A Characterization of Anonymous Truth-Revealing Position Auctions*

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Abstract

It is shown that every anonymous truth-revealing position auction is a Vickrey-Clarke-Groves (VCG) position auction. The result is proved for two independent notions of anonymity, anonymous allocation rule and utility symmetry.

1 Introduction

Sponsored search, and in particular position auctions, have become a central issue in electronic commerce and in the literature that analyzes it (see e.g., (Lahaie, 2006; Varian, 2007; Edelman, Ostrovsky, & Schwarz, 2007)). In a position auction a set of merchants, which we refer to as players, bid on a specific keyword. The positions are sold for a fixed period of time.[1] Merchants' advertisements are shown to users, which search for this keyword. The allocation rule of the auction determines which advertisements and in what order they will be presented according to the submitted bids for this keyword. In addition the auctioneer charges a merchant when a user "clicks" on its advertisement, where the charged price is determined by the payment rules of the auction.

The positions are not identical and players may experience different number of clicks. Typically an ad located at a higher position attracts more customers than an ad located at a lower position. The click-through rate, which is the expected number of clicks a player experiences, depends on both the player and her position in a separable way. That is, the click through rate that player i experiences at position q equals $\beta_i\alpha_q$, where β_i is called

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[1] In real-life the time period ends whenever agents change their bids.

the quality factor of player i (see e.g., (Lahaie, 2006; Varian, 2007)). In this paper we deal with position auctions in which players have identical quality factors, that is $\beta_i = 1$ for all i.[2]

A position auction is naturally modeled as a game with incomplete information, in which players' valuations are private. A strategy of every player maps every valuation to a bid. The leading solution concept for such games is ex post equilibrium, which is the natural generalization of Nash equilibrium to setting with incomplete information without probabilistic information.

A position auction is truth-revealing if the strategy profile in which every player reports its true valuation is an ex post equilibrium.[3] Equivalently, since we deal with the independent private-value model, a position auction is truth-revealing if and only if for every player, the truth-revealing strategy is a weakly dominant strategy.

Currently used position auctions, called next-price auctions[4] run by Google and Yahoo, are not truth-revealing (see e.g., Edelman, Ostrovsky, & Schwarz (2007)).[5] In contrast, the Vickrey-Clarke-Groves (VCG) position auctions Vickrey (1961); Clarke (1971); Groves (1973) are truth-revealing. An intriguing question is whether there exist other, not VCG, truthful position auctions, and what are their explicit structure. In this paper we show that no such anonymous position auctions exist. This is shown with two notions of anonymity, which are shown to be independent (i.e., neither of these notions implies the other one). The first notion is that of an anonymous allocation rule, and the second one is of utility symmetry. It is proved that a position auction, which is anonymous in either of the above meanings must be a VCG position auction.

We want to stress that the main novelty in our results follows from the fact that we do not require the allocation rule to be a welfare maximizer, but rather prove that anonymity implies that the allocation rule is a welfare maximizer. If the allocation rule is assumed to be a welfare maximizer then by (Holmstrom, 1979) it must be a VCG position auction.[6] As was shown in (Lavi, Mu'alem, & Nisan, 2007), our result is not extended to anonymous combinatorial auctions with convex domains of valuations.[7]

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[2]All our definitions and results can be naturally extended to the general case by introducing notions of "weighted" properties, and discussing weighted VCG position auctions

[3]By the revelation principle, if there exists an ex post equilibrium, there exists an equivalent truth-revealing position auction.

[4]Next-price auction are also called generalized second price (GSP) auctions. There are two versions for these auctions, the Overture version, and the Google version, which differ in their allocation rules (see e.g Lahaie (2006); Gagan, Goel, & Motwani (2006).)

[5]It was shown in Varian (2007); Edelman, Ostrovsky, & Schwarz (2007) that next-price auctions have equilibrium in a complete information setup in which every player knows all other players' valuations.

[6] In (Holmstrom, 1979) it was proved that when the set of valuations of every player is differentiable-path connected (and in particular when it is convex) the allocation rule in a truth-revealing mechanism determines each of the payment functions up to an additive constant.

[7]When the domain of valuations is further restricted to the non-convex domain of single-minded val-

For the sake of fluent reading, almost all proofs appear in Section 6.

