AXIOMS FOR DEFERRED ACCEPTANCE

FUHITO KOJIMA AND MIHAI MANEA

Department of Economics, Stanford University, fuhitokojima1979@gmail.com

Department of Economics, Harvard University, mmanea@fas.harvard.edu

Abstract. The deferred acceptance algorithm is often used to allocate indivisible objects when monetary transfers are not allowed. We provide two characterizations of agent-proposing deferred acceptance allocation rules. Two new axioms, individually rational monotonicity and weak Maskin monotonicity, are essential to our analysis. An allocation rule is the agent-proposing deferred acceptance rule for some acceptant substitutable priority if and only if it satisfies non-wastefulness and individually rational monotonicity. An alternative characterization is in terms of non-wastefulness, population monotonicity and weak Maskin monotonicity. We also offer an axiomatization of the deferred acceptance rule generated by an exogenously specified priority structure. We apply our results to characterize efficient deferred acceptance rules.

1. Introduction

In an assignment problem, a set of indivisible objects that are collectively owned need to be assigned to a number of agents, with each agent entitled to at most one object. Student placement in public schools and university housing allocation are examples of important assignment problems in practice. Agents are assumed to have strict preferences over objects (and being unassigned). An allocation rule specifies an assignment of objects to agents for each preference profile. No monetary transfers are allowed.

In many assignment problems each object is endowed with a priority over agents. For example, schools in Boston give higher priority to students who live nearby or have siblings

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already attending. An allocation rule is stable with respect to a given priority profile if there is no agent-object pair (i, a) such that (1) i prefers a to his assigned object, and (2) either i has higher priority for a than some agent who is assigned a, or a is not assigned to other agents up to its quota. In the school choice settings of Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003) priorities represent a social objective–e.g, it may be desirable that in Boston students attend high-schools within walking distance from their homes or that in Turkey students with excellent achievements in mathematics and science go to the best engineering universities–and stability can be regarded as a normative fairness requirement in the following sense. An allocation is stable if no student has justified envy–any school that a student prefers to his assigned school is attended (up to capacity) by students who enjoy higher priority for it.

The deferred acceptance algorithm of Gale and Shapley (1962) determines a stable allocation which has many appealing properties. The agent-proposing deferred acceptance allocation Pareto dominates any other stable allocation. Moreover, the agent-proposing deferred acceptance rule makes truthful reporting of preferences a dominant strategy for every agent. Consequently, the deferred acceptance rule is used in many practical assignment problems such as student placement in New York City and Boston (Abdulkadiroğlu, Pathak, and Roth 2005, Abdulkadiroğlu, Pathak, Roth, and Sönmez 2005) and university house allocations in MIT and the Technion (Guillen and Kesten 2008, Perach, Polak, and Rothblum 2007), to name some concrete examples. There are proposals to apply the rule to other problems, such as course allocation in business schools (Sönmez and Ünver 2009) and assignment of military personnel to positions (Korkmaz, Gökçen, and Çetinyokuş 2008).

Despite the importance of deferred acceptance rules in both theory and practice, no axiomatization has yet been obtained in an object allocation setting with unspecified priorities. Our first results (Theorems 1 and 2) offer two characterizations of deferred acceptance rules with acceptant substitutable priorities.

For the first characterization, we introduce a new axiom, individually rational (IR) monotonicity. We say that a preference profile R' is an IR monotonic transformation of a preference profile R at an allocation μ if for every agent, any object that is acceptable and preferred to μ under R' is preferred to μ under R. An allocation rule φ satisfies IR monotonicity if every agent weakly prefers $\varphi(R')$ to $\varphi(R)$ under R' whenever R' is an IR monotonic transformation of R at $\varphi(R)$. If R' is an IR monotonic transformation of R *at* $\varphi(R)$, then the interpretation of the change in reported preferences from R to R' is that all agents place fewer claims on objects they cannot receive, in the sense that each agent's set of acceptable objects that are preferred to $\varphi(R)$ shrinks. Intuitively, the IR monotonicity axiom requires that all agents be weakly better off when some agents claim fewer objects. The IR label captures the idea that each agent effectively places claims only on acceptable objects; an agent may not be allocated unacceptable objects because he can opt to remain unassigned, so the relevant definition of an upper contour set includes the IR constraint. IR monotonicity requires allocations be monotonic in the IR constrained upper contour sets. IR monotonicity resembles Maskin monotonicity (Maskin 1999), but the two axioms are independent.

We also define a weak form of efficiency, the non-wastefulness axiom. An allocation rule is non-wasteful if at every preference profile, any object that an agent prefers to his assignment is allocated up to its quota to other agents. Our first characterization states that an allocation rule is the deferred acceptance rule for some acceptant substitutable priority if and only if it satisfies non-wastefulness and IR monotonicity (Theorem 1).

In order to further understand deferred acceptance rules, we provide a second characterization based on axioms that are mathematically more elementary and tractable than IR monotonicity. An allocation rule is population monotonic if for every preference profile, when some agents deviate to declaring every object unacceptable (which we interpret as leaving the market unassigned), all other agents are weakly better off (Thomson 1983a, Thomson 1983b). Following Maskin (1999), R' is a monotonic transformation of R at μ if for every agent, any object that is preferred to μ under R' is preferred to μ under R. An allocation rule φ satisfies weak Maskin monotonicity if every agent prefers $\varphi(R')$ to $\varphi(R)$ under R' whenever R' is a monotonic transformation of R at $\varphi(R)$. Our second result shows that an allocation rule is the deferred acceptance rule for some acceptant substitutable priority if and only if it satisfies non-wastefulness, weak Maskin monotonicity and population monotonicity (Theorem 2).

We also study allocation rules that are stable with respect to an exogenously specified priority profile Ch (Section 6). We show that the deferred acceptance rule at Ch is the only stable rule at Ch that satisfies weak Maskin monotonicity (Theorem 3).

In addition to stability, efficiency is often a goal of the social planner. We apply our axiomatizations to the analysis of efficient deferred acceptance rules. The Maskin monotonicity axiom plays a key role. Recall that an allocation rule φ satisfies Maskin monotonicity if $\varphi(R') = \varphi(R)$ whenever R' is a monotonic transformation of R at $\varphi(R)$ (Maskin 1999). We prove that an allocation rule is an efficient deferred acceptance rule if and only if it satisfies Maskin monotonicity, along with non-wastefulness and population monotonicity; an equivalent set of conditions consists of Pareto efficiency, weak Maskin monotonicity and population monotonicity (Theorem 4).

Priorities are not primitive in our model except for Section 6, and our axioms are "priority-free" in the sense that they do not involve priorities. The IR monotonicity axiom conveys the efficiency cost imposed by stability with respect to some priority structure.[1] Whenever some agents withdraw claims for objects that they prefer to their respective assignments, all agents benefit. In the context of the deferred acceptance algorithm, the inefficiency is brought about by agents who apply for objects that tentatively accept them, but subsequently reject them. While it is intuitive that deferred acceptance rules satisfy IR monotonicity, it is remarkable that this "priority-free" axiom fully describes the theoretical contents of the deferred acceptance algorithm (along with the requirement of non-wastefulness).

