End-Term Examination
Mathematical Programming with Applications to Economics
Total Score: 60

1. Suppose $G = (N, E, w)$ is a weighted strongly connected directed graph with $w : E \to \mathbb{R}$. Denote the shortest path from node i to node j in G as $s(i, j)$. Show that G has a potential if and only if $s(i, j) + s(j, i) \geq 0$. (**5 marks**)

   Answer: Suppose the graph has a potential. Take any two nodes i and j. Any cycle involving i and j must have non-negative length. Take the shortest path from i to j and the shortest path from j to i. This defines a cycle involving i and j. Hence, $s(i, j) + s(j, i) \geq 0$. For the other side, if $s(i, j) + s(j, i) \geq 0$, the sum of any paths from i to j and from j to i will have length larger than $s(i, j)+s(j, i)$ by the definition of shortest path. Hence, any cycle will have non-negative length. This implies that a potential exists.

2. Let $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq \frac{1}{x_1}\}$ and $T = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$. Can S and T be strictly separated? Explain your answer. (**5 marks**)

   Answer: Suppose S and T can be strictly separated. Then, there exists $p_1, p_2, \alpha \in \mathbb{R}$ such that for all $(x_1, 0) \in T$, $p_1x_1 < \alpha$ and for all $(x_1, x_2) \in S$ $p_1x_1 + p_2x_2 > \alpha$. Choose $(\bar{x}_1 > 0, 0) \in S$ and $(\bar{x}_1, \frac{1}{\bar{x}_1}) \in T$. Since $p_1x_1 < \alpha$ for all $(x_1, 0) \in S$, we can choose $\bar{x}_1$ large enough so that $p_1\bar{x}_1 + p_2\frac{1}{\bar{x}_1} < \alpha$. This violates the fact that $p_1\bar{x}_1 + p_2\frac{1}{\bar{x}_1} > \alpha$.

3. Consider the following dictionary which appears while solving a linear program using the simplex method.

$$x_3 = 3 - 2x_1 + x_4$$
$$x_2 = \frac{1}{2} + x_1 - 2x_4$$

$$z = 10 - 2x_1 - 4x_4.$$

   - Write down the set of basic and non-basic variables in this dictionary. (**2 marks**)

     Answer: The basic variables are x_3 and x_2 and non-basic variables are x_1 and x_4.

- Find the optimal solution of this linear program. (**2 mark**)

  Answer: The optimal solution is 10 with $x_1 = 0, x_2 = \frac{1}{2}, x_3 = 3, x_4 = 0$ since the current dictionary has coefficients of non-basic variables negative in the objective function row.

4. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be two sets of positive integers, each containing n positive integers. Suppose $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_n$. Consider the problem of associating with each $i = 1, \ldots, n$ a distinct index $j(i)$ such that the sum $\sum_{i=1}^{n} a_i b_{j(i)}$ is maximized.

   (a) Formulate this problem as an assignment problem. (**2 marks**)

   Answer: Define $v_{ij} = a_i b_j$ and let x_{ij} be the variable denoting of integer a_i is associated with integer b_j. Then, the formulation is (because of TU constraint matrix, we do not need the binary constraints of variables)

$$\max \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} x_{ij}$$

$$\text{s.t.}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \forall\ j = 1, \ldots, n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall\ i = 1, \ldots, n$$

$$x_{ij} \geq 0 \qquad \forall\ i, j = 1, \ldots, n.$$

   (b) Write its dual and complementary slackness conditions. (**3+3 marks**)

   Answer: The dual of this problem is

$$\min \sum_{i=1}^{n} \pi_i + \sum_{j=1}^{n} p_j$$

$$\text{s.t.}$$

$$\pi_i + p_j \geq v_{ij} = a_i b_j \qquad \forall\ i, j = 1, \ldots, n.$$

   The CS condition is given as follows. Suppose x is a feasible solution to the primal problem and (π, p) is a feasible solution to the dual problem. They are optimal if and only if

$$x_{ij}\big[\pi_i + p_j - a_i b_j\big] = 0 \qquad \forall\ i, j \in \{1, \ldots, n\}$$

(c) Use linear programming duality theory to show that the optimal solution is to set $j(i) = i$ for all $i = 1, \ldots, n$. (**7 marks**)