2 Preliminaries

2.1 Position Auctions

A position auction *environment* is defined by three parameters (k, n, α), where $k \geq 1$ is the number of positions for sale at a given unit of time, $n \geq k$ is the number of players, and $\alpha = (\alpha_1, \cdots, \alpha_k)$ is a vector of positive numbers, which is called the *position vector*; α_q is called the *click-through rate* of position q, and it is interpreted as the expected number of clicks on an ad located at position q. It is assumed that $\alpha_1 \geq \alpha_2 \cdots \geq \alpha_k > 0$. For convenience, we add a *dummy* position, position $k+1$ with $\alpha_{k+1} = 0$. The set of non-dummy positions is denoted by K, that is, $K = \{1, ..., k\}$. The set of players is denoted by N, that is $N = \{1, ..., n\}$. For the rest of our discussion the environment will be fixed.

If i holds a position then every click of a visitor to this position gives i a revenue of $v_i \geq 0$, where v_i is called the *valuation* of i. The set of possible valuations of i is $V_i = [0, \infty)$. Let $\mathbf{V} = V_1 \times V_2 \times \cdots \times V_n$ be the set of vectors of valuations. Let $\mathbf{v} \in \mathbf{V}$. For every $S \subseteq N$ we let $\mathbf{v}_S = (v_j)_{j \in S}$, $\mathbf{v}_{-S} = \mathbf{v}_{N \setminus S}$ and $\mathbf{v}_{-i} = \mathbf{v}_{-\{i\}}$.

An *allocation* is an assignment of players to positions. We focus on position auctions in which all positions are allocated to the players. More precisely, an allocation is a vector of positive integers, $\mathbf{q} = (q_1, q_2, \cdots, q_n)$ with $1 \leq q_i \leq k+1$, such that for every non-dummy position q there exists a unique player i with $q_i = q$. The set of all allocations is denoted by $\mathbf{Q}$. A *position auction* is defined by a pair $(s, \mathbf{p})$, where $s = (s_1, s_2, \cdots, s_n) : \mathbf{V} \to \mathbf{Q}$ is the *allocation rule*, and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is the payment scheme, where $p_i : \mathbf{V} \to \mathbb{R}$. That is, when each player i submits the bid $b_i \in V_i$, player i receives the position $s_i(\mathbf{b})$, and she pays $p_i(\mathbf{b})$, where $\mathbf{b} = (b_1, \cdots, b_n) \in \mathbf{V}$ is the profile of bids.[8] [9]

We assume that players are risk neutral and have quasi-linear utilities. Therefore the

uations it was already shown in (Lehmann, O'Callaghan, & Shoham, 1999) that there exist reasonable anonymous auctions that are not VCG auctions.

[8]In real-life position auctions, as well as in some of the literature, the payments are defined per-click. In our model, the payment per-click of i is $\frac{p_i(\mathbf{b})}{\alpha_{s_i(\mathbf{b})}}$, if $s_i(\mathbf{b})$ is a non-dummy position. In a model that directly deals with per-click payments, it is implicitly assumed that a player pays nothing if she does not receive a position. Our model is slightly more general because it allows to charge non-zero payments from participants who do not get a position (i.e., participants who get the dummy position). The two models coincide under the assumptions of individual rationality and seller rationality, which is discussed in Section 5.

[9]What we call a position auction should be actually called a direct position auction. In a more general, non-direct auction the players are required to submit messages, which are not necessarily non-negative numerical bids. The restriction to direct position auctions is innocent in our context due to the revelation principle.

utility function of player i, $u_i : V_i \times \mathbf{V} \to \mathbb{R}$ is defined as follows:

$$u_i(v_i, \mathbf{b}) = \alpha_{s_i(\mathbf{b})} v_i - p_i(\mathbf{b}). \tag{1}$$

2.2 Truth-Revealing Position Auctions

We are interested in position auctions in which players have incentives to bid truthfully. A position auction $(s, \mathbf{p})$ is *truth-revealing* if bidding truthfully is a weakly dominant strategy for every player. That is, if for every player i, for every v_i, and for every fixed vector of valuations of all other players, $\mathbf{b}_{-i}$ the following holds:

$$u_i(v_i, (v_i, \mathbf{b}_{-i})) \geq u_i(v_i, (b_i, \mathbf{b}_{-i})) \ \ \forall b_i \in V_i. \tag{2}$$

We say that an allocation rule s is *implementable* if there exist a payment scheme $\mathbf{p}$ such that $(s, \mathbf{p})$ is a truth-revealing position auction. In our proofs We extensively use a characterization of truth-revealing position auctions by a monotonicity condition.

