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Electoral competition with policy-motivated candidates

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Abstract

In the multi-dimensional spatial model of elections with two policy-motivated candidates, we prove that the candidates must adopt the same policy platform in equilibrium. Moreover, when the number of voters is odd, if the gradients of the candidates' utility functions point in different directions, then they must locate at some voter's ideal point and a strong symmetry condition must be satisfied: in particular, it must be possible to pair some voters so that their gradients point in exactly opposite directions. If the number of dimensions is more than two, then our condition is knife-edge. When the number of voters is even, the situation is worse: such equilibria never exist, regardless of the dimensionality of the policy space.

JEL classification: D72; C72

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1. Introduction

What policy positions should candidates adopt in running for office? Will they choose identical positions? Or will stable choices even exist? These questions have been the central

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focus of the literature on spatial modeling that dates back to the famous work of (Downs, 1957). Downs' assumption that candidates care about winning and not about policies is standard in this literature. With such office-motivated candidates, the main findings of the spatial model of voting are well known. If the policy space is one dimensional, then the Median Voter Theorem holds: candidates choose identical positions at the median of the voters' ideal points (Downs, 1957; Black, 1958). On the other hand, if the issue space is multidimensional, then there is almost always no such unbeatable position, or "core point," and therefore equilibria almost never exist (Plott, 1967). In this paper, we reconsider the basic questions posed by spatial theory under the alternative assumption that candidates are policy-motivated.

A central paper in the literature that addresses these issues is Calvert (1985). Building on work by Wittman (1977, 1983), Calvert considers policy-motivated candidates and shows that in one dimension, convergence to the median still holds, and more generally (in any number of dimensions), if a core point exists, then the unique electoral equilibrium is for both candidates to locate at the core point. However, the assumption that a core point exists severely restricts the applicability of Calvert's result. As is well known, the existence of a core point entails a symmetry condition on voter preferences that is extremely demanding in two or more dimensions: Plott (1967) shows that a core point must be the ideal point of some voter, and the gradients of the other voters' utility functions must be paired so that, for every voter with a gradient pointing in one direction, there is exactly one voter whose gradient points in the opposite direction.[1] As a consequence, core points almost always fail to exist, and when one does exist, it will be vulnerable to even slight variations in preferences.[2] Calvert's result also assumes that voters have Euclidean preferences (circular indifference curves). He conjectures (pp. 78–79) that, if the assumption of Euclidean preferences is weakened, then other types of equilibria, in which candidates do not locate at the core point, may be created. The questions of existence and location of equilibrium points with policy-motivated candidates are left open in the general cases of non-Euclidean preferences and an empty core. We provide answers to these questions.

Under office-motivation, candidates must locate at core points in equilibrium: if one candidate were to locate at a beatable position, the other would move to exploit that opportunity. Thus, in the absence of a core point, there will be no equilibrium of the game between the candidates. Why might the assumption of policy-motivation yield different answers for the multidimensional case? The answer lies in the observation that a majority-preferred position may have undesirable policy implications for a candidate, mitigating the incentive to locate there. In other words, a change to a winning position that is beneficial to an office-motivated candidate, by definition, may not be so to a policy-motivated candidate if the winning position is a less desirable policy. Therefore, a model with policy-motivated candidates offers fewer potential profitable deviations and this suggests that we may find equilibria where none were present under office-motivation. We show that this is true only to a very limited extent. In particular, the symmetry conditions required for existence are

1 See also McKelvey and Schofield (1987).

2 See Rubinstein (1979), Schofield (1983), Cox (1984), Le Breton (1987). When the number of voters is even, the results are almost as negative: existence of core points may be robust to variations in preferences in two dimensions, but not in more.

weaker than Plott's. They are still demanding enough that equilibria will usually fail to exist in high-dimensional policy spaces, but now "high" means at least three dimensions, rather than two.

Our main results develop necessary conditions that must be satisfied by the equilibrium platforms of the candidates. We first show that in any equilibrium with neither candidate at her own ideal point, the candidates must take identical policy positions. This phenomenon is called "policy coincidence" or "policy convergence." We next consider the smaller set of equilibria in which the candidates' gradients point in different directions, so that the candidates have distinct policy preferences near the equilibrium point. Theorem 2 gives necessary conditions for existence of such an equilibrium when the voters are odd in number. Specifically, in equilibrium, the candidates must locate at the ideal point of some voter, and a type of symmetry on the voters' gradients must hold: for every voter whose gradient lies between the candidates' gradients, there must be exactly one voter whose gradient points in exactly the opposite direction.

Somewhat surprisingly, the restrictiveness of this symmetry condition turns out to depend on the dimensionality of the policy space. Indeed, for a two-dimensional issue space, we give a simple sufficient condition under which there exists an electoral equilibrium with policy-motivated candidates that is robust to small changes in the preferences of voters and candidates, even though the core may be empty. Thus, in two dimensions, the negative conclusions of Plott (1967) for office-motivated candidates do not carry over with full force. For three or more dimensions and an odd number of voters, however, we show in Theorem 3 that the existence of equilibria is knife-edge. In particular, the following symmetry condition is necessary: for every voter whose gradient does not lie on the plane spanned by the candidates' gradients, there must be exactly one voter whose gradient points in the opposite direction. In other words, if we remove the voters whose gradients lie on that plane, then the equilibrium platform must be a core point of the modified majority voting game. Because the plane is a lower-dimensional subspace, we would not expect it to contain the gradients of all voters. Typically, therefore, we must have some pairs of voters with diametrically opposed gradients, and this suggests that electoral equilibria will be rare and that, when existence does obtain, it will be vulnerable to even slight variations of voter or candidate preferences. Thus, with three or more dimensions, we conclude that equilibria with policy-motivated candidates almost never exist.

Theorem 4 takes up the case of an even number of voters and shows that existence is not even knife-edge: equilibria of the type we consider do not ever exist. Thus, the result in this case is even stronger than the result with an odd number of voters. This finding is worth noting because an even number of voters is the "optimistic" case in models of office-motivated candidates: core points and thus equilibria may be robust. But with policy-motivated candidates, these observations no longer hold.

The results we have discussed are proved in the framework of pure policy-motivation and deterministic voters, which is of course a stylized view of real elections. We focus on this polar case for several reasons. First, as our results are mainly negative, we seek to strengthen them by considering an environment amenable to existence, in contrast to the "mixed motivation" case: when office-motivation has positive weight in the candidates' payoffs, a significant (even if small) additional discontinuity is introduced into the game, and we then run the risk that nonexistence is an artifact of this discontinuity. Second, taking our model as a benchmark, we are able to show that our negative conclusions carry over even if we introduce a small amount of probabilistic voting into the model, smoothing out the payoff functions of the candidates, and even if we allow for a small benefit of winning the election (which may take a quite arbitrary form). Thus, though our main results are stated in terms of a particular model, they inform us about a "neighborhood" of models containing it. As a byproduct of this robustness result, we conclude that nonexistence of equilibrium in our model is not the product of discontinuities in candidate payoffs, but rather is the product of nonconvexities, which are unavoidable when candidates are well-informed about the behavior of voters. Last, our focus facilitates comparison to the literature on electoral competition.

The assumption of policy-motivation has been used in a significant number of applications. Surveys of this literature include Wittman (1990), Shepsle (1991), and Osborne (1995). Another line of literature combines policy-motivation with uncertainty, usually via probabilistic voting (Wittman, 1983; Calvert, 1985). These papers show that policy coincidence breaks down as soon as uncertainty about voting behavior is introduced, and that the extent of the divergence of the candidates' platforms varies continuously with the amount of policy-motivation added to the objective functions of office-motivated candidates. Our robustness results for probabilistic voting point to an issue that has gone somewhat unnoticed in this literature: equilibria need not exist in these models when there are multiple policy dimensions; indeed, when voting is close to deterministic and weight on office is small, equilibria will almost never exist. Finally, in the literature on "citizen candidates," candidates are assumed, along with other voters, to possess policy preferences.[3] But these models differ from the spatial model of elections in that candidates cannot commit to policies prior to an election; rather, office holders choose policies optimally given their preferences and, in some models, given the effects of policy choices on future electoral prospects. In contrast, our paper contributes to the understanding of the effects of policy motivation by maintaining the other basic assumptions, commitment among them, of the spatial model.

The remainder of the paper is organized as follows. In Section 2, we present the model of elections with policy-motivated candidates. In Section 3, we give two-dimensional examples of robust equilibria in the model with policy-motivated candidates, and we give a simple sufficient condition that generalizes the examples. In Section 4, we state our results on necessary conditions for existence of equilibria of two types: equilibria in which neither candidate locates at her ideal point, and the subset of equilibria in which the candidates' gradients point in different directions. In Section 5, we give conditions under which there are no other equilibria. In Section 6, we establish the robustness of our negative conclusions, showing that equilibrium nonexistence extends if a small amount of probabilistic voting and office-motivation are introduced. In Section 7, we briefly consider a simple model of mixed motives, where candidates put a fixed weight on holding office, in addition to policy concerns. The final section concludes, and Appendix A contains proofs of our results.

[3] See Osborne and Slivinski (1996), Besley and Coate (1997, 1998), Duggan (2000), Banks and Duggan (2000).

2. The model

We consider two candidates, A and B, competing for the votes of an electorate, N, containing a number n of voters. The candidates simultaneously choose policy platforms from X, a nonempty, convex subset of d-dimensional Euclidean space, $\mathbb{R}^d$.[4] We denote candidate C's platform choice by x_C. Each voter i has a preference relation on X represented by a strictly quasi-concave, differentiable utility function $u_i : X \to \mathbb{R}$, with interior *ideal point* $\tilde{x}_i$ that uniquely satisfies $\nabla u_i(x) = 0$. We assume that no two voters have the same ideal point: $\nabla u_i(x) = \nabla u_j(x) = 0$ for no x, i, and $j \neq i$. We say voter i's preferences are *Euclidean* if i has an ideal point $\tilde{x}_i$ and, for some strictly decreasing function $f : \mathbb{R}_+ \to \mathbb{R}$, $u_i(x) = f(\|x - \tilde{x}_i\|)$, i.e., voter i has circular indifference curves.

We use the notation R for weak majority preference, P for strict preference, and I for indifference: xRy if and only if $u_i(x) \geqslant u_i(y)$ for at least half of the voters; xPy if and only if $u_i(x) > u_i(y)$ for more than half of the voters (i.e., not yRx); and xIy if and only if xRy and yRx. We denote the number of voters who strictly prefer x to y by $n_A(x, y)$, the number who strictly prefer y to x by $n_B(x, y)$, and the number who are indifferent by $n_I(x, y)$. Thus, xPy if and only if $n_A(x, y) > n/2$, for example. In Appendix A, we state a lemma on the "star-shapedness" of majority preferences: if xRy, then any point between x and y will be weakly majority-preferred to y, strictly so if the number of voters is odd.

We define the *core* as the set of platforms x weakly majority-preferred to all other platforms: for all $y \in X$, xRy. If the number of voters is odd, then a standard result under our assumptions is that the core, when nonempty, consists of a single point, say x^*, and that, for all $y \neq x^*$, x^*Py. Moreover, x^* is the ideal point of some voter, say i^*. If all voters have Euclidean preferences, it is known that the majority preference relation coincides with the preferences of the "core voter" i^*, i.e., xRy if and only if $u_{i^*}(x) \geqslant u_{i^*}(y)$ (Davis et al., 1972). Thus, in that case, the majority weak preference relation is complete and transitive, with circular indifference curves. None of these conclusions holds generally when n is even.

We assume each candidate C has a preference relation on X represented by a strictly quasi-concave, differentiable utility function $u_C : X \to \mathbb{R}$, with interior ideal point $\tilde{x}_C$ that uniquely satisfies $\nabla u_C(x) = 0$. We assume that the candidates are policy-motivated, which means that a candidate may face a tradeoff between desirable and successful policy platforms. As is standard, we assume that candidates evaluate this tradeoff using expected utility.[5] Specifically, when A chooses platform x and B chooses platform y, A's expected utility is

$$U_A(x, y) = P(x, y)u_A(x) + \big(1 - P(x, y)\big)u_A(y) \tag{1}$$

[4] We use the notation C for an arbitrary candidate; i, j, k, etc., for an arbitrary voter; and x, y, z, etc., for arbitrary policies.

[5] See Duggan and Fey (2001) for a version of this model with more general assumptions on candidate preferences that do not impose the expected utility form.

(and similarly for B), where $P(x, y)$ is the probability that candidate A wins. When voters with strict preferences vote deterministically, $P(x, y)$ is equal to 1 if xPy and 0 if yPx. When xIy, its value is normally specified by some assumptions on how ties are broken and how voters make choices when indifferent, such as flipping fair coins.

Because our equilibrium existence results are ultimately negative, it is important to maintain a degree of generality with respect to the behavior of indifferent voters: otherwise, we would leave open the possibility that our conclusions were an artifact of our assumptions on voter behavior. Therefore, instead of choosing a particular specification of tie-breaking probabilities, we allow for quite arbitrary voting behavior when voters are indifferent.[6] Our assumptions are formalized in the following condition.

(A1) $P(x, y) = 1$ if xPy; $P(x, y) = 0$ if yPx; and $0 < P(x, y) < 1$ if xIy.

