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Monotonic Assignment Rules and Common Pricing

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In this paper we study the production and pricing of a good by a single supplier (such as a monopolist or government) under some given optimality criterion—for example, profit maximization or social benefit maximization. In general, this may require discriminatory pricing. The primary focus here is on the pricing policy and whether it is possible to achieve the same objective value with common pricing—where each individual acquiring the good pays the same price. We consider the case of declining (marginal) cost and show that for a large class of problems, optimality is achievable with common pricing. Because the environment is one of incomplete information, incentive and participation constraints are important restrictions on the problem. We frame the discussion in terms of interim expected utility. When ex post restrictions are considered, the problem is altered substantially, and the value of the objective may be lower under common pricing.

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1. Introduction. We consider optimal mechanism design for a class of allocation problems where a monopolistic supplier, private or public, produces and allocates units of a good among potential consumers with different valuations. We assume that the supplier has some objective such as profitability or efficiency. For example, a private monopolist may produce and sell goods to maximize profit, or a government agency may provide the good with the intent of maximizing social benefits net of costs. The framework for the discussion is an incomplete information environment where informational constraints, in particular the incentive and individual rationality constraints, restrict the optimization problem. Within this framework we consider the impact, if any, that a restriction of common pricing, where all consumers pay the same price (which may vary depending on the realization of consumers' types), has on the optimization problem and the value of the objective.

Our model is different from the standard auction model in one important aspect. In the auction environment the goods available for sale are already produced and the costs are sunk. Here, the production decision is an integral part of the optimal decision problem—the supplier must decide how many units of good to produce, to whom the goods are sold, and at what price they are sold. In fact, the standard auction model can be viewed as a particular case of our general model in which the marginal costs of the first several units of the good are zero and then infinite afterward. The optimal solution of the general model with zero fixed cost and increasing marginal cost of production turns out to be an easy extension of the classical Myerson's auction result (Bergin and Zhou [1], Segal [5]). Using a direct mechanism, the supplier should produce up to the level where the marginal (production) cost, MC, equals the marginal (virtual) revenue, MR, and charge all buyers the same price of MC.

In this paper we study the case of decreasing marginal costs. This is an important case to investigate for at least two reasons. First, without artificial entry barriers, monopoly is consistent with decreasing marginal costs only. If the production technology exhibits increasing marginal costs, competition eventually will lead to the proliferation of firms. Hence, if an industry is dominated by a monopolist, it makes more sense to assume that its marginal costs are decreasing rather than increasing. Second, with decreasing marginal costs one can no longer use the simple local condition, MC = MR, to determine the output level.

With increasing marginal cost, the canonical optimal solution (Myerson [4]) has many nice properties—for example, the direct mechanism has an equilibrium in dominant strategies with common pricing. In contrast, with decreasing marginal costs, that optimal solution generally involves discriminatory pricing. To see this, consider a situation where the marginal cost of the first unit is large and zero thereafter and suppose that there are just two individuals, one with a high valuation and one with a low valuation. Whenever the sum of these (virtual) valuations exceeds the production cost, the canonical optimal mechanism requires that the good be produced and supplied to both, with the high-valuation individual subsidizing the low-valuation individual. However, the canonical pricing scheme is only one particular optimal mechanism—it does not follow that every optimal mechanism requires price discrimination. The primary purpose of this paper is to show that it is always possible to construct an optimal mechanism with common pricing, even when marginal costs are decreasing. (The particular common-pricing mechanism constructed in the proof of our main result is a direct mechanism and, for the usual reasons, it may not be satisfactory as a descriptive model. The sole purpose here is to demonstrate that common pricing is compatible with optimality.)

In this class of mechanism design problem involving production and sale of goods, a solution typically has two components: allocation and pricing. Who should be supplied? How much should each recipient pay? The allocation decision directly determines the level of production—enough is produced to satisfy all of the recipients. Moreover, it turns out that the allocation procedure also implicitly determines the expected price any recipient must pay, up to a constant, indirectly through the incentive and participation constraints. Consequently, the assignment-allocation rule alone can be determined first and the appropriate prices determined afterward so that they satisfy other desirable properties—as long as they respect the incentive and participation constraints. This crucial fact is noted in §2 where the assignment-expected pricing connection is explained. Therefore, the scope for achieving optimality while having a common price across all individuals rests on the possibility of finding a common price function with the property that the interim expected payoffs coincide with those associated with the individual specific canonical prices. This in turn depends directly on the shapes of the allocation regions (and, for example, profit maximizing monopoly with decreasing marginal cost produces assignment regions that admit common pricing). From a technical perspective the paper addresses an interesting mathematical question concerning whether one can construct a function f on a region C in R^n with given marginals on all axes when C is not a product set. Here, we provide a general result of this type for regions that are monotonic in R^n. (Strassen [6] discusses a somewhat similar problem on the existence of measures with given marginals.)

In §4 we state the main theorem that gives a set of sufficient conditions for an assignment rule to admit a common pricing procedure. We call an assignment region regular if it satisfies this set of conditions. Thus, for any optimization problem, if the optimal assignment regions are regular, then we can find an optimal pricing procedure such that at any valuation profile, all individuals who receive the good pay the same price. In §5 we consider two applications. The first is a public good allocation problem where optimal decisions are based on cost-benefit calculations. We show that the optimal cost-benefit allocation can be achieved while charging a common price to all recipients. The second application is a monopolist profit maximization problem where it is shown that the restriction to common pricing does not lower the monopolist's expected profit. In §6 we consider the implications of the additional condition of an ex post participation or individual rationality constraint, in conjunction with the requirement of common pricing. This leads to an interesting open question that is described in §7. Section 8 concludes the paper.

2. The environment. The environment consists of a single producer supplying multiple users of an indivisible good. The marginal cost of producing the kth unit is $c(k)$ with $c(0) = 0$ and $c(k) \geq 0$ for all k, and the total cost of producing l units is $C(l) = \sum_{k=0}^{l} c(k)$. So, for example, if there is a fixed cost, $\bar{c}$, and zero marginal cost, then $c(1) = \bar{c}$ and $c(j) = 0$, $j \geq 2$. In this paper, we focus primarily on the case of decreasing marginal cost where $c(k) \geq c(k+1)$ for all $k \geq 1$.

On the demand side there are n potential consumers, $i \in I = \{1, \ldots, n\}$. Each consumer i demands at most one unit of the good and has a utility function given by:

$$u_i(y_i, p_i, m_i) = (t_i - p_i)y_i + m_i,$$

in which $y_i = 0$ or 1, depending on whether the good is consumed or not, p_i is the payment for the good, and $m_i \in R$ is the consumer's wealth. The parameter t_i is consumer i's private valuation of the good. It is assumed that t_i is private information, and the t_is are independently drawn from a common distribution with density $f(\cdot)$ that is continuous and positive on $[0, 1]$, with cumulative distribution function $F(\cdot)$. (We use standard notation: $t = (t_1, \ldots, t_n)$, $t = (t_i, t_{-i})$ where $t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$, $T = \times_{i=1}^n T_i$, $T_{-i} = \times_{j \neq i} T_j$, $f(t) = \times_{i=1}^n f(t_i)$, $f_{-i}(t_{-i}) = \times_{j \neq i} f(t_j)$, $dt = dt_1 \cdots dt_n$, and $dt_{-i} = dt_1 \cdots dt_{i-1} dt_{i+1} \cdots dt_n$. Also, given t, $t_{(i)}$ is the ith order statistic: $t_{(i)} \geq t_{(i+1)}$, $i = 1, \ldots, n-1$.)

If the good is supplied to a group K of consumers with consumer $k \in K$ paying p_k, then total revenue is $\sum_{k \in K} p_k$ and the associated cost is $C(\#K)$, where $\#K$ is the number of individuals in group K, so the net revenue is $\sum_{k \in K} p_k - C(\#K)$, or with the requirement of a common price, p: $(\#K) \cdot p - C(\#K)$.

