Multi-unit Auctions with Budget Limits*

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Abstract

We study multi-unit auctions where the bidders have a budget constraint, a situation very common in practice that has received relatively little attention in the auction theory literature. Our main result is an impossibility: there is no incentive-compatible auction that always produces a Pareto-optimal allocation. In contrast, we show that when the budgets are public knowledge there exists a *unique* auction that is incentive-compatible and Pareto-optimal. This auction additionally has good revenue properties.

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1 Introduction

The starting point of almost all of auction theory is the set of players' *valuations*: how much value (measured in some currency unit) does each of them assigns to each possible outcome of the auction. When attempting actual implementations of auctions, a mismatch between theory and practice emerges immediately: *budgets*. Players often have a maximum upper bound on their possible payment to the auction – their budget.[1] Budgets are central elements in most of economic theory, but relatively little attention has been paid to them in auction theory. A concrete example is Google's and Yahoo's ad-auctions, where budgets are an important part of a user's bid, and are perhaps even more real for the users than the rather abstract notion of a valuation.[2] Addressing budgets properly breaks down the usual quasi-linear setting, and because of this the VCG mechanism loses its incentive-compatibility. The design of incentive-compatible mechanisms becomes significantly more involved.

The few relatively recent works that study this issue focus on several different directions. A first branch of works (Che and Gale, 1998; Benot and Krishna, 2001) analyzes how budgets change the classic results on "standard" auction formats, showing for example that first-price auctions raise more revenue than second-price auctions when bidders are budget-constrained, and that the revenue of a sequential auction is higher than the revenue of a simultaneous ascending auction. A second branch of works (Laffont and Robert, 1996; Pai and Vohra, 2008) constructs single-item auctions that maximize the seller's revenue, and a third branch (Maskin, 2000) considers the problem of "constrained efficiency": maximizing the expected social welfare under Bayesian incentive compatibility constraints. A forth branch (Borgs et al., 2005; Abrams, 2006), taken by the computer science community, tries to design incentive-compatible multi-unit auctions that approximate the optimal revenue.

Our model in this paper is simple: There are m identical indivisible units for sale, and each bidder i has a private value v_i for each unit, as well as a budget limit b_i on the total amount he may pay. We also consider the limiting case where m is large by looking at auctions of a single infinitely-divisible good. Our assumption is that bidders are utility-maximizers, where i's utility from acquiring x_i units (or a fraction of x_i of the good, in the infinitely divisible good case) and paying p_i is $u_i = x_i \cdot v_i - p_i$, as long as the price is within budget, $p_i \leq b_i$, and is negative infinity (infeasible) if $p_i > b_i$. [3]

We study the fundamental question of how to produce efficient allocations in an incentive-compatible way. As the setting is not quasi-linear, allocational efficiency is not uniquely defined since different allocations are preferred by different players[4]. We thus focus at a weak efficiency requirement: Pareto-optimality, i.e., allocations where it is impossible to strictly improve the utility of

[1] The nature of what this budget limit means for the bidders themselves is somewhat of a mystery since it often does not seem to simply reflect the true liquidity constraints of the bidding firm. There seems to be some risk control element to it, some purely administrative element to it, some bounded-rationality element to it, and more.

[2] See the paper of Nisan et al. (2009) for a more detailed discussion on Google's auction structure.

[3] This model naturally generalizes to any type of multi-item auction: bidders have a valuation $v_i(\cdot)$ and a budget b_i, and their utility from acquiring a set S of items and paying p_i for them is $v_i(S) - p_i$ as long as $p_i \leq b_i$ and negative infinity if the budget has been exceeded $p_i > b_i$. It is interesting to note that the "demand-oracle model" (see e.g. Blumrosen and Nisan (2007)) represents such bidders as well. Analyzing combinatorial auctions with budget limits, even in simple settings such as additive valuations, is clearly a direction for future research.

[4] In quasi-linear settings any Pareto-optimal allocation must optimize the "social-welfare" – the sum of bidders valuations – and thus efficiency is justifiably interpreted as maximizing social-welfare.

some players without hurting those of others. We ask whether there exists an incentive-compatible[5] mechanism that always outputs a Pareto-optimal allocation in our setting.

Main results. Our main result is an impossibility: *there is no incentive-compatible and Pareto-optimal auction*, for any finite number $m > 1$ of units of an indivisible good and any $n \geq 2$ number of players.[6] The cornerstone of the analysis is a characterization result for the case where budgets are public information. For this case we show that an adaptive version of Ausubel's "clinching auction" (Ausubel, 2004) is Pareto-optimal and incentive-compatible. Moreover we show that it is the *unique* (up to tie-breaking) such auction when there are exactly two bidders.[7] We also analyze the revenue properties of this mechanism. We show that, as the number of items increases and the "dominance" of each bidder decreases, the revenue of this mechanism approaches the revenue of a non-discriminatory monopoly, that knows the values and budgets of the players and determines a single unit-price in order to maximize revenue. Thus this auction *simultaneously obtains* allocational efficiency and high revenue, while maintaining incentive-compatibility, and this is the unique such auction. In this respect we view it as a useful *positive* result.

Our characterization sheds light on the type of effects that budget limitations create. Recall that Ausubel's auction gradually increases a price parameter, and bidders keep decreasing their demands for items at this price. Whenever the combined demand of the other bidders decreases strictly below available supply, bidder i "clinches" the remaining quantity at the current price. Thus different amounts of units are acquired by bidders at different prices, and the total payment of a bidder is the sum of the prices of all units that he clinched throughout the auction. Ausubel shows that, in the quasi-linear setting, this auction yields exactly the VCG outcome and is thus incentive-compatible. The key property for incentive-compatibility is that the demands for future items are fixed and independent of the prices at which previous items were acquired. With budgets, this property no longer holds, and demand for future items changes as a function of the remaining budget. If bidder A slightly delays to report a demand decrease, bidder B will pay as a result a slightly higher price for his acquired items, which reduces his future demand. In turn, the fact that bidder B now has a lower demand implies that bidder A pays a lower price for future items, and the contradiction to incentive-compatibility becomes evident. Thus with private budgets this auction is no longer incentive-compatible, and our analysis implies that this difficulty is inherent to all Pareto-optimal allocation schemes. With public budgets (and private values), on the other hand, this manipulation is not possible, and we show that the adaptive clinching auction is the unique incentive-compatible and Pareto-optimal auction.

While in the quasi-linear setting, exact formulas for the outcome of the auction can be described (this is essentially the VCG mechanism), in our setting it is quite hard to come up with a parallel closed-form solution, especially in the infinitely-divisible good case for which the auction is a continuous time process. (This once again demonstrates the relative flexibility of ascending auctions versus direct mechanisms when one slightly changes the model). Nevertheless we present exact closed-form descriptions for an infinitely-divisible item and two players. These were certainly

[5] We consider the most basic solution concept of dominant-strategies. A natural extension of our work would be to examine the Bayesian-Nash solution concept.

[6] This theorem assumes "individual rationality" and "no positive transfers", i.e. that bidders are not paid by the auction nor do they pay more than their value or budget. Without this, the budget limits can be easily side-stepped, e.g., by using a VCG mechanism that pays losers the total value of the others.

[7] The assumption of public budgets was made many times before, e.g. by Laffont and Robert (1996) and in Maskin (2000).

surprising for us, as they do not seem to resemble any previously considered auction format. In all cases, once the exact form is found, it is a straight forward exercise to verify incentive-compatibility and Pareto-optimality. For example, if both players have equal budgets, i.e. w.l.o.g $b_1 = b_2 = 1$ and $v_1 \le v_2$, then if $\min(v_1, v_2) \le 1$ then the high-value player gets everything and pays the second highest value, and otherwise, the low-value player gets $1/2 - 1/(2 \cdot v_1^2)$ and pays $1 - 1/v_1$ and the high-value player gets $1/2 + 1/(2 \cdot v_1^2)$ and pays 1. This unfamiliar format has of-course an underlying reasoning that we explain in the body of the paper. In parallel to the indivisible case, we show for the divisible case as well that when budgets are public, this auction is the unique *anonymous* Pareto-optimal and incentive-compatible auction. In a follow-up to our work, Bhattacharya, Conitzer, Munagalaz and Xiax (2010) further analyze the divisible case, showing additional interesting properties. For example, if budgets are private, then the only profitable manipulation is to over-state one's budgets.

As a last note, we point out that the impossibility for private budgets crucially depends on the assumption that players demand *multiple* items. Indeed, recently we have seen several works that present positive results for unit-demand players with budgets. For example, Aggarwal, Muthukrishnan, Pal and Pal (2009) show that an extension of the Demange-Gale-Sotomayor ascending auction is incentive-compatible and Pareto-optimal. Hatfield and Milgrom (2005) study a more abstract unit-demand model for players with non-quasi-linear utilities that generalizes both the Gale-Shapley stable-matching algorithm as well as the Demange-Gale-Sotomayor ascending auction, showing incentive-compatibility and (in the context of our setting) Pareto-optimality. Ashlagi, Braverman, Hassidim, Lavi and Tennenholtz (2010) extend the generalized English auction to settings with budget-constraints, again showing incentive-compatibility and Pareto-optimality.

The rest of the paper is organized as follows. We start with the basic definitions in section 2, and by describing a "proportional share auction" that obtains a competitive equilibrium in section 3. The adaptive version of Ausubel's clinching auction is defined in section 4, where we also analyze its basic properties: Pareto-optimality, incentive-compatibility, and revenue. Section 5 then shows the uniqueness of this auction. Relying on this uniqueness result, section 6 then proves the impossibility result for private budgets. Section 7 describes the closed-form mechanism mentioned above.

2 Preliminaries and Notation

2.1 Allocations

We will be considering auctions of m identical indivisible items as well as the limiting case of a single infinitely divisible good.

We have n bidders, where each bidder i has a value v_i for each unit he gets, and has a budget limit b_i on his payment. Rather than explicitly declaring a bidder's utility of going over-budget to be negative infinity, we will equivalently directly declare such cases to be infeasible.

