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EFFICIENT DESIGN WITH INTERDEPENDENT VALUATIONS

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We study efficient, Bayes-Nash incentive compatible mechanisms in a social choice setting that allows for informational and allocative externalities. We show that such mechanisms exist only if a congruence condition relating private and social rates of information substitution is satisfied. If signals are multi-dimensional, the congruence condition is determined by an integrability constraint, and it can hold only in nongeneric cases where values are private or a certain symmetry assumption holds. If signals are one-dimensional, the congruence condition reduces to a monotonicity constraint and it can be generically satisfied. We apply the results to the study of multi-object auctions, and we discuss why such auctions cannot be reduced to one-dimensional models without loss of generality.

KEYWORDS: Efficient mechanisms, multi-object auctions, interdependent valuations, multidimensional information

1. INTRODUCTION

THERE EXISTS AN EXTENSIVE LITERATURE on efficient auctions and mechanism design. A lot of attention has been devoted to the case where each agent i has a quasi-linear utility function that depends on the chosen social alternative, on information (or signal) privately known to i, and on a monetary transfer, but does not depend on information available to other agents. In this framework, a prominent role is played by the Clarke-Groves-Vickrey (CGV) mechanisms (see Clark (1971), Groves (1973), Vickrey (1961)). These are mechanisms that ensure both that an efficient decision is taken and that truthful revelation of privately held information is a dominant strategy for each agent. The result holds for arbitrary dimensions of signal spaces and for arbitrary signals' distributions.[2]

In this paper we study the case where each agent has a quasi-linear utility function having as arguments signals received by *all* agents and the chosen social alternative. Hence, besides allocative externalities, we allow for informational externalities, and we speak of "interdependent valuations." Signals may be multi-dimensional, but we assume that they are independently drawn across agents. (Signal independence is the most seriously restrictive assumption; observe though that this assumption does not bite for the "principal-agent"

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[2] It is well known that, generally, CGV mechanisms cannot simultaneously satisfy conditions such as budget-balancedness and individual rationality (for example, Myerson and Satterthwaite's (1983) impossibility result can be obtained as a corollary of this fact).

framework of Example 4.4, and it is not required for the result in the one-dimensional case of Section 5.)

For an illustration, consider an auction where a set M of heterogeneous objects is divided among $n+1$ agents (agent zero is the seller; the rest are potential buyers). An alternative is a partition u of M, $u=\{u_i\}_{i=0}^N$, where u_i is the set of objects allocated to bidder i, $i=1,2,\dots N$, and u_0 is the set of unsold objects. Agent i receives a signal s_b^i for each possible bundle $b \in 2^M$, and has a valuation function V_u^i for each partition u. Different models are obtained by varying the dependence of valuations on partitions and signals. Consider the following examples: (i) V_u^i only depends on u_i and $s_{u_i}^i$. This is a pure "private values" model; (ii) V_u^i depends on the entire partition u and on $s_{u_i}^i$. This is a "private values" model which allows for allocative externalities. (iii) V_u^i depends on u and on $\{s_{u_i}^j\}_{j=0}^n$, or V_u^i depends on u and on $\{s_{u_j}^j\}_{j=0}^n$. These are models which allow for both allocative and informational externalities.[3]

For our present purpose, the main common feature of the above examples is that the information available to each agent is multi-dimensional (one signal per bundle) and that different signals affect valuations in different alternatives. This is to be contrasted to the case where several signals affect valuations in a single alternative.

There are many auction papers that go beyond the private values case (e.g., the large literature following Milgrom and Weber (1982)), but almost all of them restrict attention to situations where there is one unit (or there are several identical units), signals are one-dimensional, agents are ex-ante symmetric and do not care about what other agents receive at the auction.[4] In particular, symmetry plays an important role for the efficiency properties of the studied auction procedures.

There are several important works on efficient auctions for asymmetric bidders with interdependent valuations and one-dimensional signals. Crémer and McLean (1985) are mainly concerned with methods to extract surplus when types are correlated,[5] but their Lemma 2 contains an efficient, direct revelation

[3]For example, consider an auction where the bidders are firms in an oligopoly. Independence of signals across bidders is plausible if i's private information concerns the modification of its cost structure (fixed and variable costs) induced by the acquisition of a bundle u_i. Together with the final allocation of objects (e.g., licenses, patents, plants), this information affects the profit of all firms through the oligopolistic equilibrium (which is assumed to be outside the scope of the auction's designer).