3. Preliminaries. Incentive and participation constraints connect assignment rules (the y_is) to prices (the p_is). This connection is reviewed in §3.1. In fact, the assignment rule determines a canonical pricing rule as the solution to the assignment problem (the choices of y_is); the construction of the rule is developed in §3.2 and illustrated with an example in §3.2.1. While the canonical prices for a given assignment rule are uniquely determined, there is a family of pricing rules consistent with the same assignment rule. We utilize this fact in §4 to construct common price rules—for any i, j, $(y_i y_j) \cdot p_i = (y_i y_j) \cdot p_j$.

3.1. Mechanisms: Incentive compatibility and participation. Since the concern is with identifying feasible outcomes subject to incentive compatibility and participation constraints (rather than with providing descriptive schemes), we focus on reduced form mechanisms that are characterized in terms of assignment rules and payments, $\psi = \{y_i, p_i\}_{i=1}^n$. If, at profile $t = (t_1, \ldots, t_n)$, individual i obtains an item with probability $y_i(t)$, paying $p_i(t)$, then the net expected benefit to i is $[t_i y_i(t) - p_i(t) y_i(t)] = [t_i - p_i(t)] y_i(t)$. We take $y_i(t) \in \{0, 1\}$ for each t: at any profile, i either does or does not get an item. Let $x_i(t) = p_i(t) y_i(t)$ so that $\bar{x}_i(t_i)$, the expected payment of i type t_i, is $\bar{x}_i(t_i) = \int_{T_{-i}} x_i(t) f_{-i}(t_{-i})\, dt_{-i} = \int_{T_{-i}} p_i(t) y_i(t) f_{-i}(t_{-i})\, dt_{-i} = E\{p_i(t) y_i(t) \mid t_i\}$.

A mechanism satisfies (voluntary) *participation* if each individual's expected benefit is nonnegative:

$$E\{[t_i - p_i(t)] y_i(t) \mid t_i\} \geq 0, \quad \forall\, i, t_i \qquad \text{or} \qquad t_i \bar{y}_i(t_i) - \bar{x}_i(t_i) \geq 0, \quad \forall\, i, t_i.$$

A mechanism $\psi = \{y_i, p_i\}_{i=1}^n$ is *incentive-compatible* if for all i and all t_i:

$$t_i E\{y_i(t_i, t_{-i}) \mid t_i\} - E\{p_i(t_i, t_{-i}) y_i(t_i, t_{-i}) \mid t_i\} \geq t_i E\{y_i(\tilde{t}_i, t_{-i}) \mid t_i\} - E\{p_i(\tilde{t}_i, t_{-i}) y_i(\tilde{t}_i, t_{-i}) \mid t_i\}, \quad \forall\, \tilde{t}_i.$$

This may also be written in a more compact form:

$$\forall\, i, \ \ \forall\, t_i, \quad t_i \bar{y}_i(t_i) - \bar{x}_i(t_i) \geq t_i \bar{y}_i(\tilde{t}_i) - \bar{x}_i(\tilde{t}_i), \quad \forall\, \tilde{t}_i.$$

For any mechanism ψ, incentive compatibility implies that: (1) for each consumer, the conditional probability of obtaining an object is increasing in valuation, and (2) this conditional probability determines the conditional expected payment up to a constant (Myerson [4]). These properties are summarized in the following proposition.

Proposition 3.1. *A mechanism is incentive compatible if and only if the following two conditions hold*:
(1.1) $\forall\, i$, $\bar{y}_i(t_i)$ *is nondecreasing in* t_i, *and*
(1.2) $\bar{x}_i(t_i) = \bar{x}_i(0) + t_i \bar{y}_i(t_i) - \int_0^{t_i} \bar{y}_i(s_i)\, ds_i$.

Participation requires that $\bar{x}_i(0) \leq 0$, provided that low valuation buyers are not subsidized, $\bar{x}_i(0) = 0$, so that (1.2) becomes:

$$\bar{x}_i(t_i) = t_i \bar{y}_i(t_i) - \int_0^{t_i} \bar{y}_i(s_i)\, ds_i.$$

Figure 1 depicts the expected revenue from individual i with $\bar{x}_i(0) = 0$.

In what follows, we assume $y_i(t_i, t_{-i})$ is monotonic in t_i, noting that incentive compatibility implies only that $\bar{y}_i(t_i)$ is monotonic in t_i.

3.2. Canonical prices. If for each i, $\bar{x}_i(t_i) = E\{p_i(t) y_i(t) \mid t_i\}, \forall\, t_i$, then $\psi = \{y_i, p_i\}$ is an incentive-compatible reduced form mechanism. In general there are many different price functions, $\{p_i\}$, satisfying this condition, but one can be derived directly—the canonical price function. Define $\varphi_i(t_{-i}) = \inf\{t_i \mid y_i(t_i, t_{-i}) = 1\}$ and expand the expression for expected payment:

$$\begin{aligned} t_i \bar{y}_i(t_i) - \int_0^{t_i} \bar{y}_i(s)\, ds &= \int_{t_{-i}} \left\{ t_i y_i(t_i, t_{-i}) - \int_0^{t_i} y_i(s, t_{-i})\, ds \right\} f(t_{-i})\, dt_{-i} \\ &= \int_{t_{-i}} \left\{ t_i \cdot \chi_{\{t_i \geq \varphi_i(t_{-i})\}} - \int_{\min\{t_i, \varphi_i(t_{-i})\}}^{t_i} ds \right\} f(t_{-i})\, dt_{-i} \end{aligned}$$

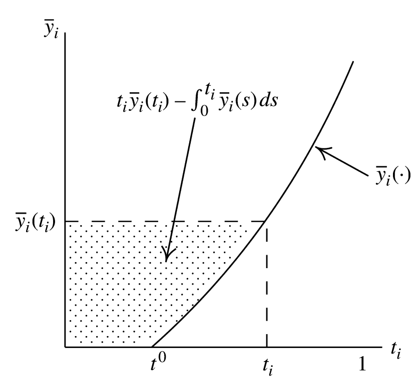

Figure 1. Win probability and expected payment.

$$= \int_{t_{-i}} \chi_{\{t_i \geq \varphi_i(t_{-i})\}} \cdot \left\{ t_i - \int_{\varphi_i(t_{-i})}^{t_i} ds \right\} f(t_{-i})\, dt_{-i}$$
$$= \int_{t_{-i}} \chi_{\{t_i \geq \varphi_i(t_{-i})\}} \cdot \varphi_i(t_{-i}) f(t_{-i})\, dt_{-i}$$
$$= \int_{t_{-i}} y_i(t_i, t_{-i}) \varphi_i(t_{-i}) f(t_{-i})\, dt_{-i}$$

where $\chi_{\{t_i \geq \varphi_i(t_{-i})\}}$ is the indicator function of the event $\{t_i \geq \varphi_i(t_{-i})\}$.

Let $p_i(t_i, t_{-i}) = \varphi_i(t_{-i}) y_i(t_i, t_{-i})$; i.e.,

$$p_i(t_i, t_{-i}) = \begin{cases} \inf\{\tilde{t}_i \mid y_i(\tilde{t}_i, t_{-i}) = 1\} & \text{if } y_i(t_i, t_{-i}) = 1, \\ 0 & \text{if } y_i(t_i, t_{-i}) = 0. \end{cases}$$

We call this price, p_i, the canonical price with respect to y_i. Given y_i, define $S_i = \{t \mid y_i(t) = 1\}$, the set of valuation profiles at which individual i obtains a good. Thus, $\varphi_i(t) = \inf\{t_i \mid (t_i, t_{-i}) \in S_i\}$. In view of Proposition 3.1, payments up to a constant are determined by the assignment rule, y_i, or the assignment region, S_i. Figure 2 illustrates the case of two buyers and decreasing marginal cost. See the example below for further discussion. (For future reference, $t_i^* = \min\{t_i \mid \exists t_{-i},\, y_i(t_i, t_{-i}) = 1\}$.)

