MID-TERM EXAMINATION SOLUTION
MATHEMATICAL PROGRAMMING WITH APPLICATIONS TO ECONOMICS
TOTAL SCORE: 50; TIME: 3 HOURS

1. Consider an undirected graph $G = (N, E)$ such that every vertex has degree greater than 1. Is it possible that G contains no cycles? Explain your answer. **(5 marks)**

   Solution: Any graph G can be partitioned into components. Each component is connected. If the original graph has no cycles, then no component will have a cycle, i.e., each component will be a tree. But we know that every tree has at least two vertices of degree 1, which is a contradiction.

2. Suppose $G = (N, E, w)$ be a weighted undirected graph which is connected. Consider a cycle $C = (i^1, i^2, \dots, i^k, i^1)$ in G. Let $\{i^1, i^2\}$ be the unique maximum weight edge of the cycle C and $\{i^2, i^3\}$ be the unique minimum weight edge of the cycle C.

   (a) Show that no minimum weight spanning tree of G contains $\{i^1, i^2\}$. **(5 marks)**

   Solution: Assume for contradiction that a minimum weight spanning tree $G' = (N, E')$ contains $\{i^1, i^2\}$. Consider the graph $G'' = (N, E' \setminus \{i^1, i^2\})$. The graph G'' will have exactly two components, one containing i^1 and another containing i^2. Let V be the set of vertices in one of the components, and $N \setminus V$ be the set of vertices in the other component. So, $\{i^1, i^2\}$ is the light edge of the cut $(V, N \setminus V)$ (it is the unique edge of MCST G' which crosses this cut). By definition, another edge of the cycle C must cross this cut. But $\{i^1, i^2\}$ has the highest weight amongst all edges in C. This is a contradiction to the fact that it is the light edge of the cut $(V, N \setminus V)$.

   (b) Will $\{i^2, i^3\}$ be **always** included in a minimum weight spanning tree? Explain your answer. **(3 marks)**

   Solution: $\{i^2, i^3\}$ need not be included in a minimum weight spanning tree. For example, consider a graph where $\{i^2, i^3\}$ is involved in two cycles - in one cycle it is the unique maximum weight edge and in the other cycle it is the unique minimum weight edge. By our earlier result, $\{i^2, i^3\}$ cannot be part of any minimum weight spanning tree.

3. Consider a directed graph $G = (N, E)$ with two unique vertices s and t such that there is a directed path from s to t. Let φ be a function $\varphi : N \to \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers) such that for every edge $(i, j) \in E$, $\varphi(j) - \varphi(i) \leq 1$.

   (a) Show that such a φ function always exists.

   Solution: φ can be chosen to be any constant function. For a non-constant function, we associate with G a weighted graph $G' = (N, E, w)$, where $w(\{i, j\}) =$

1 for all $\{i,j\} \in E$. Note that φ defines a potential of G'. Further, since weights of edges in G' are all positive, a potential will always exist.

Now, consider a graph G'' such that $G'' = (N, E'', w)$, where $E'' = E \cup \{\{s,i\} : \{s,i\} \notin E\}$ and extend w to E'' by assigning $w(\{s,i\}) = 0$ if $\{s,i\} \notin E$. Since all the weights are non-negative in G'', G'' also has a potential. Moreover, a potential of G'' is also a potential of G'. Consider the potential φ of G'' where $\varphi(s) = 0$ and $\varphi(i)$ for all $i \in N \setminus \{s\}$ is the shortest path from s to i in G''. Clearly, this is also a potential of G'.