The allocation rule s is called *monotone* if for every player i, for every $\mathbf{b}_{-i}$ the function $b_i \to \alpha_{s_i(b_i, \mathbf{b}_{-i})}$ is non-decreasing. That is, a player's click-through rate cannot decrease by raising its bid given that the other players' bids are fixed. Equivalently, s is monotone if and only if for every fixed $\mathbf{b}_{-i}$

$$(b_i - b_i')(\alpha_{s_i(b_i, \mathbf{b}_{-i})} - \alpha_{s_i(b_i', \mathbf{b}_{-i})}) \geq 0, \ \forall b_i, b_i' \in V_i. \tag{3}$$

The proof of the following proposition is derived from (Bikhchandani *et al.*, 2006):

Proposition 2.1 ((Bikhchandani *et al.*, 2006)) *For position auctions: An allocation rule is implementable if and only if it is monotone.*

Proof: The sufficiency of the monotonicity condition for deterministic (non-randomized) and finite-valued mechanisms was proved in (Bikhchandani *et al.*, 2006) for the case in which the set of valuations of each player is R_+^m for some $m \geq 1$, which is the case in our model in which $m = 1$.[10] ■

2.3 VCG Position Auctions

A well-known class of truth-revealing position auctions are the VCG position auctions. In order to describe them we need the following terminology: An allocation $\mathbf{q} \in \mathbf{Q}$ is *consistent with the vector of bids* $\mathbf{b}$ if for every $1 \leq i, j \leq n$

$$b_i > b_j \ \Rightarrow \ q_i \leq q_j.$$

[10] In the case $m = 1$ the sufficiency of the monotonicity holds also for randomized allocation rules as can be derived from (Myerson, 1981).

An allocation rule s is called a *welfare maximizer* if for every $\mathbf{b} \in \mathbf{V}$ the allocation $s(\mathbf{b})$ is consistent with $\mathbf{b}$.

For a vector of real numbers $\mathbf{b} = (b_1, \ldots, b_n)$ we denote by $b_{(j)}$ the j^{th} largest number including ties. For example, if $\mathbf{b} = (1, 2, 3, 2)$, then $b_{(1)} = 3, b_{(2)} = 2, b_{(3)} = 2, b_{(4)} = 1$.

Definition 2.2 *Let $(s, \mathbf{p})$ be a position auction.*

$(s, \mathbf{p})$ *is called a* standard VCG position auction[11] *if the following holds:*

a. s is a welfare maximizer.

b. For every $1 \le i \le n$,

$$p_i(\mathbf{b}) = \sum_{m=s_i(\mathbf{b})+1}^{k+1} b_{(m)}(\alpha_{m-1} - \alpha_m), \tag{4}$$

if $s_i(\mathbf{b}) \in K$ and $p_i(\mathbf{b}) = 0$ if $s_i(\mathbf{b}) = k + 1$.

Note that there exist many standard VCG position auctions, each of them uniquely determined by a tie-breaking rule.

Non-standard VCG position auctions share the same allocation rules with the standard ones, and their payment schemes are obtained from the standard payment schemes by adding to the payment of each player an additional payment, which depends only on the other players' bids. More precisely, $(s, \hat{\mathbf{p}})$ is a VCG position auction if there exists a standard VCG position auction, $(s, \mathbf{p})$, with the same allocation rule, s, and there exist n functions, $g_i : \mathbf{V}_{-i} \to \mathbb{R}$, such that for every i

$$\hat{p}_i(\mathbf{b}) = p_i(\mathbf{b}) + g_i(\mathbf{b}_{-i}) \quad \forall \mathbf{b} \in \mathbf{V}, \tag{5}$$

where $p_i(\mathbf{b})$ is the standard VCG payment function of i given in (4).

3 Anonymous Allocation Rules

In this section we show that every truth-revealing position auction with an anonymous allocation rule is a VCG position auction.