The canonical pricing scheme $\{p_i\}_{1 \leq i \leq n}$ satisfies the strong incentive property of strategy-proofness. It is, in fact, the unique strategy-proof mechanism that is consistent with the assignment rule $\{y_i\}_{1 \leq i \leq n}$. Since it generally requires that buyers of different types pay different prices for the same good, there is no mechanism that is both strategy-proof and nondiscriminatory. At best, a common pricing scheme can only be Bayesian incentive compatible.

3.2.1. Canonical prices: An example. To illustrate the construction of the canonical price function, consider a monopolistic profit-maximizing seller. The optimization problem is to maximize the following objective:

$$\int_T \left[\sum_{i=1}^n p_i(t_i, t_{-i}) y_i(t_i, t_{-i}) - C\left(\sum_{i=1}^n y_i(t_i, t_{-i}) \right) \right] f(t)\, dt,$$

subject to participation and incentive-compatibility constraints. If $\{y_i, p_i\}_{i=1}^n$ satisfies these constraints, then with some calculations it may be shown (Myerson [4]) that:

$$\int_T p_i(t_i, t_{-i}) y_i(t_i, t_{-i}) f(t)\, dt = \int_T J(t_i) y_i(t_i, t_{-i}) f(t)\, dt,$$

where $J(s) = s - [(1 - F(s))/f(s)]$. (Note that $J(0) < 0$ and $J(1) = 1$.) We assume the standard regularity condition that J is increasing. Therefore, an optimal selling mechanism is a solution to:

$$\max_{\{y_i\}_{i=1}^n} \int_T \left[\sum_{i=1}^n J(t_i) y_i(t_i, t_{-i}) - C\left(\sum_{i=1}^n y_i(t_i, t_{-i}) \right) \right] f(t)\, dt,$$

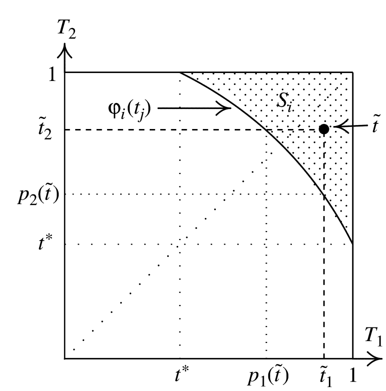

FIGURE 2. Two potential consumers and decreasing marginal cost: $c(1) > 1$, $c(2) = 0$.

subject to the incentive-compatibility and participation constraints. Define functions $\{y_i(t)\}_{t=1}^n$ such that for each realization of t, $\{y_i(t)\}_{t=1}^n$ maximizes:

$$V(t;y) = \sum_{i=1}^n J(t_i)y_i - C\left(\sum_{i=1}^n y_i\right).$$

Provided incentive compatibility is satisfied, this gives the optimal solution to the problem, and the optimal production policy solves $\max_k \sum_{j=1}^k J(t_{(j)}) - C(k)$. The simplest case to consider is where there is a cost $c(1) > 0$ and additional units are produced at no cost: $c(j) = 0$, $j \geq 2$. Then, individual i obtains a good if $t_i \geq t^0$ (where t^0 is defined by $J(t^0) = 0$), and $\sum_{j=1}^n \max\{J(t_j), 0\} \geq c(1)$.

For a simple numerical example, take $n = 2$ with $c(1) = 1$, $c(j) = 0$ for $j \geq 2$, and let the valuations of the buyers be uniformly distributed on $[0, 1]$: $f(s) = 1$, and $J(s) = s - (1-s)/1 = 2s - 1$. Therefore,

$$V(t;y) = y_1(2t_1 - 1) + y_2(2t_2 - 1) - \max\{y_1, y_2\}.$$

Since $2s - 1 \leq 1 < \frac{3}{2}$ for all $s \in [0, 1]$, it is never optimal to supply the good to only one buyer. Hence, either both buyers are supplied, or neither is. Both individuals are supplied on the region in $T_1 \times T_2$ where $J(t_1) + J(t_2) > 1$, or $t_1 + t_2 > \frac{3}{2}$ and where $\varphi_i(t_j) = \frac{3}{2} - t_j$.

The associated prices for 1 and 2 are:

$$p_1(t_1, t_2) = \begin{cases} \frac{3}{2} - t_2, & t_1 + t_2 > \frac{3}{2}, \\ 0, & t_1 + t_2 \leq \frac{3}{2}, \end{cases} \quad \text{and} \quad p_2(t_1, t_2) = \begin{cases} \frac{3}{2} - t_1, & t_1 + t_2 > \frac{3}{2}, \\ 0, & t_1 + t_2 \leq \frac{3}{2}. \end{cases}$$

Thus, on the region where they are supplied, the individuals pay the same price only if $t_1 = t_2$.

3.2.2. Canonical prices and common pricing. As the example shows, with decreasing marginal costs, the canonical price scheme is discriminatory in the sense that different individuals pay different prices in general (at $\tilde{t}$, $p_2(\tilde{t}) < p_1(\tilde{t})$). This is in sharp contrast to the case of increasing marginal cost when the canonical scheme requires that all recipients of a good pay the same price (Bergin and Zhou [1], Segal [5]). The logic for discriminatory pricing with decreasing marginal costs may seem natural. Once a unit of the good is produced and sold to a high-value consumer, the next unit of the good can be produced more cheaply and sold at a reduced price to another consumer with a lower valuation. Informational constraints aside, optimality would require that the good be produced and supplied to both, with the high-valuation individual subsidizing the low-valuation individual. However, this reasoning ignores the incentive compatibility and participation considerations. These incentive and participation constraints for the assignment allocation already constrain the optimization problem so that the addition of common pricing may not add any extra restrictions. While discriminatory pricing typically appears in the canonical pricing scheme for an assignment rule, it turns out that the same assignment rule can also be supported by a common pricing scheme.

4. The main theorem. For a common price mechanism to mimic a discriminatory pricing mechanism in the sense that they support the same assignment regions, one must replace the canonical prices $\{p_i\}$ with a common pricing function p. Of course, whether this can be achieved depends on the structure of the assignment functions, $\{y_i\}_{i=1}^n$. Because the assignment function, y_i, determines the assignment region $S_i = \{t \mid y_i(t) = 1\}$, we can examine the scope for common pricing in terms of the sets S_i. Definition 4.1 defines a set of conditions under which an assignment region is called regular, and we show in Theorem 4.1 that any mechanism having regular assignment regions admits a common pricing procedure.

Definition 4.1. A mechanism is called regular if for each i, S_i satisfies the following conditions.

(i) Symmetry: If $t \in S_i$ and σ is a permutation of $\{1, 2, \ldots, n\}$, then $\sigma(t) \in S_{\sigma(i)}$.
(ii) Monotonicity: If $t \in S_i$ and $t' \geq t$, then $t' \in S_i$.
(iii) Weak efficiency: If $t \in S_i$ and $t_j > t_i$, then $t \in S_j$.
(iv) Closure: S_i is closed.
(v) Substitutability: If $t \in S_i$, then for any pair j and k with $t_i \leq t_k < t_j$, and any $\varepsilon > 0$, there exists some $\tilde{t} \in S_i$ in which $t_k < \tilde{t}_k < t_k + \varepsilon$, $\tilde{t}_j < t_j$, and $\tilde{t}_l = t_l$ for all $l \neq j, k$.

We discuss these conditions in turn. Symmetry requires that the assignment rule is the same for ex ante or observably identical individuals and is a natural requirement when considering common pricing. Monotonicity asserts that increases of individuals' valuations do not lead to the exclusion of some individual previously supplied. (This condition is not necessary, as the increasing cost example in §3.2.1 illustrates. However, in that example and (more generally with n potential buyers) on the region where a fixed set of individuals receive the good, higher values for that subset leave all in the set of individuals assigned a good.) With declining marginal costs the condition is natural because there is no crowding-out effect. Weak efficiency requires that individuals with higher valuations are supplied before those with lower valuations. The condition of closure is just a technical condition. Finally, the condition of substitutability requires that individual valuations are "substitutable" in the sense that if at a valuation profile i is supplied with the good, then taking two individuals with distinct valuations at least as high as that of i, the higher valuation of the two may be lowered and the lower valuation raised to yield a profile where i is still supplied.