We need to argue that φ is non-constant. For this, notice that $\varphi(t) \neq \varphi(s) = 0$. To see this, note that there is a path from s to t in G. Hence, there is no direct edge $\{s,t\}$ in G'' such that $w(\{s,t\}) = 0$. This implies that $\varphi(t) \neq 0$. **(2 marks)**

(b) Let P be a path from s to t such that the number of edges in P, denoted by $e(P)$, is less than or equal to the number of edges in any path from s to t. Show $\varphi(t) - \varphi(s) \leq e(P)$. **(3 marks)**

Solution: Let $P' = (s, i^1, \ldots, i^k, t)$ be any arbitrary path from s to t. By definition, the number of edges in P', $e(P') \leq e(P)$. But $e(P') = [\varphi(i^1) - \varphi(s)] + [\varphi(i^2) - \varphi(i^1)] + \ldots + [\varphi(t) - \varphi(i^k)] = \varphi(t) - \varphi(s)$. Hence, $\varphi(t) - \varphi(s) \leq e(P)$.

(c) Find such a φ function for the graph in Figure 1. **(2 marks)**

Solution: One solution is to let $\varphi(s) = 0$ and take shortest path from s to every other vertex. This gives us $\varphi(1) = 1, \varphi(2) = 2, \varphi(3) = 1, \varphi(4) = 3, \varphi(t) = 3$.

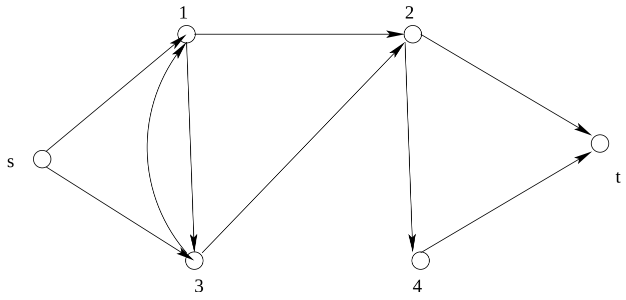

Figure 1: A directed graph

4. Does the following set of inequalities with variables x_1, x_2, x_3, x_4 have a solution? If

yes, then find a solution. **(5 marks)**

$$\begin{aligned} &x_1^2 - x_2^2 \le -2, \quad x_1^2 - x_3^2 \le 3 \\ &x_2^2 - x_1^2 \le 4, \quad x_2^2 - x_3^2 \le -1 \\ &x_2^2 - x_4^2 \le -1, \quad x_3^2 - x_1^2 \le 0 \\ &x_3^2 - x_4^2 \le 1, \quad x_4^2 - x_2^2 \le 2 \end{aligned}$$

Solution: These inequalities do not have a solution. The underlying directed graph has a negative length cycle.

5. Consider any $m \times n$ matrix. Show that the maximum number of non-zero entries such that no two entries are in the same line (i.e., same column or row) is equal to the minimum number of lines that include all non-zero entries. **(5 marks)**

   Solution: Consider a bipartite graph with with a vertex for every row and a vertex for every column. Denote these set of vertices as R and C respectively. There is an edge between $i \in R$ and $j \in C$ if and only if the ith row and jth column entry is non-zero. Denote this bipartite graph as G. Note that a matching in G is a set of non-zero entries of the matrix such that no two entries are in the same line. Also, a vertex cover of G is a set of lines (rows or columns) such that each entry of the matrix belongs to one of the lines. Since the size of maximum matching equals the size of the minimum vertex cover, the result follows.

6. Find a maximum matching and a minimum vertex cover for the bipartite graph shown in Figure 2. **(3+2 marks)**

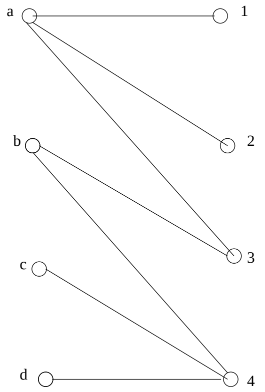

Figure 2: A bipartite graph

Solution: A maximum matching in the bipartite graph of Figure 2 is $\{\{a,1\},\{b,3\},\{c,4\}\}$. A minimum vertex cover in the bipartite graph of Figure 2 is $\{a,b,4\}$.