Definition 2.1 *An allocation is a vector of quantities $x_1, ..., x_n$ and a vector of payments $p_1, ..., p_n$ with the following properties:*

1. ***(Feasibility)*** *In the case of finite m, x_i must be a non-negative integer and $\sum_i x_i \le m$. In the case of an infinitely divisible good, x_i must be non-negative real and $\sum_i x_i \le 1$.*

2. ***(No Positive Transfers)*** $\sum_i p_i \ge 0$.

3. ***(Individual Rationality)*** $p_i \leq x_i \cdot v_i$.
4. ***(Budget Limit)*** $p_i \leq b_i$.

Our "no positive transfers" property is weak, in the sense that it allows the allocation to hand in payments to players. The only restriction is that, overall, the auctioneer does not hand money to the players. All our auctions satisfy the stronger version of the "no positive transfers" property, where for every player i we have $p_i \geq 0$, i.e., no player gets money from the auction.[8]

2.2 Auctions and Incentives

We will be formally considering only direct revelation auctions where bidders submit their value and budget to the auction, that based on this input $v_1, ..., v_n$ and $b_1, ..., b_n$ calculates the allocation $x_1, ..., x_n$ and $p_1, ..., p_n$. Our auctions have a very natural interpretation as dynamic ascending auctions, an interpretation that maintains incentive compatibility[9], but for simplicity we will just consider the auction mechanism as a black-box direct-revelation one.

Definition 2.2 *A mechanism is incentive compatible (in dominant strategies) if for every $v_1, ..., v_n$ and $b_1, ..., b_n$ and every possible manipulation v'_i and b'_i, we have that $u_i = x_i \cdot v_i - p_i \geq x'_i \cdot v_i - p'_i = u'_i$, where (x_i, p_i) are the allocation and payment of i for input (v_i, b_i) and (x'_i, p'_i) are the allocation and payment of i for input (v'_i, b'_i).*

A mechanism is incentive compatible for the case of publicly known budgets if the definition above holds for all v'_i, having fixed $b'_i = b_i$.

2.3 Pareto-optimality

We start with the classic notion of Pareto optimality:

Definition 2.3 *An allocation $\{(x_i, p_i)\}$ is Pareto-optimal if for no other allocation $\{(x'_i, p'_i)\}$ are all players better off, $x'_i v_i - p'_i \geq x_i v_i - p_i$, including the auctioneer $\sum_i p'_i \geq \sum_i p_i$, with at least one of the inequalities strict.*

In our setting, the notion of Pareto optimality if equivalent to a "no trade" condition that is much easier to work with. It essentially states that no money is "left on the table", in the sense that no player can re-sell the items he received and make a profit:

Proposition 2.4 *An allocation $\{(x_i, p_i)\}$ is Pareto-optimal in the infinitely divisible case if and only if (a) $\sum_i x_i = 1$, i.e. the good is completely sold, and (b) for all i such that $x_i > 0$ we have that for all j with $v_j > v_i$, $p_j = b_j$. I.e. a player may get a non-zero allocation only if all higher value players have exhausted their budget.*

[8] The weak version is necessary for the uniqueness result. Consider, for example, the following mechanism for one item and two players with infinite budgets: the item is allocated to player 1 if $v_1 > 0$, and otherwise to player 2. No payments are made. One can verify that this is truthful. It is also Pareto-optimal if one requires the strong NPT property, since if $v_2 > v_1 > 0$, the only allocation that Pareto-dominates the one chosen by the mechanism is an allocation in which player 1 receives a payment of v_1, and player 2 receives the item and pays v_1. The sum of payments here is 0, so with weak NPT the allocation is not Pareto-optimal, and the mechanism can be ruled out.

[9] As usual, the incentive compatibility of the iterative versions is only in the ex-post-Nash sense.

Proof: We first show that if either (a) or (b) do not hold then the allocation is not Pareto. If $\sum_i x_i < 1$ we simply add an additional quantity to some player for no additional charge, thus making him strictly better off while not harming any other player. Otherwise $\sum_i x_i = 1$ and there exists a player i with $x_i > 0$ and a player j with $v_j > v_i$ and $p_j < b_j$. Fix some ϵ such that $\epsilon \cdot v_i < b_j - p_j$. Construct an allocation (x', p') such that $x'_i = x_i - \epsilon$, $x'_j = x_j + \epsilon$, $p'_i = p_i - \epsilon \cdot v_i$, and $p'_j = p_j - \epsilon \cdot v_i$. All other players get the same quantity and pay the same price. Notice that $\sum_i p'_i = \sum_i p_i$ and that (x', p') is indeed a valid allocation. It is straight-forward to verify that i's utility remains the same while j's utility strictly increases.

For the other direction, fix an allocation (x, p) that satisfies (a) and (b). We will show that any other allocation (x', p') cannot be a Pareto improvement to (x, p) (as in Def. 2.3), implying that (x, p) is Pareto. Since (a) holds then $\sum_i x_i = 1$. Rename the players such that $v_1 \geq v_2 \geq \cdots \geq v_n$. Property (b) implies that there exists an index $1 \leq k \leq n$ such that, for any index $i < k$, $x_i > 0$ and $p_i = b_i$, for any index $i > k$, $x_i = 0$, and at k itself, $x_k > 0$. Let $\Delta = \sum_{i=1}^{k-1}(x_i - x'_i)$. For any i we need $u'_i \geq u_i$, which implies $p'_i - p_i \leq v_i \cdot (x'_i - x_i)$. We make several observations. First,

$$\sum_{i=k}^{n}(p'_i - p_i) \leq v_k(x'_k - x_k) + \sum_{i=k+1}^{n} v_i(x'_i - x_i) \leq v_k \sum_{i=k}^{n}(x'_i - x_i) = \Delta \cdot v_k$$

where the second inequality follows since $x_i = 0$ for any $i > k$, and the third inequality follows since $\sum_{i=1}^{k-1}(x_i - x'_i) - \sum_{i=k}^{n}(x'_i - x_i) = 0$. Second,

$$\sum_{i=1}^{k-1}(p_i - p'_i) \geq \sum_{1\leq i\leq k-1 \,:\, x_i \geq x'_i}(p_i - p'_i) \geq \sum_{1\leq i\leq k-1 \,:\, x_i \geq x'_i}(x_i - x'_i)v_i \geq$$

$$\sum_{1\leq i\leq k-1 \,:\, x_i \geq x'_i}(x_i - x'_i)v_k \quad \geq \quad v_k \sum_{i=1}^{k-1}(x_i - x'_i) = \Delta \cdot v_k$$

where the first inequality follows since $p_i = b_i \geq p'_i$ for any $i < k$. Now, if there exists $1 \leq i \leq k-1$ such that $x_i < x'_i$ then the above argument yields $\sum_{i=1}^{k-1}(p_i - p'_i) > \Delta \cdot v_k$. We then get $\sum_{i=1}^{k-1}(p_i - p'_i) - \sum_{i=k}^{n}(p'_i - p_i) > 0$. In other words, $\sum_i p_i > \sum_i p'_i$, a contradiction to the definition of a Pareto improvement. Therefore assume that $x_i \geq x'_i$ for any $1 \leq i \leq k-1$. This implies that

$$\sum_{i=1}^{k-1}(x_i - x'_i)v_i \geq \Delta \cdot v_k \geq \sum_{i=k}^{n}(x'_i - x_i)v_i.$$

Putting together these four inequalities, we get

$$\sum_i (u_i - u'_i) = \sum_{i=1}^{k-1}(p_i - p'_i) - \sum_{i=k}^{n}(p'_i - p_i) + \sum_{i=1}^{k-1}(x_i - x'_i)v_i - \sum_{i=k}^{n}(x'_i - x_i)v_i \geq 0.$$

As a result, $u_i = u'_i$ for any player i, hence (x', p') is not a Pareto improvement for (x, p) since there does not exist a player i with $u'_i > u_i$. $\square$

A similar "no trade" property is equivalent to Pareto-optimality also in the case of finite m (the proof is similar to the above proof and is therefore omitted):

Proposition 2.5 *An allocation $\{(x_i, p_i)\}$ is Pareto-optimal in the case of finite m if and only if (a) $\sum_i x_i = m$, i.e., all the units are sold, and (b) for all i such that $x_i > 0$ we have that for all j with $v_j > v_i$, $p_j > b_j - v_i$. I.e. a player may get a non-zero allocation only if there is no player with higher value that has larger remaining budget.*

3 The Proportional Share Auction

Recall that our main goal is to show the impossibility of constructing a mechanism that is Pareto-optimal and incentive compatible when budgets are private. Before that, we wish to point out that the source of this difficulty is the fact that values and budgets may be very close to one another. If values are guaranteed to be sufficiently large with respect to the budgets then a simple mechanism exists:

Definition 3.1 *The proportional share auction for an infinitely divisible good allocates to each bidder i a fraction $x_i = b_i / \sum_j b_j$ of the good and charges him his total budget $p_i = b_i$.*

Proposition 3.2 *Let $\alpha_i = b_i / \sum_j bj$ be the budget share of player i. The proportional-share auction with $x_i = b_i / \sum_j b_j$ and $p_i = b_i$ is Pareto Optimal and is Incentive Compatible in the range $v_i \geq \sum_j b_j/(1-\alpha)$ for all i.*

Proof: Pareto-optimality is trivial from proposition 2.4 since we charge bidders their full budget. We now prove incentive compatibility in the specified range. Since the values v_i do not affect the payment or the allocation, it suffices to show that no manipulation of b_i is profitable. Since we charge each bidder his total declared budget, it is clear that declaring $b'_i > b_i$ will lead to the bidder exceeding his budget. Thus it suffices to prove that no smaller declaration $b'_i < b_i$ is profitable. Let $u(z)$ be the utility obtained by bidder i if he declares a budget of $b'_i = z$. Thus $u(z) = v_i \cdot z/(z + \sum_{j \neq i} b_j) - z$. It suffices to show that u is monotonically increasing with z. To verify this, take the derivative with respect to z: $u'(z) = v_i \sum_{j \neq i} b_i / (\sum_j bj)^2 - 1$. This derivative is non-negative, $u'(z) \geq 0$, as long as $v_i \geq (\sum_j b_j)^2 / \sum_{j \neq i} b_j = \sum_j b_j/(1-\alpha)$, as is specified. □

4 The Adaptive Clinching Auction

We now describe the adaptive clinching ascending auction, and show that it satisfies Pareto optimality (PO), individual rationality (IR), and incentive compatibility (IC), when the budgets are known. In the next section we show that it is in fact the *unique* such auction (for two players and any number of items), which enables us to then conclude that when the budgets are private no such auction exists.