THEOREM 4.1. *If $\{(y_i)_{i=1}^n, (p_i)_{i=1}^n\}$ is a regular incentive-compatible discriminatory mechanism, then there is a common price mechanism $\{(\hat{y}_i)_{i=1}^n, p\}$ such that $\hat{y}_i = y_i$, $\forall\, i$ and $E\{py_i \mid t_i\} \equiv E\{p_i y_i \mid t_i\}$, $\forall\, i$, $\forall\, t_i$.*

PROOF. Let $\{(y_i)_{i=1}^n, (p_i)_{i=1}^n\}$ be a regular incentive discriminatory mechanism. Recall that S_i is the region in which person i will be provided with the good and y_i is the function that defines S_i: $S_i = \{t \mid y_i(t) = 1\}$. By symmetry, y_i has the same functional form for each i. Let t_i^* be defined as $t_i^* = \min\{t_i \mid \exists t_{-i},\ y_i(t_i, t_{-i}) = 1\}$, so that $(t_i, t_{-i}) \in S_i$ implies $t_i \geq t_i^*$. (By symmetry, t_i^* is the same for each i.) We must find a function $p(\cdot)$ defined on $S = \bigcup_{i=1}^n S_i$ such that for all i and all $t_i \geq t_i^*$,

$$\begin{aligned}\bar{p}_i(t_i) &= \int_{T_{-i}} p_i(t_i, t_{-i})\chi_{S_i}(t_i, t_{-i}) f_{-i}(t_{-i})\, dt_{-i} \\ &= \int_{T_{-i}} p(t_i, t_{-i})\chi_{S_i}(t_i, t_{-i}) f_{-i}(t_{-i})\, dt_{-i} = E(p\chi_{S_i} \mid t_i).\end{aligned} \tag{D}$$

The proof is given in two separate cases.

Case 1. The first case is when there is some α^* with $\alpha^* < 1$ such that

$$\Big(\alpha^*, \alpha^*, \ldots, \underbrace{t_i^*}_{i\text{th}}, \ldots, \alpha^*\Big) \in S_i.$$

By monotonicity, we can assume that $t_i^* < \alpha^*$. Denote

$$Q_i = [\alpha^*, 1] \times \cdots \times [\alpha^*, 1] \times \underbrace{[t_i^*, 1]}_{i\text{th}} \times [\alpha^*, 1] \times \cdots \times [\alpha^*, 1].$$

The situation is illustrated in Figure 3 for two individuals. From the figure, the good may be supplied to just one individual with a sufficiently high valuation. Recall that t^* is defined by $t^* = \min\{t_i \mid \exists t_{-i},\ y_i(t_i, t_{-i}) = 1\}$. When an individual's valuation is above $\tilde{t}$, that individual is supplied regardless of the others' valuations. In the region $[t^*, \tilde{t}]$, a person is supplied only if the others' valuation is sufficiently high. So $S_1 = B \cup C$ and $S_2 = A \cup C$. By weak efficiency, $Q_i \subseteq \hat{S} = \bigcap_{i=1}^n S_i$ for all i. This is a crucial point for the construction of a uniform price function on S.

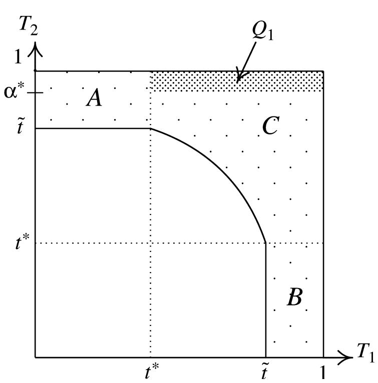

FIGURE 3. Assignment region: Case 1.

Let us begin with a constant on S: $c = \int_{t^*}^1 \bar{p}_i(t_i)f(t_i)\,dt_i / \int \chi_{S_i}(t)f(t)\,dt$. Symmetry implies the expression on the right is independent of i (for any set A, χ_A stands for the indicator function of A), so

$$\begin{aligned}\int_{t^*}^1 [\bar{p}_i(t_i) - cE(\chi_{S_i} \mid t_i)]f(t_i)\,dt_i &= \int_{t^*}^1 \bar{p}_i(t_i)f(t_i)\,dt_i - c\int_{t^*}^1\left(\int_{t_{-i}} \chi_{S_i} f(t_{-i})\,dt_{-i}\right)dt_i \\ &= \int_{t^*}^1 \bar{p}_i(t_i)f(t_i)\,dt_i - c\int_T \chi_{S_i}(t)f(t)\,dt \\ &= \int_{t^*}^1 \bar{p}_i(t_i)f(t_i)\,dt_i - \left[\frac{\int_{t^*}^1 \bar{p}_i(t_i)f(t_i)\,dt_i}{\int \chi_{S_i}(t)f(t)\,dt}\right]\int_T \chi_{S_i}(t)f(t)\,dt = 0.\end{aligned}$$

Define

$$\eta_i(t) = \frac{1}{[1-F(\alpha^*)]^{n-1}}\chi_{Q_i}(t)[\bar{p}_i(t_i) - cE(\chi_{S_i} \mid t_i)].$$

Since $Q_i \subseteq S_j$, $\forall\, j$,

$$\eta_i \cdot \chi_{S_i} = \frac{1}{[1-F(\alpha^*)]^{n-1}}\chi_{Q_i} \cdot [\bar{p}_i(t_i) - cE(\chi_{S_i} \mid t_i)].$$

Thus,

$$\begin{aligned}E(\eta_i \cdot \chi_{S_i} \mid t_i) &= \int_{T_{-i}} \eta_i \cdot \chi_{S_i} f(t_{-i})\,dt_{-i} \\ &= \frac{1}{[1-F(\alpha^*)]^{n-1}}\int_{T_{-i}} \chi_{Q_i} \cdot [\bar{p}_i(t_i) - cE(\chi_{S_i} \mid t_i)]f(t_{-i})\,dt_{-i} \\ &= \frac{1}{[1-F(\alpha^*)]^{n-1}} \cdot \chi_{[t^*,\,1]} \cdot [\bar{p}_i(t_i) - cE(\chi_{S_i} \mid t_i)]\int_{[\alpha^*,\,1]^{n-1}} f(t_{-i})\,dt_{-i} \\ &= \chi_{[t^*,\,1]} \cdot [\bar{p}_i(t_i) - cE(\chi_{S_i} \mid t_i)],\end{aligned}$$

and again, since $Q_i \subseteq S_j$, $\forall\, j$,

$$\begin{aligned}E(\eta_i \cdot \chi_{S_j} \mid t_j) &= \int_{T_{-j}} \eta_i \cdot \chi_{S_j} \cdot f(t_{-j})\,dt_{-j} \\ &= \frac{1}{[1-F(\alpha^*)]^{n-1}}\int_{T_{-j}} \chi_{Q_i} \cdot [\bar{p}_i(t_i) - cE(\chi_{S_i} \mid t_i)]f(t_{-j})\,dt_{-j} \\ &= \frac{1}{[1-F(\alpha^*)]^{n-1}} \underbrace{\int_{\alpha^*}^1 \cdots \int_{\alpha^*}^1}_{n-2 \text{ times}} \left(\int_{t^*}^1 [\bar{p}_i(t_i) - cE(\chi_{S_i} \mid t_i)]f(t_i)\,dt_i\right) f(t_{-ij})\,dt_{-ij} = 0,\end{aligned}$$

where the last equality follows from

$$\int_{t^*}^1 [\bar{p}_i(t_i) - cE(\chi_{S_i} \mid t_i)]f(t_i)\,dt_i = 0.$$

Consequently,

$$E(\eta_i\chi_{S_i} \mid t_i) = [\bar{p}_i(t_i) - cE(\chi_{S_i} \mid t_i)] \qquad \text{and} \qquad E(\eta_i\chi_{S_j} \mid t_j) = 0 \quad \text{for all } j \neq i.$$