Essentially, we require of indifferent voters only that they vote for each candidate with positive probability. These probabilities may vary arbitrarily with the particular platforms over which the voter is indifferent.[7] When n is even, majority indifference may hold even if no voters are themselves indifferent, so we impose an additional condition on the form of $P(x, y)$ in this case.

(A2) If n is even and $n_A(x, y) = n_B(x, y) = n/2$ and $n_A(z, y) = n_B(z, y) = n/2$, then $P(x, y) = P(z, y)$. If n is even and $n_B(x, y) = n_B(z, y) = n/2$ and $n_A(x, y) < n_A(z, y)$, then $P(x, y) < P(z, y)$ (and likewise for B).

This assumption requires two things. First, all ties in which there are no indifferent voters are broken the same way. Second, if exactly half of the voters strictly prefer B's position, then the chance that A wins is increasing in the number of voters with a strict preference for A.[8]

The game between the candidates is thus defined by the strategy sets X for each candidate and the payoff functions given by Eq. (1). We use (pure strategy) Nash equilibrium as our equilibrium concept.[9] We say that an equilibrium (x_A, x_B) is a *nonsatiated equilibrium* if x_A and x_B are interior to X and neither candidate's chosen platform is at her ideal point: $\nabla u_A(x_A) \neq 0$ and $\nabla u_B(x_B) \neq 0$. We say an equilibrium (x_A, x_B) is a *nonaligned equilibrium* if the platforms are interior and the candidates' gradients do not point in the same direction: there do not exist $\alpha, \beta \geqslant 0$, at least one nonzero, such that $\alpha \nabla u_A(x_A) = \beta \nabla u_B(x_B)$. Note that every nonaligned equilibrium is nonsatiated.[10]

6 This approach is similar to that of Simon and Zame (1990).

7 In fact, we even allow indifferent voters to abstain from voting with any probability (possibly one), as long as the winner in case of a tie is determined randomly with each candidate receiving positive probability.

8 Again, see Duggan and Fey (2001) for a version of this model with more general assumptions on candidate preferences in the case of ties.

9 In other words, (x_A, x_B) is an equilibrium if neither candidate C can deviate to a different platform to produce a preferred pair: there does not exist $x'_A \in X$ such that $U_A(x'_A, x_B) > U_A(x_A, x_B)$ (and likewise for B).

10 To see this, suppose (x_A, x_B) is an equilibrium with $\nabla u_A(x_A) = 0$. Then choosing $\alpha > 0$ and $\beta = 0$ implies $\alpha \nabla u_A(x_A) = \beta \nabla u_B(x_B)$. So (x_A, x_B) is not a nonaligned equilibrium.

3. Sufficient conditions

We begin by illustrating that, unlike the case of office-motivated candidates, equilibria can exist in the absence of a core point. In Fig. 1, the ideal points of three voters are arranged in a triangle, and we give the voters and candidates Euclidean preferences. It is a (nonaligned) equilibrium for the candidates to locate at voter 3's ideal point in this example, because the weakly majority-preferred platforms are those weakly preferred by voters 1 and 2. This set, being the intersection of two circles, is sufficiently kinked—so that no such platforms are preferred by either candidate—as long as 1's and 2's ideal points are far enough apart. Obviously, as this configuration of voter ideal points has no majority core, no such equilibrium exists in the case of office-motivated candidates. Moreover, it is easy to see that this equilibrium is also robust to small variations in the preferences of the players.

Policy motivation can have a substantial effect even when the core is nonempty. In this case, when candidates are office-motivated, there can be no equilibria other than at the core point. This is not true when candidates are policy-motivated, as illustrated in Fig. 2. In this example, we give voters 1 and 3 Euclidean preferences but, as evidenced by voter 2's indifference curve, we give that voter non-Euclidean preferences. Voter 2's ideal point is the core point, but it is a nonaligned equilibrium for both candidates to locate at voter 3's ideal point: none of the platforms weakly majority-preferred to $\tilde{x}_3$, in the region described by hash marks, are preferred to $\tilde{x}_3$ by either candidate. Once again, note that the equilibrium in this example is robust, in the sense that it survives small enough variations in the gradients of the voters and candidates. The non-Euclidean preferences of voter 2 are necessary in this example, as Calvert (1985) shows that when voters' preferences are Euclidean and the core is nonempty, there can be no nonaligned equilibria other than the core. Thus, Fig. 2

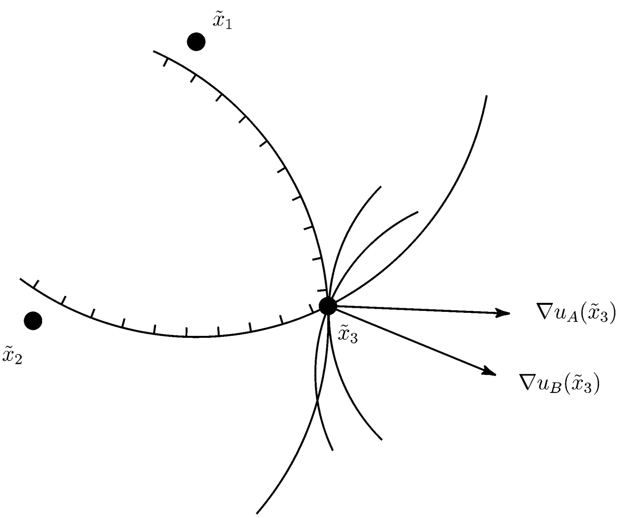

Fig. 1. A nonaligned equilibrium with no core.

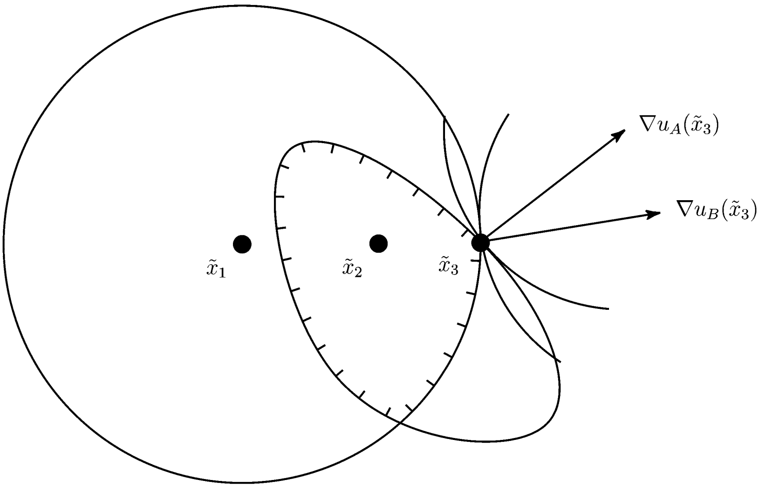

Fig. 2. A nonaligned equilibrium not at the core.

confirms Calvert's conjecture that equilibria can be supported at points other than the core when preferences are non-Euclidean.

Next, we move away from these specific examples and present a simple condition that ensures the existence of an equilibrium in a two-dimensional policy space. This condition requires that the two candidates locate at the ideal point of some voter and that each voter can be paired with another voter whose preferences are generally opposed. In order to formally state the result, we need the following definitions. For vectors $p, q \in \mathbb{R}^d$, we use the notation $\text{cone}\{p, q\} = \{\alpha p + \beta q \mid \alpha, \beta \geqslant 0 \text{ and } \alpha + \beta > 0\}$ to denote the convex cone generated by p and q and we refer to the cone generated by $-p$ and $-q$ as the "opposed cone" of p and q. For any nonempty set $G \subseteq N$, a function $\pi : G \to G$ is a *pairing* on G if π is one-to-one and, for all $i \in G$, $\pi(\pi(i)) = i$.

Proposition 1. *Assume n is odd, $d = 2$, and assume* (A1). *If $x_A = x_B = \hat{x}$, where $\nabla u_k(\hat{x}) = 0$ for some voter k, and there exists a pairing π on $N \setminus \{k\}$ such that, for all $i \in N \setminus \{k\}$ and for all $C = A, B$,*

$$\nabla u_i(\hat{x}) \cdot \nabla u_C(\hat{x}) \neq 0 \quad \Rightarrow \quad \nabla u_{\pi(i)}(\hat{x}) \in \text{cone}\big\{-\nabla u_C(\hat{x}), -\nabla u_i(\hat{x})\big\}, \tag{2}$$

then (x_A, x_B) is an equilibrium.

The restriction expressed in condition (2) is illustrated in Fig. 3. In the figure, the opposed cone of the gradients of candidate C and voter i is pictured, and the condition requires that the gradient of the voter paired with i must lie in this opposed cone. In other words, voter i must be "blocked" by some voter $\pi(i)$, in that any alternative that candidate C and voter i prefer to $\hat{x}$ must make voter $\pi(i)$ worse off. Thus, this condition is equivalent to requiring that $\hat{x}$ be Pareto optimal relative to voters i and $\pi(i)$ and candidate C. It is easy to see that this sufficient condition is satisfied by the two examples presented above, and it is satisfied at the core point, if it exists.

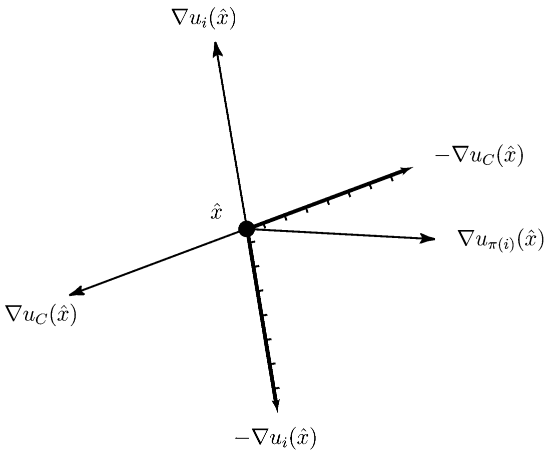

Fig. 3. Opposed cone of $\nabla u_C(\hat{x})$ and $\nabla u_i(\hat{x})$.

The proof of the proposition is as follows. Suppose that $x_A = x_B = \hat{x}$ is not an equilibrium. Then, as n is odd, there must be an alternative y that is majority-preferred to $\hat{x}$ such that $u_C(y) > u_C(\hat{x})$ for some candidate C. If we denote the vector from $\hat{x}$ to y by q, then by strict quasi-concavity, any alternative z a sufficiently small distance from $\hat{x}$ in direction q must also satisfy $zP\hat{x}$ and $u_C(z) > u_C(\hat{x})$. Clearly, voter k prefers $\hat{x}$ to z, and thus $(n+1)/2$ of the remaining $n-1$ voters must prefer z to $\hat{x}$. This implies that for any pairing π on $N \setminus \{k\}$, there must be a pair of voters, j and $\pi(j)$, that both prefer z to $\hat{x}$. But then $\hat{x}$ is not Pareto optimal relative to voters j and $\pi(j)$ and candidate C, as required by the condition of the proposition.

While Proposition 1 gives conditions sufficient for existence of equilibria, it is possible that some of these equilibria may be fragile, in the sense that arbitrarily small perturbations of voter or candidate preferences may lead to nonexistence. However, if we strengthen the condition of Proposition 1 so that blocking gradients are required to be in the "open" opposed cone (where α and β are restricted to be strictly positive), then it is clear that the equilibria established in the proposition will be robust to such perturbations.

The proposition requires that condition (2) holds for both candidates. That is, the gradient of the voter paired with i must lie in the intersection of the opposed cones of i and A and i and B. Now, it is easy to see that if the gradient of voter i is between the gradients of the two candidates, then the opposed cone of i and A intersects with the opposed cone of i and B in exactly one direction, namely, $-\nabla u_i(\hat{x})$. In other words, the sufficient condition requires that voters whose gradients lie between the gradients of the candidates must be paired with voters whose gradients point in exactly the opposite direction. As we show in the next section, this condition on voter gradients in this region is actually necessary for nonaligned equilibria to exist.

4. Necessary conditions

In this section, we present necessary conditions for the existence of particular types of equilibria in our model. By doing so, we shed light on whether such equilibria are

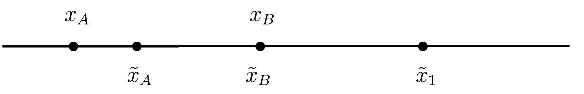

Fig. 4. A "satiated" equilibrium without policy coincidence.

likely to exist or not for a typical choice of preferences. We first establish that, in every nonsatiated equilibrium, the candidates must choose the same platform, a phenomenon termed "policy coincidence." Thus, if neither candidate is at her optimal position, then the incentives of electoral competition lead to a unique policy choice for the voters, even though the candidates might have starkly different policy preferences.

Theorem 1. *Assume* (A1) *and* (A2). *If (x_A, x_B) is a nonsatiated equilibrium, then $x_A = x_B$.*

As a consequence of this theorem, in *any* equilibrium of the candidate positioning game, either one (or both) of the candidates is at her ideal point or they choose identical positions. It is easy to find examples of the first sort of "satiated" equilibria that violate policy coincidence. Fig. 4 gives an example of such an equilibrium with one dimension and one voter with Euclidean preferences. Here, candidate A's ideal point, $\tilde{x}_A$, is to the left of candidate B's, which is to the left of the voter's ideal point, $\tilde{x}_1$. If candidate B's platform, x_B, is at her ideal point, $\tilde{x}_B$, and if candidate A locates anywhere to the left of B, then neither candidate can deviate profitably.[11] In this example of a satiated equilibrium, one candidate happens to lose with probability one; in fact, this can be shown to be a general feature of satiated equilibria. Proposition 2, in the next section, gives a condition that rules out the possibility of such equilibria when n is odd. In the one-dimensional case, the condition is simply that the candidates' ideal points lie on opposite sides of the median ideal point.