The auction keeps for every player i the current number of items q_i already allocated to i, the current total price for these items p_i, and her remaining total budget $B_i = b_i - p_i$. The auction also keeps the global unit-price p and the global remaining number of items q. The price p gradually ascends as long as the total demand is strictly larger than the total supply, where the demand of player i is defined by:

$$D_i(p) = \begin{cases} \lfloor \frac{B_i}{p} \rfloor & v_i > p \\ 0 & \text{otherwise.} \end{cases}$$

If we were to keep the price ascending until total demand would be smaller or equal to the number items, and only then allocate all items according to the demands, then a player could sometimes gain by performing a "demand reduction", thus harming incentive compatibility. Instead, following Ausubel's method, we allocate items to player i as soon as the total demand of the *other* players decreases strictly below the number of currently available items, q. In particular, if at some price p we have $x = q - \sum_{j \neq i} D_j(p) > 0$ then we allocate x items to player i for a unit price p. At this point in the auction, the relevant variables are updated as follows: $q_i \leftarrow q_i + x$, $p_i \leftarrow p_i + p \cdot x$, $b_i \leftarrow b_i - p \cdot x$, and $q \leftarrow q - x$. This will ensure incentive compatibility. The global picture of such an auction is:

The Adaptive Clinching Auction (preliminary version):

1. Initialize all variables appropriately.
2. While $\sum_i D_i(p) > q$,
   (a) If there exists a player i such that $D_{-i}(p) = \sum_{j \neq i} D_j(p) < q$ then allocate $q - D_{-i}(p)$ items to player i for a unit price p. Update all running variables, and repeat.
   (b) Otherwise increase the price p, recompute the demands, and repeat.
3. Otherwise (hopefully $\sum_i D_i(p) = q$): allocate to each player her demand, at a unit-price p, and terminate.

Note that step 2a does not change the amount of over demand, since both the total demand and the total supply are reduced by the same quantity (the number of items that player i gets). Therefore the only factor that affects the over demand is the price; as the price ascends the total over demand decreases. Thus, one would hope that when we reach step 3 we would indeed get $\sum_i D_i(p) = q$, which will enable us to allocate all items at the end (a necessary condition for achieving Pareto optimality). However clearly this is not quite the case, because the demand functions are not continuous. The demand drops integrally, by definition, and may drop by several items at once. In particular, there are two potentially problematic change points: when the price reaches the value v_i, and when the price reaches the remaining budget B_i. The latter point is identified by using:

$$D_i^+(p) = \lim_{x \to p^+} D_i(x),$$

as, for $p = B_i < v_i$, we have $D_i(p) > 0$ and $D_i^+(p) = 0$. Similarly, the former point is identified by using:

$$D_i^-(p) = \lim_{x \to p^-} D_i(x),$$

as, for $p = v_i \leq B_i$, we have $D_i^-(p) > 0$ and $D_i(p) = 0$. We modify the above definition of the auction to use these more refined conditions: (1) the over demand is computed using $D_i^+(p)$, since this ensures that we do not terminate with a price that is just a bit higher than the remaining budget of a player to whom we wish to allocate one last item, and (2) just before termination, if we are left with some non-allocated items, then this must have happened because the final price reached the value of some players (for such a player i we have $D_i^-(p) > 0$ and $D_i(p) = 0$), which caused an abrupt decrease in her demand. These players are indifferent between receiving or not receiving an item, and so we can allocate to them all remaining items.

The Adaptive Clinching Auction (complete version):

1. Initialize all variables appropriately.
2. While $\sum_i D_i^+(p) > q$,
   (a) If there exists a player i such that $D_{-i}^+(p) = \sum_{j \neq i} D_j^+(p) < q$ then allocate $q - D_{-i}^+(p)$ items to player i for a unit price p. Update all running variables (including the allocated and available quantities, the remaining budgets, and the current demands), and repeat.
   (b) Otherwise increase the price p, recompute the demands, and repeat.
3. Otherwise ($\sum_i D_i^-(p) \geq q \geq \sum_i D_i^+(p)$):
   (a) For every player i with $D_i^+(p) > 0$, allocate $D_i^+(p)$ units to player i for a unit-price p and update all running variables.
   (b) While $q > 0$ and there exists a player i with $D_i(p) > 0$, allocate $D_i(p)$ units to player i, for a unit-price p, and update the running variables.
   (c) While $q > 0$ and there exists a player i with $D_i^-(p) > 0$, allocate $D_i^-(p)$ units to player i, for a unit-price p.
   (d) Terminate.

Let us consider a short example to illustrate the above process. Suppose three items and three players with $v_1 = \infty, b_1 = 1, v_2 = \infty, b_2 = 1.9, v_3 = 1, b_3 = 1$. When the price is below 0.5, each player demands at least two items, and so, for every player, the other players demand more than three items. Therefore no allocations will take place, and the price will keep ascending. At $p = 0.5$, $D_1^+(0.5) = D_3^+(0.5) = 1$ (note that $D_1(0.5)$ and $D_3(0.5)$ are still 2). Thus, player 2 "clinches" one item for a price 0.5. Immediately after that, the demand of player 2 is updated to be 2. The available number of items is 2, and so no player can get any items. At a price 0.7 the demand of player 2 reduces to 1, but this still does not enable the auction to allocate any item to any player. The price keeps ascending until $p = 1$. At this point, $D_1^+(1) = 0, D_2^+(1) = 1, D_3^+(1) = 0$, and so the total demand reduces to be strictly below the number of available items (which is still 2). Thus we enter step 3. In 3a player 2 gets one item and in 3b player 1 gets one item. Note that we do not allocate any item to player 3, though $D_3^-(1) = 1$. Indeed, moving an item from 2 to 3, for example, will violate the Pareto optimality.

The following basic property of the auction implies almost immediately that IR and IC hold:

Claim 4.1 *The marginal utility of an item that is clinched at price p is non-negative if and only if $p \leq v_i$.*

Proof: If $p > v_i$ then by definition, since a player pays p for the clinched item, its marginal utility is negative. Now assume that $p \leq v_i$. Whenever player i gets x items at a unit-price p in steps 2a, 3a, and 3b in the auction, it follows that $x \leq D_i(p)$, where the demand is computed with respect to the remaining available budget. The definition of the demand function then implies that $B_i \geq x \cdot p$, hence the marginal utility is non-negative. If player i gets an item in step 3c then $D_i(p) = 0$ and $D_i^-(p) > 0$. The structure of the demand function implies that this can happen only if $p = v_i$, and in addition the available budget at price p is at least $D_i^-(p)$ times p. Thus in this case the player's additional utility from those items is exactly zero. $\square$

Corollary 4.2 *The adaptive clinching auction satisfies Individual Rationality (IR), i.e. every truthful player obtains a non-negative utility.*

Proof: By declaring the true value, a player guarantees that the marginal utility of every clinched item is non-negative, hence the total utility is non-negative as well. $\square$

Corollary 4.3 *The adaptive clinching auction satisfies Incentive Compatibility (IC), i.e. a truthful player cannot increase her utility by declaring any value different than her true value.*

Proof: Observe that declaring a value in this auction is equivalent to deciding on the exact price in which to completely drop from the auction. By the above claim, any items that are clinched after price $p = v_i$ have strictly negative utility, so declaring $\tilde{v}_i > v_i$ can only decrease the total utility. Similarly, any items that are clinched before price $p = v_i$ have non-negative utility, so declaring $\tilde{v}_i > v_i$ can only decrease the total utility as well. $\square$

To prove Pareto-optimality, we first need to show that all items are indeed allocated:

Claim 4.4 *The adaptive clinching auction always allocates all items.*

Proof: Define $D(p) = \sum_i D_i(p)$ and define $D^+(p)$ and $D^-(p)$ similarly. Observe that these three functions are monotone non-increasing, and that $D^-(p) = D(p) = D^+(p)$ for any continuity point of $D(p)$. Moreover, if p^* is a discontinuity point of $D(p)$ and $D^+(p) > q$ for any $p < p^*$ then $D^-(p^*) \geq q$.

Suppose that the auction enters step 3 at a price p^*. We wish to argue that $D^-(p^*) \geq q$. Indeed, for any $p < p^*$, at the beginning of step 2 we had $D^+(p) > q$, and after step 2a this inequality is maintained (since if we allocate Δ units to player i then the total demand and the number of available items both drop by Δ). Therefore after step 2b we have $D^+(p) \geq q$ (if p is a continuity point) or $D^+(p) < q$ and $D^-(p) \geq q$ (if p is a discontinuity point). In any case, if the auction enters step 3 then $D^-(p^*) \geq q$, and the claim follows. $\square$

Claim 4.5 *The adaptive clinching auction satisfies Pareto optimality (PO).*

Proof: We will check the "no trade" condition of Prop. 2.5. We already showed property (a) ($\sum_i x_i = m$) and it remains to show property (b). Fix any two players i and j. We need to verify that, if j received at least one item, then i's remaining budget at the end of the auction is smaller than j's value. Consider the last price p at which player j received an item.

First suppose that p is not the price that ended the auction. In this case (step 2a), since j received an item, the auction rules imply that $D^+_{-j}(p)$ exactly equals the number of items left after player j was allocated her items. Since the auction allocates all items, and since it is IR, we get that each player $i \neq j$ received after price p exactly $D_i(p)$, her demand at p. In particular, this means that the remaining available budget of i is at most p (otherwise the demand of i at p was higher – she could have bought one more item at a price lower than her value). On the other hand, $v_j > p$, since j demanded items at p, and we are done.