Hence, we can take the following function as the uniform price function:

$$p(t) = c\chi_S(t) + \sum_{j=1}^n \eta_j(t).$$

For this price function, (D) is satisfied since

$$\begin{aligned}E(p\chi_{S_i} \mid t_i) &= \int_{T_{-i}} \left[c\chi_S(t_i, t_{-i}) + \sum_{j=1}^n \eta_j(t_i, t_{-i})\right]\chi_{S_i}(t_i, t_{-i})f_{-i}(t_{-i})\,dt_{-i} \\ &= cE(p\chi_{S_i} \mid t_i) + \sum_{j=1}^n E(\eta_j\chi_{S_i} \mid t_i) = cE(p\chi_{S_i} \mid t_i) + [\bar{p}_i(t_i) - cE(\chi_{S_i} \mid t_i)] = \bar{p}_i(t_i).\end{aligned}$$

Case 2. The more delicate case is when there is no α^* with $\alpha^* < 1$, such that

$$\Big(\alpha^*, \alpha^*, \ldots, \underbrace{t^*}_{i\text{th}}, \ldots, \alpha^*\Big) \in S_i.$$

In this case, substitutability implies that

$$\Big(1, 1, \ldots, \underbrace{t^*}_{i\text{th}}, \ldots, 1\Big)$$

is the unique point in S_i when $t_i = t^*$.

The substitutability property implies that for any $\varepsilon > 0$, there is some $t^*(\varepsilon) < t_i^* + \varepsilon$ and some $\alpha^*(\varepsilon) < 1$ such that

$$\Big(\alpha^*(\varepsilon), \alpha^*(\varepsilon), \ldots, \underbrace{t^*(\varepsilon)}_{i\text{th}}, \ldots, \alpha^*(\varepsilon)\Big) \in S_i.$$

Pick $\alpha^*(\varepsilon) = \min_\alpha\{(\alpha, \ldots, t^*(\varepsilon), \ldots, \alpha) \mid (\alpha, \ldots, t^*(\varepsilon), \ldots, \alpha) \in S_i\}$. Since S_i is closed, $\alpha^*(\varepsilon) \to 1$ when $\varepsilon \to 0$. Therefore, for some fixed $\varepsilon^* > 0$,

$$\Big(\alpha^*(\varepsilon^*), \alpha^*(\varepsilon^*), \ldots, \underbrace{t^*(\varepsilon^*)}_{i\text{th}}, \ldots, \alpha^*(\varepsilon^*)\Big) \in S_i$$

implies $\alpha^*(\varepsilon^*) > t^*(\varepsilon^*)$.

This construction is illustrated in Figure 4 where the set, S_1 (or S_2), is drawn when there are two individuals. Recall that t^* is defined by $t^* = \min\{t_i \mid \exists t_{-i},\ y_i(t_i, t_{-i}) = 1\}$.

Let

$$U_i = S_i \cap \{t \mid t_i \le t^*(\varepsilon^*)\}$$

so, $t \in U_i$ implies $t_i \le t^*(\varepsilon^*)$ and $t_j \ge \alpha^*(\varepsilon^*)$ and, hence, all U_i are disjoint.

We now can define the uniform price function p. First, let $p(t) = p_i(t)$ on each region U_i. Hence, (D) is satisfied for all i and all $t_i < t^*(\varepsilon)$. Second, we need to define the price function on $\tilde{S} = S\backslash\bigcup_{j=1}^n U_j$. Let $\tilde{S}_i = S_i\backslash\bigcup_{j=1}^n U_j$. For $t \in \tilde{S}$, define:

$$p_i'(t) = \left[p_i(t) + \sum_{j\neq i} \frac{E((p_i - p_j)\chi_{S_i\cap U_j} \mid t_i)}{E(\chi_{\tilde{S}_i} \mid t_i)}\right]\chi_{\tilde{S}_i}(t).$$

Repeat what we did in the first case for $\tilde{S}$ and construct a common price $p(t)$ on $\tilde{S}$ so that for all $t_i > t^*(\varepsilon)$,

$$\int_{T_{-i}} \chi_{\tilde{S}_i} \cdot p'(t_i, t_{-i}) f(t_{-i})\, dt_{-i} = \int_{T_{-i}} \chi_{\tilde{S}_i} \cdot p_i'(t_i, t_{-i}) f(t_{-i})\, dt_{-i}.$$

Hence, the overall price function on $\bigcup_{j=1}^n U_j$ is

$$p(t) = p'(t)\cdot\chi_{\tilde{S}}(t) + \sum_{j=1}^n p_j(t)\cdot\chi_{U_j}(t).$$

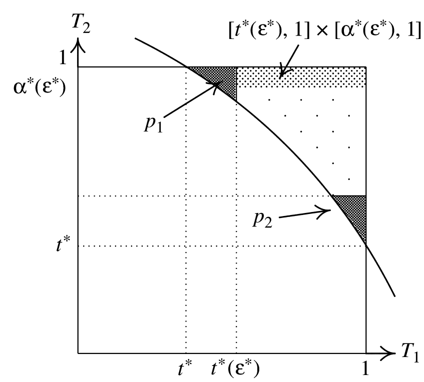

FIGURE 4. Assignment region: Case 2.

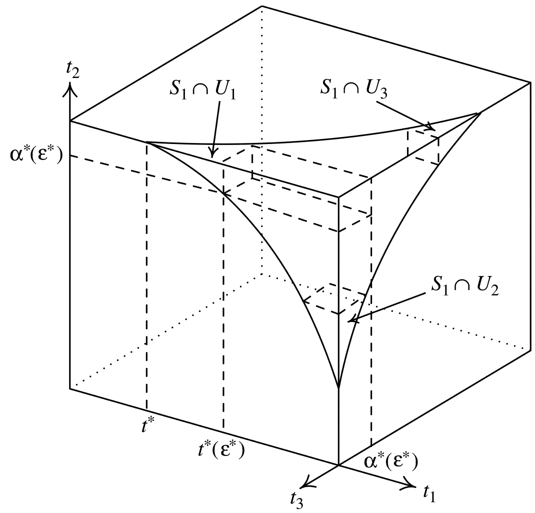

FIGURE 5. Assignment region: Case 2, three individuals.

Now we can verify (D) for player i with $t_i > t^*(\varepsilon)$:

$$\begin{aligned}
&\int_{T_{-i}} p(t)\chi_{S_i}(t) f(t_{-i})\, dt_{-i} \\
&\quad = \int_{T_{-i}} \left[p'(t)\cdot \chi_{\tilde{S}}(t) + \sum_{j=1}^{n} p_j(t)\cdot \chi_{U_j}(t)\right] \chi_{S_i}(t) f(t_{-i})\, dt_{-i} \\
&\quad = \int_{T_{-i}} p'(t)\cdot \chi_{\tilde{S}}(t)\chi_{S_i}(t) f(t_{-i})\, dt_{-i} + \int_{T_{-i}} \left[\sum_{j\neq i} p_j(t)\cdot \chi_{U_j}(t)\right]\chi_{S_i}(t) f(t_{-i})\, dt_{-i} \\
&\quad = \int_{T_{-i}} \left[p_i(t) + \sum_{j\neq i} \frac{E((p_i - p_j)\chi_{S_i\cap U_j} \mid t_i)}{E(\chi_{\tilde{S}_i} \mid t_i)}\right] \chi_{\tilde{S}_i}(t) f(t_{-i})\, dt_{-i} + \int_{T_{-i}} \left[\sum_{j\neq i} p_j(t)\cdot \chi_{U_j}(t)\right]\chi_{S_i}(t) f(t_{-i})\, dt_{-i} \\
&\quad = \int_{T_{-i}} p_i(t)\chi_{\tilde{S}_i}(t) f(t_{-i})\, dt_{-i} + \int_{T_{-i}} \left[\sum_{j\neq i} p_i(t)\cdot \chi_{U_j}(t)\right]\chi_{S_i}(t) f(t_{-i})\, dt_{-i} = \int_{T_{-i}} p_i(t)\chi_{S_j}(t) f(t_{-i})\, dt_{-i}.
\end{aligned}$$

Figure 5 depicts the three person case where $S_1 = S_2 = S_3$; at each type-value profile, either all individuals receive the good or no one receives the good. Here, on $S_1 \cap U_i$, the common price is set equal to p_i, and on $S_1 \backslash (\bigcup U_i)$ the price is constructed according to the first stage of the proof.