We can say considerably more about equilibria in which the gradients of the candidates do not point in the same direction. We establish that, when the number of voters is odd, the candidates must locate at some voter's ideal point, say $\hat{x}$. Moreover, a limited version of Plott's (1967) symmetry condition must hold: it must be possible to pair voters whose gradients are between the candidates' gradients with voters whose gradients point in exactly opposite directions. For vectors $p, q \in \mathbb{R}^d$, we use the notation $\operatorname{cone}^\circ\{p, q\} = \{\alpha p + \beta q \mid \alpha, \beta > 0\}$ to denote the open cone generated by p and q.

Theorem 2. *Assume n is odd, and assume* (A1). *If (x_A, x_B) is a nonaligned equilibrium, then $x_A = x_B = \hat{x}$, where $\nabla u_k(\hat{x}) = 0$ for some voter k. If $\nabla u_A(\hat{x})$ and $\nabla u_B(\hat{x})$ are linearly independent, then for every $p \in \operatorname{cone}^\circ\{\nabla u_A(\hat{x}), \nabla u_B(\hat{x})\}$,*

$$\left|\left\{i \in N \mid \exists \alpha > 0\colon\ \nabla u_i(\hat{x}) = \alpha p\right\}\right| = \left|\left\{i \in N \mid \exists \alpha < 0\colon\ \nabla u_i(\hat{x}) = \alpha p\right\}\right|. \tag{3}$$

If $\nabla u_A(\hat{x})$ and $\nabla u_B(\hat{x})$ are linearly dependent, then Eq. (3) *holds for all $p \in \mathbb{R}^d$.*

In this theorem, Eq. (3) is the formal expression requiring voters to be matched with other voters with opposing gradients. This requirement is limited, in that it need only hold

[11] A similar example with n even can be constructed simply by placing a second voter's ideal point to the right of voter 1's.

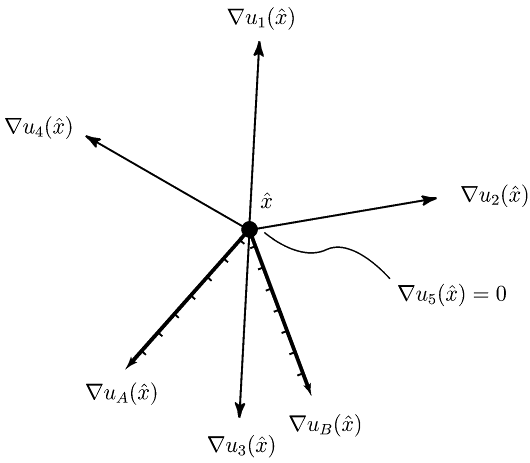

Fig. 5. The symmetry condition of Theorem 2.

for voters with gradients in a prescribed region. This is depicted in Fig. 5. Here, the candidates locate at voter 5's ideal point. The gradients of voters 1 and 3 point in opposite directions. The gradients of voters 2 and 4 are not matched in this way, but, because neither gradient (or its opposite) lies in the open cone generated by the candidates' gradients, the symmetry condition of the theorem is preserved.[12]

By the first part of the theorem, the candidates must locate at some ideal point, say $\hat{x}$, in a nonaligned equilibrium. The proof of the remainder of the theorem is largely concerned with the case in which the candidates' gradients are linearly independent. We show that the set of platforms weakly majority-preferred to $\hat{x}$, the region described by hash marks in Fig. 6, must lie below the hyperplanes defined by the gradients of the candidates. This implies a kind of "kink" in the boundary of that set, one that is not possible when the core is nonempty and the preferences of the voters are Euclidean. Under those conditions, the majority preference relation would coincide with the preference relation of the core voter, so the majority indifference curves would simply be circles and obviously could not have kinks. Thus, in Calvert's (1985) model, the only platform weakly preferred to $\hat{x}$ is $\hat{x}$ itself, i.e., the candidates must locate at the core point, and then symmetry of the voters' gradients follows from Plott's (1967) theorem. In the proof of Theorem 2, we show, without assuming Euclidean preferences or the existence of a core point, that the boundary of the set of platforms weakly majority-preferred to $\hat{x}$ is "kinked enough" only if the symmetry condition of the theorem holds.

Figure 7 demonstrates that the necessary conditions for equilibrium presented in Theorem 2 are not sufficient. In particular, if the candidates are located at voter 3's ideal point with candidate gradients as depicted, then the conditions of the theorem are satisfied, but candidate A can move to a more desirable platform preferred by voters 1 and 2. Therefore, this choice of candidate positions is not an equilibrium.

[12] In fact, the condition of Proposition 1 is satisfied, so that it is indeed an equilibrium to locate at $\hat{x}$.

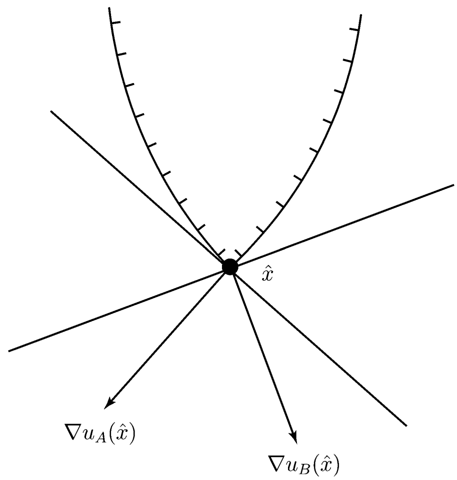

Fig. 6. A kink in the boundary of the majority-preferred-to set.

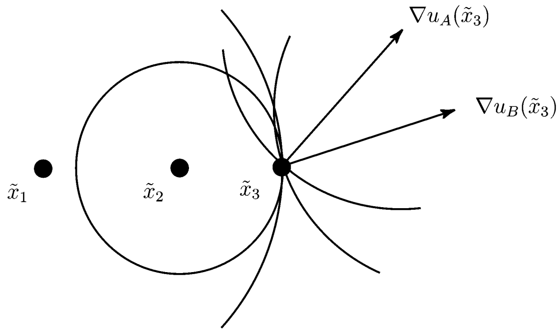

Fig. 7. Disequilibrium satisfying the necessary condition.

Theorem 2 applies only to nonaligned equilibria. That it cannot be applied to “aligned equilibria,” even those in which the candidates adopt the same platform, can be seen by modifying the example of Fig. 4. Suppose both candidates have the same platform, say $\hat{x}$, anywhere between candidate B’s ideal point, $\tilde{x}_B$, and the voter’s, $\tilde{x}_1$. This is a nonsatiated, aligned equilibrium: for each candidate, the only platforms majority-preferred to $\hat{x}$ are less desirable than $\hat{x}$. Clearly, the candidates are not located at the ideal point of any voter, and the symmetry condition of the theorem is violated. Proposition 3, in the next section, gives a condition under which no such aligned equilibria will exist. In the one-dimensional case, the condition there is simply that the candidates’ ideal points lie on opposite sides of the median.

There is a limitation of Theorem 2: the symmetry condition, Eq. (3), applies only to voters with gradients in the plane defined by the candidates’ gradients. Thus, in more than two dimensions, this condition only applies to voters with gradients that are precisely co-planar with the gradients of the candidates, an event that generically never occurs. In such multidimensional spaces, then, a direct application of Theorem 2 yields a negligible restriction on voter's preferences. However, we can use this theorem to prove the next, which imposes a severe restriction on voters with gradients that are *not* co-planar with the candidates' gradients. Precisely, Theorem 3 says that, given a nonaligned equilibrium $(\hat{x}, \hat{x})$, for every voter whose gradient does not lie on the plane spanned by the candidates' gradients, there must be a voter whose gradient points in exactly the opposite direction. Alternatively, if we delete the voters whose gradients lie on the plane spanned by the candidates' gradients, but leaving the $\hat{x}$ voter, then the platform $\hat{x}$ must be a core point of the resulting majority preference relation.

Theorem 3. *Assume n is odd, and assume* (A1). *If (x_A, x_B) is a nonaligned equilibrium, then $x_A = x_B = \hat{x}$, where $\nabla u_k(\hat{x}) = 0$ for some voter k. Moreover, for every $p \in \mathbb{R}^d$ such that $p \notin \operatorname{span}\{\nabla u_A(\hat{x}), \nabla u_B(\hat{x})\}$,*

$$\left|\left\{i \in N \mid \exists \alpha > 0\colon\ \nabla u_i(\hat{x}) = \alpha p\right\}\right| = \left|\left\{i \in N \mid \exists \alpha < 0\colon\ \nabla u_i(\hat{x}) = \alpha p\right\}\right|.$$

As with Theorem 2, Theorem 3 applies only to nonaligned equilibria and it cannot be extended to aligned equilibria, even those in which the candidates adopt the same platform.[13] An example is given in Fig. 8. Here, we assume three voters and Euclidean preferences over a multidimensional policy space, with the ideal points of the voters arranged in an isosceles triangle, voter 1's ideal point at the apex. Candidate B's ideal point is above that, and candidate A's ideal point is above that, both coplanar with the voters' ideal points. In this example, it is an equilibrium for both candidates to adopt the same platform anywhere between voter 1's and candidate B's ideal points. One possible location is indicated in the figure. Clearly, in this equilibrium the candidates locate at no voter's ideal point. Moreover, the span of the candidate's gradients is the line through their ideal points, and neither voter

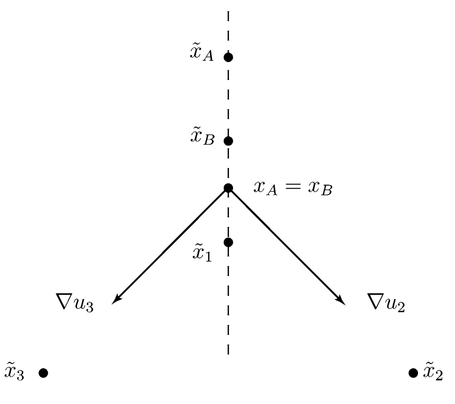

Fig. 8. A nonsatiated aligned equilibrium violating the conditions of Theorem 3.

[13] To be clear, Theorem 2 limits voters' gradients that are in the open cone of (and thus co-planar with) the candidates' gradients, and Theorem 3 limits the gradients that are not co-planar with the gradients of the candidates. The latter theorem is therefore nonvacuous only in more than two dimensions.

2's nor voter 3's gradients can be opposed to voter 1's in the required way, violating the symmetry condition of the theorem.

The following corollary of Theorem 3 gives a general condition on the gradients of voters under which nonaligned equilibria fail to exist. The condition holds quite widely when the dimension of the policy space is at least three. It suggests that, for "most" specifications of differentiable, strictly quasi-concave voter utility functions, we would not expect nonaligned equilibria to exist—and that, if existence did obtain, it would be sensitive to even slight variations of voter or candidate preferences.

Corollary 1. *Assume n is odd, and assume* (A1). *Assume that for all voters i, the dimension of* $\mathrm{span}\{\nabla u_j(\tilde{x}_i) \mid j \in N\}$ *is at least three. And assume that, for all voters j and k, $\nabla u_j(\tilde{x}_i)$ and $\nabla u_k(\tilde{x}_i)$ are linearly independent. Then there does not exist a nonaligned equilibrium.*

The proof the corollary is simple. Theorem 3 tells us that, given a nonaligned equilibrium (x_A, x_B), the candidates must locate at the ideal point of some voter, say i. Since $\mathrm{span}\{\nabla u_A(\tilde{x}_i), \nabla u_B(\tilde{x}_i)\}$ is a two-dimensional space and the dimension of $\mathrm{span}\{\nabla u_j(\tilde{x}_i) \mid j \in N\}$ is at least three, there is some voter j such that $\nabla u_j(\tilde{x}_i) \notin \mathrm{span}\{\nabla u_A(\tilde{x}_i), \nabla u_B(\tilde{x}_i)\}$. But, under the assumptions of the corollary, there is no voter whose gradient points in the direction opposite that of voter j's, a contradiction.

Thus, with n odd, the "typical" case is that no nonaligned equilibria exist. An even stronger result holds if n is even: nonaligned equilibria never exist. Existence in this case hinges on the possibility that the candidates' gradients point in exactly the same direction in equilibrium (as in Fig. 8, if we add a voter below voter 1), or, as shown in the next section, both candidates locate at their own ideal point.

Theorem 4. *Assume n is even, and assume* (A1) *and* (A2). *There does not exist a nonaligned equilibrium.*

In the proof of the theorem, we first verify that, as in Theorem 2, the candidates would have to locate at the ideal point, say $\hat{x}$, of some voter, say i. Deleting that voter from N, we are left with an electorate, N', with an odd number of voters. Furthermore, there is no voter in N' with ideal point $\hat{x}$, violating a necessary condition in Theorem 2 for equilibrium in the reduced model. Thus, one of the candidates can move to a better platform, say x', preferred by a majority of voters in N' to $\hat{x}$. Adding i back to the electorate, x' still *weakly* beats $\hat{x}$. Under condition (A2), this still gives the candidate a profitable deviation, and we conclude that nonaligned equilibria cannot exist when n is even.