Answer: First take a feasible solution of primal: $x_{ii} = 1$ for all $i \in \{1, \ldots, n\}$ and $x_{ij} = 0$ if $i \neq j$ for all $i, j \in \{1, \ldots, n\}$. We will then construct a dual feasible solution. For this, we note that if we construct $p_1, \ldots, p_n$, then $\pi_1, \ldots, \pi_n$ can be obtained from it as $\pi_i = \max_{j \in \{1,\ldots,n\}}[a_i b_j - p_j]$, and this will be dual feasible. Now, consider the following p vector:

$$\begin{aligned} p_1 &= 0 \\ p_j &= p_{j-1} + a_j(b_j - b_{j-1}) \qquad \forall\, j \in \{2, \ldots, n\}. \end{aligned}$$

Define $b_0 = b_1$. The p vector can then be calculated as $p_j = \sum_{k=1}^{j} a_k[b_k - b_{k-1}]$. We next verify that it satisfies CS conditions. By the definition of x, $x_{ij} = 1$ if and only if $i = j$. Hence, we need to show that $\pi_i = \max_{j \in \{1,\ldots,n\}}[a_i b_j - p_j] = a_i b_i - p_i$. To show this, pick any $j \neq i$. We consider two cases.

Case 1: If $i > j$, then

$$\begin{aligned} p_i - p_j &= \sum_{k=j+1}^{i} a_k[b_k - b_{k-1}] \\ &\leq a_i \sum_{k=j+1}^{i} [b_k - b_{k-1}] \\ &= a_i b_i - a_i b_j. \end{aligned}$$

Hence, $a_i b_i - p_i \geq a_i b_j - p_j$.

Case 2: If $j > i$, then

$$\begin{aligned} p_j - p_i &= \sum_{k=i+1}^{j} a_k[b_k - b_{k-1}] \\ &\geq a_i \sum_{k=i+1}^{j} [b_k - b_{k-1}] \\ &= a_i b_j - a_i b_i. \end{aligned}$$

Hence, $a_i b_i - p_i \geq a_i b_j - p_j$.

Thus, in both cases $\pi_i = \max_{j \in \{1,\ldots,n\}}[a_i b_j - p_j] = a_i b_i - p_i$. Hence, the CS conditions are satisfied, which implies that x is an optimal solution.

5. Consider the linear fractional program.

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & \frac{cx + \gamma}{dx + \delta} \\ \text{s.t.} \ & \\ & Ax \le b \end{aligned} \tag{FP}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $\gamma, \delta \in \mathbb{R}$. Assume that the polyhedron $P = \{x \in \mathbb{R}^n : Ax \le b\}$ is bounded and $dx + \delta > 0$ for all $x \in P$.

Show that the linear fractional program (**FP**) can be solved by solving the following linear program (**LFP**). (**10 marks**)

$$\begin{aligned} \min_{y \in \mathbb{R}^n, z \in \mathbb{R}} \ & cy + \gamma z \\ \text{s.t.} \ & \\ & Ay - bz \le 0 \\ & dy + \delta z = 1 \\ & z \ge 0. \end{aligned} \tag{LFP}$$

More precisely, suppose $\tilde{y} \in \mathbb{R}^n$ and $\tilde{z} \in \mathbb{R}$ are an optimal solution of (**LFP**), then show that (a) $\tilde{z} > 0$ and (b) $\tilde{x} = \frac{\tilde{y}}{\tilde{z}}$ is an optimal solution of (**FP**).

Answer: First, we show that in any optimal solution of (**LFP**), $z > 0$. Assume for contradiction that this is not true. Then, there exists a $y \in \mathbb{R}^n$ which satisfies

$$\begin{aligned} Ay &\le 0 \\ dy &= 1. \end{aligned}$$

Hence, $y \neq 0$. Moreover, for any $\lambda > 0$, we have

$$A(\lambda y) \le 0.$$

Choose any $x \in P$. Then, $A(x + \lambda y) \le Ax + 0 = Ax \le b$. Hence, if $x \in P$, then $x + \lambda y \in P$. Since λ can be chosen arbitrarily, this contradicts the fact that P is bounded. Hence, in any optimal solution $z > 0$.