The weak Maskin monotonicity axiom is mathematically similar to–and is weaker than (i.e., implied by)–Maskin monotonicity. We establish that weak Maskin monotonicity is sufficient, along with non-wastefulness and population monotonicity, to characterize deferred acceptance rules. At the same time, if we replace weak Maskin monotonicity by

[1]We do not regard IR monotonicity as a normative (either desirable or undesirable) requirement, but as a positive comprehensive description of the deferred acceptance algorithm. The reason is that priorities often reflect social objectives, and "priority-free" statements such as IR monotonicity may lack normative implications for priority-based assignment problems. The present welfare analysis disregards the social objectives embedded in the priorities. Nonetheless, as already mentioned, for a given priority structure, the corresponding deferred acceptance rule attains constrained efficiency subject to stability.

Maskin monotonicity in the list of axioms above we obtain a characterization of efficient deferred acceptance rules. The contrast between these two results demonstrates that we can attribute the inefficiency of some deferred acceptance rules entirely to instances where weak Maskin monotonicity is satisfied, but Maskin monotonicity is violated.

Our analysis focuses on substitutable priorities because priorities may be non-responsive but substitutable in applications. Such priorities arise, for example, in school districts concerned with balance in race distribution (Abdulkadiroğlu and Sönmez 2003) or in academic achievement (Abdulkadiroğlu, Pathak, and Roth 2005) within each school. In fact, substitutability of priorities is an "almost necessary" condition for the non-emptiness of the core.[2] [3] Since the relevant restrictions on priorities vary across applications, allowing for substitutable priorities is a natural approach.

Special instances of deferred acceptance rules have been characterized in the literature. Svensson (1999) axiomatizes the serial dictatorship allocation rules. Ehlers, Klaus, and Papai (2002), Ehlers and Klaus (2003), Ehlers and Klaus (2006), and Kesten (2006a) offer various characterizations for the mixed dictator-pairwise-exchange rules. Mixed dictator-pairwise-exchange rules correspond to deferred acceptance rules with acyclic priority structures. For responsive priorities, Ergin (2002) shows that the only deferred acceptance rules that are efficient correspond to acyclic priority structures.

Other allocation mechanisms have been previously characterized. Papai (2000) characterizes the hierarchical exchange rules, which generalize the priority-based top trading cycle rules of Abdulkadiroğlu and Sönmez (2003). In the context of housing markets, Ma (1994) characterizes the top trading cycle rule of David Gale described by Shapley and Scarf (1974). Kesten (2006b) shows that the deferred acceptance rule and the top trading

[2]Formally, suppose there are at least two proper objects a and b. Fix a non-substitutable priority for a. Then there exist a preference profile for the agents and a responsive priority for b such that, regardless of the priorities for the other objects, the core is empty. The first version of this result, for a slightly different context, appears in Sönmez and Ünver (2009). The present statement is due to Hatfield and Kojima (2008).

[3]When priorities are substitutable, the core coincides with the set of stable allocations.

cycle rule for some fixed priority profile are equivalent if and only if the priority profile is acyclic.[4]

When the priority structure is a primitive of the model as in Section 6, alternative characterizations of the deferred acceptance rule are known. The classic result of Gale and Shapley (1962) implies that the deferred acceptance rule is characterized by constrained efficiency subject to stability. Alcalde and Barbera (1994) characterize the deferred acceptance rule by stability and strategy-proofness. Balinski and Sönmez (1999) consider allocation rules over the domain of pairs of responsive priorities and preferences. An allocation rule respects improvements if an agent is weakly better off when his priority improves for each object. Balinski and Sönmez (1999) show that the deferred acceptance rule is the only stable rule that respects improvements.

2. Framework

Fix a set of **agents** N and a set of **(proper) object types** O. There is one **null object** type, denoted $\emptyset$. Each object $a \in O \cup \{\emptyset\}$ has **quota** q_a; $\emptyset$ is not scarce, $q_\emptyset = |N|$. Each agent i is allocated exactly one object in $O \cup \{\emptyset\}$. An **allocation** is a vector $\mu = (\mu_i)_{i \in N}$ assigning object $\mu_i \in O \cup \{\emptyset\}$ to agent i, with each object a being assigned to at most q_a agents. We write $\mu_a = \{i \in N | \mu_i = a\}$ for the set of agents who are assigned object a.

Each agent i has a strict (complete, transitive and antisymmetric) **preference relation** R_i over $O \cup \{\emptyset\}$.[5] We denote by P_i the asymmetric part of R_i, i.e., aP_ib if only if aR_ib and $a \neq b$. An object a is **acceptable** to agent i if $aP_i\emptyset$, and **unacceptable** to agent i if $\emptyset P_i a$. Let $R = (R_i)_{i \in N}$ be the preference profile of all agents. For any $N' \subset N$, we use the notation $R_{N'} = (R_i)_{i \in N'}$.[6] We write $\mu R \mu'$ if and only if $\mu_i \; R_i \; \mu'_i$ for all $i \in N$.

We denote by $\mathcal{A}$ and $\mathcal{R}$ the sets of allocations and preference profiles, respectively. An **allocation rule** $\varphi : \mathcal{R} \to \mathcal{A}$ maps preference profiles to allocations. At R, agent i is assigned object $\varphi_i(R)$, and object a is assigned to the set of agents $\varphi_a(R)$.

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[4]Kesten's acyclicity condition is stronger than Ergin's.

[5]The null object may represent off-campus housing in the context of university housing allocation, or private schools in the context of student placement in public schools. Allowing for preferences that rank the null object above some proper objects is natural in such applications.

[6]Our analysis carries through if we do not allow preferences to rank pairs of unacceptable objects, and regard all preferences that coincide in the ranking of acceptable objects as identical.

3. Deferred Acceptance

A **priority** for a proper object $a \in O$ is a correspondence $Ch_a : 2^N \to 2^N$, satisfying $Ch_a(N') \subset N'$ and $|Ch_a(N')| \leq q_a$ for all $N' \subset N$. $Ch_a(N')$ is interpreted as the set of high priority agents in N' "chosen" by object a. Ch_a is **substitutable** if agent i is chosen by object a from a set of agents N' whenever i is chosen by a from a set N'' that includes N'; formally, for all $N' \subset N'' \subset N$ we have $Ch_a(N'') \cap N' \subset Ch_a(N')$. Ch_a is **acceptant** if object a accepts each agent when its quota is not entirely allocated; formally, for all $N' \subset N$, $|Ch_a(N')| = \min(q_a, |N'|)$.[7] Let $Ch = (Ch_a)_{a \in O}$ denote the priority profile. Ch is substitutable/acceptant if Ch_a is substitutable/acceptant for all $a \in O$.

The allocation μ is **individually rational** at R if $\mu_i \; R_i \; \emptyset$ for all $i \in N$. The allocation μ is **blocked** by a pair $(i, a) \in N \times O$ at (R, Ch) if $aP_i\mu_i$ and $i \in Ch_a(\mu_a \cup \{i\})$. An allocation μ is **stable** at (R, Ch) if it is individually rational at R and is not blocked by any pair $(i, a) \in N \times O$ at (R, Ch). When Ch is substitutable, the following (agent-proposing) **deferred acceptance rule**, denoted φ^{Ch}, produces a stable allocation $\varphi^{Ch}(R)$ at (R, Ch) (Gale and Shapley (1962), extended to the case of substitutable priorities by Roth and Sotomayor (1990)).

- Step 1: Every agent applies to his most preferred acceptable object under R (if any). Let $\tilde{N}_a^1$ be the set of agents applying to object a. Object a tentatively accepts the agents in $N_a^1 = Ch_a(\tilde{N}_a^1)$, and rejects the agents in $\tilde{N}_a^1 \setminus N_a^1$.
- Step t ($t \geq 2$): Every agent who was rejected at step $t-1$ applies to his next preferred acceptable object under R (if any). Let $\tilde{N}_a^t$ be the new set of agents applying to object a. Object a tentatively accepts the agents in $N_a^t = Ch_a(N_a^{t-1} \cup \tilde{N}_a^t)$, and rejects the agents in $(N_a^{t-1} \cup \tilde{N}_a^t) \setminus N_a^t$.