5. Additional types of equilibria

In the preceding section, we gave several results for nonsatiated and nonaligned equilibria. But what about equilibria of this game that are not nonaligned or nonsatiated? For example, Theorem 1 establishes that nonsatiated equilibria exhibit policy coincidence, but as the example in Fig. 4 demonstrates, there can exist equilibria that are not nonsatiated and that do not exhibit policy coincidence. In this case, then, without a more detailed equilibrium selection argument, we cannot state unequivocally that policy coincidence will or will not occur. To deal with these issues, in this section we provide conditions under which satiated and aligned equilibria will not exist. Specifically, we present two results. The first gives a sufficient condition under which all equilibria must be nonsatiated, and the second gives stronger conditions under which all equilibria must be nonaligned.

Our first result is Proposition 2. It uses the condition that, given either candidate's ideal point, there exists a majority-preferred platform that the other candidate also prefers. This extends the condition, frequently assumed in one-dimensional models, that the candidates' ideal points are on opposite sides of the median (or medians, when n is even). We discuss the plausibility of a stronger condition at the end of the section.

Proposition 2. *Assume* (A1) *and* (A2). *Assume that there exists platforms $x, y \in X$ such that $x P \tilde{x}_A$ and $u_B(x) > u_B(\tilde{x}_A)$ and that $y P x_B$ and $u_A(y) > u_A(\tilde{x}_B)$. If (x_A, x_B) is an interior equilibrium, then either it is nonsatiated or*: *n is even and $x_A = \tilde{x}_A$ and $x_B = \tilde{x}_B$.*

Returning to the issue discussed at the beginning of this section, this proposition and Theorem 1 imply that policy coincidence must hold when n is odd and the condition given in the proposition holds. Obviously, the example in Fig. 4 does not satisfy this condition, as both candidates' ideal points are to the left of the voter's ideal point.

When n is even, Proposition 2 leaves open the possibility of a satiated equilibrium, as long as both candidates locate at their ideal points. This possibility is depicted in Fig. 9, where the ideal points of the two voters are between those of the candidates. It is easy to see that the condition of Proposition 2 is satisfied: the ideal point of voter 1, $\tilde{x}_1$, is preferred to $\tilde{x}_A$ by both voters and by candidate B; similarly, $\tilde{x}_2$ is preferred to $\tilde{x}_B$ by the voters and by candidate A. Note that there exist open intervals Y and Z around $\tilde{x}_A$ and $\tilde{x}_B$, respectively, such that every platform in Y is majority-indifferent to every platform in Z. Thus, because there are no small moves for either candidate to platforms that will beat her opponent, our argument for Theorem 1 (in Appendix A) that one candidate will have a profitable deviation does not go through. Indeed, there is no compelling reason why one of the candidates must have a profitable deviation in this situation—that will depend on the exact specification of the candidates' utility functions.

The next proposition gives a condition, strengthening that of Proposition 2, under which all equilibria are nonaligned. Once again, the condition extends the familiar one from one-dimensional models that the candidates' ideal points are on opposite sides of the median. We will say that an interior platform x satisfies the *alignment condition* if $\alpha \nabla u_A(x) = \beta \nabla u_B(x)$ for some $\alpha, \beta \geqslant 0$, at least one nonzero.

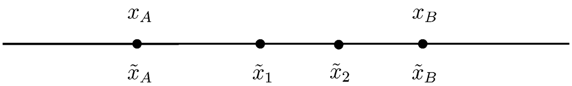

Fig. 9. A satiated equilibrium with n even, as in Proposition 2.

Proposition 3. *Assume* (A1) *and* (A2). *Assume that, for each $x \in X$ satisfying the alignment condition, there exists a platform $y \in X$ such that yPx and, for some candidate C, $u_C(y) > u_C(x)$. If (x_A, x_B) is a nonsatiated equilibrium, then it is nonaligned.*

The proof is trivial and omitted. To see that the condition in this proposition is indeed stronger than that of Proposition 2, set $x = \tilde{x}_A$; then the condition of Proposition 3 yields C and y such that $u_C(y) > u_C(x)$; and then, of course, we must have $C = B$, fulfilling the condition of Proposition 2. Therefore, when n is odd, *all* equilibria are nonaligned under the condition of Proposition 3.

The condition of Proposition 3 is not completely transparent, and so it is of interest to understand when it (and therefore the condition of Proposition 2) might hold. As an illustration, we give a sufficient condition for the antecedent condition in Proposition 3 to apply. In doing so, we establish that if preferences are "close" to having a core point, then all nonsatiated equilibria will be nonaligned and thus the stringent symmetry conditions of Theorem 3 must be satisfied.

To begin, suppose that $d \geqslant 2$, that n is odd, and that voter and candidate preferences are Euclidean. Let $Y \subseteq X$ denote the *yolk*, the smallest closed ball intersecting all median hyperplanes (McKelvey, 1986). Thus, if the hyperplane

$$H_{u,v} = \left\{z \in \mathbb{R}^d \mid 2z \cdot (u - v) = (u + v) \cdot (u - v)\right\}$$

bisecting two platforms, u and v, does not intersect Y, then majority indifference between u and v cannot hold. Whether uPv or vPu depends on whether Y is on the u-side or v-side of $H_{u,v}$. Suppose further that there exists $t \in \mathbb{R}^d$ such that, for all $w \in Y$,

$$t \cdot \tilde{x}_A < t \cdot w < t \cdot \tilde{x}_B.$$

For simplicity, we normalize t so that $\|t\| = 1$. Note that, since Y is compact, the minimum value of $t \cdot w$ over Y, denoted $\min t \cdot Y$, exists and $t \cdot \tilde{x}_A < \min t \cdot Y$. Likewise, $\max t \cdot Y < t \cdot \tilde{x}_B$. Also note the implication that $t \cdot (\tilde{x}_B - \tilde{x}_A) > 0$. Obviously, this situation, depicted in Fig. 10, is more plausible when the yolk is small, i.e., when the core is "close" to being

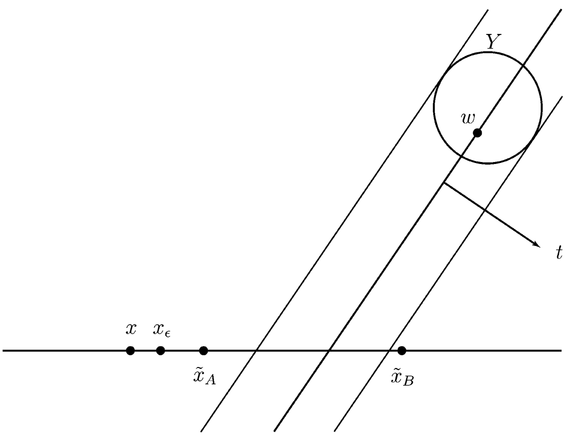

Fig. 10. The yolk "separating" the candidates' ideal points.

nonempty. When the core is nonempty, it is equal to the yolk and the above condition holds as long as the candidates' ideal points are not colinear with (and to the same side of) the core point.

When such a t exists, the assumption of Proposition 3 holds. To see this, note that the set of platforms that satisfy the alignment condition must lie on the line $\operatorname{span}\{\tilde{x}_A - \tilde{x}_B\} + \tilde{x}_A$ spanned by the candidates' ideal points, but not strictly between them. Letting x be such a platform, that means

$$x = \alpha\tilde{x}_A + (1-\alpha)\tilde{x}_B = \tilde{x}_B + \alpha(\tilde{x}_A - \tilde{x}_B)$$

for $\alpha \geqslant 1$ or $\alpha \leqslant -1$. If $\alpha \geqslant 1$, then

$$\begin{aligned} t \cdot x &= t \cdot \tilde{x}_B + t \cdot (\tilde{x}_A - \tilde{x}_B) + (1-\alpha)t \cdot (\tilde{x}_B - \tilde{x}_A) \\ &= t \cdot \tilde{x}_A + (1-\alpha)t \cdot (\tilde{x}_B - \tilde{x}_A) \\ &\leqslant t \cdot \tilde{x}_A. \end{aligned}$$

Similarly, $t \cdot x \geqslant t \cdot \tilde{x}_B$ if $\alpha \leqslant -1$. Suppose without loss of generality that $\alpha \geqslant 1$, as in Fig. 10. Define $x_\epsilon = x + \epsilon(\tilde{x}_B - \tilde{x}_A)$, and pick $\epsilon > 0$ small enough that $t \cdot x_\epsilon < \min t \cdot Y$. With $t \cdot x \leqslant t \cdot \tilde{x}_A < \min t \cdot Y$, this implies that the bisecting hyperplane H_{x,x_ϵ} does not intersect the yolk. And since the yolk is on the x_ϵ-side of the hyperplane, we have $x_\epsilon P x$. Finally, note that

$$u_B(x_\epsilon) - u_B(x) = \epsilon(2\alpha - \epsilon)(\tilde{x}_B - \tilde{x}_A) \cdot (\tilde{x}_B - \tilde{x}_A),$$

which is positive for small enough $\epsilon > 0$, as required.

6. Local robustness of nonexistence

Our analysis has so far been confined to environments in which voters vote in a deterministic fashion (with only indifferent voters possibly randomizing between the candidates) and in which candidates are motivated solely by policy preferences. This model is, of course, a stylized representation of real-world elections, and it is best viewed as a benchmark, rather than taken literally. It is therefore important to consider whether our results on equilibrium nonexistence persist when the model is subject to perturbations, of which we consider two types: we allow for uncertainty in voting behavior, as in the literature on probabilistic voting, and we allow for more general candidate incentives.

The introduction of noise into voting behavior alters the structure of the electoral game, smoothing the candidates' payoffs and eliminating discontinuities present in the deterministic model. Nonconvexities in the candidates' payoffs may remain, however, and existence of (pure strategy) equilibria is not guaranteed. Indeed, we show that when equilibria fail to exist in our benchmark model, as is often the case, equilibria will also fail to exist in probabilistic voting models "close" to the benchmark. This remains true even if we give the candidates a small positive benefit from holding office, even if that benefit can vary with the platforms of the candidates. Thus, adding a small amount of randomness into voter behavior and perturbing the incentives of the candidates will not solve the nonexistence problem of Corollary 1 and Theorem 4. An added insight from the result is that it is the nonconvexities—not discontinuities in candidate payoffs—that drive the problem of nonexistence in the benchmark model.

To extend our analysis, we imbed our model in a space Λ, where each model $\lambda \in \Lambda$ corresponds to a function $P(x, y|\lambda)$, which represents candidate A's probability of winning, and functions $w_A(x, y|\lambda)$ and $w_B(x, y\lambda)$, which represent any benefits of winning to the candidates. We assume these functions take non-negative values, but we do not impose continuity or any other restrictions. These benefits could capture the prestige of holding office, or monetary rents due to salary or bribes, or the cooperation of interest groups or party members. More generally, $w_A(x, y|\lambda)$ could be interpreted as reflecting the preferences of constituency groups that the office holder, as a representative, may feel obligated to serve. When A chooses platform x and B chooses platform y in model λ, A's expected utility is then

$$U_A(x, y|\lambda) = P(x, y|\lambda)\big(u_A(x) + w_A(x, y|\lambda)\big) + \big(1 - P(x, y|\lambda)\big)u_A(y)$$

(and similarly for B). The definitions of equilibrium for an arbitrary model λ remain as above. We designate the model λ^* as the model with pure policy motivation and deterministic voting studied above, so that $P(x, y|\lambda^*) = P(x, y)$ and $w_A(x, y|\lambda^*) = w_B(x, y|\lambda^*) = 0$ for all $x, y \in X$, and we let P^* denote the strict majority preference relation in λ^*.

We say a sequence $\{\lambda^m\}$ *approximates* λ^* if

(i) for all $x, y \in X$, we have $0 < P(x, y|\lambda^m) < 1$,
(ii) $w_A(\cdot|\lambda^m) \to 0$ and $w_B(\cdot|\lambda^m) \to 0$ uniformly, and
(iii) for every $x, y \in X$ such that $x P^* y$, there exist open neighborhoods G of x and H of y such that $P(\cdot|\lambda^m) \to 1$ uniformly on $G \times H$ and $P(\cdot|\lambda^m) \to 0$ uniformly on $H \times G$.

While condition (i) formalizes the idea that voting is indeed probabilistic, condition (ii) requires that benefits of winning become negligible in the limit, as they are in the model λ^*. Condition (iii) stipulates that the candidates' probability of winning satisfies a certain continuity condition. In contrast to (ii), uniform convergence is required only in the case of a majority strict preference, and then only in an open set around the candidates' platforms.[14] Though technical in nature, the condition is weak: we show later in the context of the two most widely used models of probabilistic voting that our definition captures the intuitive meaning of being "close" to deterministic.

