FINAL EXAMINATION

MATHEMATICAL PROGRAMMING WITH APPLICATIONS TO ECONOMICS

TOTAL SCORE: 100

NOTE: All matrix multiplications are done by taking appropriate transposes, which is not shown in notations. You may draw the figures in pencil but use a pen otherwise.

PART A

This part contains multiple choice questions. Each question may contain multiple correct answers. Mark all the correct choices. Each question carries **4 marks**.

1. Two linear programs (**P1**) and (**P2**) are given with the same set of variables and the same set of constraints except that (**P1**) has all variables non-negative and (**P2**) has all variables free. The dual of (**P2**) is unbounded. Then, which of the following are possible?

   a) The dual of (**P1**) is infeasible.

   b) The dual of (**P1**) is unbounded. (**Answer**)

   c) The dual of (**P1**) has a feasible solution. (**Answer**)

   d) The dual of (**P1**) has an optimal solution.

2. If the simplex method cycles then which of the following are true for two consecutive degenerate iterations?

   a) The objective function value does not change. (**Answer**)

   b) The values of variables do not change. (**Answer**)

   c) The values of variables may change but the objective function value does not change.

   d) The set of basic variables change. (**Answer**)

3. The first phase of the two-phase simplex method terminated with an objective function value $z \neq 0$ for a linear program (**P**). Then,

   a) (**P**) is unbounded.

   b) (**P**) is infeasible. (**Answer**)

   c) (**P**) has an optimal solution.

   d) Dual of (**P**) is either unbounded or infeasible. (**Answer**)

4. Consider the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, where A is an $m \times n$ matrix and b is a $m \times 1$ matrix. Suppose P has an extreme point z. Define A_z to be the set of rows of A such that $a_i z = b_i$ for every row i of A_z. Then, which of the following are true.

   a) $m \leq n$.
   b) $m \geq n$. (**Answer**)
   c) $rank(A_z) = n$. (**Answer**)
   d) $rank(A_z) = m$.

5. Which of the following are true about a minimum cost spanning tree (MCST) of a graph?

   a) The minimum cost edge always belongs to an MCST. (**Answer**)
   b) An MCST contains no cycles. (**Answer**)
   c) There may be multiple paths between a pair of vertices in an MCST.
   d) The number of edges in an MCST will change if the weights of edges are changed.

6. A relaxation of an integer program (in maximization form) must satisfy which of the following requirements.

   a) It must have the same objective function as the integer program.
   b) Its feasible region must include the feasible region of the integer program. (**Answer**)
   c) For every point in the feasible region of the integer program, the objective function of the relaxed problem must have higher value than the objective function of the integer program. (**Answer** - I will give full marks if you have not chosen this)
   d) The optimal solution of the relaxed problem must coincide with the optimal solution of the integer program.

7. Suppose A is an $m \times n$ matrix. Which of the following are convex sets?

   a) Cone of A. (**Answer**)
   b) Convex hull of columns of A. (**Answer**)
   c) Union of cone of A and $\{x \in \mathbb{R}^m : \|x\| \leq 1\}$.
   d) Intersection of cone of A and $\{x \in \mathbb{R}^m : \|x\| \leq 1\}$. (**Answer**)

8. A linear program (**P**) is solved using the two-phase simplex method (with smallest subscript rule for choosing leaving and entering variables). Which of the following are NOT possible in any iteration in the second phase of the simplex method?

a) The coeffcients of the non-basic variables in the final row (corresponding to the objective function) are all negative.

b) An entering variable can be increased by an arbitrarily large amount.

c) The second phase never terminates, i.e., cycles. (**Answer**)

d) We conclude that (**P**) is infeasible. (**Answer**)

9. In a weighted directed graph the shortest path from any node i to a node j is denoted as $s(i, j)$. Suppose $s(i, j) + s(j, i) < 0$ for some nodes i and j. Then which of the following are true.

   a) A potential exists in this graph.

   b) No potential exists in this graph. (**Answer**)

   c) A unique potential exists in this graph.

   d) The set of potentials form a 45-degree line in this graph.

10. Let (IP) be an integer program (in maximization form) and (LP) be its linear programming relaxation. Suppose (LP) has an integral solution. Then which of the following are always true?

    a) The optimal solution value of (LP) is strictly higher than the optimal solution value of (IP).

    b) The optimal solution value of (LP) is equal to the optimal solution value of (IP). (**Answer**)

    c) The optimal solution value of dual of (LP) is equal to the optimal solution value of (IP). (**Answer**)

    d) The dual of (LP) may be unbounded.