The algorithm terminates when each agent is either tentatively accepted by some object or has been rejected by every object that is acceptable to him. Each agent tentatively accepted by a proper object at the last step is assigned that object, and all other agents

[7] Any linear order $\succ_a$ on N defines an acceptant substitutable priority Ch_a, with $Ch_a(N')$ equal to the set of $\min(q_a, |N'|)$ top ranked agents in N' under $\succ_a$. Hence the class of acceptant responsive priorities is a subset of the class of acceptant substitutable priorities. Studying substitutable priorities is important because priorities may often be non-responsive but substitutable in practice, as discussed in the introduction.

are assigned the null object. The allocation reached by the deferred acceptance rule is the **agent-optimal stable allocation** at (R, Ch)–it is stable at (R, Ch), and it is weakly preferred under R by every agent to any other stable allocation at (R, Ch) (Theorem 6.8 in Roth and Sotomayor (1990)).

Remark 1. It can be easily shown that no two distinct priorities induce the same deferred acceptance rule. Therefore, the subsequent characterization results lead to unique representations.

4. First Characterization of Deferred Acceptance Rules

We introduce two axioms, non-wastefulness and IR monotonicity, that characterize the set of deferred acceptance rules. Priorities are not primitive in our model except for Section 6, and our axioms are "priority-free" in the sense that they do not involve priorities.

Definition (Non-wastefulness)**.** An allocation rule φ is **non-wasteful** if

$$\varphi_i(R)\ R_i\ a,\ \forall R \in \mathcal{R}, i \in N, a \in O \cup \{\emptyset\} \text{ with } |\varphi_a(R)| < q_a.$$

Non-wastefulness is a weak requirement of efficiency. An object is not assigned to an agent who prefers it to his allocation only if the entire quota of that object is assigned to other agents. Note that if φ is non-wasteful then $\varphi(R)$ is individually rational for every $R \in \mathcal{R}$, as the null object is not scarce.

To introduce the main axiom, we say that R'_i is an **individually rational (IR) monotonic transformation** of R_i at $a \in O \cup \{\emptyset\}$ (R'_i *i.r.m.t.* R_i *at* a) if any object that is ranked above both a and $\emptyset$ under R'_i is ranked above a under R_i, i.e.,

$$b\, P'_i\, a\ \&\ b\, P'_i\, \emptyset \Rightarrow b\, P_i\, a,\ \forall b \in O.$$

R' is an IR transformation of R at an allocation μ (R' *i.r.m.t.* R *at* μ) if R'_i *i.r.m.t.* R_i at μ_i for all i.

Definition (IR monotonicity)**.** An allocation rule φ satisfies **individually rational (IR) monotonicity** if

$$R' \text{ } i.r.m.t. \text{ } R \text{ } at \text{ } \varphi(R) \Rightarrow \varphi(R')\ R'\ \varphi(R).$$

In words, φ satisfies IR monotonicity if every agent weakly prefers $\varphi(R')$ to $\varphi(R)$ under R' whenever R' is an IR monotonic transformation of R at $\varphi(R)$. If R' *i.r.m.t.* R *at* $\varphi(R)$, then the interpretation of the change in reported preferences from R to R' is that all agents place fewer claims on objects they cannot receive, in the sense that each agent's set of acceptable objects that are preferred to $\varphi(R)$ shrinks. Intuitively, the IR monotonicity axiom requires that all agents be weakly better off when some agents claim fewer objects. The IR label captures the idea that each agent effectively places claims only on acceptable objects. An agent may not be allocated unacceptable objects because he can opt to remain unassigned ($\emptyset$ represents the outside option), so the relevant definition of an upper contour set includes the IR constraint. Hence IR monotonicity requires allocations be monotonic in the IR constrained upper contour sets (ordered according to set inclusion).

Theorem 1. *An allocation rule* φ *is the deferred acceptance rule for some acceptant substitutable priority* Ch*, i.e.,* $\varphi = \varphi^{Ch}$*, if and only if* φ *satisfies non-wastefulness and IR monotonicity.*

The proof appears in the Appendix. Example 1 below, borrowed from Ergin (2002), illustrates an instance where a deferred acceptance rule satisfies IR monotonicity, and provides some intuition for the "only if" part of the theorem.

IR monotonicity resembles Maskin (1999) monotonicity. R_i' is a **monotonic transformation** of R_i at $a \in O \cup \{\emptyset\}$ (R_i' *m.t.* R_i *at* a) if any object that is ranked above a under R_i' is also ranked above a under R_i, i.e., $b\, P_i'\, a \;\Rightarrow\; b\, P_i\, a$, $\forall b \in O \cup \{\emptyset\}$. R' is a monotonic transformation of R at an allocation μ (R' *m.t.* R *at* μ) if R_i' *m.t.* R_i at μ_i for all i.

Definition (Maskin monotonicity)**.** An allocation rule φ satisfies **Maskin monotonicity** if

$$R' \textit{ m.t. } R \textit{ at } \varphi(R) \Rightarrow \varphi(R') = \varphi(R).$$

On the one hand, IR monotonicity has implications for a larger set of preference profile pairs (R, R') than Maskin monotonicity, as R' *m.t.* R *at* $\varphi(R) \Rightarrow R'$ *i.r.m.t.* R *at* $\varphi(R)$. On the other hand, for every preference profile pair (R, R') for which both axioms have implications (R' *m.t.* R *at* $\varphi(R)$), Maskin monotonicity imposes a stronger restriction than IR monotonicity, as $\varphi(R') = \varphi(R) \Rightarrow \varphi(R')\ R'\ \varphi(R)$. Example 1 establishes the independence of the IR monotonicity and Maskin monotonicity axioms. The example also shows that deferred acceptance rules do not always satisfy Maskin monotonicity (observation made previously by Kara and Sönmez (1996)), and some top trading cycle rules violate IR monotonicity, but satisfy Maskin monotonicity.

Example 1. Let $N = \{i, j, k\}, O = \{a, b\}, q_a = q_b = 1$. The priorities are given by maximizing the strict orderings $i \succ_a j \succ_a k$ and $k \succ_b i \succ_b j$ over each subset of agents, i.e., for $c \in O$ and nonempty $N' \subset N$, $Ch_c(N')$ is the highest ranked agent in N' under $\succ_c$.

Consider the following set of preferences for the agents.

R_i	R''_i	R_j	R'_j	R_k
b	$\emptyset$	a	$\emptyset$	a
a	b	$\emptyset$	a	b
$\emptyset$	a	b	b	$\emptyset$

Let $R = (R_i, R_j, R_k), R' = (R_i, R'_j, R_k), R'' = (R''_i, R_j, R_k)$. In the first step of the deferred acceptance algorithm for (R, Ch), i applies to b, and j and k apply to a, then k is rejected by a. In the second step, k applies to b, and i is rejected by b. At the third step, i applies to a, and j is rejected by a. The algorithm terminates after the third step, and the allocation is given by $\varphi^{Ch}(R) = (\varphi^{Ch}_i(R), \varphi^{Ch}_j(R), \varphi^{Ch}_k(R)) = (a, \emptyset, b)$. In the first step of the deferred acceptance algorithm for (R', Ch), i applies to b and k applies to a. The algorithm terminates at the first step, and the allocation is given by $\varphi^{Ch}(R') = (b, \emptyset, a)$.