The next proposition establishes, essentially, that if there is no equilibrium in the benchmark model, as we have shown is often the case, then there is an open set of models containing λ^* in which equilibria fail to exist. For simplicity, we have chosen to phrase the result in terms of equilibria, rather than nonsatiated or nonaligned equilibria, but the logic of the proof holds fairly generally: by a similar proof, for example, we can show that if there is no nonaligned equilibrium in the deterministic model, then there are no

[14] These restrictions are critical for the interpretation of our results, as a sequence of continuous functions cannot converge uniformly to a discontinuous function. Without them, since the probability of winning function is discontinuous in model λ^*, we would not be able to approximate λ^* with a sequence of continuous probability of winning functions.

nonaligned equilibria in nearby probabilistic voting models.[15] Likewise, for simplicity we restrict attention to the case of an odd number of voters.[16]

Theorem 5. *Assume n is odd and X is compact. Let $\{\lambda^m\}$ approximate λ^*. If there is no equilibrium in λ^*, then, for m high enough, there is no equilibrium in λ^m.*

Put contrapositively, the proof of Theorem 5 establishes the closed graph property of the equilibrium correspondence at the benchmark model: the limit point of equilibria in models close to λ^* must be an equilibrium in λ^*. Ordinarily, this property of equilibrium correspondences is to be expected. In our case, however, the limiting model is discontinuous, and then the conventional wisdom does not apply. We use the structure of candidate and voter utilities, along with some uniform convergence along the sequence of probabilistic voting models (which, as we see next, is quite natural), to prove the result. In these respects, Theorem 5 is similar to Corollary 8 from Banks and Duggan (2005), who show that, when the core is empty, equilibria in probabilistic voting models close to deterministic do not exist.[17]

We have formulated the idea of "approximation" in abstract terms in order to capture the intuitive meaning of "close" to the benchmark model. While conditions (i) and (ii) are not controversial, condition (iii) is less transparent. Next, we establish that the condition is permissive in one of the most commonly used probabilistic voting frameworks, which captures uncertainty about voters' preferences for nonpolicy characteristics of the candidates: in the additive bias model, the voters' utilities from candidate platforms are subject to random utility increments. We show that a sequence of additive bias models in which voting behavior becomes arbitrarily close to deterministic, in intuitive terms, will necessarily satisfy our condition.

In the *additive bias model*, each voter i has policy preferences given by u_i, as in Section 2. In addition, each voter's utilities are modified by an additive utility shock to each candidate. Without loss of generality, we normalize the shock for candidate A to zero and consider only a "bias," denoted β_i, for candidate B. The bias term β_i is stochastic and independent of the other voters' biases and the platforms of the candidates. Given the candidates' platforms and bias β_i, we assume voter i votes for candidate B if $u_i(x_A) < u_i(x_B) + \beta_i$, votes for candidate A if this inequality is reversed, and votes for each candidate with probability one half if equality holds. Here, a model λ is identified with a distribution function $F_i(\cdot|\lambda)$ for each voter i from which the voter's bias term is drawn. We assume $F_i(\cdot|\lambda)$ is continuous and strictly increasing for all λ. Thus, the probability voter i votes for candidate A is

$$P_i(x_A, x_B|\lambda) = F_i\big(u_i(x_A) - u_i(x_B)|\lambda\big),$$

[15] We would then need to add some technical conditions. We would require that utility functions have continuous gradients, and we would restrict attention to sequences of equilibria that do not converge to a boundary point of the policy space and such that the candidates' gradients do not become arbitrarily close to aligned.

[16] A similar result holds for n even, but the appropriate definition of approximation becomes somewhat more involved.

[17] Banks and Duggan (2005) restrict attention to a specific model of probabilistic voting, the "additive bias model," and they consider expected plurality maximizing candidates.

and the probability that A wins is

$$P(x_A, x_B|\lambda) = \sum_{C \in \mathcal{M}} \left(\prod_{i \in C} P_i(x_A, x_B|\lambda) \right) \left(\prod_{i \notin C} (1 - P_i(x_A, x_B|\lambda) \right),$$

where $\mathcal{M}$ denotes the subsets of voters with greater than $n/2$ members.

Proposition 4. *Let $\{\lambda^m\}$ be a sequence of additive bias models such that, for each voter i, the sequence $F_i(\cdot|\lambda^m)$ converges weak* to the point mass on zero.*[18] *Then $\{\lambda^m\}$ satisfies condition* (iii) *in the definition of approximation.*

We also illustrate the role of condition (iii) in another common probabilistic voting framework, which captures uncertainty about voters' policy preferences: in the random preference model, policy preferences of voters are themselves random variables. Again, we show that condition (iii) is consistent with the intuitive meaning of "close" to deterministic voting.

In the *random preference model,* each voter's policy preferences are given by $u_i(x|\theta)$, where θ is a preference parameter lying in a metric space Θ, and where $u_i : X \times \Theta \to \Re$ is jointly continuous. Here, a model λ is identified with a Borel probability measure over Θ, which in turn generates probabilistic voter preferences. For each voter i, define

$$P_i(x_A, x_B|\lambda) = \lambda\big(\big\{\theta \mid u_i(x_A|\theta) > u_i(x_B|\theta)\big\}\big) + \frac{1}{2}\lambda\big(\big\{\theta \mid u_i(x_A|\theta) = u_i(x_B|\theta)\big\}\big),$$

and define candidate A's probability of winning as we have above.

Proposition 5. *Let $\{\lambda^m\}$ be a sequence of random preference models such that λ^m converges to λ^* in the weak* topology, where λ^* puts probability one on some $\theta^* \in \Theta$.*[19] *Then $\{\lambda^m\}$ satisfies condition* (iii) *in the definition of approximation.*

Condition (iii) captures the notion of "close" to deterministic voting in other frameworks as well, extending the scope of Theorem 5. For example, in the quantal response voting model, considered by McKelvey and Patty (2003), when the distribution on voters' error terms converges to zero, voting behavior approximates voting in our deterministic model, and again condition (iii) is satisfied. For another example, if each voter observes the candidates' platforms with some noise (and votes as though the observed platforms were correct), then condition (iii) is satisfied as the noise goes to zero.[20]

7. Mixed motivations

Theorem 5 of the previous section demonstrated a neighborhood containing our original model in which our negative results hold: despite small perturbations of the model

[18] That is, for every bounded, continuous $\phi : \mathbb{R} \to \mathbb{R}$, the integrals $\int \phi(z) F_i(\mathrm{d}z|\lambda^m)$ converge to $\phi(0)$.

[19] That is, for every bounded, continuous $\phi : \Theta \to \mathbb{R}$, the integrals $\int \phi(\theta)\lambda^m(\mathrm{d}\theta)$ converge to $\phi(\theta^*)$.

[20] This claim does not hold if the voters are strategic, as in the model of Lagerlöf (2003). There, because candidate deviations are unobservable, candidates must locate at their ideal points in equilibrium.

in a rich variety of directions (allowing for probabilistic voting and office benefits of a quite arbitrary form), equilibrium nonexistence carries over. Here, we introduce a degree of office-motivation in the simplest way possible, in the form of a fixed, positive benefit of winning. We let $w > 0$ denote the benefit of winning, in which case candidate A's expected utility is given by

$$U_A(x, y) = P(x, y)\bigl(u_A(x) + w\bigr) + \bigl(1 - P(x, y)\bigr)u_A(y). \tag{4}$$

As a consequence of our results for policy-motivated candidates, this functional form permits a global characterization of nonsatiated equilibria. For simplicity, we assume the number of voters is odd.

Proposition 6. *Assume n is odd, preferences are given by Eq.* (4)*, and* (A1) *holds. Then (x_A, x_B) is a nonsatiated equilibrium if and only if $x_A = x_B = x^*$, where x^* is a core point.*

The proof is straightforward. Clearly, it is an equilibrium for both candidates to locate at the core point. To prove the converse, the arguments of Theorem 1 can be modified to obtain the result that, in a nonsatiated equilibrium, the candidates must adopt the same platform, say $\hat{x}$. To show that $\hat{x}$ must be a core point, suppose not. Then there is some y majority-preferred to it. That platform may be a worse policy outcome from a candidate's point of view, but every platform between $\hat{x}$ and y is also majority-preferred to $\hat{x}$. By picking such a platform close enough to $\hat{x}$, the candidate can make the disutility of the policy change less than w, the utility from winning, a contradiction. Thus, in this mixed model, an equilibrium must exhibit the symmetry of the voters' gradients from Plott's (1967) theorem, and we again conclude that equilibria will rarely exist. Given the policy coincidence result of Theorem 1, the argument for this case is drastically simplified by the discontinuity implied by the fixed reward w. Our results for pure policy-motivation show, however, that the negative conclusion is not merely an artifact of this discontinuity.

In the n even case, no strong symmetry condition is required of core points, and thus equilibria with purely office-motivated candidates need not be rare or fragile. Under mixed motivations, however, this observation no longer holds. In particular, if we impose some additional structure on $P(x, y)$ in the case of majority indifference, such as the assumption that all ties are broken equiprobably, then we can prove that no nonaligned equilibria exist. The argument is similar to the proof of Theorem 4. Thus, the robustness of equilibria possible with office-motivation does not extend to the mixed case, at least when considering nonaligned equilibria.

8. Conclusion

Although our conclusions are negative, they nevertheless have important consequences for formal models of politics. Our results illustrate how the findings of the standard spatial model carry over to a setting with a natural alternative assumption about candidate preferences. Indeed, the equilibrium existence problem runs much deeper than previously realized: even after we remove many of the discontinuities created by pure office-motivation in multiple dimensions, policy-motivated candidates typically have a sufficient number of deviations to break any potential equilibrium. The nonexistence problem persists even if we smooth the candidates' payoffs by adding a small amount of uncertainty about voting behavior, demonstrating the role of nonconvexities in the failure of existence. Our results emphasize the importance of modeling elections in richer detail, whence equilibria may emerge from additional structure, whether institutional (parties, interest groups, the media), informational (through reputational concerns), or dynamic (within or across elections). As these modeling approaches will likely include a component of policy-motivation, the techniques developed in this paper may inform future research by shedding light on the intricacies of policy-oriented incentives.

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Appendix A. Proofs of results

Many of the arguments of this appendix will use the following standard lemma, which follows in a straightforward way from the strict quasi-concavity of the voters' utility functions.

Lemma 1. *If xRy then, for all $\alpha \in (0,1)$, $\alpha x + (1-\alpha)yRy$; if n is odd, moreover, then $\alpha x + (1-\alpha)yPy$.*

We now state and prove the results of Sections 3 through 6.

Theorem 1. *Assume* (A1) *and* (A2). *If (x_A, x_B) is a nonsatiated equilibrium, then $x_A = x_B$.*

Proof. Suppose (x_A, x_B) is a nonsatiated equilibrium and that $x_A \neq x_B$. Without loss of generality, suppose $x_A R x_B$. By (A1), this implies that $P(x_A, x_B) > 0$. We first deal with the case of n odd. We begin by establishing that $u_A(x_A) > u_A(x_B)$. If $u_A(x_B) > u_A(x_A)$, then, because $P(x_A, x_B) > 0$, A can gain by moving to x_B. This contradicts the supposition that (x_A, x_B) is an equilibrium. If $u_A(x_A) = u_A(x_B)$, then $U_A(x_A, x_B) = u_A(x_A)$. Let $x' = (1/2)x_A + (1/2)x_B$, so by Lemma 1, $x' P x_B$. Thus (A1) implies that $U_A(x', x_B) = u_A(x')$. Since u_A is strictly quasi-concave, we have $u_A(x') > u_A(x_A) = u_A(x_B)$. So deviating to x' is profitable for A, a contradiction. Therefore, it must be that $u_A(x_A) > u_A(x_B)$.

Next, we rule out the case $x_A I x_B$. In this case, (A1) requires that $P(x_A, x_B) < 1$. Let $\{\alpha_m\}$ be a sequence increasing to one, and define $x_m = \alpha_m x_A + (1-\alpha_m)x_B$. By Lemma 1, $x_m P x_B$ for all m, and thus $U_A(x_m, x_B) = u_A(x_m)$. As $x_m \to x_A$ and $u_A(x_A) > u_A(x_B)$, we have $U_A(x_m, x_B) > U_A(x_A, x_B)$ for large enough m, a contradiction. Therefore, $x_A P x_B$ must hold.

By continuity of the u_i's, there is an open set $Y \subseteq \mathbb{R}^d$ containing x_A such that, for all $x \in X \cap Y$, $x P x_B$. Since (x_A, x_B) is nonsatiated, $\nabla u_A(x_A) \neq 0$. Letting $x_\epsilon = x_A + \epsilon \nabla u_A(x_A)$ for $\epsilon > 0$, and choosing ϵ close enough to zero, we have $x_\epsilon \in X$ because x_A is interior to X, and $u_A(x_\epsilon) > u_A(x_A)$ and $x_\epsilon P x_B$. Therefore, A can gain by deviating to x_ϵ, a contradiction. So $x_A \neq x_B$ cannot hold.

We now deal with the n even case, maintaining the supposition that $x_A R x_B$. Again, we start by showing that $u_A(x_A) > u_A(x_B)$. The same argument as above rules out $u_A(x_B) > u_A(x_A)$. If $u_A(x_A) = u_A(x_B)$, then, as above, let $x' = (1/2)x_A + (1/2)x_B$, and note that $u_A(x') > u_A(x_A) = u_A(x_B) = U_A(x_A, x_B)$. In the n even case, Lemma 1 implies only that $x' R x_B$. But (A1) still implies that $P(x', x_B) > 0$ and so $U_A(x', x_B) > U_A(x_A, x_B)$. So A can gain by deviating to x'. Therefore, $u_A(x_A) > u_A(x_B)$ holds.