Now suppose that p is the price at which the auction ended. The auction rules imply that if i had $D^+_i(p) > 0$ then she received all this demand, and so by the same argument as above she does not have any remaining budget to buy an item from j. A second case is $D^+_i(p) = 0$ and $D_i(p) > 0$. This implies that the remaining budget of player i at this step is $B_i = p$. If player i received her demand $D_i(p)$ then the argument of above still holds. If not, it must be that player j received her items in step 3a or 3b (but not in 3c, since not all players in 3b were awarded their demand). Thus $D_j(p) > 0$ hence $v_j > p = B_i$ and a Pareto improvement cannot take place. The last case is $D_i(p) = 0$ and $D_i^-(p) > 0$. Hence $p = v_i$, and since $v_j \geq p$ this again rules out the possibility of a Pareto improvement. □

4.1 Revenue Considerations

We now examine the revenue properties of the adaptive clinching auction. Interestingly, we will show that it extracts a large fraction of the revenue of a non-discriminatory monopoly that knows the budgets and values of the players, and has to determine a single unit-price at which items will be sold. This shows that the adaptive clinching auction is not only efficient, but also has excellent revenue properties.

Let us start by defining our benchmark more precisely. To strengthen our result and simplify the analysis at the same time, we allow the monopoly (but not the mechanism!) to sell also fractions of the good, and not just integer quantities. Towards this end, let a *fractional assignment* be a real vector $x = (x_1, \ldots, x_n)$, where for each i, $x_i \geq 0$, and $\Sigma_i x_i \leq m$.

Fix the budgets and values of the bidders. Given a fractional assignment $x = (x_1, \ldots, x_n)$ of the items to the bidders, define the *monopoly revenue* from x to be $\Sigma_i x_i \cdot p^*(x)$, where $p^*(x)$ is the largest price that satisfies, for each i with $x_i > 0$, $v_i \geq p^*(x)$ and $b_i \geq x_i \cdot p^*(x)$. Define the *optimal monopoly revenue* to be the supremum over all fractional assignments x of the monopoly revenue from x. Let x^* be the fractional allocation that obtains this optimal monopoly revenue, and $p^* = p^*(x^*)$.

The fraction of the optimal monopoly price that the adaptive clinching auction collects depends on two parameters. The first parameter is the ratio $m/(m+n)$ (recall that m is the number of items and n is the number of players), which quantifies the item-divisibility of the setting. The second parameter is a "bidder dominance" parameter:

$$\alpha = \max_{i=1,\ldots,n} \frac{x_i^*}{\sum_{j=1}^n x_j^*}. \tag{1}$$

The term α represents the competition among the players. If it is small then the monopoly obtains its revenue from many bidders, while if it is large then there exist only few bidders that dominate the revenue of the monopoly. In the extreme, if $\alpha = 1$ then all items are sold to one single player. In this case it is intuitively clear that we cannot hope to extract a large revenue, since there is no competition. In general, we will show that the revenue of the adaptive clinching auction is at least $\frac{m}{m+n} \cdot (1-\alpha) \cdot R$, where R is the optimal monopoly revenue. Thus, when the number of items is much larger than the number of bidders and no single bidder dominates the monopoly's revenue, the revenue of our auction is very close to the monopoly's revenue.

The approach of comparing an auction's revenue to the optimal fixed-price revenue was initiated by Goldberg, Hartline, Karlin, Saks and Wright (2006). In the context of auctions with budget limitations it was used by Borgs et al. (2005) and Abrams (2006). In particular, Abrams (2006) showed that the optimal monopoly revenue is always at least half of the optimal *multi-price* revenue, that may charge different prices from different players.[10] Thus, comparing the revenue of the auction

[10] The argument is based on the following claim: if in the competitive equilibrium there is more than a single winner, then the revenue of this allocation is at least half of the optimal revenue (the maximal payment that satisfies

to *any other revenue criteria* can yield a ratio which may be smaller by a constant factor of at most 1/2.

We now begin to formally analyze the revenue of the adaptive clinching auction. In the rest of this section, we denote the optimal monopoly price by p^*, and the fractional assignment that maximizes the optimal monopolist price by $x^* = (x_1^*, \ldots, x_n^*)$.

Claim 4.6 *It can be assumed without loss of generality that all items are allocated in the fractional assignment that maximizes the optimal monopolist price. I.e., $\sum_i x_i^* = m$.*

Proof: Assume that $\sum_i x_i^* < m$. Let $W = \{\ i \mid v_i \geq p^*\ \}$ and $B = \sum_{i \in W} b_i$. Since the unit-price is p^*, any player i with $v_i < p^*$ must have $x_i^* = 0$, hence the optimal monopoly price is at most B. Additionally, for any $i \in W$ we must have $x_i^* = b_i/p^*$ since otherwise we can increase the quantity that i gets, contradicting the fact that x^* maximizes the revenue. This implies that $\sum_{i \in W} b_i/p^* = \sum_{i \in W} x_i^* < m$, hence $p^* > B/m$. Now, by setting $p = B/m$ and $x_i = b_i/p$ for any $i \in W$ (note that $v_i \geq p^* > B/m = p$), we get revenue exactly B, and $\sum_i x_i = m$, thus the claim follows. □

Claim 4.7 *No player clinches an item before the price reaches $\tilde{p} = \frac{m}{m+n} \cdot (1-\alpha) \cdot p^*$.*

Proof: We will show that, for each player i, $\sum_{j \neq i} D_j(\tilde{p}) \geq m$, which implies the claim. Let $W = \{\ j \mid x_j^* > 0\ \}$, and $W_{-i} = W \setminus \{i\}$. For any $j \in W$, $v_j \geq p^* > \tilde{p}$, hence $D_j(\tilde{p}) = \lfloor \frac{b_j}{\tilde{p}} \rfloor$. We therefore have

$$\sum_{j \neq i} D_j(\tilde{p}) \geq \sum_{j \in W_{-i}} D_j(\tilde{p}) = \sum_{j \in W_{-i}} \lfloor \frac{b_j}{\tilde{p}} \rfloor \geq \sum_{j \in W_{-i}} (\frac{b_j}{\tilde{p}} - 1) \geq \sum_{j \in W_{-i}} \frac{b_j}{\tilde{p}} - n$$

We next note that $\sum_{j \in W_{-i}} x_j^* = m - x_i^* \geq m - \alpha \cdot m = m(1-\alpha)$. This gives us:

$$\sum_{j \neq i} D_j(\tilde{p}) \geq \sum_{j \in W_{-i}} \frac{b_j}{\tilde{p}} - n = \frac{m+n}{m} \cdot \frac{1}{1-\alpha} \cdot \sum_{j \in W_{-i}} \frac{b_j}{p^*} - n \geq$$

$$\geq \frac{m+n}{m} \cdot \frac{1}{1-\alpha} \cdot \sum_{j \in W_{-i}} x_j^* - n \geq \frac{m+n}{m} \cdot \frac{1}{1-\alpha} \cdot m(1-\alpha) - n = m$$

which proves the claim. □

Theorem 4.8 *The revenue of the adaptive clinching auction is at least a fraction of $\frac{m}{m+n} \cdot (1-\alpha)$ of the revenue of the optimal monopoly price.*

Proof: By claim 4.6 we may assume that the optimal revenue is achieved by allocating all items and thus the optimal monopoly revenue is at most $m \cdot p^*$. The adaptive clinching auction sells all items (by claim 4.4), and by claim 4.7 each item is sold for a price of at least $\tilde{p} = \frac{m}{m+n} \cdot (1-\alpha) \cdot p^*$. Thus the revenue of the adaptive clinching auction is at least $m \cdot \tilde{p} = \frac{m}{m+n} \cdot (1-\alpha) \cdot (m \cdot p^*)$. □

individual rationality: $p_i \leq b_i$ and $p_i \leq x_i \cdot v_i$). Let us sketch the proof of this. Let p be the equilibrium price. Split the bidders to those with $v_i > p$ and those with $v_i \leq p$. The equilibrium revenue is $m \cdot p$. All bidders in the first set pay their full budget anyway in the equilibrium. We can never get more than a total of payment $m \cdot p$ from all bidders in the second set (since $v_i \leq p$). Thus the optimal revenue is at most $2m \cdot p$.

While it is clear why the $(1-\alpha)$ factor is essential in the statement of the theorem, it might be useful to see the role of the indivisibility factor. To demonstrate this, consider the following example. Suppose the number of items and bidders is equal, and all bidders have budget of 1 and value ∞. The monopoly sells one item to each player for a price of 1. The adaptive clinching auction sells one item to each player, for a price of 1/2, since at this price $D_i^+(1/2) = 1$ for every player i. Thus, there is a ratio of 1/2 between our revenue and the monopoly revenue, as theorem 4.8 predicts.

5 Uniqueness of the Clinching Auction

In this section we show that the ascending clinching mechanism is essentially the only mechanism that is truthful, individually rational, and Pareto optimal for the setting of publicly known budgets. In the next section we utilize this result to show that there is no mechanism if the budgets are private.

Strictly speaking, we do not prove uniqueness for all possible budgets b_1 and b_2, but for "almost" all budgets. This is in a sense the best we can hope for, as, for example, for one item and $b_1 = b_2$ there are indeed multiple possible auctions (which are identical up to tie breaking). The following technical definition attempts to deal with this issue.