All agents prefer $\varphi^{Ch}(R')$ to $\varphi^{Ch}(R)$ under R' (the preference is strict for i and k, and weak for j) as a consequence of R' *i.r.m.t.* R *at* $\varphi^{Ch}(R)$. Indeed, in the deferred acceptance algorithm for (R, Ch), there is a chain of rejections–k is rejected by a because j claims his higher priority to a, then i is rejected by b because k claims his higher priority to b, then j is rejected by a because i claims his higher priority for a; j is assigned the null object in spite of his initial priority claim to a that starts off the rejection chain. If j does not claim his higher priority to a, and reports R'_j instead of R_j, the rejection chain does not occur, weakly benefitting everyone (with respect to R').

The rule φ^{Ch} violates Maskin monotonicity since R' *m.t.* R *at* $\varphi^{Ch}(R)$, and $\varphi^{Ch}_i(R) \neq \varphi^{Ch}_i(R')$.

The top trading cycle rule associated with the priorities $(\succ_a, \succ_b)$ (Abdulkadiroğlu and Sönmez 2003) violates IR monotonicity. At R, i and k trade their priorities for a and b; the top trading cycle allocation is $\mu = (b, \emptyset, a)$. At R'', k cannot trade his priority for b with i since i does not place claims for b (as i declares b unacceptable); and j has higher priority than k for a, hence k does not receive a; the top trading cycle allocation is $\mu'' = (\emptyset, a, b)$. IR monotonicity is violated, as R'' *i.r.m.t.* R at μ and agent k strictly prefers μ to μ'' under R_k. The top trading cycle rule considered here satisfies Maskin monotonicity by Papai (2000) and Takamiya (2001).

The following examples show that non-wastefulness and IR monotonicity are independent axioms if $|N|, |O| \geq 2$, and there is at least one scarce object, that is, $q_a < |N|$ for some $a \in O$.

Example 2. Consider the allocation rule that allocates the null object to every agent for all preference profiles. This rule trivially satisfies IR monotonicity, but violates non-wastefulness.

Example 3. Let $N = \{1, 2, \ldots, n\}$. Suppose that a is one of the scarce objects ($q_a < |N|$) and b is a proper object different from a (such a and b exist by assumption). Let R denote a (fixed) preference profile at which every agent ranks a first and $\emptyset$ second. Consider the allocation rule under which (1) at any preference profile where agent q_a reports R_{q_a}, the assignment is according to the serial dictatorship with the ordering of agents $1, 2, \ldots, n$, that is, agent 1 picks his most preferred object, agent 2 picks his most preferred object among the remaining ones (objects perviously picked by a number of agents smaller than their respective quotas), and so on, and (2) at any other preference profile, the assignment is specified by the serial dictatorship with the agent ordering $1, 2, \ldots, q_a - 1, q_a + 1, q_a, q_a + 2, \ldots, n$, defined analogously to (1). The allocation rule described above clearly satisfies non-wastefulness, but violates IR monotonicity. Indeed, let R'_{q_a} be a preference relation for agent q_a that ranks a first and b second. The profile $(R'_{q_a}, R_{N\setminus\{q_a\}})$ *i.r.m.t.* R at the allocation for R, but agent q_a is assigned a at R and b at $(R'_{q_a}, R_{N\setminus\{q_a\}})$, and $a \; P'_{q_a} \; b$.

5. Second Characterization of Deferred Acceptance Rules

We offer an alternative characterization of deferred acceptance rules in terms of more elementary axioms. These axioms are mathematically more tractable, and further help our understanding of deferred acceptance rules. For instance, in Section 7 we obtain a characterization of Pareto efficient deferred acceptance rules via a simple alteration in the new collection of axioms.

We first define the weak Maskin monotonicity axiom. Recall that R'_i is a **monotonic transformation** of R_i at $a \in O \cup \{\emptyset\}$ (R'_i *m.t.* R_i *at* a) if any object that is ranked above a under R'_i is also ranked above a under R_i, i.e., $b\, P'_i\, a \;\Rightarrow\; b\, P_i\, a,\ \forall b \in O \cup \{\emptyset\}$. R' is a monotonic transformation of R at an allocation μ (R' *m.t.* R *at* μ) if R'_i *m.t.* R_i at μ_i for all i.

Definition (Weak Maskin Monotonicity)**.** An allocation rule φ satisfies **weak Maskin monotonicity** if

$$R' \ m.t. \ R \ at \ \varphi(R) \Rightarrow \varphi(R') \ R' \ \varphi(R).$$

To gain some perspective, note that the implication of R' *m.t.* R *at* $\varphi(R)$ is that $\varphi(R') = \varphi(R)$ under Maskin monotonicity, but only that $\varphi(R')\ R'\ \varphi(R)$ under weak Maskin monotonicity. Therefore, any allocation rule that satisfies the standard Maskin monotonicity axiom also satisfies weak Maskin monotonicity.

We next define the population monotonicity axiom (Thomson 1983a, Thomson 1983b). As a departure from the original setting, suppose that the collection of all objects (q_a copies of each object type $a \in O \cup \{\emptyset\}$) needs to be allocated to a subset of agents N', or equivalently, that the agents outside N' receive $\emptyset$ and are removed from the assignment problem. It is convenient to interpret the new setting as a restriction on the set of preference profiles, whereby the agents in $N \setminus N'$ are constrained to report every object as unacceptable. Specifically, let $R^{\emptyset}$ denote a fixed preference profile that ranks $\emptyset$ first for every agent. For any $R \in \mathcal{R}$, we interpret the profile $(R_{N'}, R^{\emptyset}_{N\setminus N'})$ as a deviation from R generated by restricting the assignment problem to the agents in N'.

Definition (Population Monotonicity)**.** An allocation rule φ is **population monotonic** if

$$\varphi_i(R)\ R_i\ \varphi_i(R_{N'}, R^{\emptyset}_{N\setminus N'}), \forall i \in N', \forall N' \subset N, \forall R \in \mathcal{R}.$$

IR monotonicity clearly implies both weak Maskin monotonicity and population monotonicity. We prove that the latter two axioms, along with non-wastefulness, are sufficient to characterize deferred acceptance rules.

Theorem 2. *An allocation rule φ is the deferred acceptance rule for some acceptant substitutable priority Ch, i.e., $\varphi = \varphi^{Ch}$, if and only if φ satisfies non-wastefulness, weak Maskin monotonicity, and population monotonicity.*

The proof appears in the Appendix.

We show that the three axioms from Theorem 2 are independent if $|N|, |O| \geq 2$ and $q_a < |N| - 1$ for at least one object $a \in O$.[8] The rule described in Example 2 satisfies weak Maskin monotonicity and population monotonicity, and violates non-wastefulness. The rule from Example 3 satisfies non-wastefulness and population monotonicity, but not weak Maskin monotonicity. Lastly, the following example defines a non-wasteful and weakly Maskin monotonic rule, which is not population monotonic.