The following notation will be useful. For every profile $\mathbf{b} \in \mathbf{V}$ denote by $\mathbf{b}^{ij}$ the bid profile obtained from $\mathbf{b}$ by exchanging player i's and j's valuations, that is $\mathbf{b}^{ij} = (b_1, \ldots, b_{j-1}, b_i, b_{j+1}, \ldots, b_{i-1}, b_j, b_{j+1}, \ldots, b_n)$.

A natural requirement from an anonymous allocation rule would be that $s_j(\mathbf{b}^{ij}) = s_i(\mathbf{b})$ for every profile of bids $\mathbf{b}$ and every pair of players i, j. However, such a requirement is not well defined due to tie issues. Before we define an anonymous allocation rule we need the following:

[11] Some authors (see e.g., (Holzman *et al.*, 2004)) call the standard VCG mechanism, the VC mechanism. However, we decided to use the more common terminology.

Let $\mathbf{b} \in \mathbf{V}$ be a bid profile. b_i is *distinct* in $\mathbf{b}$ if $b_i \neq b_j$ for every player $j \neq i$. When it is clear from the context we will just say that b_i (or i's bid) is distinct. We further say that $\mathbf{b}$ is *generic* if b_i is distinct for every player i.

Definition 3.1 *An allocation rule s is* anonymous *if the following holds: For every player i, and for every for every $\mathbf{b} \in \mathbf{V}$ such that b_i is distinct in $\mathbf{b}$, $s_j(\mathbf{b}^{ij}) = s_i(\mathbf{b})$ for every player j.*

Hence, if i's bid is distinct in $\mathbf{b}$, and i switches bids with player j, j receives the previous position of i, but i does not necessarily receives the previous position of j, unless j's bid is also distinct.

Our main result in this section is:

Theorem 3.2 *A truth-revealing position auction with an anonymous allocation rule is a VCG position auction.*

The next example shows that if we slightly weaken the definition of anonymous allocation rules by requiring that the anonymity condition in Definition 3.1 holds only for generic bid profiles, Theorem 3.2 does not hold.

Definition 3.3 *An allocation rule s is* weakly anonymous *if for every generic $\mathbf{b} \in \mathbf{V}$ and every player i, $s_j(\mathbf{b}^{ij}) = s_i(\mathbf{b})$ for every player j, $j \neq i$.*

In the following example we show that there exists a truth-revealing position auction with a weakly anonymous allocation rule which is not a VCG-position auction.

Example 1 We define an allocation rule s as follows. Let $\mathbf{b}$ be a bid profile. If $\mathbf{b}$ is generic then $s(\mathbf{b})$ is consistent with $\mathbf{b}$. Otherwise, let $z(\mathbf{b})$ be the highest non distinct bid (in a tie) in $\mathbf{b}$. Let $T(\mathbf{b}) = \{i : b_i \geq z(\mathbf{b})\}$ be the set of players with valuation larger than or equal $z(\mathbf{b})$. s allocates positions $1, 2, \ldots, \min\{k, |T(\mathbf{b})|\}$ positions to players in $T(\mathbf{b})$ in decreasing order while breaking ties in lexicographically order (giving priority to players with a smaller index). All other positions are allocated to players in $N \setminus T(\mathbf{b})$ in lexicographically order. To illustrate, let $k = 4$ and $n = 6$. In the bid profile $(5, 3, 4, 4, 5, 6)$, $z = 5$, and the allocation rule s allocates position 1, 2 and 3 to players 6, 1 and 5 respectively, and position 4 to player 2. By definition s is weakly anonymous. It is easily verified that s is monotone, and therefore by Proposition 2.1 it is implementable. Moreover note that s is not welfare maximizer, and therefore the position auction is not a VCG auction.

4 Utility Symmetric Position Auctions

So far we have dealt with anonymous allocation rules. This type of symmetry is natural in position auctions and in other auction settings, but it does not have an analogue in many general mechanisms. Another type of symmetry, which is commonly used in mechanism design theory is utility symmetry. Roughly speaking a position auction is utility symmetric if whenever a pair of players who report truthfully their bids exchange their valuations and their bids, their utilities will also be exchanged. More specifically:

Definition 4.1 *A position auction is called utility symmetric if for every two distinct players, i, j, for every bids of the other players, $\mathbf{b}_{-\{i,j\}}$, and for every v_i, v_j*

$$u_j(v_i, \mathbf{b}^{ij}) = u_i(v_i, \mathbf{b}),$$

where $\mathbf{b} = (v_i, v_j, \mathbf{b}_{-\{i,j\}})$.