Let $S = (S_1, S_2)$ be a partition of $\{1, \dots, m\}$. Given $b_1, b_2 \geq 0$, define $b_i^{k,S}$ recursively, for each $1 \leq k \leq m$: for $k = m$, $b_1^{m,S} = b_1, b_2^{m,S} = b_2$. For each $1 \leq k \leq m-1$, if $k \in S_1$ then: $b_1^{k,S} = b_1^{k+1,S}$, $b_2^{k,S} = b_2^{k+1,S} - \frac{b_1^{k+1,S}}{k+1}$. if $k \in S_2$then: $b_1^{k,S} = b_1^{k+1,S} - \frac{b_2^{k+1,S}}{k+1}$, $b_2^{k+1,S} = b_2^{k+1,S}$. We say that b_1 and b_2 are *S-generic* if for each $1 \leq k \leq m$ we have that $b_1^k \neq b_2^k$. We say that b_1 and b_2 are *generic* if they are S-generic for all S.

Notice that given any b_1 and b_2, a small perturbation will make them generic.

Theorem 5.1 *Let A be a truthful, Pareto optimal, and individually rational mechanism for m items and 2 players with known budgets b_1 and b_2 that are generic. If $v_1 \neq v_2$ then the output of A coincides with the clinching auction.*

The proof shows that all truthful, Pareto optimal, and individually rational mechanisms has the same output under the conditions of the lemma. Since the adaptive clinching auction is truthful, Pareto optimal, and individually rational, all other mechanisms coincide with it. We start with a useful lemma:

Lemma 5.2 *If $v_j < v_i$ and $v_j \leq \frac{b_i}{m}$ then player i receives all items and pays $p_i = m \cdot v_j$ in any truthful mechanism that satisfies IC, PO, and IR. In this case j's payment, p_j, is exactly zero.*

Proof: First consider the case $v_j < v_i < \frac{b_i}{m}$. In this case if player i receives $x < m$ items then since by IR he pays at most $x \cdot v_i < \frac{m-1}{m} b_i$ he has left enough money to buy an item from player j and pay him $v_j + \epsilon < v_i$, which contradicts PO. Thus player i receives all items. Standard monotonicity arguments (see e.g. footnote 12 below) now imply that i receives all items for any $v_i \geq \frac{b_i}{m}$ (when $v_j < \frac{b_i}{m}$).

If $v_j = \frac{b_i}{m}$ then for $v_i < \frac{m-1}{m-2} \cdot \frac{b_i}{m}$ it must be that player i receives $x \geq m-1$ items, otherwise if $x \leq m-2$ then by IR $p_i \leq x \cdot v_i \leq \frac{(m-1)b_i}{m}$ and $b_i - p_i \geq \frac{b_i}{m} = v_j$, and by lemma 2.5 this contradicts PO since $v_i > v_j$. If $x = m-1$ then by monotonicity player i receives $m-1$ items for any value in the interval $(\frac{b_i}{m}, v_i]$, therefore by IC her payment p_i is at most $\frac{(m-1)b_i}{m}$. But then again this contradicts PO as above. Thus player i receives all items in this case as well.

To prove that the payments are as claimed first suppose that $v_j = 0$. By IR $p_j \leq 0$. For any declaration $v_i' > 0$ player i receives all items (as argued above) and pays at most $p_i' \leq m \cdot v_i'$. Thus by IC if $v_j = 0$ then $p_i \leq 0$. NPT requires $p_i + p_j \geq 0$ which implies $p_i = p_j = 0$ for the case $v_i > v_j = 0$.

For a general value v_j, since j receives no items here as well, then IC implies $p_j = 0$. Using the standard argument of the second-price auction we finally get that $p_i = m \cdot v_j$, and the claim follows. □

We continue with the main proof. Without loss of generality we assume throughout that $b_1 < b_2$. The proof is by induction on the number of items m, and we start with the base case $m = 1$.

Lemma 5.3 *All truthful, Pareto optimal, and individually rational mechanisms for one item and 2 players with known generic budgets have the same output if $v_1 \neq v_2$.*

Proof: We show that the only possible mechanism is the following: the winner is the player i that maximizes $\min(b_i, v_i)$. The winner pays the mechanism $\min(b_j, v_j)$, where j is the other player, and the loser's payment is exactly zero.[11].

It is easy to verify that the above mechanism satisfies the required properties. We now prove that this is the only possible mechanism. If $\min(v_1, v_2) \leq b_1$ then the claim follows from lemma 5.2. Otherwise assume $v_1, v_2 > b_1$.

We show that player 2 must win the item. First observe that if $v_1 < \min(v_2, b_2)$ then the only Pareto optimal allocation allocates the item to 2 (in the other allocation player 2 can buy the item from 1, and they are both better off). Suppose that there exists some value $v_1' > b_1$ such that 1 wins the item even though $v_2 > b_1$. By feasibility 1's payment in this case is at most b_1, and 1 has positive utility from declaring v_1'. Thus when 1's true value is $b_1 < v_1 < \min(v_2, b_2)$ he can declare v_1' and improve his utility, contradicting IC.

Therefore for any $v_2 > b_1$ player 2 must be the winner. Player 1's payment must be exactly zero by IC since his payment must be equal to the case when he declares $v_1' < b_1$. This also implies that player 2's payment is the minimal possible value he needs to declare in order to win, i.e. $\min(b_1, v_1)$, and the claim follows. □

We now continue the induction, assuming uniqueness for $m-1$ items, and proving uniqueness for m items. The logic is as follows. We start with some mechanism A for m items that is truthful, Pareto optimal, and individually rational. We then explicitly describe the output (and payments) of A on all inputs, except for inputs of the form $v_1, v_2 \geq \frac{b_1}{m}$. To characterize A's behavior in this domain, we use A to construct a new mechanism A_{m-1} for $m-1$ items *and different budgets.* At the beginning A_{m-1} will only be defined on $v_1, v_2 \geq \frac{b_1}{m}$. We will show that the output of A on inputs where $v_1, v_2 \geq \frac{b_1}{m}$ is defined by the output of A_{m-1}.

Now we would like to finish the proof by claiming that A_{m-1} is unique, by the induction hypothesis. However, since A_{m-1} is not defined on all the domain of possible valuations, we cannot

[11]Notice that if the b_1 and b_2 are not generic, i.e., $b_1 = b_2$, then indeed this auction is not uniquely defined as if $v_1, v_2 > b_1 = b_2$ we can break ties in favor of both players, and still get a valid output. Also notice that this mechanism is indeed identical to the clinching auction.

directly apply the induction hypothesis, as there might be other mechanisms if the domain of possible valuations is restricted. To overcome this, we will extend A_{m-1}, and define its output on all valuations in the domain. Then we will show that A_{m-1} is Pareto optimal, individually rational, and truthful, hence it is unique by the induction hypothesis. Now we can uniquely determine the output of A on all possible valuations, and in particular in the domain $v_1, v_2 \geq \frac{b_1}{m}$, as needed.

Let us now define the mechanism A_{m-1}. A_{m-1} works on budgets $b'_1 = b_1$ and $b'_2 = b_2 - \frac{b_1}{m}$. Notice that b'_1 and b'_2 are generic, and that now it is not necessarily true that $b'_1 \leq b'_2$. We start by defining A_{m-1} on inputs where $v_1, v_2 > \frac{b_1}{m}$: denote the output of A given inputs v_1 and v_2 by $(\vec{x}, \vec{p})$, where x_i is the amount that i gets, and p_i is his payment. Let the output of A_{m-1} be (x_1, p_1) for player 1 (i.e., as in A), and for player 2 set the output to $(x_2 - 1, p_2 - \frac{b_1}{m})$. In particular, observe that given the output of A_{m-1} on valuations in this domain, we can deduce the output of A on the same valuations.

We now extend the definition of A_{m-1} for valuations where $\min(v_1, v_2) \leq \frac{b_1}{m}$. In this case we allocate all items to the bidder with the highest value, and his payment is $m - 1$ times the value of the other player.

Lemma 5.4 *A_{m-1} outputs a feasible allocation, and is Pareto optimal, individually rational, and truthful.*

Before proving this lemma itself we will require he following helpful lemmas:

Lemma 5.5 *Let A be a mechanism for m items that is Pareto optimal, individually rational, and truthful. Suppose that $\min(v_1, v_2) > \frac{b_1}{m}$. Then, a player that wins x items pays at least $x \cdot \frac{b_1}{m}$.*

Proof: Suppose by contradiction that there exist (v_1, v_2) in which some player i gets $x \geq 1$ items and pays $t < x \cdot \frac{b_1}{m}$. Consider now a different valuation v'_i such that $t/x < v'_i < \frac{b_1}{m}$. By Lemma 5.2 i is allocated no items when he declares are v'_i and the other player declares the same as before. Here i will be better off by declaring v_i instead of v'_i, since he will be allocated x items and will get a positive utility: $x \cdot v'_i - t > 0$, contradicting incentive-compatibility. □

Lemma 5.6 *Let A be a mechanism for m items that is Pareto optimal, individually rational, and truthful. Suppose that $v_2 > \frac{b_1}{m}$. Then, player 2 wins at least one item.*

Proof: Suppose that there is a declaration v_1 such that, when the players declare (v_1, v_2), player 1 win all items. By Lemma 5.5 the payment of player 1 is at least $m \cdot \frac{b_1}{m} = b_1$. His payment is exactly b_1 since this is his budget. By incentive-compatibility, in any declaration $v'_1 > \frac{b_1}{m}$ he must still win all items (player 2 still declares v_2). Fix v'_1 such that $\min(v_2, b_2) > v'_1 > \frac{b_1}{m}$. From above we get that player 1 gets all items when the declarations are (v'_1, v_2). However this contradicts pareto-optimality, using claim 2.5, since $v_2 > v'_1$ but $p_2 = 0 < b_2 - v'_1$. □

Lemma 5.7 *Let A be a mechanism for m items that is Pareto optimal, individually rational, and incentive-compatible. Suppose that $v_1 > \frac{b_1}{m}$. Then, if player 2 wins exactly one item he pays exactly $\frac{b_1}{m}$.*

Proof: Fix some v_2 such that, when the declaration is (v_1, v_2), player 2 gets $x_2 = 1$ and pays some p_2. By claim 5.5, $p_2 \geq \frac{b_1}{m}$. Now fix some v'_2 such that $v_2 > v'_2 > \frac{b_1}{m}$. Suppose that in the declaration (v_1, v_2') player 2 gets x_2' and pays p_2'. It is well-known[12] that IC implies that $x_2' \le x_2$. By claim 5.6 $x_2' \ge 1$, and therefore we must have $x_2' = 1$. IC now implies that $p_2 = p_2'$. Therefore we have $v_2' \ge p_2 \ge \frac{b_1}{m}$. Since this is true for any $v_2' > \frac{b_1}{m}$ we get that $p_2 = \frac{b_1}{m}$, as claimed. $\square$

Proof: (of Lemma 5.4) During the proof we abuse notation a bit and identify the output of A with A, and the output of A_{m-1} with A_{m-1}. We break the proof into several claims.