Example 4. Let $N = \{1, 2, \ldots, n\}$. Consider the allocation rule under which at any preference profile where agent 1 declares every object unacceptable, the assignment is according to the serial dictatorship allocation for the ordering of agents $1, 2, \ldots, n-2, n-1, n$; otherwise, the assignment is specified by the serial dictatorship for the ordering $1, 2, \ldots, n-2, n, n-1$. The allocation rule so defined satisfies non-wastefulness and weak Maskin monotonicity, but not population monotonicity. To show that the rule violates population monotonicity, suppose that a is an object with $q_a < |N| - 1$ and b is a proper object different from a (such a and b exist by assumption). Let R be a preference profile where the first ranked objects are b for agent 1; a for agents $2, 3. \ldots, q_a, n-1, n$; and $\emptyset$

[8] If $q_a \geq |N| - 1$ for all $a \in O$, then non-wastefulness implies population monotonicity. In that case, in any market that excludes at least one agent, every non-wasteful allocation assigns each of the remaining agents his favorite object.

for the other agents. Note that agent n receives a at R and some $c \neq a$ at $(R_1^{\emptyset}, R_{N\setminus\{1\}})$, and aP_nc.

IR monotonicity implies both weak Maskin monotonicity and population monotonicity, and under the assumption of non-wastefulness, by Theorems 1 and 2, is equivalent to the conjunction of the latter two axioms. However, the following example shows that weak Maskin monotonicity and population monotonicity do not imply Maskin monotonicity if $|N|, |O| \geq 2$.

Example 5. Let $N = \{1, 2, \ldots, n\}$. Fix two proper objects a and b (such a and b exist by assumption). Consider the allocation rule under which, at preference profile R, (1) agent 1 is assigned the higher ranked object between a and $\emptyset$ under R_1; (2) agent 2 is assigned the higher ranked object between b and $\emptyset$ under R_2, except for the case $bP_1\emptyset P_1 a$, when he is assigned $\emptyset$; (3) the agents in $N \setminus \{1, 2\}$ are assigned $\emptyset$. One can check that this allocation rule satisfies weak Maskin monotonicity and population monotonicity. To show that the rule violates IR monotonicity, let R be a preference profile where agent 1 ranks b first and a second, and agent 2 ranks b first, and let R_1' be a preference for agent 1 that ranks b first and $\emptyset$ second. Then IR monotonicity is violated since $(R_1', R_{N\setminus\{1\}})$ *i.r.m.t.* R at the allocation under R, but agent 2 is assigned b at R and $\emptyset$ at $(R_1', R_{N\setminus\{1\}})$, and $bP_2\emptyset$.

6. Axioms for Stable Rules

In this section we study stable allocation rules for an exogenously specified priority structure Ch. We say that an allocation rule φ is stable at Ch if $\varphi(R)$ is stable at (R, Ch) for all R. We show that the deferred acceptance rule at Ch is the only allocation rule that is stable at Ch and satisfies weak Maskin monotonicity.

Theorem 3. *Let Ch be an acceptant substitutable priority, and φ be a stable allocation rule at Ch. Then φ is the deferred acceptance rule for Ch, i.e., $\varphi = \varphi^{Ch}$, if and only if it satisfies weak Maskin monotonicity.*

Proof. The "only if" part is a consequence of Theorem 2. The "if" part follows from Lemma 2 in the Appendix. □

7. Efficient Deferred Acceptance Rules

An allocation μ **Pareto dominates** another allocation μ' at the preference profile R if $\mu_i R_i \mu'_i$ for all $i \in N$ and $\mu_i P_i \mu'_i$ for some $i \in N$. An allocation is **Pareto efficient** at R if no allocation Pareto dominates it at R. An allocation rule φ is **Pareto efficient** if $\varphi(R)$ is Pareto efficient at R for all $R \in \mathcal{R}$. An allocation rule φ is **group strategy-proof** if there exist no $N' \subseteq N$ and $R, R' \in \mathcal{R}$ such that $\varphi_i(R'_{N'}, R_{N \setminus N'})\ R_i\ \varphi_i(R)$ for all $i \in N'$ and $\varphi_i(R'_{N'}, R_{N \setminus N'})\ P_i\ \varphi_i(R)$ for some $i \in N'$.

In general, there are deferred acceptance rules that are neither efficient nor group strategy-proof. Since deferred acceptance rules are often used in resource allocation problems where efficiency is one of the goals of the social planner, it is desirable to develop necessary and sufficient conditions for the efficiency of these rules.

Proposition 1. *Let Ch be an acceptant substitutable priority. The following properties are equivalent.*

(1) φ^{Ch} *is Pareto efficient.*
(2) φ^{Ch} *satisfies Maskin monotonicity.*
(3) φ^{Ch} *is group strategy-proof.*

The proof is given in the Appendix.

Proposition 1 generalizes part of Theorem 1 from Ergin (2002). Under the assumption that priorities are responsive, Ergin shows that a deferred acceptance rule is Pareto efficient if and only if it is group strategy-proof, and that these properties hold if and only if the priority is acyclic. Takamiya (2001) shows that Maskin monotonicity and group strategy-proofness are equivalent for any allocation rule.

Theorem 4. *Let φ be an allocation rule. The following conditions are equivalent.*

(1) *φ is the deferred acceptance rule for some acceptant substitutable priority Ch, i.e., $\varphi = \varphi^{Ch}$, and φ is Pareto efficient.*
(2) *φ satisfies non-wastefulness, Maskin monotonicity and population monotonicity.*
(3) *φ satisfies Pareto efficiency, weak Maskin monotonicity and population monotonicity.*

The proof appears in the Appendix.

In view of Proposition 1, two additional characterizations of efficient deferred acceptance rules are obtained by replacing the Pareto efficiency property in condition (1) of Theorem 4 with Maskin monotonicity and respectively group strategy-proofness.

Recall from Theorem 2 that weak Maskin monotonicity is sufficient, along with non-wastefulness and population monotonicity, to characterize deferred acceptance rules. Theorem 4 shows that if we replace weak Maskin monotonicity by Maskin monotonicity in the list of axioms above we obtain a characterization of efficient deferred acceptance rules. The contrast between these two results demonstrates that we can attribute the inefficiency of some deferred acceptance rules entirely to instances where weak Maskin monotonicity is satisfied, but Maskin monotonicity is violated.

Appendix A

Proof of Theorem 1. Since IR monotonicity implies weak Maskin monotonicity and population monotonicity, the "if" part of Theorem 1 follows from the "if" part of Theorem 2, which we establish later. We prove the "only if" part here.

We need to show that a deferred acceptance rule φ^{Ch} with acceptant substitutable priority Ch satisfies the non-wastefulness and IR monotonicity axioms. φ^{Ch} is non-wasteful since Ch is acceptant and the deferred acceptance rule is stable.

In order to prove that φ^{Ch} satisfies IR monotonicity, suppose that R' *i.r.m.t.* R at $\varphi^{Ch}(R)$. We need to show that $\varphi^{Ch}(R')\ R'\ \varphi^{Ch}(R) =: \mu^0$. Define μ^1 by assigning each agent i the higher ranked object between μ_i^0 and $\emptyset$ under R_i'.