7. Consider a flow graph $G = (N, E, c)$ with s and t being the source and terminal node respectively. Let c' be another capacity function such that $c'(i, j) = c(i, j) + \alpha$ for all $(i, j) \in E$, where $\alpha > 0$. Now consider the flow graph $G' = (N, E, c')$ with s and t being the source and the terminal node respectively. If $\nu(G)$ and $\nu(G')$ denote the maximum flows in flow graphs G and G' respectively, then show that $\nu(G') - \nu(G) \leq \alpha|E|$. **(5 marks)**

   Solution: Let $(V, N \setminus V)$ be an (s, t)-cut of G, which has the minimum capacity over all (s, t)-cuts. Let E' be the set of edges which crosses this cut. Denote the capacity of $(V, N \setminus V)$ as $\kappa^G(V)$ and $\kappa^{G'}(V)$ in G and G' respectively. By the maxflow-mincut theorem, $\nu(G) = \kappa^G(V)$ and $\nu(G') \leq \kappa^{G'}(V)$. Hence, $\nu(G') - \nu(G) \leq \kappa^{G'}(V) - \kappa^G(V) = \alpha|E'| \leq \alpha|E|$.

8. Consider a flow graph $G = (N, E, c)$ with k source nodes $(s_1, \ldots, s_k)$ and k terminal nodes $(t_1, \ldots, t_k)$. A feasible flow now (a) must satisfy capacity constraints at every edge in E and (b) flow balancing constraints at every node which is neither a source nor a terminal node.

   (a) Transform G to another flow graph G' such that it has a unique source and a unique terminal node and the maximum flow in G' equals the maximum flow in G. You have to specify the set of nodes, the set of edges, and the capacities of edges in G', and also argue why the maximum flow of G will equal the maximum flow of G'. **(5 marks)**

   Solution: We add to G two new nodes: s and t. We also add edges to G, where we put an edge from s to s_i for all $s_i \in \{s_1, \ldots, s_k\}$, and an edge from t_i to t for all $t_i \in \{t_1, \ldots, t_k\}$. Call this graph G'. The capacity of new edges are set very high - say sum of capacities of all edges in G. The capacity of edges in G remains the same in G'.

   Now, it is clear that we can have a maximum flow of G such that the flow between any pairs of sources or any pairs of terminal nodes is zero (this is because flow balancing need not hold at source and terminal nodes, and a source can send as much flow as possible). By substituting the excess flow at source s_i as the flow in edge (s, s_i) for all i and the negative of excess flow at terminal node t_i as the flow in edge (t_i, t) for all i, we get a feasible flow of G' for the maximum flow of G. Hence, maximum flow of G' is greater than or equal to the maximum flow of G. Also, the maximum flow of G' is a feasible flow of G (where we forget about flows in new edges). Hence, the maximum flow of G is greater than or equal to the maximum flow of G'. This shows that the maximum flow of G must equal the maximum flow of G'.

(b) Use this to find the maximum flow of the flow graph in Figure 3, where s_1 and s_2 are source nodes and t_1 and t_2 are terminal nodes. **(5 marks)**

Solution: The maximum flow of the flow graph in Figure 3 is 5 - the maximum flow f is given by $f(s_1, 1) = 3 = f(1, t_1)$, $f(s_2, 2) = 2 = f(2, t_2)$, and $f(1, 2) = f(2, t_1) = f(s_2, s_1) = 0$.

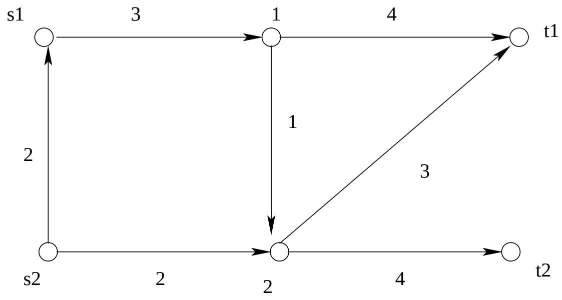

Figure 3: A Flow graph with multiple source and terminal nodes