Claim 5.8 *A_{m-1} outputs a feasible allocation and is individually rational.*

Proof: We show that A_{m-1} is individually rational, i.e., a player that receives no items pays no items. The feasibility proof is trivial. If $\min(v_1, b_1) \le \frac{b_1}{m}$, then the we conduct a second price auction, and the loser pays nothing. Else, if player 1 is allocated no items in A_{m-1}, then he pays nothing, since A is individually rational and 1 gets nothing also in A. Consider the case where player 2 is allocated no items in A_{m-1}. It means that it was allocated exactly one item in A, and by Lemma 5.7 his payment is $\frac{b_1}{m}$ in A, hence in A_{m-1} his payment is 0. $\square$

Claim 5.9 *A_{m-1} is Pareto optimal.*

Proof: Consider first the case where $v_1, v_2 > \frac{b_1}{m}$. By claim 2.5, it is enough to show two things: (1) If $v_1 > v_2$ then $p_1' > b_1' - v_2$: since A is Pareto-optimal then $p_1 > b_1 - v_2$, and since $p_1' = p_1$ and $b_1' = b_1$ the claim follows; and (2) If $v_2 > v_1$ then $p_2' > b_2' - v_1$, or, equivalently, $v_1 > b_2' - p_2'$: since A is Pareto-optimal then $v_1 > b_2 - p_2$, and since $b_2' - p_2' = b_2 - p_2$ the claim follows.

Now consider the case where $\min(v_1, v_2) \le \frac{b_1}{m}$. Let $b_i' = \min(b_1', b_2')$. First, observe that we have that if $b_i' = b_1'$ then $\frac{b_1}{m} \le \frac{b_1'}{m-1}$, since $b_i' = b_1'$. For $b_i' = b_2' = b_2 - \frac{b_1}{m}$, we also have that $\frac{b_2'}{m-1} = \frac{b_2 - \frac{b_1}{m}}{m-1} \ge \frac{b_1 - \frac{b_1}{m}}{m-1} \ge \frac{b_1}{m}$. Hence in this range, by Lemma 5.2, it is Pareto optimal to allocate all items to the bidder with the highest value, as A_{m-1} indeed does. $\square$

Claim 5.10 *A_{m-1} is incentive compatible.*

Proof: Once again we consider the several different cases. Start with the case where $v_1, v_2 > \frac{b_1}{m}$, and suppose player i declares $v_i' > \frac{b_1}{m}$ instead (and is allocated x_i' items and pays p_i'). Clearly, $i \ne 1$, as the allocation and payment of player 1 are the same as in A, and A is truthful. Suppose $i = 2$ is better off declaring v_2': $v_2(x_2) - p_2 < v_2(x_2') - p_2'$. Observe that in A we have that: $v_2(x_2 + 1) - (p_2 + \frac{b_1}{m}) < v_2(x_2' + 1) - (p_2' + \frac{b_1}{m})$, a contradiction to the truthfulness of A.

Suppose that $v_1, v_2 > \frac{b_1}{m}$, and that player i declares $v_i' < \frac{b_1}{m}$ instead. Notice that $x_i' = 0$, so i cannot increase his profit from declaring v_i'.

In the case where $\min(v_1, v_2) \le \frac{b_1}{m}$ player i is not better off declaring $v_i' < \frac{b_1}{m}$, as in this range we are essentially conducting a second price auction, which is truthful.

Finally, suppose $\min(v_1, v_2) \le \frac{b_1}{m}$. Consider player i that declares $v_i' > \frac{b_1}{m}$. Suppose $v_j > \frac{b_1}{m}$, where j is the other player. Observe that if i wins some items, then by Lemma 5.5 j has to pay at least $\frac{b_1}{m}$ for every item he wins, which is more than is value. If $v_j < \frac{b_1}{m}$, then we conduct a second price auction, regardless of what i declares, and this auction is truthful. $\square$ $\square$

By the induction hypothesis, we have that A_{m-1} is unique. By our discussion, this is enough to prove the uniqueness of A and this concludes the proof of the theorem.

[12] A short proof, based on the W-MON condition of Bikhchandani et al. (2006), is: from IC we have $v_2 \cdot x_2 - p_2 \ge v_2 \cdot x_2' - p_2'$ since when the true type is v_2 the player will not benefit from declaring v_2'. Similarly, $v_2' \cdot x_2' - p_2' \ge v_2' \cdot x_2 - p_2$. Combining, we get $v_2'(x_2' - x_2) \ge p_2' - p_2 \ge v_2(x_2' - x_2)$, and since $v_2' < v_2$ it follows that $x_2' \le x_2$.

6 An Impossibility Result for Private Budgets

Once the public-budgets case is completely analyzed, the impossibility for private budgets follows quite easily.

Theorem 6.1 *There is no truthful, incentive compatible, and Pareto optimal mechanism if the budgets are private.*

Proof: We utilize our uniqueness result for 2 players with known budgets. Since we characterized exactly how the mechanism behaves with given budgets, it suffices to show an example where a player is better off declaring a different budget than his real one. Notice that although we present the example for two bidders, the result for more bidders follows by adding more bidders with value and budget of zero.

Suppose that $b_1 = 1, v_1 = \infty, b_2 = 1 + \Sigma_{k=2}^{m} \frac{1}{k} - \delta, v_2 = \infty$, for some small $\delta > 0$. (We might add some small perturbation to make b_1 and b_2 generic.) For each of the first $m-1$ items our auction will allocate the item to player 2 and charge $\frac{1}{k}$ for the k'th item. Then, player 1's budget is finally bigger than player 2's free budget, so player 1 wins the last item with a payment of $1 - \delta$.

Suppose now that player 1 declares $b_1' = 1 + \epsilon$ instead, for small enough ϵ. The resulting allocation is the same, but player 2 is charged $\frac{1+\epsilon}{k}$ for the k'th item (for $k > 1$). Thus, when the auction allocates the last item, player 2's free budget is smaller than before: $1 - \delta - \Sigma \frac{\epsilon}{k}$. This is also the payment of player 1. Notice that player 1 is allocated one item, just as when declaring b_1, but his payment is smaller, so he better off declaring b_1' instead of b_1. $\square$

7 The Infinitely-Divisible Good Setting

While the adaptive clinching auction may be applied in the infinitely divisible setting by treating it as a continuous time process, the analysis is not straight-forward. In this section we rely on this process to obtain an explicit closed-form auction for a divisible good setting, and we directly prove that it is incentive-compatible and Pareto. We limit ourselves to the case of two bidders. We then show that if the budgets are equal then this auction is unique among all anonymous auctions, and we use this to derive a general impossibility result for anonymous mechanisms in the private-budget case.

7.1 An IC+PO mechanism for known budgets

We construct an incentive-compatible and Pareto-optimal mechanism for two bidders with publicly-known budgets. We start by analyzing two special cases, that will be used later on as building blocks for the general mechanism.

First special case: only one bidder with a budget limit. We first look at the case where only one of the players is budget-limited. Assume that $b_1 = 1$ (this is w.l.o.g) and $b_2 = \infty$. Let us overview the course of the adaptive clinching auction for this case. As long as the price p is below 1 and below $\min(v_1, v_2)$, both players demand all the quantity, and so no clinching occurs. If $\min(v_1, v_2) \leq 1$ then the player i with the minimal value will drop out when the price will reach her value, and the other player will get the entire quantity and will pay the lower value. Otherwise assume that $\min(v_1, v_2) > 1$. When the price exceeds 1, player 1 starts reducing her demand to quantities smaller than 1 (recall that $D_i(p) = b_i/p$). Therefore player 2 starts clinching the quantity that is not being demanded anymore by player 1. The total quantity clinched up to price p is $1 - D_1(p) = 1 - 1/p$ and thus player 2 clinches $d(1 - D_1(p))/dp = 1/p^2$ units at marginal price p. The total payment of player 2 up to price p is obtained by integrating the product. This continues until the price reaches $\min(v_1, v_2)$ (recall that player 2 has infinite budget, hence she never reduces her demand). Once we reach the point $p = \min(v_1, v_2)$, the lower player drops, and the larger player gets the remaining quantity at the current unit-price. This leads us to "guess" that the following mechanism will be IC+PO for this special case:

Definition 7.1 (Mechanism A)

- *If* $\min(v_1, v_2) \leq 1$ *then the high player gets everything at the second price:* $x_i = 1, p_i = v_j$ *(and* $x_j = 0, p_j = 0$*), where* $v_i > v_j$*.*
- *Otherwise, if* $v_2 \geq v_1$ *then the high non-budget-limited player gets everything* $x_2 = 1$ *and pays* $1 + \ln v_1$*.*
- *Otherwise, if* $v_1 > v_2$ *then the high player gets* $x_1 = 1/v_2$ *and pays* $p_1 = 1$*, while the non-budget-limited player gets* $x_2 = 1 - 1/v_2$ *and pays* $p_2 = \ln v_2$*.*

We give an explicit proof that Mechanism A indeed satisfies PO and IC. In the proof, we use a slightly weaker assumption instead of $b_2 = \infty$, a relaxation that will become important in the sequel.