For $t \geq 1$, if μ^t can be blocked at (R', Ch) we choose an arbitrary object a^t that is part of a blocking pair and define μ^{t+1} by

$$\mu_i^{t+1} = \begin{cases} a^t & \text{if } i \in Ch_{a^t}(\mu_{a^t}^t \cup \{j \in N | a^t P_j' \mu_j^t\}) \\ \mu_i^t & \text{otherwise.} \end{cases} \tag{A.1}$$

If μ^t cannot be blocked, then let $\mu^{t+1} = \mu^t$. Part of the next lemma establishes that each μ^t is well-defined, that is, μ^t is an allocation for all $t \geq 0$. The sequence $(\mu^t)_{t\geq 0}$ is a variant of the vacancy chain dynamics of Blum, Roth, and Rothblum (1997).[9]

Lemma 1. *The sequence* $(\mu^t)_{t\geq 0}$ *satisfies*

$$\mu^t \in \mathcal{A} \tag{A.2}$$

$$\mu^t \; R' \; \mu^{t-1} \tag{A.3}$$

$$\mu^t_a \subset Ch_a(\mu^t_a \cup \{j \in N | aP'_j \mu^t_j\}) \tag{A.4}$$

for every $a \in O, t \geq 1$. *The sequence* $(\mu^t)_{t\geq 0}$ *becomes constant in a finite number of steps* T, *and the allocation* μ^T *is stable at* (R', Ch).

Proof. We prove the claims A.2-A.4 by induction on t.

We first show the induction base case, $t = 1$. The definition of μ^1 immediately implies that $\mu^1 \in \mathcal{A}$ and $\mu^1 R' \mu^0$, proving A.2 and A.3 (at $t = 1$). To establish A.4 (at $t = 1$), fix $a \in O$. We have that

$$\mu^0_a = Ch_a(\mu^0_a \cup \{j \in N | a \; P_j \; \mu^0_j\}) \tag{A.5}$$

because μ^0 is stable at (R, Ch) and Ch_a is an acceptant and substitutable priority. By construction,

$$\mu^1_a \subset \mu^0_a. \tag{A.6}$$

Since R' *i.r.m.t.* R at μ^0, it must be that $\{j \in N | a \; P'_j \; \mu^1_j\} \subset \{j \in N | a \; P_j \; \mu^0_j\}$.[10] Therefore,

$$\mu^1_a \cup \{j \in N | aP'_j \mu^1_j\} \subset \mu^0_a \cup \{j \in N | aP_j \mu^0_j\}. \tag{A.7}$$

Ch_a's substitutability and A.5-A.7 imply

$$\mu^1_a \subset Ch_a(\mu^1_a \cup \{j \in N | aP'_j \mu^1_j\}).$$

[9]Note that the exclusion of agent i with preferences R_i from the market can be modeled as a change in i's reported preferences making every object unacceptable, which is an IR transformation of R_i at every object.

[10]Suppose that $a \; P'_j \; \mu^1_j$. Then $\mu^1_j \; R'_j \; \emptyset$ implies $a \; P'_j \; \emptyset$. By definition, $\mu^1_j \; R'_j \; \mu^0_j$, so $a \; P'_j \; \mu^0_j$. The assumption that R'_j *i.r.m.t.* R_j at μ^0_j, along with $aP'_j\emptyset$ and $aP'_j\mu^0_j$, implies that $aP_j\mu^0_j$.

To establish the inductive step, we assume that the conclusion holds for $t \geq 1$, and prove it for $t+1$. The only non-trivial case is $\mu^t \neq \mu^{t+1}$.

By the inductive hypothesis A.4 (at t), $\mu^t_{a^t} \subset Ch_{a^t}(\mu^t_{a^t} \cup \{j \in N | a^t P'_j \mu^t_j\})$. By definition,

$$\mu^{t+1}_{a^t} = Ch_{a^t}(\mu^t_{a^t} \cup \{j \in N | a^t P'_j \mu^t_j\}). \tag{A.8}$$

To prove A.2 (at t+1), first note that A.8 implies $|\mu^{t+1}_{a^t}| = |Ch_{a^t}(\mu^t_{a^t} \cup \{j \in N | a^t P'_j \mu^t_j\})| \leq q_{a^t}$. If $a \neq a^t$, then by construction $\mu^{t+1}_a \subset \mu^t_a$, and by A.2 (at t) we conclude that $|\mu^{t+1}_a| \leq |\mu^t_a| \leq q_a$. Therefore $\mu^{t+1} \in \mathcal{A}$.

To show A.3 (at $t+1$), note that $a^t = \mu^{t+1}_j P'_j \mu^t_j$ for any $j \in \mu^{t+1}_{a^t} \setminus \mu^t_{a^t}$, and each agent outside $\mu^{t+1}_{a^t} \setminus \mu^t_{a^t}$ is assigned the same object under μ^{t+1} and μ^t. Therefore $\mu^{t+1} R' \mu^t$.

We show A.4 (at $t+1$) separately for the cases $a = a^t$ and $a \neq a^t$.

By construction of μ^{t+1},

$$\mu^{t+1}_{a^t} \cup \{j \in N | a^t P'_j \mu^{t+1}_j\} = \mu^t_{a^t} \cup \{j \in N | a^t P'_j \mu^t_j\}.[11]$$

Then A.8 implies that

$$\mu^{t+1}_{a^t} = Ch_{a^t}(\mu^{t+1}_{a^t} \cup \{j \in N | a^t P'_j \mu^{t+1}_j\}).$$

For any $a \neq a^t$, we have $\mu^{t+1}_a \subset \mu^t_a$ by construction, and $\{j \in N | a P'_j \mu^{t+1}_j\} \subset \{j \in N | a P'_j \mu^t_j\}$ since $\mu^{t+1} R' \mu^t$. Therefore,

$$\mu^{t+1}_a \cup \{j \in N | a P'_j \mu^{t+1}_j\} \subset \mu^t_a \cup \{j \in N | a P'_j \mu^t_j\}. \tag{A.9}$$

Recall the inductive hypothesis A.4 (at t), $\mu^t_a \subset Ch_a(\mu^t_a \cup \{j \in N | a P'_j \mu^t_j\})$. Substitutability of Ch_a, $\mu^{t+1}_a \subset \mu^t_a$, and A.9 imply that

$$\mu^{t+1}_a \subset Ch_a(\mu^{t+1}_a \cup \{j \in N | a P'_j \mu^{t+1}_j\}),$$

completing the proof of the induction step.

By A.3, the sequence $(\mu^t)_{t \geq 0}$ becomes constant in a finite number of steps T. The final allocation μ^T is individually rational at R' and is not blocked at (R', Ch), so is stable at (R', Ch). $\square$

[11] We have $\mu^t_{a^t} \subset \mu^{t+1}_{a^t}$ by construction and $\{j \in N | a^t P'_j \mu^{t+1}_j\} \subset \{j \in N | a^t P'_j \mu^t_j\}$ since $\mu^{t+1}_j R'_j \mu^t_j$ for every $j \in N$. At the same time, an inspection of A.1 reveals that $\mu^{t+1}_{a^t} \setminus \mu^t_{a^t} = \{j \in N | a^t P'_j \mu^t_j\} \setminus \{j \in N | a^t P'_j \mu^{t+1}_j\}$.

To finish the proof of the "only if" part, let μ^T be the stable matching identified in Lemma 1. $\varphi^{Ch}(R')\ R'\ \mu^T$ because $\varphi^{Ch}(R')$ is the agent-optimal stable allocation at (R', Ch). Therefore, we have

$$\varphi^{Ch}(R')\ R'\ \mu^T\ R'\ \mu^{T-1}\ R' \ldots R'\ \mu^1\ R'\ \mu^0 = \varphi^{Ch}(R),$$

showing that φ^{Ch} satisfies IR monotonicity. □

Proof of Theorem 2. Since weak Maskin monotonicity and population monotonicity are implied by IR monotonicity, the "only if" part of Theorem 2 follows from the "only if" part of Theorem 1 shown above. We only need to prove the "if" part here.

