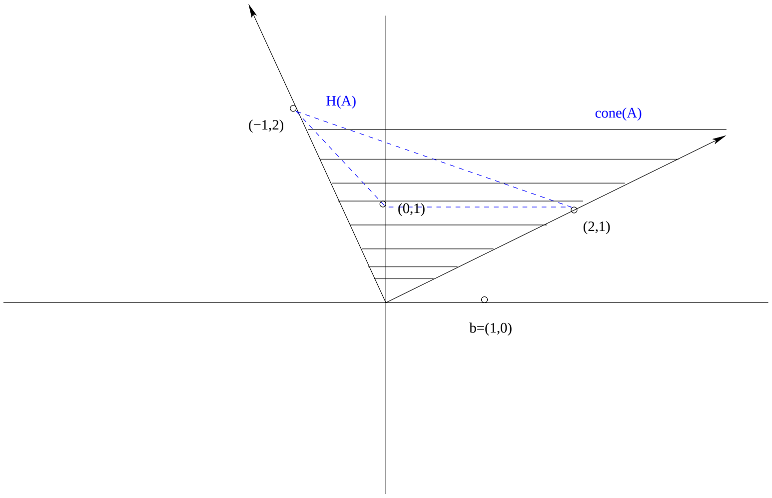

Figure 1: Cones generated by columns of $A$