[4]Auction models emphasizing the role of allocative externalities in a one-object setup are discussed in Jehiel and Moldovanu (1996) and Jehiel, Moldovanu, and Stacchetti (1996, 1999).

[5]See also Crémer and McLean (1988) and McAfee and Reny (1992). Full extraction mechanisms are, in particular, efficient. Neeman (1998) shows that the full extraction results may not hold in a model that can be interpreted as one where agents have multi-dimensional signals, and signals have some private and some common components. Aoyagi (1998) presents a general existence result of efficient, budget balanced, and incentive compatible mechanisms when agents have finitely many correlated types. None of the above papers covers the present framework (i.e., a continuum of mutually payoff relevant multi-dimensional types), but we suspect that correlation among types allows some possibility results. On the other hand, the mechanisms displayed in the literature above are not very intuitive and require potentially unlimited transfers as correlations get small.

mechanism for an auction of several identical units where bidders have interdependent valuations, downward sloping demand, and finitely many, linearly ordered types. The equilibrium of that mechanism is a so called *ex-post Nash equilibrium*, i.e., for any agent, truth-telling is optimal even after the private information of other agents is revealed.[6] Crémer and McLean also point out "single-crossing" properties of marginal valuation functions that are important for efficient implementation. Maskin (1992) shows that an English ascending auction for one unit achieves efficient outcomes for bidders with asymmetric, interdependent valuations, and with types drawn from a one-dimensional continuum. He also constructs an equivalent, direct revelation mechanism. Ausubel (1997, Appendix B) generalizes Crémer and McLean's mechanism to the case where bidders have types drawn from a one-dimensional continuum.[7] Dasgupta and Maskin (2000) study general multi-object auctions where agents have one-dimensional signals and where there are no allocative externalities.[8] They construct a mechanism, the rules of which do not depend on the functional form of the valuation functions, and that achieves efficient allocations under appropriate conditions on marginal valuations. Perry and Reny (1999a) construct an efficient bidding procedure that is less complex than Dasgupta and Maskin's mechanism, but that works only for a one-dimensional model with several identical units. In their procedure agents place many bids that depend on the unit, and on the potential competitor on that unit.

Maskin (1992) observed that, in general, no efficient, incentive-compatible one-unit auction exists if a buyer's valuation for that unit depends on a multi-dimensional signal (see further comments on this result in Section 4 below). Dasgupta and Maskin (2000) show how to transform such a framework into one where valuations depend on a one-dimensional sufficient statistic.[9] The reduced one-dimensional model admits efficient, incentive compatible mechanisms that are also constrained efficient (i.e., second-best) for the original model.

This paper is organized as follows: In Section 2 we present the social choice model. In Section 3 we obtain a characterization theorem for Bayesian incentive compatible direct mechanisms. In Section 4 we exhibit impossibility results about efficient, Bayesian incentive compatible mechanisms. We only require value maximization, and we completely ignore budget-balancedness and any other constraints. Hence, we show that providing incentives for truthful revela-

[6]This concept is stronger than Bayes-Nash implementation, but it is weaker than dominant strategy implementation. Williams and Radner (1988) show that, with interdependent valuations, efficient, dominant-strategy implementation is generally not possible.

[7]Ausubel (1997) also studies an indirect, ascending bidding procedure that is efficient for the case of interdependent valuations only if bidders are ex-ante symmetric and have constant marginal valuations up to a fixed capacity. Perry and Reny (1999b) show how to modify this procedure in order to get efficiency when agents are asymmetric and marginal valuations are decreasing.

[8]Dasgupta and Maskin allow for heterogeneous objects. But, if the units are not identical, the representation of preferences on various bundles generally requires at least one scalar signal per bundle—see the examples above.

[9]A similar reduction is performed in Jehiel, Moldovanu, and Stacchetti (1996) for purposes of revenue maximization.

tion of privately held information is not compatible even with a very weak efficiency requirement.