Fix a rule φ that satisfies the non-wastefullness, weak Maskin monotonicity and population monotonicity axioms. To show that φ is a deferred acceptance rule for some acceptant substitutable priority, we proceed in three steps. First, we construct a priority profile Ch and verify that it is acceptant and substitutable. Second, we show that for every $R \in \mathcal{R}$, $\varphi(R)$ is a stable allocation at (R, Ch). Third, we prove that $\varphi(R)$ is the agent-optimal stable allocation at (R, Ch).

For $a \in O \cup \{\emptyset\}$, let R^a be a fixed preference profile which ranks a as the most preferred object for every agent. For each $a \in O, N' \subset N$, define

$$Ch_a(N') = \varphi_a(R^a_{N'}, R^{\emptyset}_{N \setminus N'}).$$

We have that $Ch_a(N') \subset N'$ because φ is non-wasteful and the null object is not scarce.

Step 1. Ch_a is an acceptant and substitutable priority for all objects $a \in O$.

Ch_a is an acceptant priority because φ is non-wasteful.

In order to show that Ch_a is substitutable, consider $N' \subset N'' \subset N$. Assume that $i \in Ch_a(N'') \cap N'$. By definition, $\varphi_i(R^a_{N''}, R^{\emptyset}_{N \setminus N''}) = a$. Since $i \in N' \subset N''$, population monotonicity for the subset of agents N' and the preference profile $(R^a_{N''}, R^{\emptyset}_{N \setminus N''})$ implies that $\varphi_i(R^a_{N'}, R^{\emptyset}_{N \setminus N'})\ R^a_i\ \varphi_i(R^a_{N''}, R^{\emptyset}_{N \setminus N''}) = a$. Hence $\varphi_i(R^a_{N'}, R^{\emptyset}_{N \setminus N'}) = a$, which by definition means that $i \in Ch_a(N')$. This shows $Ch_a(N'') \cap N' \subset Ch_a(N')$.

Step 2. $\varphi(R)$ is a stable allocation at (R, Ch) for all $R \in \mathcal{R}$.

For all R, $\varphi(R)$ is individually rational because φ is non-wasteful and the null object is not scarce.

To show that no blocking pair exists, we proceed by contradiction. Assume that $(i, a) \in N \times O$ blocks $\varphi(R)$, i.e.,

$$a \quad P_i \quad \varphi_i(R) \tag{A.10}$$

$$i \quad \in \quad Ch_a(\varphi_a(R) \cup \{i\}). \tag{A.11}$$

Let $N' = \varphi_a(R)$. N' has q_a elements by non-wastefulness of φ and A.10. Fix a preference $R_i^{a\varphi_i(R)}$ for agent i, which ranks a first and $\varphi_i(R)$ second. Note that $(R_i^{a\varphi_i(R)}, R^a_{N'}, R_{N\setminus(N'\cup\{i\})})$ *m.t.* R at $\varphi(R)$ ($R_i^{a\varphi_i(R)}$ *m.t.* R_i at $\varphi_i(R)$ by A.10, R^a_j *m.t.* R_j at $\varphi_j(R)$ for $j \in N'$ because $\varphi_j(R) = a$ by definition of N', and the preferences of the agents in $N \setminus (N' \cup \{i\})$ are identical under the two preference profiles). As φ satisfies weak Maskin monotonicity, it follows that

$$\begin{aligned}
\varphi_j(R_i^{a\varphi_i(R)}, R^a_{N'}, R_{N\setminus(N'\cup\{i\})}) \quad & R^a_j \quad \varphi_j(R) = a, \text{ hence} \\
\varphi_j(R_i^{a\varphi_i(R)}, R^a_{N'}, R_{N\setminus(N'\cup\{i\})}) \quad & = \quad a, \forall j \in N'.
\end{aligned}$$

Using φ's population monotonicity for the subset of agents $N' \cup \{i\}$ and the preference profile $(R_i^{a\varphi_i(R)}, R^a_{N'}, R_{N\setminus(N'\cup\{i\})})$ we obtain that

$$\begin{aligned}
\varphi_j(R_i^{a\varphi_i(R)}, R^a_{N'}, R^\emptyset_{N\setminus(N'\cup\{i\})}) \quad & R^a_j \quad \varphi_j(R_i^{a\varphi_i(R)}, R^a_{N'}, R_{N\setminus(N'\cup\{i\})}) = a, \text{ hence} \\
\varphi_j(R_i^{a\varphi_i(R)}, R^a_{N'}, R^\emptyset_{N\setminus(N'\cup\{i\})}) \quad & = \quad a, \forall j \in N'. \qquad \text{(A.12)}
\end{aligned}$$

From the construction of Ch_a, A.11 is equivalent to $\varphi_i(R^a_{N'\cup\{i\}}, R^\emptyset_{N\setminus(N'\cup\{i\})}) = a$. Note that $(R_i^{a\varphi_i(R)}, R^a_{N'}, R^\emptyset_{N\setminus(N'\cup\{i\})})$ *m.t.* $(R^a_{N'\cup\{i\}}, R^\emptyset_{N\setminus(N'\cup\{i\})})$ at $\varphi(R^a_{N'\cup\{i\}}, R^\emptyset_{N\setminus(N'\cup\{i\})})$ ($R_i^{a\varphi_i(R)}$ *m.t.* R^a_i at $\varphi_i(R^a_{N'\cup\{i\}}, R^\emptyset_{N\setminus(N'\cup\{i\})}) = a$, and the preferences of all other agents are identical under the two preference profiles). As φ satisfies weak Maskin monotonicity, it follows that

$$\begin{aligned}
\varphi_i(R_i^{a\varphi_i(R)}, R^a_{N'}, R^\emptyset_{N\setminus(N'\cup\{i\})}) \quad & R_i^{a\varphi_i(R)} \quad \varphi_i(R^a_{N'\cup\{i\}}, R^\emptyset_{N\setminus(N'\cup\{i\})}) = a, \text{ hence} \\
\varphi_i(R_i^{a\varphi_i(R)}, R^a_{N'}, R^\emptyset_{N\setminus(N'\cup\{i\})}) \quad & = \quad a. \qquad \text{(A.13)}
\end{aligned}$$

By A.12 and A.13, $\varphi_a(R_i^{a\varphi_i(R)}, R^a_{N'}, R^\emptyset_{N\setminus(N'\cup\{i\})}) \supset N' \cup \{i\}$, hence φ allocates a to at least $|N'| + 1 = q_a + 1$ agents, which is a contradiction with the feasibility of φ.

Step 3. $\varphi(R) = \varphi^{Ch}(R)$ for all $R \in \mathcal{R}$.

We state and prove the main part of this step as a separate lemma in order to invoke it in the proof of Theorem 3 as well.

Lemma 2. *Let Ch be an acceptant substitutable priority, and suppose that φ is a stable allocation rule at Ch that satisfies weak Maskin monotonicity. Then φ is the deferred acceptance rule for Ch, i.e., $\varphi = \varphi^{Ch}$.*

Proof. Fix a preference profile R. For each $i \in N$, let R'_i be the truncation of R_i at $\varphi_i^{Ch}(R)$, that is, R_i and R'_i agree on the ranking of all proper objects, and any object less preferred than $\varphi_i^{Ch}(R)$ under R_i is unacceptable under R'_i.