Part B

1. Let $G = (N, E)$ be a directed graph with weight $w : E \to \mathbb{Z}$, i.e., integer weights. Assume that G contains no cycle of negative weight. Consider the following inequalities, defining the non-negative potentials of G.

$$\begin{aligned} p(j) - p(i) \leq w(i,j) \qquad &\forall\ (i,j) \in E \\ p(i) \geq 0 \qquad &\forall\ i \in N. \end{aligned} \tag{\textbf{P}}$$

Show that every extreme point of (**P**) is integral. In other words, if weights of a directed graph is integral, then extreme non-negative potentials are integral. **(15 marks)**

Answer: The polyhedra described by the potential inequalities can be written as $P = \{p \in \mathbb{R}^{\#N} : p(j) - p(i) \leq w(i,j)\ \forall\ (i,j) \in E, p(i) \geq 0\ \forall\ i \in N\}$. In matrix form, we can write this as $P = \{p \in \mathbb{R}^{\#N} : Ap \leq w, p \geq 0\}$, where A is the constraint matrix and w is a column of integers. We notice that:

- Every entry of A is 0, -1, or 1.
- Each row containts exactly two non-zero entries and both entries are of opposite sign.

Hence, A is totally unimoduar by the result we proved in the class.

Now any extreme point z of P can be written as $A_z z = w_z$, where A_z corresponds to the rows of A that are tight. We know that $rank(A_z) = n$. Hence, A_z is invertible. So, $z = w_z A_z^{-1}$. Since A is TU, A_z^{-1} is integral. Since w_z is integral, we get that z is integral.

2. Here is a dictionary of a linear program in the simplex method.

$$\begin{aligned} x_2 &= 2 - x_1 + x_4 \\ x_3 &= 4x_1 - 3x_4 \\ Z &= 5 - 2x_1 - \frac{3}{2}x_4. \end{aligned}$$

- Identify the basic and non-basic variables in this dictionary. **(5 Marks)**

  Answer: x_1 and x_4 are non-basic variables. x_2 and x_3 are basic variables.

- Does this dictionary correspond to the optimal solution of the linear program? Explain your answer. **(5 marks)**

  Answer: Since the coefficients of non-basic variables in the objective function row are negative, this corresponds to an optimal dictionary.

- Find the optimal solution (values of variables and objective function) of this linear program and its dual (x_3 and x_4 are slack variables). **(10 marks)**

  Answer: The value of the optimal solution is $z = 5$, $x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 0$. The dual will contain a variable corresponding to each slack variable. Let these be y_1 (corresponding to x_4) and y_2 (corresponding to x_3). We can read the optimal solution of the dual from the objective function row of the primal optimal dictionary. More precisely, negative of coefficients of slack variables equal the optimal value of dual variables. Hence, $y_1 = \frac{3}{2}, y_2 = 0$ is the optimal dual solution. The optimal dual objective function value is equal to 5 by strong duality.

  To get the same solution in detail, we can write the problem in terms of original variables x_1 and x_2 as

$$Z = \max 5 - 2x_1 - \frac{3}{2}[x_2 + x_1 - 2] = \max 8 - \frac{7}{2}x_1 - \frac{3}{2}x_2$$

$$\begin{aligned} \text{s.t.} \quad -x_1 - x_2 &\leq -2 \\ -x_1 + 3x_2 &\leq 6 \\ x_1, x_2 &\geq 0. \end{aligned}$$

  Note that the slack variable x_4 corresponds to the first constraint and x_3 corresponds to the second constraint. The dual of this linear program is

$$\begin{aligned} &8 + \min -2y_1 + 6y_2 \\ &\text{s.t.} \quad -y_1 - y_2 \geq -\frac{7}{2} \\ &\qquad -y_1 + 3y_2 \geq -\frac{3}{2} \\ &\qquad y_1, y_2 \geq 0. \end{aligned}$$

  Set $y_1 = \frac{3}{2}$ and $y_2 = 0$. We see that this is a feasible solution. The objective function value of dual for this feasible solution is 5. Hence, by strong duality, this is also an optimal solution.