Note that definition 4.1 implies that truthful players with the same valuation should have the same utility. It is immediate to verify that every standard VCG position auction is utility symmetric. Therefore a standard VCG position auction is both utility symmetric and has an anonymous allocation rule. In general, a utility symmetric position auction does not necessarily have an anonymous allocation rule and vice-versa. Constructing a position auction with an anonymous allocation rule, which is not utility symmetric is immediate: a position auction with at least two positions in which the allocation rule is a welfare maximizer and for every i $p_i(\cdot) \equiv i$.

In the following example we show a utility symmetric position auction with a non anonymous allocation rule.

Example 2 Let $n = 2$, $k = 1$, and $\alpha_1 = 1$. Let $s_1(\mathbf{b}) = 1$ for every bid profile $\mathbf{b}$, i.e., player 1 always gets position 1. The payment scheme is defined as follows: For every bid profile $\mathbf{b}$ $p_1(\mathbf{b}) = b_1$ and $p_2(\mathbf{b}) = 0$. Note that whenever both players bid truthfully, the utility of both players equals 0. Therefore the auction is utility symmetric and obviously the allocation rule is not anonymous.

Our main result in this section is:

Theorem 4.2 *A truth-revealing utility symmetric position auction is a VCG position auction.*

5 Individual Rationality and Seller Rationality

A truth-revealing position auction $(s, \mathbf{p})$ is *individually rational* if reporting the true valuation guarantees every player a non-negative utility. That is, for every player i and for every v_i the following holds:

$$u_i(v_i, (v_i, \mathbf{b}_{-i})) \geq 0 \;\; \forall \mathbf{b}_{-i} \in \mathbf{V}_{-i}. \tag{6}$$

A truth-revealing position auction $(s, \mathbf{p})$ is *seller rational* if for every player i

$$p_i(\mathbf{b}) \geq 0 \quad \forall \mathbf{b} \in \mathbf{V}. \tag{7}$$

It is easily verified that every standard VCG position auction is both individually rational and seller rational. However, a non-standard VCG position auction does not necessarily satisfy any of the rationality conditions because the functions g_i in (5) can be arbitrarily chosen . We show that

Theorem 5.1 *Every truth-revealing, individually rational and seller rational position auction which is either utility symmetric or has a symmetric allocation rule is necessarily a standard VCG position auction.*

6 Proofs

In this section we prove our main results, Theorems 3.2, 4.2, and 5.1. For simplicity we will prove our results for the case in which $\alpha_1 > \alpha_2 > \cdots > \alpha_k$.
Throughout the proofs we frequently use the fact proved in Proposition 2.1, that if $(s, \mathbf{p})$ is truth-revealing, s is monotone (see Section 2.2).

In our proofs we will use the following proposition derived from (Holmstrom, 1979).[12]

Proposition 6.1 ((Holmstrom, 1979)) *Let s be a welfare maximizer. If the allocation rule, s is implementable by a payment scheme $\mathbf{q}$, $(s, \mathbf{q})$ is a VCG position auction.*

Proof: In (Holmstrom, 1979) it was proved that when the set of valuations of every player is differentiable-path connected (and in particular when it is convex) the allocation rule in a truth-revealing mechanism determines each of the payment functions in the payment scheme up to an additive constant. Since every V_i is a convex set the proof follows. ■

In the proof of Theorem 4.2 we will use the following lemma whose proof can be deduced from (Myerson, 1981), where it is stated for position auctions with a single position. The extension to general position auctions is given in (Archer, 2004) (section 2.4.2)

Lemma 6.2 ((Myerson, 1981)) *If a position auction $(s, \mathbf{p})$ is truth-revealing then for every $v_i \in V_i$ and every $\mathbf{b_{-i}} \in \mathbf{V}_{-i}$,*

$$u_i(v_i, (v_i, \mathbf{b}_{-i})) = u_i(0, (0, \mathbf{b}_{-i})) + \int_0^{v_i} \alpha_{s_i(x, \mathbf{b}_{-i})} dx. \tag{8}$$

In the proof of Theorem 5.1 we will use the following two lemmas:

Lemma 6.3 *Let $(s, \mathbf{p})$ be a standard VCG position auctions, where p_i is defined in (4) for every player i. The following holds for every i and for every bid profile $\mathbf{b}$:*

[12] See (Heydenreich *et al.*, 2007; Müller, Perea, & Wolf, 2007), where the results in (Holmstrom, 1979) are further generalized.