Once again, we next rule out the case $x_A I x_B$. In this case, by definition, $n_A(x_A, x_B) + n_I(x_A, x_B) \geqslant n/2$. For sufficiently large m (with x_m defined as above), the $n_A(x_A, x_B)$ voters who prefer x_A to x_B will also prefer x_m to x_B and the $n_I(x_A, x_B)$ indifferent voters will strictly prefer x_m to x_B, by strict quasi-concavity. Therefore, if $n_A(x_A, x_B) + n_I(x_A, x_B) > n/2$, deviating to x_m results in A winning for sure, which is profitable for x_m close enough to x_A. So it must be that $n_A(x_A, x_B) + n_I(x_A, x_B) = n/2$ which implies $n_B(x_A, x_B) = n/2$. If $n_A(x_A, x_B) < n/2$, then assumption (A2) and the argument just given imply that $P(x_m, x_B) > P(x_A, x_B)$ for sufficiently large m. So this is a profitable deviation for A. If $n_A(x_A, x_B) = n/2$, then continuity yields an open set $Y \subseteq \mathbb{R}^d$ such that for all $x \in X \cap Y$ and all $i \in N$, $u_i(x_A) > u_i(x_B)$ if and only if $u_i(x) > u_i(x_B)$. Defining x_ϵ as above, (A2) then requires that $P(x_\epsilon, x_B) = P(x_A, x_B)$, for sufficiently small ϵ. Therefore, since (x_A, x_B) is nonsatiated, A can profitably deviate to x_ϵ. Therefore, $x_A P x_B$, and the final contradiction follows as in the n odd case. □

Theorem 2. *Assume n is odd, and assume* (A1). *If (x_A, x_B) is a nonaligned equilibrium, then $x_A = x_B = \hat{x}$, where $\nabla u_k(\hat{x}) = 0$ for some voter k. If $\nabla u_A(\hat{x})$ and $\nabla u_B(\hat{x})$ are linearly independent, then for every $p \in \operatorname{cone}^\circ\{\nabla u_A(\hat{x}), \nabla u_B(\hat{x})\}$:*

$$\left|\left\{i \in N \mid \exists \alpha > 0\colon\ \nabla u_i(\hat{x}) = \alpha p\right\}\right| = \left|\left\{i \in N \mid \exists \alpha < 0\colon\ \nabla u_i(\hat{x}) = \alpha p\right\}\right|. \tag{3}$$

If $\nabla u_A(\hat{x})$ and $\nabla u_B(\hat{x})$ are linearly dependent, then Eq. (3) *holds for all $p \in \mathbb{R}^d$.*

Proof. Consider any nonaligned equilibrium (x_A, x_B). As every nonaligned equilibrium is nonsatiated, we know from Theorem 1 that $x_A = x_B = \hat{x}$ for some $\hat{x} \in X$. To simplify notation, let $p_A = \nabla u_A(\hat{x})$ and $p_B = \nabla u_B(\hat{x})$, and normalize both vectors so that $\|p_A\| = \|p_B\| = 1$. We first claim that, for both candidates C, we must have $p_C \cdot y < p_C \cdot \hat{x}$ for all platforms $y \neq \hat{x}$ such that $y R \hat{x}$. Otherwise, we would have $p_C \cdot y \geqslant p_C \cdot \hat{x}$ for some $y \neq \hat{x}$ such that $y R \hat{x}$. It follows from Lemma 1 that $x_\alpha = \alpha \hat{x} + (1-\alpha) y P \hat{x}$ for all $\alpha \in (0, 1)$. Also, $p_C \cdot x_\alpha \geqslant p_C \cdot \hat{x}$. Using the assumption that $\hat{x}$ is interior to X, we take α close enough to one that x_α is also interior to X. Since the u_i are continuous, there is an open set $Y \subseteq X$ containing x_α such that, for all $z \in Y$, $z P \hat{x}$. Defining $z_\beta = x_\alpha + \beta p_C$, we take β small enough that $z_\beta \in Y$, and therefore $z_\beta P \hat{x}$. By construction,

$$p_C \cdot (z_\beta - \hat{x}) = p_C \cdot (x_\alpha - \hat{x}) + \beta p_C \cdot p_C > 0.$$

Finally, define $w_\gamma = \gamma\hat{x} + (1-\gamma)z_\beta$. Again using Lemma 1, $w_\gamma P\hat{x}$ for all $\gamma \in (0,1)$. Since $p_C \cdot (w_\gamma - \hat{x}) > 0$, we may take γ close enough to one that $u_C(w_\gamma) > u_C(\hat{x})$. But then, by assumption (A1), we have $U_A(w_\gamma, \hat{x}) > U_A(\hat{x}, \hat{x})$ or $U_B(\hat{x}, w_\gamma) > U_B(\hat{x}, \hat{x})$ (depending on the identity of C), a contradiction. This establishes the claim.

If the gradients of the candidates are linearly dependent at the equilibrium platform $\hat{x}$, then, since they are nonzero and do not point in the same direction, it follows that the gradients point in opposite directions: $\alpha p_A = p_B$ for some $\alpha < 0$. Take any $y \neq \hat{x}$ such that yRx. From the above claim, $p_A \cdot y < p_A \cdot \hat{x}$ and $p_B \cdot y < p_B \cdot \hat{x}$. But, since the gradients of the candidates point in opposite directions, the latter yields $p_A \cdot y > p_A \cdot \hat{x}$, a contradiction. Therefore, $\hat{x}Py$ for all $y \neq \hat{x}$, which implies $\hat{x}$ is a core point. Then Plott's (1967) theorem implies that $\hat{x}$ is the ideal point of at least one voter and that the symmetry condition holds for all $p \in \mathbb{R}^d$.

Now consider the case in which the candidates' gradients are linearly independent, and suppose that, for all voters i, $\nabla u_i(\hat{x}) \neq 0$. Let $p \in \text{cone}^\circ\{p_A, p_B\}$ be any vector satisfying $p = \alpha p_A + \beta p_B$ for some $\alpha, \beta > 0$. To deduce a contradiction, we will first find a vector $q \in \mathbb{R}^d$ such that $p \cdot q = 0$, $p_A \cdot q > 0$, and $p_B \cdot q < 0$. Construct q as follows. Since p_A and p_B are linearly independent, we have $p_A \cdot p < \|p\|$. Let q be p_A minus the projection of p_A onto the one-dimensional subspace spanned by p, i.e.,

$$q = p_A - \frac{(p_A \cdot p)}{(p \cdot p)} p.$$

Then, since $p_A \cdot p_A = 1$ and $(p_A \cdot p)/\|p\| < 1$, we have $p_A \cdot q > 0$. Furthermore, $q \cdot p = 0$, implying $p_B \cdot q = -(\alpha/\beta) p_A \cdot q < 0$. This gives us a vector q with the desired properties. In fact, there is an open set Q containing q such that, for all $s \in Q$, $p_A \cdot s > 0$ and $p_B \cdot s < 0$. Because N is finite and $\nabla u_i(\hat{x}) \neq 0$ for all voters i, we may choose $r \in Q$ so that $r \cdot \nabla u_i(\hat{x}) \neq 0$ for all i. Therefore, since the voters are odd in number, either

$$\bigl\{i \in N \mid r \cdot \nabla u_i(\hat{x}) > 0\bigr\} \quad \text{or} \quad \bigl\{i \in N \mid r \cdot \nabla u_i(\hat{x}) < 0\bigr\}$$

contains a majority of voters. Suppose, without loss of generality, that this is true for the first group of voters, and define $x_\epsilon = \hat{x} + \epsilon r$ for $\epsilon > 0$. Since $\hat{x}$ is interior to X, we may choose ϵ small enough that $x_\epsilon \in X$. Furthermore, since $\nabla u_i(\hat{x}) \cdot (x_\epsilon - \hat{x}) > 0$ for a majority of voters, $x_\epsilon P \hat{x}$ for ϵ close enough to zero. And, since $p_A \cdot (x_\epsilon - \hat{x}) = \epsilon p_A \cdot r > 0$, we have $u_A(x_\epsilon) > u_A(\hat{x})$ for ϵ close enough to zero. But then there is a small enough ϵ such that A can profitably deviate to x_ϵ, a contradiction. Therefore, $\nabla u_k(\hat{x}) = 0$ for some voter k.

Now take any $p \in \text{cone}^\circ\{p_A, p_B\}$, and suppose the symmetry condition of the theorem is violated. We will show that one of the candidates has a profitable deviation, a contradiction. Let $\sigma = 1$ if

$$\bigl|\bigl\{i \in N \mid \exists \alpha > 0\colon\ \nabla u_i(\hat{x}) = \alpha p\bigr\}\bigr| > \bigl|\bigl\{i \in N \mid \exists \alpha < 0\colon\ \nabla u_i(\hat{x}) = \alpha p\bigr\}\bigr|,$$

and let $\sigma = -1$ if the opposite inequality holds. As above, pick $q \in \mathbb{R}^d$ such that $p \cdot q = 0$, $p_A \cdot q > 0$, and $p_B \cdot q < 0$. Let Q be an open set containing q on which the two strict inequalities hold, and let $Q' = \{s \in Q \mid p \cdot s = 0\}$ be the elements of that set orthogonal to p. For $r \in Q'$, let $O(r) = \{s \in \mathbb{R}^d \mid s \cdot r = 0\}$ denote the subspace orthogonal to r. We claim that $\bigcap_{r \in Q'} O(r) = \text{span}\{p\}$. To see this, let $\{b_1, \ldots, b_{d-1}\}$ be a basis for the $(d-1)$-dimensional subspace orthogonal to p, and take $r \in Q'$ and $\epsilon > 0$ such that

$\{r+\epsilon b_1,\dots,r+\epsilon b_{d-1}\}$ is linearly independent and contained in Q'. By linear independence, the dimension of

$$\bigcap_{h=1}^{d-1} O(r+\epsilon b_h)$$

is one. Of course, $p \in O(r)$ for all $r \in Q'$, establishing the claim.

Then, since N is finite and k is the only voter with ideal point $\hat{x}$, choose $r \in Q'$ so that $r \cdot \nabla u_i(\hat{x}) = 0$ if and only if $i = k$ or, for some $\alpha \neq 0$, $\nabla u_i(\hat{x}) = \alpha p$. Partition $N \setminus \{k\}$ into four sets,

$$\begin{aligned}
I &= \bigl\{i \in N \mid r \cdot \nabla u_i(\hat{x}) > 0\bigr\}\\
J &= \bigl\{i \in N \mid r \cdot \nabla u_i(\hat{x}) < 0\bigr\}\\
K &= \bigl\{i \in N \mid \exists \alpha > 0\colon\ \nabla u_i(\hat{x}) = \sigma\alpha p\bigr\}\\
L &= \bigl\{i \in N \mid \exists \alpha < 0\colon\ \nabla u_i(\hat{x}) = \sigma\alpha p\bigr\},
\end{aligned}$$

and note that $|K| > |L|$. Without loss of generality, suppose $|I| \geqslant |J|$. Since $N \setminus \{k\}$ contains $n-1$ voters, we have $|K| + |I| > (n-1)/2$, and this implies $|K| + |I| \geqslant (n+1)/2 > n/2$. We will use r to construct a profitable deviation for candidate A. (If the inequality $|I| < |J|$ held instead, we would use $-r$ to construct a profitable deviation for B.) Let $x_\delta = \hat{x} + \delta r$ for $\delta > 0$. Then $\nabla u_i(\hat{x}) \cdot (x_\delta - \hat{x}) = \delta \nabla u_i(\hat{x}) \cdot r > 0$ for all $i \in I$, and $p_A \cdot (x_\delta - \hat{x}) = \delta p_A \cdot r > 0$. Choose δ close enough to zero that x_δ is interior to X. Define $x_\epsilon = x_\delta + \epsilon\sigma p$ for $\epsilon > 0$, and choose ϵ close enough to zero that, for all $i \in I$, $\nabla u_i(\hat{x}) \cdot (x_\epsilon - \hat{x}) > 0$; and small enough that $p_A \cdot (x_\epsilon - \hat{x}) > 0$. Note that, since $\nabla u_i(\hat{x}) \cdot r = 0$ for all $i \in K$, we have

$$\nabla u_i(\hat{x}) \cdot (x_\epsilon - \hat{x}) = \delta \nabla u_i(\hat{x}) \cdot r + \epsilon \nabla u_i(\hat{x}) \cdot \sigma p > 0$$

for all $i \in K$. Picking ϵ close enough to zero, we have $x_\epsilon \in X$ and, for all $i \in I \cup K$, $u_i(x_\epsilon) > u_i(\hat{x})$, which implies $x_\epsilon P \hat{x}$. Furthermore, $u_A(x_\epsilon) > u_A(\hat{x})$. But then once again there is a small enough ϵ such that A can profitably deviate to x_ϵ, a contradiction. Therefore, the symmetry condition of the theorem must hold. □

Theorem 3. *Assume n is odd, and assume* (A1). *If (x_A, x_B) is a nonaligned equilibrium, then $x_A = x_B = \hat{x}$, where $\nabla u_k(\hat{x}) = 0$ for some voter k. Moreover, for every $p \in \mathbb{R}^d$ such that $p \notin \operatorname{span}\{\nabla u_A(\hat{x}), \nabla u_B(\hat{x})\}$,*

$$\Bigl|\bigl\{i \in N \mid \exists \alpha > 0\colon\ \nabla u_i(\hat{x}) = \alpha p\bigr\}\Bigr| = \Bigl|\bigl\{i \in N \mid \exists \alpha < 0\colon\ \nabla u_i(\hat{x}) = \alpha p\bigr\}\Bigr|.$$