Relatively simple results are obtained for situations where incentive compatible mechanisms cannot condition on some signal that is relevant for efficiency considerations. Theorem 4.1 shows impossibility for the case where there is at least one agent possessing information that affects other agents, but does not directly affect the owner of that information. A similar argument is used in Example 4.2, which shows that efficient, incentive compatible mechanisms may not exist if there are an alternative k and an agent i such that agent i's signal affecting her valuation in alternative k is multi-dimensional (this corresponds to Maskin's (1992) example). The basic intuition behind these results is that a one-dimensional instrument (agent i's transfer in alternative k) is not sufficient to extract multi-dimensional information relevant for an efficient choice of alternative k.

Our main impossibility result is Theorem 4.3. We consider there a framework where each agent i has a K-dimensional signal s^i (K is the number of alternatives). The coordinate s^i_k is a *one-dimensional* signal affecting the valuations of *all* agents for alternative k. This framework is critical since, a-priori, incentive compatible mechanisms may condition on all signals, and since, in contrast to the above case, the one-dimensional transfer associated with alternative k should, in principle, be sufficient to extract the one-dimensional signal s^i_k.

To understand the insight behind Theorem 4.3, consider a situation where there are $K \geq 2$ alternatives and where only agent i obtains a private K-dimensional signal. Keep this signal constant in all but two coordinates k and k', and imagine the locus in the $(s^i_k, s^i_{k'})$ subspace where alternatives k and k' yield the same highest social welfare (see Figure 1 in Section 4). At each point, the slope of this curve equals the social (i.e., with respect to social welfare) marginal rate of substitution among i's signals in alternatives k and k'. In order to make i choose efficiently, we must ensure that i's types along that curve are indifferent between alternatives k and k'. This means that, along the curve, i's value in alternative k plus the transfer he obtains in this alternative must equal his value in alternative k' plus the transfer in k'. But, for any given transfers, the locus in the $(s^i_k, s^i_{k'})$ subspace where i is indifferent between k and k' is given by a different curve whose slope equals at each point i's private marginal rate of substitution among his signals in alternatives k and k'. Efficient, incentive compatible mechanisms exist only in the nongeneric situation where the two curves coincide. Theorem 4.3 generalizes this intuition to the more complex setting where several agents obtain private signals. For the linear model detailed in the paper, we can exhibit a simple global necessary condition that needs to be satisfied for incentive-compatible, efficient mechanisms to exist. The condition relates private and social rates of informational substitution, and it holds only for a closed, zero-measure set of parameters.[10]

[10] We show that the congruence condition is satisfied in situations where either a certain symmetry condition, or the private values assumption, hold.

The proof of Theorem 4.3 is based on the following technical observation: an incentive compatible mechanism generates for each agent a vector field. This field associates to each type a vector of expected probabilities with which the various alternatives are chosen. A generalization of the standard one-dimensional envelope argument shows that this vector field is the gradient of the equilibrium expected utility function. Since it is a gradient, the vector field must satisfy an integrability condition involving its cross-derivatives.[11] The impossibility results follow by showing that the vector fields generated by efficient mechanisms satisfy the required conditions only under very restrictive conditions.

Since the integrability constraint bites in any multi-dimensional model, results similar to Theorem 4.3 hold as soon as there is at least one agent whose signal is of dimension $d \geq 2$.

In Section 5 we study the remaining case where signal spaces are one-dimensional. We construct a mechanism that is efficient and incentive compatible if several inequalities relating private and social marginal valuations are satisfied. The main idea of the construction is to make i's transfer equal to the cumulative effect of i's action (here a signal report) on all other agents.[12] Since i's effect on others depends here on i's signal, incentive compatible transfers must neutralize this influence.

To get an intuition for the result, consider again a situation where only one agent receives a private signal, and consider a type s^* of this agent where alternatives k and k' yield the same highest social welfare.[13] As above, in order to induce the agent to choose efficiently, the transfers in alternatives k and k' must make type s^* indifferent between the two.[14] This relation fixes the difference between the two transfers, and, given a condition on *private* marginal valuations, all types can be induced to correctly choose among k and k'. The final step is to find a condition (relating *private* and *social* marginal valuations) that allows a consistent aggregation of the transfer differences obtained for each pair of alternatives.[15]

Concluding comments are gathered in Section 6. In particular, we comment on the difficulty of finding constrained efficient (i.e., second-best) mechanisms in the general multi-dimensional setup.