5. Applications. This section considers two applications that illustrate the applicability of the main theorem. The first is an optimal cost-benefit problem, the second a monopoly profit-maximization problem. In both cases we show that the optimal mechanisms are regular. Therefore, both problems admit common pricing mechanisms that are optimal.

5.1. Optimal cost-benefit analysis. Consider the optimal provision of a good by a public agency where cost-benefit analysis is used to determine the appropriate production and distribution of the good. The agency wishes to maximize the difference between the total benefits of individuals who obtain the good and the cost of providing them with the good.

To achieve this goal, one should rank all individuals according to their valuations and assign the good in order of descending valuation, up to the point where benefit net of cost is maximized. With increasing marginal costs, one simply stops at the first point when the marginal cost exceeds the marginal benefit. More precisely, for each profile $t = (t_1, \ldots, t_n)$, let $(t_{(1)}, \ldots, t_{(n)})$ be a permutation of t, in which $t_{(i)}$ is the ith order valuation: $t_{(i)} \geq t_{(i+1)}$. Then one should stop at k such that $t_{(k)} \geq c(k)$ and $t_{(k+1)} < c(k+1)$. It is no longer this simple with decreasing marginal costs because the decline of marginal costs after this point could be much bigger than the reduction of the marginal benefits. In general, the agency should choose to produce k units of the good such that k maximizes the expression:

$$V(t,k) = \left(\sum_{i=1}^{k} t_{(i)}\right) - C(k).$$

(In the case of a tie, choose the largest k that maximizes the expression. Ties only occur with zero probability.)

DEFINITION 5.1. A mechanism is cost-benefit (C-B) optimal if, at all profiles t, exactly $k(t)$ units of good are produced, where $k(t)$ is the largest number that maximizes

$$V(t,k)=\left(\sum_{i=1}^{k} t_{(i)}\right)-C(k).$$

It is easy to characterize the region S_i for any C-B optimal mechanism. Since individual i with valuation t_i gets an object only if the valuation t_i is as large as the marginal cost of production at the optimum, $c(k(t))$, it follows that:

PROPOSITION 5.1. *For any C-B optimal mechanism, $t\in S_i$ if and only if $t_i\geq c(k(t))$.*

Provided S_i is regular, an optimal C-B mechanism admits a common pricing rule for the good: Each individual can be charged the same price. The next proposition confirms that S_i is indeed regular.

PROPOSITION 5.2. *Any C-B optimal mechanism is regular.*

PROOF. It is clear that a C-B optimal mechanism must be symmetric and weakly efficient. To see that it is monotonic, suppose that $t\in S_i$ and $t'\geq t$. Note that $t'\geq t$ implies $((t)'_{(1)},\dots,(t)'_{(n)})\geq (t_{(1)},\dots,t_{(n)})$. Hence, for any $k<k(t)$,

$$\begin{aligned}
V(t',k(t))-V(t',k) &= \left(\sum_{i=1}^{k(t)}(t)'_{(i)}\right)-C(k(t))-\left(\sum_{i=1}^{k}(t)'_{(i)}\right)+C(k)\\
&=\left(\sum_{i=k+1}^{k(t)}(t)'_{(i)}\right)-C(k(t))+C(k)\\
&\geq \left(\sum_{i=k+1}^{k(t)}t_{(i)}\right)-C(k(t))+C(k)\\
&=\left(\sum_{i=1}^{k(t)}t_{(i)}\right)-C(k(t))-\left(\sum_{i=1}^{k}t_{(i)}\right)+C(k)\\
&=V(t,k(t))-V(t,k)\\
&\geq 0.
\end{aligned}$$

This means $k(t')\geq k(t)$. In other words, at least as many units of the good are produced at t'. The fact that $t\in S_i$ implies $t_i\geq c(k(t))$. Hence, $t'_i\geq t_i\geq c(k(t))\geq c(k(t'))$. Then, by the previous proposition, $t'\in S_i$.

To see that S_i is closed, let $t^l\to t$ and $t^l\in S_i$. Taking subsequences if necessary, $k(t^l)\to k^*$ and so, for l sufficiently large (say $l\geq \bar{l}$), $k(t^l)=k^*$. Thus, for $l\geq\bar{l}$,

$$\sum_{i=1}^{k^*}t^l_{(i)}-C(k^*)=\sum_{i=1}^{k(t^l)}t^l_{(i)}-C(k(t^l))\geq\sum_{i=1}^{k(t)}t^l_{(i)}-C(k(t)),$$

where the inequality follows because at profile t^l, production level $k(t^l)$ is optimal. Taking limits,

$$\sum_{i=1}^{k^*}t_{(i)}-C(k^*)\geq\sum_{i=1}^{k(t)}t_{(i)}-C(k(t))$$

so that $k^*\leq k(t)$ (because $k(t)$ is the largest value of k maximizing $\sum_{i=1}^{k}t_{(i)}-C(k)$). Thus, $c(k^*)\geq c(k(t))$ and because $t^l_i\geq c(k^*)$ for $l\geq\bar{l}$, in the limit $t_i\geq c(k(t))$ and by Proposition 5.2, $t\in S_i$.

It remains to show that substitutability is satisfied. Let $(t_i,t_{-i})\in S_i$ and suppose that $t_k<t_j$. Let $\tilde{t}_j$ and $\tilde{t}_k$ satisfy $t_j>\tilde{t}_j>\tilde{t}_k>t_k\geq t_i$ and $\tilde{t}_j+\tilde{t}_k=t_j+t_k$, and for $l\neq j,k$, $\tilde{t}_l=t_l$. Thus, $\sum_{m=1}^{k(t)}\tilde{t}_{(i)}-c(k(t))=\sum_{m=1}^{k(t)}t_{(i)}-c(k(t))$. Also, no value $r<k(t)$ gives a higher C-B value: for $r\leq k(t)$,

$$\sum_{m=1}^{r}\tilde{t}_{(i)}-c(r)\leq\sum_{m=1}^{k(t)}\tilde{t}_{(i)}-c(k(t)).$$

This follows because, relative to t, with $\tilde{t}$ a reduction occurs at index k that is only recovered at index l, so the C-B value of $\tilde{t}$ is never higher than that of t, which is maximized at $k(t)$ and the $\tilde{t}$ profile attains the same C-B value as t at $k(t)$. Thus, at $\tilde{t}$, $k(t)$ is optimal ($=k(\tilde{t})$). Consequently, since $t_i\geq c(k(t))$, $\tilde{t}\in S_i$. □

Because the mechanism is regular, we have the following theorem.

THEOREM 5.1. *There is a common price optimal C-B mechanism. All recipients at any valuation profile pay the same price.*

PROOF. The proof follows from Proposition 5.2.

It should be noted that this problem can be formulated as a problem of a public-good provision with quasilinear preferences. Hence, an optimal C-B mechanism is also a Groves mechanism (Groves [3]). Although there is a large body of literature on Groves mechanisms, the possibility of constructing a Groves mechanism with common pricing has not been considered. In fact, most scholars have emphasized the importance of cross-subsidizing from buyers of higher valuations to buyers of lower valuations in achieving optimality in such models.