Proposition 7.2 *Fix any two budgets* $b_1 \leq b_2$*. Then, mechanism A is Pareto-optimal and individually-rational, and,*

1. *It is a dominant-strategy for player* 1 *to declare her true value.*
2. *If* $v_2 \leq e^{b_2 - 1}$ *then it is a dominant-strategy for player* 1 *to declare her true value. More precisely, let* $u_2(z)$ *denote player* 2*'s resulting utility when she declares* z*. Then* $u_2(v_2) \geq u_2(z)$ *for any real number* z*.*

Proof: Pareto-optimality follows directly from proposition 2.4 since in the first two cases the low bidder gets allocated 0, and in the last case, the high bidder has his budget exhausted.

Let us start by looking at the incentives of bidder 1. If $v_2 \leq 1$ then he is faced with exactly two possibilities $x_1 = 1, p_1 = v_2$ and $x_1 = 0, p_1 = 0$. It is clear that he prefers the former if and only if $v_1 \geq v_2$, which is what happens with the truth. If $v_2 > 1$ then he is faced with two possibilities: either declare some $z \leq v_2$ in which case he gets $x_1 = 0, p_1 = 0$ or declare some $z > v_2$ and get allocated $x_1 = 1/v_2, p_1 = 1$. His utility in the first case is $u_i = 0$ and in the second $u_i = v_1/v_2 - 1$, which is positive iff $v_1 > v_2$ and given to him by the mechanism when telling the truth $z = v_1$.

Now for bidder 2. The case $v_1 \leq 1$ is as before. Otherwise he may declare either $z < v_1$ getting $x_2 = 1 - 1/z, p_2 = \ln z$ or declaring $z \geq v_1$ getting $x_2 = 1, p_2 = 1 + \ln v_1$. In the first case his utility is at most $u_2(z) = v_2 - v_2/z - lnz$ (it is exactly this term if $p_2 \leq b_2$, otherwise it is smaller). This term for $u_2(z)$ is maximized for $z = v_2$ (by solving for $du_2/dz = 0$). Thus in the first case his utility is at most $v_2 - 1 - \ln v_2$. In the second case his utility at most $u_2 = v_2 - 1 - \ln v_1$. If $v_2 < v_1$ then the former term is larger than the latter term, and indeed by declaring $z = v_2$ the player obtains a utility exactly equal to $v_2 - 1 - \ln v_2$ since when $z = v_2$ we have $p_2 = \ln v_2 < \ln e^{b_2 - 1} < b_2$. If

$v_2 \geq v_1$ then the latter term is better, and indeed by declaring $z = v_2$ the player obtains a utility exactly equal to $v_2 - 1 - \ln v_1$ since in this case $p_2 = 1 + \ln v_1 \leq 1 + \ln v_2 \leq 1 + \ln e^{b_2 - 1} = b_2$. Thus declaring $z = v_2$ obtains maximal utility, no matter what is v_1.

Individual-rationality follows from incentive-compatibility, since a player can always obtain a zero utility by declaring $v_i = 0$. $\square$

Corollary 7.3 *Mechanism A is Pareto-optimal and incentive-compatible, assuming only one bidder is budget-constrained.*

Second special case: bidders with equal budgets. The second special case we analyze is when the budgets are equal. Assume without loss of generality that $b_1 = b_2 = 1$ and $v_1 \leq v_2$. In addition, it will be useful for the sequel to explicitly denote the initial quantity by Q (and not to assume $Q = 1$).

We again "guess" the correct mechanism by looking at the course of the adaptive clinching auction. Similarly to before, while $p \leq 1/Q$ no clinching occurs since each player demands all available quantity. At this point, the demand of both players is equal to available quantity, and hence from this point on both players will start clinching. Calculating the exact rate at which the clinching occurs is slightly more involved in this case. Let $D_i(p), b_i(p)$ denote the current demand and remaining budget of player i at price p, and let $q_i(p)$ denote the total quantity that player i have received up to price p. When the price reaches $\min(v_1, v_2)$, the lower player drops and the higher player receives the remaining quantity, but before this point the two players are completely identical, so we can remove the subscript i from the three functions. We have

$$D(p) = \frac{b(p)}{p} \ , \ b'(p) = -q'(p) \cdot p$$

directly from the definition of the adaptive clinching auction. It will turn out useful to construct the three functions so that clinching will continuously occur, for all prices $p \geq 1/Q$. For this to happen, we need that the current demand of each player will always be exactly equal to the current available quantity (since in such a case, and only in such a case, when a player decreases her demand, the other player performs clinching). This means:

$$D(p) = Q - 2 \cdot q(p)$$

Solving these three equations, we get:

$$q(p) = \frac{Q}{2} - \frac{1}{2 \cdot Q \cdot p^2} \ , \ b(p) = \frac{1}{Q \cdot p}$$

We next show explicitly that using these functions will indeed yield Pareto optimality and incentive compatibility. Moreover, in the sequel (Theorem 7.9) we show that this is the *unique* anonymous mechanism that is PO and IC.

Definition 7.4 (Mechanism B) *Assume that $b_1 = b_2 = 1$ and $v_1 \leq v_2$. Assume also that the initial available quantity is Q.*

- *If $v_1 \leq 1/Q$ then the high player gets everything at the second price: $x_2 = Q, p_2 = v_1 \cdot Q$ (and $x_1 = 0, p_1 = 0$).*

- *Otherwise, the low player gets* $x_1 = Q/2 - 1/(2 \cdot Q \cdot v_1^2)$ *and pays* $p_1 = 1 - 1/(Q \cdot v_1)$ *and the high player gets* $x_2 = Q/2 + 1/(2 \cdot Q \cdot v_1^2)$ *and pays* $p_2 = 1$*.*

Proposition 7.5 *Mechanism B is Pareto-optimal, individually-rational, and incentive-compatible, in the case of publically known and equal budgets.*

Proof: Pareto-optimality follows directly from proposition 2.4: in the first case the high player gets all the quantity, and in the second case the budget of the high player is exhausted.

Let us consider the incentives of one bidder with value v_i when the other bids a fixed value v_j. If $v_j \leq 1/Q$ then bidder i can choose between declaring $z \leq v_j$ in which case $x_i = 0, p_i = 0$ and thus $u_i = 0$ (in case of tie, if $x_i = 1, p_i = v_j$ then we still have $u_i = 0$) to bidding $z > v_j$ in which case $x_i = Q, p_i = v_j \cdot Q$ and thus $u_i = (v_i - v_j)Q$. The latter is better if and only if $v_i > v_j$, and by bidding $z = v_i$ player i gets the better option.

If $v_j > 1/Q$, then bidder i can choose between declaring $z < v_j$ in which case $x_i = Q/2 - 1/(2 \cdot Q \cdot z^2), p_i = 1 - 1/(Q \cdot z)$ to bidding $z > v_j$ in which case $x_i = Q/2 + 1/(2 \cdot Q \cdot v_j^2), p_i = 1$. Thus the utility when bidding $z < v_j$ is $v_i(Q/2 - 1/(2 \cdot Q \cdot z^2)) - (1 - 1/(Q \cdot z))$, and this is maximized by $z = v_i$. Thus the utility when bidding $z < v_j$ is at most $v_i(Q/2 - 1/(2 \cdot Q \cdot v_i^2)) - (1 - 1/(Q \cdot v_i))$ (call this $u^{(L)}$, and the utility when bidding $z > v_j$ is exactly $v_i(Q/2 + 1/(2 \cdot Q \cdot v_j^2)) - 1$ (call this $u^{(H)}$).

A short calculation shows that $u^{(L)} > u^{(H)}$ if and only if $v_i < v_j$. Therefore: (1) if $v_i < v_j$ then a player will maximize his utility by obtaining a utility equal to $u^{(L)}$, which can be obtained by declaring $z = v_i$, and (2) if $v_i > v_j$ then a player will maximize his utility by obtaining a utility equal to $u^{(H)}$, which can be obtained by declaring $z = v_i$. Thus no matter what is v_j, declaring v_i will maximize player i's utility. This proves incentive-compatibility.

Individual-rationality follows from incentive-compatibility, since a player can always obtain a zero utility by declaring $v_i = 0$. $\square$

The general case: bidders with arbitrary budgets. We now reach the case of general budgets, and again wish to examine the course of the adaptive clinching auction before constructing the closed-form mechanism. Assume that $b_1 = 1 < b_2$. When the price just crosses the point $p = 1$ the situation is similar to the first special case from above: player 2 still demands all quantity so player 1 does not perform clinching, and player 1 starts reducing her demand, so player 2 starts to clinch. Using the equations found in the first special case from above, the total clinched quantity of player 2 at price p is $q_2(p) = 1 - 1/p$, and her remaining budget is $b_2(p) = b_2 - \ln p$. This situation continues until the point where the available quantity at price p equals the demand of player 2 at that price, which can be found by solving:

$$\frac{b_2 - \ln p}{p} = \frac{b_2(p)}{p} = D_2(p) = 1 - q_2(p) = \frac{1}{p}$$

and the solution is $p^* = e^{b_2 - 1}$. To verify, note that at this price the available quantity is $1/p^*$, and the remaining budget of player 2 is $b_2(p^*) = 1$. Hence player 2 demands exactly the remaining quantity. Looking at player 1 we can see that, since she did not clinch anything up to p^*, her remaining budget is equal to her original budget, which was $b_1 = 1$. Thus the demand of player 1 at p^* is also $1/p^*$, again exactly equal to the remaining quantity. Therefore at p^* we have switched to a situation very similar to the second special case from above: both players have remaining budgets that are equal to 1, and at an initial price p^* simultaneously demand exactly the available quantity. Thus, the calculations of the second special case of above, setting $Q = 1/p^*$, describe the course of the auction from this point until the end. In other words, we see that the general construction is simply a combination of the two special cases studied above. Note that the course of the above auction stops whenever the price reaches the point $\min(v_1, v_2)$, and this can be in any of the three parts of the auction – at $p < 1$, at $1 < p \leq p^*$, or at $p > p^*$. This description gives us the general mechanism:

Definition 7.6 (General Mechanism) *Assume* $b_1 = 1 \leq b_2$ *and initial quantity of* 1*. Let* $p^* = e^{b_2 - 1}$*.*

- *If* $\min(v_1, v_2) < p^*$ *then run Mechanism A.*
- *Otherwise, allocate to player* 2 *an initial quantity of* $1 - 1/p^*$ *for a total price* $b_2 - 1$*. Allocate the remaining quantity* $Q = 1/p^*$ *using Mechanism B, where the initial budget of player* 2 *at the mechanism is* $b_2 = 1$*, and the rest of the parameters are unchanged.*

Proposition 7.7 *The General Mechanism is Pareto-optimal and incentive-compatible in the case of publically known budgets.*

Proof: We first prove Pareto-optimality. If $\min(v_1, v_2) < p^*$ then the outcome is determined by mechanism A, hence is Pareto-optimal by proposition 7.2. If $\min(v_1, v_2) \geq p^*$, then mechanism B is run, and inside it we always enter the second option, which implies that the high-value player pays 1. If this is player 1 then this exhausts her budget, and if this is player 2 then her total payment is $(b_2 - 1) + 1 = b_2$, so her budget exhausted as well. Thus by proposition 2.4 the outcome is indeed Pareto-optimal.