(i) if $s_i(\mathbf{b}) = k$ *then* $p_i(\mathbf{b}) = b_{(k+1)}$.

(ii) if $s_i(\mathbf{b}) > s_j(\mathbf{b})$ *then* $p_i(\mathbf{b}) \geq p_j(\mathbf{b})$.

(iii) $p_i(\mathbf{b}) \leq b_{(i)}$.

Proof: The lemma states obvious properties of a standard VCG position auction. Its explicit proof can be derived from Lemma 1 in (Ashlagi, Monderer, & Tennenholtz, 2007). ■

Lemma 6.4 *Every individually rational and seller rational VCG position auction is a standard VCG position auction.*

Proof: Let $(s, \hat{\mathbf{p}})$ be a VCG position auction where $\hat{p}_i$ is defined as in (5) for every player i. We have to prove that for every i, $g_i(\mathbf{b}_{-i}) = 0$ for every $\mathbf{b}_{-i}$.

Suppose in negation that there exist a player i and $\mathbf{b}_{-i} \in \mathbf{V}_{-i}$ such that $g_i(\mathbf{b}_{-i}) \neq 0$. Assume first that $g_i(\mathbf{b}_{-i}) > 0$. Let $b_i = \inf\{x_i \in V_i | s_i(x_i, \mathbf{b}_{-i}) \in K\}$. Since s is monotone in b_i (Proposition 2.1) $s_i(b_i + \epsilon, \mathbf{b}_{-i}) \geq s_i(b_i, \mathbf{b}_{-i})$ for every $\epsilon > 0$. Hence, by the first two parts in Lemma 6.3 and by the definition of b_i $p_i(b_i + \epsilon, \mathbf{b}_{-i}) \geq b_i$ for every $\epsilon > 0$. Therefore, for every $0 < \epsilon < g_i(\mathbf{b}_{-i})$ we obtain that $\hat{p}_i(b_i + \epsilon, \mathbf{b}_{-i}) = p_i(b_i + \epsilon, \mathbf{b}_{-i}) + g_i(\mathbf{b}_{-i}) \geq b_i + \epsilon$, i.e., a player with valuation $v_i = b_i + \epsilon > 0$ would pay more than his value per-click in a non-dummy position - a contradiction.
Suppose next that $g_i(\mathbf{b}_{-i}) < 0$. Then for $v_i < g_i(\mathbf{b}_{-i})$ we obtain by the third part of Lemma 6.3 that $\hat{p}_i(v_i, \mathbf{b}_{-i}) \leq v_i + g_i(\mathbf{b}_{-i}) < 0$ - a contradiction. ■

Proof of Theorem 3.2:

Let $(s, \mathbf{p})$ be a truth-revealing position auction in which s is anonymous. By Proposition 6.1 it suffices to show that s is a welfare maximizer, i.e., that for every bid profile, $\mathbf{b}$, $s(\mathbf{b})$ is consistent with $\mathbf{b}$. that not all Let $\tilde{\mathbf{V}} = \{\mathbf{b} \in \mathbf{V} : b_1 = b_2 = \cdots = b_n\}$. The proof is trivial for every $\mathbf{b} \in \tilde{\mathbf{V}}$.