The next application considers a monopoly profit maximization problem where the restriction to common pricing is shown to cause no reduction in profit relative to the unconstrained (discriminatory) program.

5.2. Monopoly profit maximization. Here we consider profit maximizing mechanisms for a monopolist. The seller's optimization problem is to maximize the following objective:

$$\int_T \left[\sum_{i=1}^n p_i(t_i, t_{-i}) y_i(t_i, t_{-i}) - C\left(\sum_{i=1}^n y_i(t_i, t_{-i})\right)\right] f(t)\, dt$$

subject to participation and incentive-compatibility constraints. If $\{y_i, p_i\}_{i=1}^n$ satisfy these conditions, then with some calculations it may be shown (Myerson [4]) that

$$\int_T p_i(t_i, t_{-i}) y_i(t_i, t_{-i}) f(t)\, dt = \int_T J(t_i) y_i(t_i, t_{-i}) f(t)\, dt,$$

where $J(s) = s - [(1 - F(s))/f(s)]$. (Note that $J(0) < 0$ and $J(1) = 1$.) We assume the standard regularity condition that J is increasing. Therefore, an optimal selling mechanism is a solution to

$$\max_{\{y_i\}_{i=1}^n} \int_T \left[\sum_{i=1}^n J(t_i) y_i(t_i, t_{-i}) - C\left(\sum_{i=1}^n y_i(t_i, t_{-i})\right)\right] f(t)\, dt,$$

subject to the incentive-compatibility and participation constraints. Define functions $\{y_i(t)\}_{t=1}^n$ such that for each realization of t, $\{y_i(t)\}_{t=1}^n$ maximizes:

$$V(t; y) = \sum_{i=1}^n J(t_i) y_i - C\left(\sum_{i=1}^n y_i\right).$$

This gives the canonical optimal solution to the problem.

Let $\{S_i\}_{i=1}^n$ be the assignment regions (determined by $\{y_i\}$).

PROPOSITION 5.3. *The canonical profit-maximizing mechanism is regular.*

PROOF. From the discussion in §3, incentive compatibility is satisfied if y_i is monotone (weakly) increasing, and participation is satisfied if the individual with the lowest valuation has nonnegative utility. Note, however, that the unconstrained solution to the problem yields a monotonic y_i (because J is weakly increasing and C weakly decreasing); hence, $S_i = \{t \mid y_i(t) = 1\}$ is monotonic. It is also weakly efficient since higher t_is imply higher J values and closed because we assume that the density function f is continuous with $f(0) > 0$. Symmetry also follows directly, as does substitutability, because for any l, k with $0 < \tilde{t}_l, t_k < 1$, there are $\tilde{t}_k, \tilde{t}_l$ with $1 \geq \tilde{t}_k > t_k$, $0 < \tilde{t}_l < t_l$ and $J(t_k) + J(t_l) = J(\tilde{t}_k) + J(\tilde{t}_l)$, so the revenue at the $\tilde{t}$ profile is at least as large as at t. In this case, i with type t_i continues to receive a good (although l may not). Thus S_i satisfies all of the regularity conditions.

Because S_i is regular, from Theorem 4.1, we come to the following conclusion.

THEOREM 5.2. *There is a profit-maximizing mechanism with common pricing.*

6. Ex post individual rationality. Our discussion thus far has formulated incentive compatibility and individual rationality using *interim* expected utilities, defined in terms of win probabilities and expected payments given one's valuation. However, this leaves open the possibility that ex post an individual might have to pay more for the good than the good is worth to him.

Recall from §3 the requirement that $t_i\bar{y}_i(t_i) - \bar{x}_i(t_i) \geq 0$ for all t_i, so the interim expected payment is never more than a buyer's valuation—the individual rationality constraint. However, even if the interim expected payment never exceeds the buyer's valuation, the ex post payment could because this condition admits the possibility that $t_i y_i(t) - x_i(t) < 0$ holds at some profiles $t = (t_i, t_{-i})$. When the transfer made depends on others' types or valuations, an individual may at some states face a transfer that yields negative net utility. For example, in the buyer-seller context the price a buyer has to pay is determined by all valuations, which may be larger than the buyer's valuation. Then a buyer who freely entered a purchase agreement before the valuations of others become known may subsequently wish to renege on the agreement at realizations where $t_i < p(t)$ and $y_i(t) = 1$. Thus, the interim criterion might be considered appropriate only when each individual's cost of withdrawal from the agreement is greater than the loss where the price exceeds the individual's valuation. However, when no such agreement can be made, the more appropriate condition is the ex post individual rationality. (Chung and Ely [2] also consider the imposition of ex post individual rationality in a class of mechanism design models.)

An assignment-pricing mechanism $\{p_i, y_i\}_{i=1}^n$ is ex post individually rational if

$$p(t) \leq \min\{t_i \mid y_i(t) = 1\}, \quad \forall\, t.$$

In other words, whenever an individual acquires the good, the price paid never exceeds the value of the good to the individual.

In what follows, we focus on the monopoly profit maximization framework and provide two examples. In Example 1, we first find the canonical discriminatory pricing mechanism, and then provide a common price mechanism that also satisfies ex post individual rationality. However, this is not always possible as Example 2 shows. Hence, the imposition of ex post individual rationality in conjunction with common pricing lowers monopoly profit relative to the discriminatory mechanism.

Example 1. There is a fixed cost to produce the first unit of the good, and the marginal costs of additional units are zero. Specifically, let $c(1) = c > 1$ and $c(r) = 0$ for $r \neq 1$. There are two potential buyers, and their types are uniformly distributed on $[0, 1]$ with density $f(s) = 1$, and $J(s) = s - (1-s)/1 = 2s - 1$. To maximize the expected profit, as we explained in §4, the monopolist should choose, for each profile $t = (t_1, t_2)$, y_1 and y_2 that maximize the expression $y_1 J(t_1) + y_2 J(t_2) - c$. Hence, the canonical optimal mechanism is to provide goods to both players when $t_1 + t_2 > 1 + c/2$ (see Figure 6) and charge buyer 1, $p_1(t_1, t_2) = 1 + c/2 - t_2$, and buyer 2, $p_2(t_1, t_2) = 1 + c/2 - t_1$.

The canonical optimal mechanism is ex post individual rational but discriminatory. Our main result guarantees that there is another optimal mechanism with common pricing. However, the mechanism developed in the proof of the theorem satisfies interim individually rational but might not be ex post individually rational. Here, with the linear structure in this example, we can construct an optimal common price mechanism that satisfies individual rationality ex post. While the assignment rule is still the same, the single price function is $p(t_1, t_2) = t_1 + t_2 - 1$. Because $t_1 \leq 1$, and $t_2 \leq 1$, this price function is ex post individually rational. This price function also respects incentive compatibility because

$$\int_{1+c/2-t_1}^{1} p_1(t_1, t_2) f(t_2)\, dt_2 = \int_{1+c/2-t_1}^{1} \left[1 + \frac{c}{2} - t_2\right] dt_2 = \frac{1}{2}\left(t_1 + \frac{c}{2}\right)\left(t_i - \frac{c}{2}\right) = \frac{1}{2} t_i^2 - \frac{c^2}{8}.$$

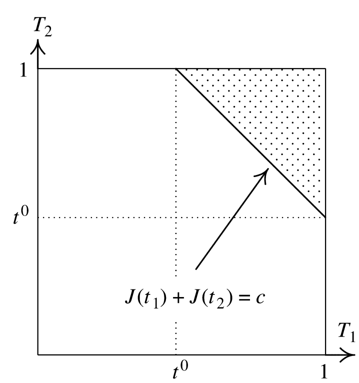

Figure 6. The assignment region.