We now prove incentive-compatibility. Consider first the incentives of player 1. If $v_2 < p^*$ then mechanism A is used, no matter what player 1 reports, and the claim follows from proposition 7.2. Otherwise $v_2 > p^*$. If $v_1 < p^*$ then by the properties of mechanism B player 1 prefers receiving zero utility to receiving some quantity as a result of declaring some $z > p^*$, since, in mechanism B, when $v_1 < p^*$ player 1 gets nothing. Thus in this case player 1 maximizes utility by the truthful declaration. If $v_1 > p^*$ then if she declares some $z < p^*$ she gets zero utility while if she declares v_1 she gets a non-negative utility since mechanism B is individually rational. Thus she prefers to declare some $z > p^*$ and since mechanism B is incentive-compatible it must be that $z = v_1$. This proves incentive-compatibility for player 1.

Now consider player 2. If $v_1 < p^*$ then the proof is a before. Otherwise $v_1 > p^*$. If $v_2 < p^*$ then player 2 prefers getting nothing from mechanism B to getting some positive quantity as a result of declaring some $z > p^*$, and she prefers getting from mechanism A a quantity that results from declaring v_2 to getting $1 - 1/p^*$ and paying $b_2 - 1$ (which results from declaring $z = p^*$). Thus player 2 prefers to declare v_2 over declaring some $z > p^*$, and therefore by the incentive-compatibility of mechanism A she prefers to declare v_2 over any other declaration z. If $v_2 > p^*$ then player 2 prefers getting some quantity from mechanism B according to the declaration $z = v_2$ over not getting anything from mechanism B, since mechanism B is individually rational. Additionally, player 2 prefers the outcome $x_2 = 1 - 1/p^*, p_2 = b_2 - 1$ over any other outcome that results from mechanism A by declaring some $z < p^*$, since $v_2(1 - 1/p^*) - lnp^* > v_2(1 - 1/z) - lnz$. Thus player 2 prefers the outcome resulting from declaring v_2 over any other outcome that results from declaring some $z < p^*$. By the incentive compatibility of mechanism B, declaring v_2 will maximize player 2's utility. Therefore incentive-compatibility for player 2 follows.

Individual-rationality follows from incentive-compatibility, since a player can always obtain a zero utility by declaring $v_i = 0$. $\square$

7.2 Uniqueness for equal (and known) budgets

To show uniqueness we cannot simply use similar arguments to the ones of the discrete case, since there we used induction on the number of items, while here the number of items is fixed, in some sense. Thus we use completely different arguments, and rely on the additional property of anonymity. As defined, mechanism B is not really anonymous, breaking the tie $v_1 = v_2$ "in favor" of v_2. An anonymous mechanism with the same properties can be obtained by "splitting" in case of a tie:

Definition 7.8 (Mechanism C)

- *If $v_1 = v_2 = v \leq 1$ then $x_1 = x_2 = 1/2$ and $p_1 = p_2 = v/2$.*
- *If $v_1 = v_2 = v > 1$ then $x_1 = x_2 = 1/2$ and $p_1 = p_2 = 1 - 1/(2v)$.*
- *If $v_1 \neq v_2$ then run mechanism B.*

It is not hard to verify that mechanism C maintains the properties IC and PO of mechanism B. Moreover, we show:

Theorem 7.9 *Mechanism C is the only anonymous mechanism for the divisible good setting that satisfies incentive compatibility (IC) and Pareto-optimality (PO).*

Proof: Let us fix a mechanism that satisfies the above properties and reason about it. In the rest of the proof we denote the smaller value by v_i, thus $v_i \leq v_j$.

Step 1: We first handle the case of $v_i \leq 1$. If also $v_j < 1$ then $p_j \leq v_j < 1$ and thus PO implies $x_i = 0$ and $x_j = 1$. By the usual arguments of IC we must have $p_j = v_i$. Now for values $v_j \geq 1$, if $x_j = 1$ then by IC p_j is determined by x_j and thus is $p_j = v_i$. Otherwise $x_i > 0$ and thus by PO $p_j = 1$ but this is a contradiction to IC since declaring a value $v_i < v'_j < 1$ both increases x_j and decreases p_j.

Step 2: We will now show that there exist functions $q(t)$ and $p(t)$ such that whenever $v_i < v_j$ then $x_i = q(v_i)$, $p_i = p(v_i)$, and $x_j = 1 - q(v_i)$, $p_j = 1$. I.e. the low player's value determines the allocation between the two players as well as his own payment, while the high player exhausts his budget. First assume to the contrary that for some $1 < v_i < v_j$, $p_j < 1$, and thus by PO $x_i = 0$, $p_i = 0$, and $x_j = 1$. But then a bidder with $p_j < v'_j < 1 < v_i$ that, according to step 1, gets nothing, would be better off declaring v_j and getting positive utility, in contradiction to IC. Thus $p_j = 1$ whenever $1 < v_i < v_j$. Thus, by IC, for a fixed v_i, different values of v_j must get the same x_j, i.e. x_j depends only on v_i. By PO, $x_i = 1 - x_j$ and thus it also only depends on V_i, and then by IC p_i must be determined uniquely by x_i and thus depends only on v_i.

Step 3: Using IC as usual, we have that for any $1 < t < t' < v_j$: $t(q(t') - q(t)) \leq p(t') - p(t) \leq t'(q(t') - q(t))$. As usual this implies that $dp/dt = t \cdot dq/dt$ or, more precisely, since we do not know that q is differentiable or even continuous, that $p(t) = tq(t) - \int_1^t q(x)dx$, where integrability of q is a direct corollary of its monotonicity. (This already takes into account the boundary condition that for t approaching 1 from above, $q(x)$ must approach 0, as otherwise for the fixed limit $\delta > 0$ we will have that for every value of $v_2 > v_1 > 1$, we will have $x_2 \leq 1 - \delta$, which by IR implies $p_2 < 1$ and thus contradicts PO.)

Step 4: Using IC we have that for $1 < t < v_j < t'$: $tq(t) - p(t) \geq t(1 - q(t')) - 1$ and $t'q(v_j) - p(v_j) \geq t'(1 - q(v_j)) - 1$ Letting t, t' approach v_j we have that $tq(t) - p(t) = t(1 - q(t)) - 1$, i.e. $p(t) = 1 + t(2q(t) - 1)$ for all t except for at the at most countably many points of discontinuity of q.

Step 5: Combining the last two steps we have $1 + t(2q(t) - 1) = tq(t) - \int_1^t q(x)dx$, i.e. $q(t) = 1 - 1/t - (\int_1^t q(x)dx)/t$, except for at most the countably many points of discontinuity of q. The solution to this differential equation, is $q(t) = 1/2 - 1/(2t^2)$, which gives $p(t) = 1 - 1/t$. The uniqueness of solution is implied since if another function satisfies the equation everywhere except for countably many points, then the difference function $d(t)$ would satisfy $d(t) = -(\int_1^t d(x)dx)/t$ everywhere except for countably many points, which only holds for $d(t) = 0$. □

7.3 The impossibility for private budgets

From theorem 7.9 we rather easily deduce:

Theorem 7.10 *There exists no anonymous, incentive compatible, and Pareto-optimal mechanism for the divisible good setting, for the case of privately known budgets b_1, b_2.*

Proof: We first note that by direct scaling of theorem 7.9 we have that that the only anonymous IC+PO mechanism for the case of a publically known budget $b_1 = b_2 = B$ gives $x_i = (1 - B^2/v_i^2)/2$, $p_i = B(1 - B/v_i)$, $x_j = (1 + B^2/v_i^2)/2$, $p_j = 1$ for the case $1 < v_i < v_j$, and $x_j = 1$, $p_j = v_i$, $x_i = 0$, $p_i = 0$ for the case $v_i < 1$ and $v_i < v_j$.

Let us now assume to the contrary that an anonymous IC+PO auction existed, then for any fixed values of b_1, b_2 it must be identical to the scaled version of mechanism C. Now let us look at a few cases with $v_1 = 2$, $v_2 = 2 + \epsilon$. First let us look at the case $b_1 = b_2 = 1$. The previous theorem mandates that in this case $x_1 = 3/8$, $p_1 = 1/2$ and $x_2 = 5/8$, $p_2 = 1$, (and thus $u_2 = 1/4 + O(\epsilon)$.)

Now let us look at the case where $b_1 = b_2 = 2 - \epsilon$. Again the theorem 7.9 with scaling mandates that $x_1 > 0$ and also $u_1 > 0$.

Now let us look at the case of $b_1 = 1$ and $b_2 = 2 - \epsilon$. If $x_2 < 1$ then, by PO, $p_2 = b_2 = 2 - \epsilon$, and thus $u_2 < 2\epsilon$, which means that player 2 has a profitable lie stating $b_2 = 1$. Thus $x_2 = 1$ and $x_1 = 0$, but then player 1 has a profitable lie stating that $b_1 = 2 - \epsilon$. □

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