We need the following notations. Let $\mathbf{b} \in \mathbf{V} \setminus \tilde{\mathbf{V}}$. For every position $q \in K$ we denote by $i(\mathbf{b}, q)$ the player assigned to position q in $\mathbf{b}$, that is $s_{i(\mathbf{b},q)}(\mathbf{b}) = q$. Let $H(\mathbf{b}) = \{i \in N | b_i = b_{(1)}\}$ be the set of highest bidders in $\mathbf{b}$ and let $h(\mathbf{b})$ be the position with the largest index of a highest bidder in the bid profile $\mathbf{b}$, i.e.,

$$h(\mathbf{b}) = \max\{q \in K \cup \{k+1\} | \exists i \in H(\mathbf{b}) \quad s_i(\mathbf{b}) = q\}.$$

Let $v(\mathbf{b})$ be the position with a lowest index of a non-highest bidder in the bid profile $\mathbf{b}$. That is $v(\mathbf{b}) = \min\{q \in K \cup \{k+1\} | b_{i(\mathbf{b},q)} < b_{(1)}\}$. Finally, let $V(\mathbf{b})$ be the set of players which have the same bid as the player in position, i.e., $V(\mathbf{b}) = \{j \in N | b_j = b_{i(\mathbf{b},v(\mathbf{b}))}\}$.

We show that for every bid profile $\mathbf{b}$, $v(\mathbf{b}) \geq h(\mathbf{b})$, i.e., a highest bidder can not get a position with a higher index than a non-highest bidder. This will complete the proof since by similar arguments we can show that for every $\mathbf{b}_{-H(\mathbf{b})}$, $v(\mathbf{b}_{-H(\mathbf{b})}) \geq h(\mathbf{b}_{-H(\mathbf{b})})$, and the proof continues recursively.

Suppose in negation that there exist a bid profile $\mathbf{b}$ such that $v(\mathbf{b}) < h(\mathbf{b})$. Note that $v(\mathbf{b}) \in K$. We distinguish between the following two cases:

1. $|V(\mathbf{b})| = 1$: Let i be the player in position $v(\mathbf{b})$, i.e. i is distinct in $\mathbf{b}$. Note that if $s_j(\mathbf{b}) < s_i(\mathbf{b})$ for $j \neq i$ then $j \in H(\mathbf{b})$.

   Let $\hat{\mathbf{b}} = (b_{(1)}, \mathbf{b}_{-i})$. That is $\hat{\mathbf{b}}$ is the bid profile obtained from $\mathbf{b}$ by increasing player i's bid to $b_{(1)}$. By Proposition 2.1 s is monotone and therefore $s_i(\hat{\mathbf{b}}) \leq s_i(\mathbf{b})$.

   Let $j \in H(\mathbf{b})$. Since s is monotone $s_j(\hat{\mathbf{b}}) \leq s_j((b_i, \hat{\mathbf{b}}_{-j}))$. Note that the bid profile $(b_i, \hat{\mathbf{b}}_{-j})$ is obtained from $\mathbf{b}$ by switching the bids of players i and player j. Since s is anonymous and i is distinct in $\mathbf{b}$ we have that $s_j((b_i, \hat{\mathbf{b}}_{-j})) = s_i(\mathbf{b})$. Recall that $s_i(\mathbf{b}) = v(\mathbf{b})$. We obtained that every player in $H(\mathbf{b}) \cup \{i\}$ must be in the the first $v(\mathbf{b})$ positions contradicting $|H(\mathbf{b}) \cup \{i\}| \geq v(\mathbf{b}) + 1$.

2. $|V(\mathbf{b})| > 1$: Let i be the player in position $v(\mathbf{b})$. Let $d_i = b_i + \frac{b_{(1)} - b_{(i)}}{2}$, and let $\mathbf{b}^1 = (d_i, \mathbf{b}_{-i})$. Since s is monotone $s_i(\mathbf{b}^1) \leq s_i(\mathbf{b})$. In addition note that player i is the only player that bids d_i in the profile $\mathbf{b}^1$. Therefore $v(\mathbf{b}^1) \leq v(\mathbf{b})$. Moreover, since the highest bidders in $\mathbf{b}$ still bid $b_{(1)}$ in $\mathbf{b}^1$ we have that $v(\mathbf{b}^1) < h(\mathbf{b}^1)$.

   If $|V(\mathbf{b}^1)| = 1$ then by case 1 we obtain a contradiction.