3. Write down the dual of the following linear programs: **(10 marks)**

$$\begin{aligned} &\max 2x_1 - 3x_2 \\ &\text{s.t.} \quad x_1 + 3x_2 \geq 6 \\ &\qquad -3x_1 + x_2 \leq 2 \\ &\qquad 2x_1 + 4x_2 = 3 \\ &\qquad x_1 \geq 0. \end{aligned}$$

Answer: After writing the primal in standard form, one can write the dual as

$$\begin{aligned} \min\ & -6y_1 + 2y_2 + 3y_3 \\ \text{s.t.}\ & -y_1 - 3y_2 + 2y_3 \geq 2 \\ & -3y_1 + y_2 + 4y_3 = -3 \\ & y_1, y_2 \geq 0. \end{aligned}$$

$$\begin{aligned} \min\ & 2x_1 - 3x_2 \\ \text{s.t.}\ & x_1 + 3x_2 \geq 6 \\ & -3x_1 + x_2 \leq 2 \\ & 2x_1 + 4x_2 = 3 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Answer: After writing the primal in standard form, one can write the dual as

$$\begin{aligned} \max\ & 6y_1 - 2y_2 + 3y_3 \\ \text{s.t.}\ & y_1 + 3y_2 + 2y_3 \leq 2 \\ & 3y_1 - y_2 + 4y_3 \leq -3 \\ & y_1, y_2 \geq 0. \end{aligned}$$

4. Give an example of a linear program which is infeasible and whose dual is also infeasible. **(5 marks)**

   Answer: Here is a linear program which is infeasible (infeasible since adding the two constraints give us $0 \leq -1$),

$$\begin{aligned} \max\ & 2x_1 - x_2 \\ \text{s.t.}\ & x_1 - x_2 \leq -2 \\ & x_2 - x_1 \leq 1 \\ & x_1, x_2 \geq 0. \end{aligned}$$

   Dual of this linear program is:

$$\begin{aligned} \min\ & -2y_1 + y_2 \\ \text{s.t.}\ & y_1 - y_2 \geq 2 \\ & -y_1 + y_2 \geq -1 \\ & y_1, y_2 \geq 0. \end{aligned}$$

   This dual is infeasible, since adding the two constraints give us $0 \geq 1$.

5. Consider the following integer program (**IP**):

$$
\begin{aligned}
& \min \sum_{j=1}^{5} c_j x_j \\
\text{s.t.} \quad & \frac{7}{4}x_1 - \frac{2}{3}x_2 + \frac{5}{2}x_3 - \frac{5}{12}x_4 + \frac{19}{6}x_5 = \frac{8}{3} \\
& x_i \in \mathbb{Z}_+ \qquad \forall\, i \in \{1,2,3,4,5\}.
\end{aligned} \tag{IP}
$$

Show that the problem (**IP-1**) is a relaxation of (**IP**). **(10 marks)**

$$
\begin{aligned}
& \min \sum_{j=1}^{5} c_j x_j \\
\text{s.t.} \quad & \frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5 = \frac{2}{3} + w \\
& w \in \mathbb{Z}_+ \\
& x_i \in \mathbb{Z}_+ \qquad \forall\, i \in \{1,2,3,4,5\}.
\end{aligned} \tag{IP-1}
$$

Answer: Since objective functions of both the problems are same, we only need to show that every feasible solution of (**IP**) is also a feasible solution of (**IP-1**). Let $x_1, \dots, x_5$ be a feasible solution of (**IP**). Then, we can write

$$
\begin{aligned}
& \frac{7}{4}x_1 - \frac{2}{3}x_2 + \frac{5}{2}x_3 - \frac{5}{12}x_4 + \frac{19}{6}x_5 = \frac{8}{3} \\
\Leftrightarrow\ & \left(\frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5\right) + (x_1 - x_2 + 2x_3 - x_4 + 3x_5) = 2 + \frac{2}{3}.
\end{aligned}
$$

Let $w = 2 - (x_1 - x_2 + 2x_3 - x_4 + 3x_5)$. Let $\bar{w} = \frac{3}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 + \frac{7}{12}x_4 + \frac{1}{6}x_5$. We first argue that $w \geq 0$. Assume for contradiction $w < 0$. Since $x_1, \dots, x_5$ are integers, then $2 - (x_1 - x_2 + 2x_3 - x_4 + 3x_5) \leq -1$. Hence $x_1 - x_2 + 2x_3 - x_4 + 3x_5 \geq 3$. But $x_1 - x_2 + 2x_3 - x_4 + 3x_5 = \frac{8}{3} - \bar{w}$. Hence, $\bar{w} \leq -\frac{1}{3}$. This is impossible since $x_1, \dots, x_5 \geq 0$ implies that $\bar{w} \geq 0$.

Now, since $x_1, \dots, x_5$ are integers w is also a non-negative integer. Hence $x_1, \dots, x_5, w \in \mathbb{Z}_+$ is a feasible solution of (**IP-1**).