Proof. Let (x_A, x_B) be a nonaligned equilibrium. From Theorem 2, it follows that $x_A = x_B = \hat{x}$, where $\nabla u_k(\hat{x}) = 0$ for some voter k. As shown in the proof of Theorem 2, if the gradients of the candidates are linearly dependent, then $\hat{x}$ is a core point, and the symmetry condition of the theorem is satisfied. We assume, then, that their gradients are linearly independent. As above, let $p_A = \nabla u_A(\hat{x})$ and $p_B = \nabla u_B(\hat{x})$ and normalize so that $\|p_A\| = \|p_B\| = 1$. Moreover, for every voter i, let $p_i = \nabla u_i(\hat{x})$. Given $q, r \in \mathbb{R}^d$, let $S(q, r) = \operatorname{span}\{q, r\}$ denote the subspace spanned by q and r. We will take q and r to be linearly independent, implying that $S(q,r)$ is a two-dimensional subspace, i.e., a plane. Given $p,q,r \in \mathbb{R}^d$, let

$$p(q,r) = \operatorname{proj}_{S(q,r)} p$$

denote the projection of p onto the span of $\{q,r\}$. Thus, $p_C(q,r)$ would be the projection of candidate C's gradient onto that plane. Given $p \in \mathbb{R}^d$, let

$$O(p) = \{q \in \mathbb{R}^d \mid q \cdot p = 0\}$$

denote the subspace orthogonal to p. Given $p \in \mathbb{R}^d$, let

$$S(p) = \operatorname{span}\{\operatorname{proj}_{O(p)} p_A, \operatorname{proj}_{O(p)} p_B\}$$

denote the subspace spanned by the projections of the candidates' gradients onto the space orthogonal to p. Given $p,q \in \mathbb{R}^d$, let

$$q(p) = \operatorname{proj}_{S(p)} q$$

denote the projection of q onto that plane. Note that, since $\operatorname{proj}_{O(p)} p_C \in S(p)$ and $S(p) \subseteq O(p)$, we have

$$p_C(p) = \operatorname{proj}_{S(p)} p_C = \operatorname{proj}_{O(p)} p_C,$$

so $p_C(p)$ is just the gradient of candidate C projected onto the subspace orthogonal to p. That, in turn, implies $S(p_A(p), p_B(p)) = S(p)$. Finally, note the further implication that $q(p) = q(p_A(p), p_B(p))$.

Let $q,r \in \mathbb{R}^d$ be vectors such that the gradients of the candidates, projected onto the plane $S(q,r)$, point in different directions, i.e., there do not exist $\alpha, \beta \geqslant 0$, at least one nonzero, such that $\alpha p_A(q,r) = \beta p_B(q,r)$. Clearly, if we restrict the candidates' platforms to the two-dimensional space $\hat{x} + S(q,r)$, then the pair $(\hat{x},\hat{x})$ is a nonaligned equilibrium of the restricted game. Take any $p \in \operatorname{cone}^\circ\{p_A(q,r), p_B(q,r)\}$ in the open cone generated by the candidates' projected gradients, so that the antecedent conditions of Theorem 2 hold in the restricted game. We claim that

$$\bigl|\{i \in N \mid \exists \alpha > 0\colon\ p_i(q,r) = \alpha p(q,r)\}\bigr| = \bigl|\{i \in N \mid \exists \alpha < 0\colon\ p_i(q,r) = \alpha p(q,r)\}\bigr|.$$

If not, then, by Theorem 2, one of the candidates has a profitable deviation in the restricted game, and therefore the candidate has a profitable deviation in the original game, a contradiction. This establishes the claim.

To prove the theorem, take any $p \notin \operatorname{span}\{p_A, p_B\}$, normalize so $\|p\| = 1$, let

$$I = \{i \in N \mid \exists \alpha > 0\colon\ \nabla u_i(\hat{x}) = \alpha p\}$$
$$J = \{i \in N \mid \exists \alpha < 0\colon\ \nabla u_i(\hat{x}) = \alpha p\},$$

and suppose that $|I| \neq |J|$. Without loss of generality, suppose $|I| > |J|$. In light of the above claim, a contradiction is proved if we find vectors q and r satisfying three conditions:

(1) there do not exist $\alpha, \beta \geqslant 0$, at least one nonzero, such that $\alpha p_A(q,r) = \beta p_B(q,r)$;
(2) $p(q,r) \in \operatorname{cone}^\circ\{p_A(q,r), p_B(q,r)\}$;

(3) the symmetry condition of Theorem 2 in the game restricted to $\hat{x} + S(q, r)$ is violated, specifically,

$$I = \{i \in N \mid \exists \alpha > 0\colon\ p_i(q, r) = \alpha p(q, r)\}$$
$$J = \{i \in N \mid \exists \alpha < 0\colon\ p_i(q, r) = \alpha p(q, r)\}.$$

We first consider the possibility of setting $q = p_A(p)$ and $r = p_B(p)$. As noted above, we would then have $p_A(q, r) = p_A(p)$ and $p_B(q, r) = p_B(p)$, so condition (1) is satisfied if $p_A(p)$ and $p_B(p)$ are linearly independent, and we claim that is indeed the case. To show this, note that there exist unique, nonzero α and β such that

$$p_A = p_A(p) + \alpha p \quad \text{and} \quad p_B = p_B(p) + \beta p.$$

If $p_A(p)$ and $p_B(p)$ were linearly dependent, then there would exist γ and δ, at least one nonzero, such that $\gamma p_A(p) + \delta p_B(p) = 0$. But then

$$\gamma p_A + \delta p_B = (\alpha\gamma + \beta\delta) p,$$

which implies

$$p = \left(\frac{\gamma}{\alpha\gamma + \beta\delta}\right) p_A + \left(\frac{\delta}{\alpha\gamma + \beta\delta}\right) p_B,$$

contradicting $p \notin \operatorname{span}\{p_A, p_B\}$. Therefore, the projected gradients of the candidates are linearly independent, as claimed.

We cannot simply set $q = p_A(p)$ and $r = p_B(p)$, however, because then we would have

$$p(q, r) = p\big(p_A(p), p_B(p)\big) = 0,$$

violating condition (2). Next, we establish the existence of a perturbation, s, of p such that conditions (1) and (2) are both satisfied by $q = p_A(s)$ and $r = p_B(s)$. As noted above, $p(s) = p(p_A(s), p_B(s))$ and $p_C(s) = p_C(p_A(s), p_B(s))$ for each candidate, so condition (2) can be written as $p(s) \in \operatorname{cone}^\circ\{p_A(s), p_B(s)\}$. To construct the perturbation, let $s_\epsilon = p - (\epsilon/2)(p_A + p_B)$ for $\epsilon > 0$. Note that, by linearity of the projection mapping and $s_\epsilon(s_\epsilon) = 0$,

$$\begin{aligned} p(s_\epsilon) &= \big(s_\epsilon + (\epsilon/2) p_A + (\epsilon/2) p_B\big)(s_\epsilon) \\ &= (\epsilon/2) p_A(s_\epsilon) + (\epsilon/2) p_B(s_\epsilon). \end{aligned}$$

Thus, $p(s_\epsilon) \in \operatorname{cone}^\circ\{p_A(s_\epsilon), p_B(s_\epsilon)\}$. Taking ϵ close enough to zero that $p_A(s_\epsilon)$ and $p_B(s_\epsilon)$ are linearly independent, we set $s = s_\epsilon$ for the desired perturbation.

We now wish to find perturbations, q and r, of $p_A(s)$ and $p_B(s)$ that satisfy condition (3) as well as (1) and (2). Let voter j satisfy $p_j(s) = \alpha p(s)$ for some $\alpha < 0$ but $p_j \neq \alpha p$. That is, although the voter's gradient appears to point in the $-p$ direction when projected, the voter is not a member of J. Note the immediate implication that p_j and p are linearly independent. We will find arbitrarily close vectors v and w such that $p_j(v, w) = \alpha' p(v, w)$ for no $\alpha' < 0$. Note that

$$
\begin{aligned}
p_j \cdot p_A(s) &= \big(p_j - p_j(s)\big) \cdot p_A(s) + p_j(s) \cdot p_A(s) \\
&= p_j(s) \cdot p_A(s) \\
&= \alpha p(s) \cdot p_A(s) \\
&= \alpha\big(p(s) - p\big) \cdot p_A(s) + \alpha p \cdot p_A(s) \\
&= \alpha p \cdot p_A(s),
\end{aligned}
$$

where the second equality follows from $(p_j - p_j(s)) \cdot p_A(s) = 0$ and the fourth equality from $(p(s) - p) \cdot p_A(s) = 0$. Similarly, $p_j \cdot p_B(s) = \alpha p \cdot p_B(s)$. These equalities imply

$$
\frac{p_j \cdot p_A(s)}{p_j \cdot p_B(s)} = \frac{p \cdot p_A(s)}{p \cdot p_B(s)}.
$$

Since p_j and p are linearly independent, there exists $t \in \mathbb{R}^d$ such that $p_j \cdot t > 0$ and $p \cdot t < 0$. Define $v_\epsilon = p_A(s) + \epsilon t$ and $w_\epsilon = p_B(s) - \epsilon t$ for $\epsilon > 0$, and note that

$$
\frac{p_j \cdot v_\epsilon}{p_j \cdot w_\epsilon} > \frac{p \cdot v_\epsilon}{p \cdot w_\epsilon}.
$$

Thus, $p_j(v_\epsilon, w_\epsilon) = \alpha' p(v_\epsilon, w_\epsilon)$ for no $\alpha' < 0$. That is, the gradient of voter j, projected onto the plane spanned by v_ϵ and w_ϵ, no longer appears to point in the $-p$ direction. Since conditions (1) and (2) hold on open sets around $p_A(s)$ and $p_B(s)$, we can choose ϵ small enough that (1) and (2) hold for v_ϵ and w_ϵ. Since N is finite, we can perturb v_ϵ and w_ϵ a finite number of times, if needed, so that the only voters whose projected gradients point in the $-p(v_\epsilon, w_\epsilon)$ direction are the members of J. By a similar argument, we can perturb v_ϵ and w_ϵ so that the only voters whose projected gradients point in the $p(v_\epsilon, w_\epsilon)$ direction are the members of I, fulfilling condition (3). □

Theorem 4. *Assume n is even, and assume* (A1) *and* (A2). *There does not exist a nonaligned equilibrium.*

Proof. To prove the theorem, consider any nonaligned equilibrium (x_A, x_B). By Theorem 1, the candidates must locate at the same platform, say $\hat{x} = x_A = x_B$. We claim that $\nabla u_k(\hat{x}) = 0$ for some voter k, for suppose not. As in the proof of Theorem 2, let $r \in \mathbb{R}^d$ be such that $p_A \cdot r > 0 > p_B \cdot r$ and such that $r \cdot \nabla u_i(\hat{x}) = 0$ for no voter i. Then either

$$
\big\{i \in N \mid r \cdot \nabla u_i(\hat{x}) > 0\big\} \quad \text{or} \quad \big\{i \in N \mid r \cdot \nabla u_i(\hat{x}) < 0\big\}
$$

contains at least half of the voters. Suppose, without loss of generality, that this is true for the first group of voters, and define $x_\epsilon = \hat{x} + \epsilon r$ for $\epsilon > 0$. Since $\hat{x}$ is interior to X, we may choose ϵ small enough that $x_\epsilon \in X$. Furthermore, since $\nabla u_i(\hat{x}) > 0$ for at least half of the voters, $x_\epsilon R \hat{x}$ for ϵ close enough to zero. And, since $p_A \cdot (x_\epsilon - \hat{x}) = \epsilon p_A \cdot r > 0$, we have $u_A(x_\epsilon) > u_A(\hat{x})$ for ϵ close enough to zero. But then, by assumption (A1), we have $U_A(x_\epsilon, \hat{x}) > U_A(\hat{x}, \hat{x})$, as $x_\epsilon R \hat{x}$, so candidate A has an incentive to deviate. This contradiction implies that $\nabla u_k(\hat{x}) = 0$ for some voter k.