[11] A similar condition appears in the classical demand theory for several goods (see Chapter 3 in Mas-Colell, Whinston, and Green (1995)): the matrix of price derivatives for a demand function arising from utility maximization must be *symmetric*.

[12] The idea can be traced back to Pigou. It constitutes the basis of the Clarke-Groves-Vickrey approach.

[13] In contrast to the case of multi-dimensional signals, generically we cannot vary now this signal without violating the condition that both alternatives yield the same social welfare. Hence, the locus discussed above degenerates here to a single point.

[14] Without transfers, i is indifferent when he has, say, a type $s' \neq s^*$.

[15] This is necessary when there are more than two alternatives.

2. THE MODEL

There are K social alternatives, indexed by $k = 1, \ldots, K$, and N agents, indexed by $i = 1, \ldots, N$.

Each agent i has a signal (or type) s^i that is drawn from a space $S^i \subseteq \Re^{K \times N}$ according to a continuous density $f_i(s^i) > 0$, independently of other agents' signals. Each agent i knows s^i, and the densities $\{f_j\}_{j=1}^N$ are common knowledge. The idea is that coordinate s^i_{kj} of s^i influences the utility of agent j in alternative k.[16]

We assume that the signal spaces S^i are bounded and convex,[17] and that they have a nonempty interior (given the usual topology in $\Re^{K \times N}$) and a piecewise smooth boundary. Let S denote the Cartesian product $\prod_{i=1}^N S^i$, with generic element s. Denote by S^{-i} the type space of agents other than i, with s^{-i} as generic element.

If alternative k is chosen, and if i obtains a transfer x_i, then i's utility is given by $V^i_k(s^1_{ki}, \ldots, s^N_{ki}) + x_i$, where $V^i_k(s^1_{ki}, \ldots, s^N_{ki}) = \sum_{j=1}^N a^j_{ki} s^j_{ki}$, and where the scalar parameters[18] $\{a^j_{ki}\}_{1 \le k \le K,\, 1 \le j,\, i \le N}$ are common knowledge. We assume throughout the paper that $\forall i, \forall k,\ a^i_{ki} \ge 0$.

For any $s \in S$, let $\kappa^*(s)$ denote the set of welfare-maximizing alternatives at s, i.e., $\kappa^*(s) = \arg\max_k \sum_{i=1}^N V^i_k(s^1, \ldots, s^N)$. Note that $\kappa^*(s)$ is a singleton almost everywhere.

A function p: $S \to \Re^K$ such that $\forall k, s,\ 0 \le p_k(s) \le 1$ and $\forall s, \sum_{k=1}^K p_k(s) = 1$ is called a *social choice rule*. A social choice rule (SCR) is said to be *efficient* if

$$\forall s, \quad p_k(s) \neq 0 \Rightarrow k \in \kappa^*(s).$$

Since $\kappa^*(s)$ is a singleton almost everywhere, an efficient SCR is uniquely defined and deterministic almost everywhere.

A *direct revelation mechanism* (DRM) is defined by a pair (p, x) where p is a SCR, and x: $S \to \Re^N$ is a payment scheme. For reported signals $s = (s^1, \ldots, s^N)$, the term $p_k(s)$ is the probability that alternative k is chosen, and $x_i(s)$ is the transfer to agent i. A DRM is *efficient* if the associated SCR is efficient.[19]

Given a payment scheme x and a SCR p, we now define for each agent i the conditional expected payment function y_i: $S^i \to \Re$ and the conditional expected

[16] We address below (see Example 4.2) situations where the signal of an agent i affecting the utility of agent j in alternative k is itself multi-dimensional.

[17] Convexity is assumed for convenience. If S^i is simply-connected, all results go through unchanged.

[18] The analysis directly extends to the case where the valuation functions include also a constant, i.e., $V^i_k(s^1_{ki}, \ldots, s^n_{ki}) = \sum_{j=1}^n a^j_{ki} s^j_{ki} + b^i_k$, because such constants do not affect incentives.