Now, consider an optimal solution $(\tilde{y}, \tilde{z})$ of (**LFP**). Let $\tilde{x} = \frac{\tilde{y}}{\tilde{z}}$. Hence, $A\tilde{x} \le b$ and $d\tilde{x} + \delta = \frac{1}{\tilde{z}} > 0$. Hence $\tilde{x}$ is a feasible solution of (**FP**). Suppose x is an optimal solution of (**FP**). Define $z = \frac{1}{dx+\delta}$ and $y = xz$. Since $dx + \delta > 0$, we get $z > 0$. Since $Ax \le b$, we get $Ay - bz \le 0$. Further $dx + \delta > 0$ implies that $dy + \delta z > 0$. Let $dy + \delta z = \epsilon$. Define, $y' = \frac{y}{\epsilon}$ and $z' = \frac{z}{\epsilon}$. Note that $x = \frac{y}{z} = \frac{y'}{z'}$. Further $dy' + \delta z' = 1$

and $Ay' - bz' \leq 0$. Hence, from every optimal solution x, we get a feasible solution (y', z'). But $cy' + \gamma z' = \frac{cx+\gamma}{dx+\delta} \leq \frac{c\tilde{x}+\gamma}{d\tilde{x}+\delta} = c\tilde{y} + \gamma\tilde{z} \leq cy' + \gamma z'$, where the first inequality is because $\tilde{x}$ is feasible solution of (**FP**) but x is an optimal solution of (**FP**) and the second inequality is because $(\tilde{y}, \tilde{z})$ is an optimal solution of (**LFP**) but (y', z') is a feasible solution of (**LFP**). Hence, $\tilde{x}$ is an optimal solution of (**FP**).

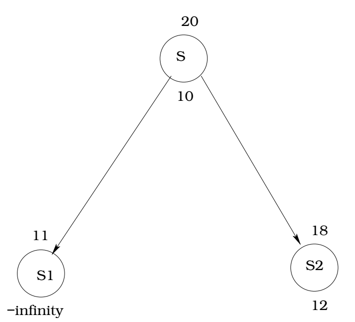
6. Consider an integer program (in standard maximization form) whose feasible region is S. Figures 1(a) and 1(b) give you two instances where the feasible region is partitioned into $S1$ and $S2$ in a branch and bound tree. The figures near each node reflect the lower bound (written below a node) and the upper bound (written above a node). For both Figures 1(a) and 1(b), answer the following.

   (a) Update the lower and upper bounds of the original integer program. (**4 marks**)

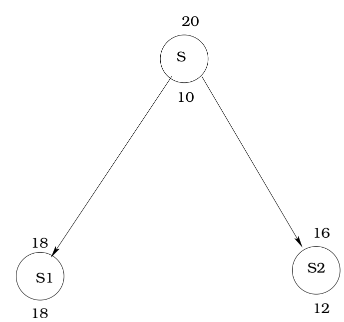
   Answer: For Figure 1(a), the new upper bound is $18 = \max(11, 18)$ and new lower bound is $12 = \max(-\infty, 12)$. For Figure 1(b), the new upper bound is $18 = \max(18, 16)$ and the new lower bound is $18 = \max(18, 12)$.

   (b) Which of the nodes can be pruned and why? (**6 marks**)

   Answer: In Figure 1(a), S_1 can be pruned by bound (the upper bound in S_1 is 11, whereas the lower bound is 12). In Figure 1(b), S_1 can be pruned due to optimality and S_2 can be pruned due to bound (the upper bound in S_2 is 16, which is less than the lower bound of the problem - 18).



(a)



(b)

Figure 1: How to do pruning

7. Consider the standard linear program (**LP-I**).

$$\begin{aligned}
\max_{x \in \mathbb{R}^n} & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \\
\sum_{j=1}^{n} a_{ij} x_j \le b_i & \qquad \forall\, i \in \{1, \ldots, m\} \\
x_j \ge 0 & \qquad \forall\, j \in \{1, \ldots, n\}.
\end{aligned} \tag{LP-I}$$

Answer the following questions.

- Write the auxiliary linear programming problem of (**LP-I**) to start the first phase of the simplex method. (**5 marks**)

  Answer: The auxiliary LP is:

$$\begin{aligned}
& \min x_0 \\
& \text{s.t.} \\
\sum_{j=1}^{n} a_{ij} x_j - x_0 \le b_i & \qquad \forall\, i \in \{1, \ldots, m\} \\
x_j \ge 0 & \qquad \forall\, j \in \{0, 1, \ldots, n\}.
\end{aligned}$$

- Show that the original LP (**LP-I**) has a feasible solution if and only if the auxiliary linear program has optimal value zero. (**5 marks**)

  Answer: If the original LP has a feasible solution, setting $x_0 = 0$ along with that feasible solution is feasible for auxiliary problem. This is the minimum value of x_0. Hence, it is optimal. Suppose the auxiliary problem has an optimal solution with value zero. Then $x_0 = 0$. Hence, this feasible solution, with x_0 removed, is clearly feasible for the original LP.