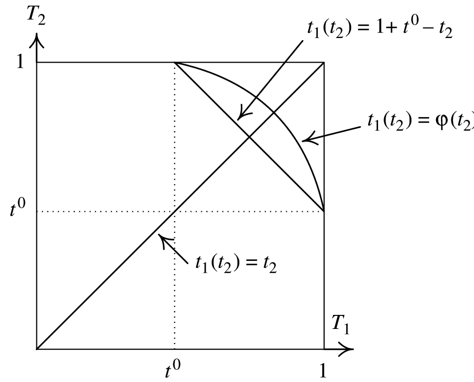

FIGURE 7. The functions $\varphi(t_2)$, $1+t^0-t_2$, t_2.

and

$$\int_{1+c/2-t_1}^{1} p(t_1,t_2)f(t_2)\,dt_2 = \int_{1+c/2-t_1}^{1} [t_1+t_2-1]\,dt_2 = \frac{1}{2}\left(t_1+\frac{c}{2}\right)\left(t_i-\frac{c}{2}\right) = \frac{1}{2}t_i^2 - \frac{c^2}{8}.$$

This price function has two other advantages over the canonical price function. First, it is strictly increasing in buyers' types while the canonical price function is actually weakly decreasing in buyers' types, with the total revenue strictly decreasing in buyers' types. Second, this price function is robust in that it does not depend on c (although the allocation region does depend on c).

In this example, the assignment rule is such that we can find a common price function that is ex post individually rational. However, this is not always the case, as the next example illustrates.

This example gives a regular incentive-compatible mechanism, where it is impossible to find a common price function that supports the assignment rule (which is optimal) and is ex post individually rational.

EXAMPLE 2. We consider a profit-maximizing mechanism with discriminatory pricing for a two-buyer model and show that the assignment rule of this mechanism cannot be supported by a common price function that is ex post individually rational. This implies that if one insists on ex post individual rationality, then the maximal expected profit that can be achieved by a discriminatory mechanism might be strictly higher than what is achievable with common price mechanisms. (Note that the canonical pricing function associated with any incentive-compatible mechanism is ex post individually rational.)

We consider a case in which $c(1)=1$ and $c(r)=0$ for $r\neq 1$. There are two potential buyers and their types are distributed on $[0,1]$ according to some distribution $f(s)$. We assume that f is linear and strictly decreasing and that the J function for f is strictly increasing and strictly convex. (It is easy to find a density function f that satisfies these assumptions. For instance, we can choose $f(s)=11/10-s/5$. Then

$$J(s) = s - \frac{1-\left(\frac{11}{10}s-\frac{1}{10}s^2\right)}{\frac{11}{10}-\frac{1}{5}s}, \qquad J'(s) = 6\frac{(37-11s+s^2)}{(-11+2s)^2}, \qquad J''(s) = -162\frac{1}{(-11+2s)^3}.$$

Because for all $s\in[0,1]$, $J'(s)>0$ and $J''(s)>0$, J is strictly increasing and strictly convex.)

The optimal profit-maximizing mechanism with price discrimination chooses (for each t), $y_1(t)$ and $y_2(t)$ to maximize

$$J(t_1)y_1 + J(t_2)y_2 - C(y_1+y_2)$$

and then charges each buyer the canonical price. Given $c(1)=1$, it is never worthwhile to provide the good to only one buyer because $1=J(1)>J(t)$ for all $t<1$. Also, the solution of $J(t^0)=0$ satisfies $J(t^0)+J(1)=1$. Hence, the assignment region for either buyer is the same and is given by $\{(t_1,t_2)\mid J(t_1)+J(t_2)\geq 1\}$, the region above the curve φ_2 in Figure 7.

Let the lower boundary of the assignment region be the curve $\{(t_1,t_2)\mid J(t_1)+J(t_2)=1\}$. It can also be written as $t_1=\varphi(t_2)$. Because the region is symmetric, we can also write the curve as $t_2=\varphi(t_1)$. Given that J is strictly increasing and strictly convex, it is easy to verify that φ is strictly decreasing and strictly concave. Note that φ also defines the canonical price functions $p_1(t_1,t_2)=\varphi(t_2)$ and $p_2(t_1,t_2)=\varphi(t_1)$ for $J(t_1)+J(t_2)\geq 1$. So the expected payment of buyer 1 with type $t_1=1$ is

$$\bar{p}_1(1) = \int_{t^0}^{1} \varphi(t_2)f(t_2)\,dt_2.$$

For any common price function $p(t_1, t_2)$ that is also ex post individually rational, the expected payment of type $t_1 = 1$ is $\int_{t^0}^1 p(1, t_2) f(t_2)\, dt_2$, and ex post individual rationality requires $t_2 \geq p(1, t_2)$; hence,

$$\int_{t^0}^1 t_2 f(t_2)\, dt_2 \geq \int_{t^0}^1 p(1, t_2) f(t_2)\, dt_2.$$

If $p(t_1, t_2)$ also supports the assignment rule, then

$$\int_{t^0}^1 p(1, t_2) f(t_2)\, dt_2 = \bar{p}_1(1).$$

Therefore, we must have

$$\int_{t^0}^1 t_2 f(t_2)\, dt_2 \geq \int_{t^0}^1 \varphi(t_2) f(t_2)\, dt_2.$$

However,

$$\int_{t^0}^1 \varphi(t_2) f(t_2)\, dt_2 > f\left(\frac{1+t^0}{2}\right) \int_{t^0}^1 \varphi(t_2)\, dt_2 > f\left(\frac{1+t^0}{2}\right) \int_{t^0}^1 [1+t^0-t_2]\, dt_2 = f\left(\frac{1+t^0}{2}\right) \int_{t^0}^1 t_2\, dt_2 > \int_{t^0}^1 t_2 f(t_2)\, dt_2.$$

However, this is a contradiction. (The first inequality follows because f and φ are strictly decreasing, and f is linear; the second inequality follows because φ is strictly concave with $\varphi(t_2) \geq 1 + t^0 - t_2$ on $[t^0, 1]$; the third equality follows from simple integration; and the last inequality follows because f is decreasing and linear, and t_2 is increasing.)

7. An open problem. In the absence of the ex post individual rationality requirement, identifying the optimal selling procedure involves two steps—first, identifying the regions in type space on which sales take place and then determining the appropriate prices. To search for a common selling price, one may also proceed in this manner. Once sale regions are determined, the interim expected price for each i, $(\bar{p}_i(t_i))$, is determined by the incentive compatibility condition. Without an ex post individual rationality requirement, as long as sale regions are regular, one can find a common price p, such that for any buyer i, $E\{p \mid t_i, y_i = 1\} = \bar{p}_i(t_i)$, ensuring that incentive compatibility is satisfied. However, when the ex post individual rationality requirement is imposed, this two-step process may no longer work. It may be that sales regions that are optimal when discriminatory pricing is allowed cannot be supported by a common pricing scheme that is also ex post individually rational. This case raises an interesting problem—directly identifying the optimal mechanism when both common pricing and ex post individual rationality are imposed. Formulating this problem yields the program:

$$\max_{(p, \{y_i\}_{i=1}^n)} \int \sum_{i=1}^n p(t) y_i(t) - C\left(\sum_{i=1}^n y_i(t)\right) f(t)\, dt,$$

subject to incentive compatibility of the (common price p) and ex post participation $(p(t) \leq \min_i\{t_i \mid y_i(t) = 1\})$. Alternatively, one can consider the problem of identifying those valuation distributions F, such that the corresponding regions S_i admit a common price satisfying ex post participation. The authors leave this as a topic for further investigation.

8. Conclusion. The preceding discussion has shown that in assignment-pricing problems in which the assignment regions satisfy a regularity condition, common pricing is optimal for the corresponding objective. When ex post individual rationality is imposed, this may no longer be true, as we show by example. That observation raises two interesting questions: identifying optimal common pricing mechanisms in such cases (which necessarily involve different assignment rules) and identifying assignment regions (such as in Example 1) that admit ex post individually rational common pricing that is optimal relative to the objective.

Finally, we have focused on the declining marginal cost case. As noted earlier, the increasing marginal cost case is much simpler to consider and, in fact, the canonical prices are already a common price in that case.

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