   Assume otherwise, i.e., $|V(\mathbf{b}^1)| > 1$. Let i_1 be the player in position $v(\mathbf{b}^1)$. Since player i is the only player to bid d_i in $\mathbf{b}^1$ we have that $i_1 \neq i$. Set $d_{i_1} = d_i + \frac{b_{(1)} - d_i}{2}$ and let $\mathbf{b}^2 = (\mathbf{b}^1_{-i_1}, d_{i_1})$. By the same arguments above we have that $v(\mathbf{b}^2) < h(\mathbf{b}^2)$. Hence, if $|V(\mathbf{b}^2)| = 1$ then a contradiction is obtained by case 1. Otherwise continue this process until obtaining a bid profile $\mathbf{b}^l$ for some $l \geq 3$ such that $|V(\mathbf{b}^r)| = 1$. This process will end after a finite number of steps; indeed, if $|V(\mathbf{b}^r)| > 1$ for some $r > 1$ in the process, then the player in position $v(\mathbf{b}^r)$ in the bid profile $\mathbf{b}^r$ will have a distinct bid for the rest of the entire process.

■

Proof of Theorem 4.2:

Let $(s, \mathbf{p})$ be a truth-revealing and utility symmetric position auction. In order to prove that $(s, \mathbf{p})$ is a VCG position auction it suffices, by Proposition 6.1, to show that s is a welfare maximizer. Let $H(\mathbf{b})$,$h(\mathbf{b})$ and $v(\mathbf{b})$ be defined as in the proof of Theorem 3.2.

As in the proof of Theorem 3.2 it is enough to show that for every bid profile $\mathbf{b}$, $v(\mathbf{b}) \geq h(\mathbf{b})$, i.e., a highest bidder can not get a position with a higher index than a non-highest bidder.

Suppose in negation that there exist a bid profile $\mathbf{b}$ such that $v(\mathbf{b}) < h(\mathbf{b})$. Let i be the player in position $v(\mathbf{b})$ in the bid profile $\mathbf{b}$. That is $s_i(\mathbf{b}) = v(\mathbf{b})$. Note that if $s_j(\mathbf{b}) < s_i(\mathbf{b})$ then $j \in H(\mathbf{b})$. Moreover by the negation assumption $|H(\mathbf{b})| \geq v(\mathbf{b})$.

Let $d = b_{(1)} - b_i$. Let $\hat{\mathbf{b}} = (b_{(1)}, \mathbf{b}_{-i})$. By (8) and since s is monotone (Proposition 2.1)

$$u_i(\hat{b}_i, \hat{\mathbf{b}}) \geq u_i(b_i, \mathbf{b}) + d\alpha_{v(\mathbf{b})}. \tag{9}$$

Again, by the monotonicity of s $s_i(\hat{\mathbf{b}}) \leq v(\mathbf{b})$. Since $H(\hat{\mathbf{b}}) = H(\mathbf{b}) \cup \{i\}$ we have that $|H(\hat{\mathbf{b}})| > v(\mathbf{b})$ which implies that there exists a player $j \in H \setminus \{i\}$ for which $s_j(\hat{\mathbf{b}}) > v(\mathbf{b})$. Since $\hat{b}_i = \hat{b}_j$ and by utility symmetry

$$u_i(\hat{b}_i, \hat{\mathbf{b}}) = u_j(\hat{b}_j, \hat{\mathbf{b}}). \tag{10}$$

By (8) and since s is monotone

$$u_j(b_i, (b_i, \hat{\mathbf{b}}_{-j})) \geq u_j(\hat{b}_j, \hat{\mathbf{b}}) - d\alpha_{s_j(\hat{\mathbf{b}})}. \tag{11}$$

Since $\alpha_{v(\mathbf{b})} > \alpha_{s_j(\hat{\mathbf{b}})}$ we have that

$$u_j(b_i, (b_i, \hat{\mathbf{b}}_{-j})) > u_j(\hat{b}_j, \hat{\mathbf{b}}) - d\alpha_{v(\mathbf{b})}. \tag{12}$$

By (9), (10) and (12) we obtain that $u_j(b_i, (b_i, \hat{\mathbf{b}}_{-j})) > u_i(b_i, \mathbf{b})$ which contradicts utility symmetry. ■

Proof of Theorem 5.1:

By Theorems 3.2 and 4.2, $(s, \mathbf{p})$ is a VCG position auction, and by Lemma 6.4 it is necessarily a standard VCG position auction. ■.

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