Now consider the model with k removed from the set of voters, i.e., let the set of voters be $N' = N \setminus \{k\}$, now odd in number. Because we assumed the voters in N had distinct ideal points, there is no voter with ideal point at $\hat{x}$ in the modified model (with k removed). Following the proof of Theorem 2, one of the candidates, say A, can move to some platform x' such that $u_A(x') > u_A(\hat{x})$ and $x'P'\hat{x}$, where P' represents the strict majority preference relation in the modified model. That is, a majority of voters in N' strictly prefer x' to $\hat{x}$. Returning to the original model, that means that at least half of the voters in N strictly prefer x' to $\hat{x}$. Therefore, we have $u_A(x') > u_A(\hat{x})$ and $x'R\hat{x}$. As above, this implies $U_A(x', \hat{x}) > U_A(\hat{x}, \hat{x})$, a contradiction. □

Proposition 2. *Assume* (A1) *and* (A2). *Assume that there exists platforms $x, y \in X$ such that $xP\tilde{x}_A$ and $u_B(x) > u_B(\tilde{x}_A)$ and that yPx_B and $u_A(y) > u_A(\tilde{x}_B)$. If (x_A, x_B) is an interior equilibrium, then either it is nonsatiated or: n is even and $x_A = \tilde{x}_A$ and $x_B = \tilde{x}_B$.*

Proof. It is sufficient to show that the only interior equilibria in which one candidate, say A, adopts her ideal point occur when n is even and B also adopts her ideal point. We first assume n is odd. Suppose $(\tilde{x}_A, x_B)$ is an interior equilibrium. There are three cases to check. First, $\tilde{x}_A P x_B$. Letting $xP\tilde{x}_A$ and $u_B(x) > u_B(\tilde{x}_A)$, candidate B can deviate to x and do strictly better, a contradiction. Second, $\tilde{x}_A I x_B$. As in the proof of Theorem 1, $u_B(x_B) > u_B(\tilde{x}_A)$ and B can gain by moving toward $\tilde{x}_A$ a small amount, a contradiction. Third, $x_B P\tilde{x}_A$. By continuity of the u_i's, there is an open set of platforms containing x_B that are majority-preferred to $\tilde{x}_A$. So candidate B can gain by moving toward his ideal point by a small amount, unless $x_B = \tilde{x}_B$. If this is true, then, just as B could in the first case, candidate A can gain by moving to a platform y such that yPx_B and $u_A(y) > u_A(x_B)$, a contradiction.

If n is even, then we need to modify the above argument only in the second case ($\tilde{x}_A I x_B$). Once again, the arguments given in the proof of Theorem 1 establish that $u_B(x_B) > u_B(\tilde{x}_A)$ and either B can win outright by moving a small amount toward $\tilde{x}_A$ or all such moves will maintain a tie. In the former case, a small enough move by B is profitable. In the latter, by assumption (A2) candidate B can gain by moving toward his ideal point by a small amount, unless $x_B = \tilde{x}_B$. So it must be the case that both candidates are at their ideal points. □

Theorem 5. *Assume n is odd and X is compact. Let $\{\lambda^m\}$ approximate λ^*. If there is no equilibrium in λ^*, then, for m high enough, there is no equilibrium in λ^m.*

Proof. If not, then we can extract a subsequence $\{(x_A^m, x_B^m)\}$ (still indexed by m, for convenience) such that (x_A^m, x_B^m) is an equilibrium in λ^m and, for some $x_A, x_B \in X$, $(x_A^m, x_B^m) \to (x_A, x_B)$. Note that $u_A(x_A) \geqslant u_A(x_B)$, for otherwise $u_A(x_A) < u_A(x_B)$. Then

$$
\begin{aligned}
&U_A\big(x_B^m, x_B^m|\lambda^m\big) - U_A\big(x_A^m, x_B^m|\lambda^m\big)\\
&\quad = P\big(x_A^m, x_B^m|\lambda^m\big)\big(u_A\big(x_B^m\big) - u_A\big(x_A^m\big)\big) + P\big(x_B^m, x_B^m|\lambda^m\big)w_A\big(x_B^m, x_B^m|\lambda^m\big)\\
&\qquad - P\big(x_A^m, x_B^m|\lambda^m\big)w_A\big(x_A^m, x_B^m|\lambda^m\big),
\end{aligned}
$$

which, using (i), is positive if and only if

$$
u_A\big(x_B^m\big) - u_A\big(x_A^m\big)
$$

$$> P\left(x_A^m, x_B^m|\lambda^m\right) w_A\left(x_A^m, x_B^m|\lambda^m\right) - \frac{P(x_B^m, x_B^m|\lambda^m)}{P(x_A^m, x_B^m|\lambda^m)} w_A\left(x_B^m, x_B^m|\lambda^m\right).$$

By continuity, we have $\lim u_A(x_B^m) - u_A(x_A^m) > 0$, and since w_A converges uniformly to zero by (ii), we see that the inequality must hold for m high enough. But this, of course, contradicts the assumption that (x_A^m, x_B^m) is a equilibrium.

We claim that either $x_A P^* x_B$ or $x_B P^* x_A$ or $x_A = x_B$. Otherwise, we have $x_A \neq x_B$ and $x_A I x_B$. For each m, given platforms (x_A^m, x_B^m), one of the candidates must win with probability less than or equal to one half. Assume without loss of generality that this is true of candidate A for infinitely many m, and consider the subsequence (still indexed by m) for which this holds. Thus, $P(x_A^m, x_B^m|\lambda^m) \leqslant 1/2$ for all m. For any $\epsilon \in (0, 1)$, define $x_A^\epsilon = (1-\epsilon)x_A + \epsilon x_B$, and note that $x_A^\epsilon P^* x_B$ by Lemma 1. Using continuity and strict quasi-concavity, $u_A(x_A) \geqslant u_A(x_B)$ implies that there exists $\epsilon \in (0, 1)$ such that

$$u_A(x_A^\epsilon) > \frac{1}{2}u_A(x_A) + \frac{1}{2}u_A(x_B). \tag{5}$$

By (iii), $P(x_A^\epsilon, \cdot|\lambda^m)$ converges to one uniformly on some open set containing x_B, so we have $P(x_A^\epsilon, x_B^m|\lambda^m) \to 1$. With (ii), it follows that

$$U_A\left(x_A^\epsilon, x_B^m|\lambda^m\right) \to u_A\left(x_A^\epsilon\right). \tag{6}$$

On the other hand, since $u_A(x_A) \geqslant u_A(x_B)$ and since A wins with probability no more than one half in λ^m, we have for all $\delta > 0$,

$$U_A\left(x_A^m, x_B^m|\lambda^m\right) - \delta \leqslant \frac{1}{2}u_A\left(x_A^m\right) + \frac{1}{2}u_A\left(x_B^m\right) + w_A\left(x_A^m, x_B^m|\lambda^m\right),$$

for m high enough. This implies

$$\limsup U_A\left(x_A^m, x_B^m|\lambda^m\right) \leqslant \frac{1}{2}u_A(x_A) + \frac{1}{2}u_A(x_B). \tag{7}$$

Combining (5), (6), and (7), we have $U_A(x_A^\epsilon, x_B^m|\lambda^m) > U_A(x_A^m, x_B^m|\lambda^m)$ for m high enough, a contradiction.

We now claim that

$$U_A\left(x_A^m, x_B^m|\lambda^m\right) \to U_A(x_A, x_B|\lambda^*). \tag{8}$$

If $x_A P^* x_B$, then $P(x_A, x_B|\lambda^*) = 1$, and (ii) implies that $P(x_A^m, x_B^m|\lambda^m) \to 1$. In this case, the claim holds. A symmetric argument addresses the case in which $x_B P^* x_A$. If $x_A = x_B$, then $\lim u_A(x_A^m) = \lim u_A(x_B^m) = u_A(x_A) = u_A(x_B)$, establishing the claim.

By assumption, (x_A, x_B) is not an equilibrium in λ^*, so some candidate, say A, as a profitable deviation, say x'_A, i.e.,

$$U_A(x'_A, x_B|\lambda^*) > U_A(x_A, x_B|\lambda^*). \tag{9}$$

We will show that for m high enough, this leads to a profitable deviation for A in λ^m, a contradiction.

Note that $u_A(x'_A) > u_A(x_B)$, for otherwise, we have $u_A(x'_A) \leqslant u_A(x_B)$, which implies $U_A(x'_A, x_B|\lambda^*) \leqslant u_A(x_B)$. But then (9) implies that $u_A(x_A) < u_A(x_B)$, contradicting our earlier claim. For $\epsilon \in (0,1)$, define $z_A^\epsilon = (1-\epsilon)x'_A + \epsilon x_B$. By strict quasi-concavity, we have $u_A(z_A^\epsilon) > u_A(x_B)$. By Lemma 1, $P(z_A^\epsilon, x_B|\lambda^*) \geqslant P(x'_A, x_B|\lambda^*)$. Then we have

$$\begin{aligned}\liminf_{\epsilon\to 0} U_A\bigl(z_A^\epsilon, x_B|\lambda^*\bigr) \qquad (10)\\ = u_A(x_B) + \liminf_{\epsilon\to 0} P\bigl(z_A^\epsilon, x_B|\lambda^*\bigr)\bigl(u_A\bigl(z_A^\epsilon\bigr) - u_A(x_B)\bigr)\\ \geqslant u_A(x_B) + P(x'_A, x_B|\lambda^*)\bigl(u_A(x'_A) - u_A(x_B)\bigr)\\ = U_A(x'_A, x_B|\lambda^*).\end{aligned}$$

Using (9) and (10), choose ϵ small enough that

$$U_A\bigl(z_A^\epsilon, x_B|\lambda^*\bigr) > U_A(x_A, x_B|\lambda^*). \qquad (11)$$

Furthermore, use Lemma 1 to choose ϵ so that $z_A^\epsilon P^* x_B$ or $x_B P^* z_A^\epsilon$.

We claim that

$$\liminf_{m\to\infty} P\bigl(z_A^\epsilon, x_B^m|\lambda^m\bigr) \geqslant P\bigl(z_A^\epsilon, x_B|\lambda^*\bigr). \qquad (12)$$

If $x_B P^* z_A^\epsilon$, then $P(z_A^\epsilon, x_B|\lambda^*) = 0$, and the claim clearly holds. If $z_A^\epsilon P^* x_B$, then, by (iii), $\lim P(z_A^\epsilon, x_B^m|\lambda^m) = 1$, establishing the claim.

Finally, we claim that

$$\liminf_{m\to\infty} U_A\bigl(z_A^\epsilon, x_B^m|\lambda^m\bigr) \geqslant U_A\bigl(z_A^\epsilon, x_B|\lambda^*\bigr). \qquad (13)$$

Using $u_A(z_A^\epsilon) > u_A(x_B)$, we apply (ii) and (12) to the expression

$$\begin{aligned}U_A\bigl(z_A^\epsilon, x_B^m|\lambda^m\bigr) &= u_A\bigl(x_B^m\bigr) + P\bigl(z_A^\epsilon, x_B^m|\lambda^m\bigr)\bigl(u_A\bigl(z_A^\epsilon\bigr) - u_A\bigl(x_B^m\bigr)\bigr)\\ &\quad + w_A\bigl(z_A^\epsilon, x_B^m|\lambda^m\bigr),\end{aligned}$$

to deduce that

$$\begin{aligned}\liminf_{m\to\infty} U_A\bigl(z_A^\epsilon, x_B^m|\lambda^m\bigr) &\geqslant u_A(x_B) + P\bigl(z_A^\epsilon, x_B|\lambda^*\bigr)\bigl(u_A\bigl(z_A^\epsilon\bigr) - u_A(x_B)\bigr)\\ &= U_A\bigl(z_A^\epsilon, x_B|\lambda^*\bigr),\end{aligned}$$

as claimed. Combining (8), (11), and (13), we find that, for m high enough, we have $U_A(z_A^\epsilon, x_B^m|\lambda^m) > U_A(x_A^m, x_B^m|\lambda^m)$. This contradiction establishes the theorem. □

Proposition 4. *Let $\{\lambda^m\}$ be a sequence of additive bias models such that, for each voter i, the sequence $F_i(\cdot|\lambda^m)$ converges weak* to the point mass on zero. Then $\{\lambda^m\}$ satisfies condition* (iii) *in the definition of approximation.*

Proof. Suppose $x_A P^* x_B$, and let $\Delta > 0$ be such that, for each voter i with $u_i(x_A) > u_i(x_B)$, we have $\Delta < u_i(x_A) - u_i(x_B)$. Let G and H be open neighborhoods around x_A and x_B, respectively, such that, for all $y \in G$ and all $z \in H$, $\Delta < u_i(y) - u_i(z)$. Since $F_i(\cdot|\lambda^*)$ is continuous at Δ, it follows that $F_i(\Delta|\lambda^m) \to F_i(\Delta|\lambda^*) = 1$. Therefore, for every i with $u_i(x_A) > u_i(x_B)$, we have

$$\lim_{m\to\infty} \inf_{y\in G, z\in H} F_i\bigl(u_i(y) - u_i(z)|\lambda^m\bigr) \geqslant \lim_{m\to\infty} F_i\bigl(\Delta|\lambda^m\bigr) = 1,$$

and then $x_A P^* x_B$ delivers the claim. □

Proposition 5. *Let $\{\lambda^m\}$ be a sequence of random preference models such that λ^m converges to λ^* in the weak* topology, where λ^* puts probability one on some $\theta^* \in \Theta$. Then $\{\lambda^m\}$ satisfies condition* (iii) *in the definition of approximation.*

Proof. Suppose $x_A P^* x_B$. Let $C \subseteq N$ consist of the voters i such that $u_i(x_A|\theta^*) > u_i(x_B|\theta^*)$. By continuity, there exist open neighborhoods G, H, and $\widehat{\Theta}$ around x_A, x_B, and θ^*, respectively, such that for all $i \in C$, all $y \in G$, all $z \in H$, and all $\hat{\theta} \in \widehat{\Theta}$, we have $u_i(y|\hat{\theta}) > u_i(z|\hat{\theta})$. By weak* convergence, $\lambda^m(\widehat{\Theta}) \to 1$. Therefore, for every $i \in C$, we have

$$\lim_{m\to\infty} \inf_{y\in G, z\in H} P_i\bigl(y, z|\lambda^m\bigr) \geqslant \lim_{m\to\infty} \lambda^m\bigl(\widehat{\Theta}\bigr) = 1,$$

and then $x_A P^* x_B$ delivers the claim. □

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