We first establish that $\varphi^{Ch}(R)$ is the unique stable allocation at (R', Ch). Since $\varphi^{Ch}(R)$ is stable at (R, Ch), it is also stable at (R', Ch). By definition, $\varphi^{Ch}(R')$ is the agent-optimal stable allocation at (R', Ch), thus $\varphi^{Ch}(R')\ R'\ \varphi^{Ch}(R)$. This leads to

$$\varphi^{Ch}(R')\ R\ \varphi^{Ch}(R),$$

as R'_i is the truncation of R_i at $\varphi_i^{Ch}(R)$ for all $i \in N$. Then the stability of $\varphi^{Ch}(R')$ at (R', Ch) implies its stability at (R, Ch). But $\varphi^{Ch}(R)$ is the agent-optimal stable allocation at (R, Ch), so it must be that

$$\varphi^{Ch}(R)\ R\ \varphi^{Ch}(R').$$

The series of arguments above establishes that

$$\varphi^{Ch}(R) = \varphi^{Ch}(R').$$

Thus $\varphi^{Ch}(R)$ is the agent-optimal stable allocation at (R', Ch).

Let μ be a stable allocation at (R', Ch). We argue that $\mu = \varphi^{Ch}(R)$. Since $\varphi^{Ch}(R)$ is the agent-optimal stable allocation at (R', Ch), we have that $\varphi_i^{Ch}(R)\ R'_i\ \mu_i$ for all $i \in N$. Since R'_i is the truncation of R_i at $\varphi_i^{Ch}(R)$, it follows that $\mu_i \in \{\varphi_i^{Ch}(R), \emptyset\}$ for all $i \in N$. If $\mu_i \neq \varphi_i^{Ch}(R)$ for some agent $i \in N$, then $|\mu_{\varphi_i^{Ch}(R)}| < |\varphi^{Ch}_{\varphi_i^{Ch}(R)}(R)| \leq q_{\varphi_i^{Ch}(R)}$ and $\varphi_i^{Ch}(R) P'_i \mu_i = \emptyset$, which is a contradiction with the stability of μ at (R', Ch) (as $Ch_{\varphi_i^{Ch}(R)}$ is acceptant). It follows that $\mu = \varphi^{Ch}(R)$, hence $\varphi^{Ch}(R)$ is the unique stable allocation at (R', Ch).

By hypothesis, φ is a stable allocation rule at Ch, thus $\varphi(R')$ is a stable allocation at (R', Ch). As $\varphi^{Ch}(R)$ is the unique stable allocation at (R', Ch), we need

$$\varphi(R') = \varphi^{Ch}(R).$$

We have that R *m.t.* R' at $\varphi(R')$ because R'_i is the truncation of R_i at $\varphi_i(R') = \varphi_i^{Ch}(R)$ for all $i \in N$. As φ satisfies weak Maskin monotonicity, it follows that $\varphi(R)\ R\ \varphi(R') = \varphi^{Ch}(R)$. Since $\varphi(R)$ is a stable allocation at (R, Ch) and $\varphi^{Ch}(R)$ is the agent-optimal stable allocation at (R, Ch), we obtain that $\varphi(R) = \varphi^{Ch}(R)$, finishing the proof of the lemma. □

We resume the proof of Step 3. By assumption, φ satisfies weak Maskin monotonicity. Step 1 shows that Ch is an acceptant substitutable priority and Step 2 proves that φ is a stable allocation at Ch. So, φ satisfies all the hypotheses of Lemma 2. Therefore $\varphi = \varphi^{Ch}$, which completes the proof of Step 3, and of the "if" part of the theorem. □

Proof of Proposition 1. We prove each of the three implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) by contradiction.

To show (1) $\Rightarrow$ (2), assume that φ^{Ch} is Pareto efficient, but not Maskin monotonic. Then there exist preference profiles R, R' such that R' *m.t.* R at $\varphi^{Ch}(R)$ and $\varphi^{Ch}(R') \neq \varphi^{Ch}(R)$. As φ^{Ch} satisfies weak Maskin monotonicity by Theorem 2, it follows that $\varphi^{Ch}(R')$ Pareto dominates $\varphi^{Ch}(R)$ at R'. Since R' *m.t.* R at $\varphi^{Ch}(R)$, this implies that $\varphi^{Ch}(R')$ Pareto dominates $\varphi^{Ch}(R)$ at R, which contradicts the assumption that φ^{Ch} is Pareto efficient.

To show (2) $\Rightarrow$ (3), assume that φ^{Ch} is Maskin monotonic, but not group strategy-proof. Then there exist $N' \subseteq N$ and preference profiles R, R' such that $\varphi_i^{Ch}(R'_{N'}, R_{N\setminus N'})\ R_i\ \varphi_i^{Ch}(R)$ for all $i \in N'$, with strict preference for some i. For every $i \in N'$, let R''_i be a preference relation that ranks $\varphi_i^{Ch}(R'_{N'}, R_{N\setminus N'})$ first and $\varphi_i^{Ch}(R)$ second.[12] Clearly, $(R''_{N'}, R_{N\setminus N'})$ *m.t.* $(R'_{N'}, R_{N\setminus N'})$ at $\varphi^{Ch}(R'_{N'}, R_{N\setminus N'})$ and $(R''_{N'}, R_{N\setminus N'})$ *m.t.* R at $\varphi^{Ch}(R)$. Then the assumption that φ^{Ch} is Maskin monotonic leads to

$$\varphi^{Ch}(R'_{N'}, R_{N\setminus N'}) = \varphi^{Ch}(R''_{N'}, R_{N\setminus N'}) = \varphi^{Ch}(R),$$

[12]If $\varphi_i^{Ch}(R'_{N'}, R_{N\setminus N'}) = \varphi_i^{Ch}(R)$ then we simply require that R''_i rank $\varphi_i^{Ch}(R'_{N'}, R_{N\setminus N'})$ first.

which is a contradiction with $\varphi_i^{Ch}(R'_{N'}, R_{N\setminus N'})\ P_i\ \varphi_i^{Ch}(R)$ for some $i \in N'$.

To show (3) ⇒ (1), suppose that φ^{Ch} is group strategy-proof, but not Pareto efficient. Then there exist an allocation μ and a preference profile R such that μ Pareto dominates $\varphi^{Ch}(R)$ at R. For every $i \in N$, let R'_i be a preference that ranks μ_i as the most preferred object. Since μ is the agent-optimal stable allocation at (R', Ch), we obtain that $\varphi^{Ch}(R') = \mu$. The deviation for all agents in N to report R' rather than R leads to a violation of group strategy-proofness of φ^{Ch}. □

Proof of Theorem 4. We prove the three implications (1) ⇒ (2) ⇒ (3) ⇒ (1).

To show (1) ⇒ (2), assume that $\varphi = \varphi^{Ch}$ for some acceptant substitutable priority Ch and that φ is Pareto efficient. By the equivalence of the properties (1) and (2) in Proposition 1, φ satisfies Maskin monotonicity. By Theorem 2, φ satisfies non-wastefulness and population monotonicity.

To show (2) ⇒ (3), suppose that φ satisfies non-wastefulness, Maskin monotonicity and population monotonicity. Since Maskin monotonicity implies weak Maskin monotonicity, Theorem 2 shows that $\varphi = \varphi^{Ch}$ for some acceptant substitutable priority Ch. As φ^{Ch} satisfies Maskin monotonicity by assumption, the equivalence of the conditions (1) and (2) in Proposition 1 implies that φ is Pareto efficient.

To show (3) ⇒ (1), assume that φ satisfies Pareto efficiency, weak Maskin monotonicity and population monotonicity. As Pareto efficiency implies non-wastefulness, by Theorem 2 we obtain that $\varphi = \varphi^{Ch}$ for some acceptant substitutable priority Ch. □

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