[19] We ignore here (as in the CGV approach) the (ex post) "budget balancedness" condition, which imposes $\sum_i x_i(s) \le 0$, $\forall s$. In other words, we abstract from efficiency losses due to potential external subsidies.

probability assignment functions $q^i\colon S^i \to \Re^K$ associated with x and p:

$$y_i(t^i) = \int_{S^{-i}} x_i(t^i, s^{-i}) f_{-i}(s^{-i})\, ds^{-i},$$

$$q_k^i(t^i) = \int_{S^{-i}} p_k(t^i, s^{-i}) f_{-i}(s^{-i})\, ds^{-i}.$$

Assume that agent i believes that all other agents report truthfully and assume that i reports type t^i when his true type is s^i. Then, i's expected utility is given by

$$\begin{aligned} \text{(2.1)} \qquad U_i(t^i, s^i) &= \int_{S^{-i}} \left[\sum_k \left(p_k(t^i, s^{-i}) \sum_{j=1}^{N} a_{ki}^j s_{ki}^j \right) \right] f_{-i}(s^{-i})\, ds^{-i} + y_i(t^i) \\ &= \sum_k a_{ki}^i s_{ki}^i q_k^i(t^i) \\ &\quad + \sum_k \int_{S^{-i}} \left[\left(p_k(t^i, s^{-i}) \sum_{j \neq i} a_{ki}^j s_{ki}^j \right) \right] f_{-i}(s^{-i})\, ds^{-i} + y_i(t^i). \end{aligned}$$

Define also

$$\text{(2.2)} \qquad \mu_i(s^i) = U_i(s^i, s^i).$$

3. INCENTIVE COMPATIBLE MECHANISMS

By the revelation principle it is enough to restrict attention to direct, incentive compatible revelation mechanisms. A DRM is (Bayes-Nash) *incentive compatible* if

$$\forall i, \forall s^i, t^i \in S^i, \quad \mu_i(s^i) = U_i(s^i, s^i) \geq U_i(t^i, s^i).$$

For the characterization of incentive compatible mechanisms we need several definitions. A vector field $\Psi\colon S^i \to \Re^{K \times N}$ is *monotone* if

$$\forall s^i, t^i \in S^i, \quad (s^i - t^i) \cdot (\Psi(s^i) - \Psi(t^i)) \geq 0.$$

A vector field Ψ is *conservative* if there exists a differentiable function ρ: $S^i \to \Re$ such that $\Psi = \nabla \rho$ (where ∇ denotes the gradient). A function ρ with the above property is called a *potential function* for Ψ. For a convex (and hence simply-connected) domain S^i, the existence of a potential function for Ψ is equivalent to the following condition: For any $s^i, t^i \in S^i$, the integral of Ψ from s^i to t^i is independent of the path of integration.[20]

THEOREM 3.1: *Let (p, x) be a DRM, and let $\{q^i\}_{i=1}^n$ be the associated conditional probability assignments. For each agent i, let $Q^i(s^i)\colon \Re^{K \times N} \to \Re^{K \times N}$ be the vector field, where, for each alternative k, the* kith *coordinate is given by $a_{ki}^i q_k^i(s^i)$*

[20] Path integrals and the equivalence result are discussed in any multivariate calculus textbook. For a particularly clear and simple exposition, see Chapter V in Lang (1973).

and the kjth *coordinate*, $j \neq i$, *is zero*. *Then* (p, x) *is incentive compatible if and only if the following conditions hold*:

1. $\forall i$, *the vector field* Q^i *is monotone and conservative*.
2. $\forall i, \forall s^i, t^i \in S^i$, $\mu_i(s^i) = \mu_i(t^i) + \int_{t^i}^{s^i} Q^i(\tau^i)\, d\tau^i$.[21,22]

PROOF: See Appendix.

4. IMPOSSIBILITY RESULTS

In an incentive compatible mechanism (p, x) we have $\mu_i(s^i) = \max_{t^i} U_i(t^i, s^i)$. The function μ_i is convex (see the proof of Theorem 3.1), and hence twice differentiable almost everywhere. Assuming that μ_i is differentiable at s^i, we obtain by the Envelope Theorem that

$$(4.1) \qquad \forall k, \quad \frac{\partial \mu_i}{\partial s^i_{ki}}(s^i) = a^i_{ki} q^i_k(s^i),$$

$$(4.2) \qquad \forall k, \forall j \neq i, \quad \frac{\partial \mu_i}{\partial s^i_{kj}}(s^i) = 0.$$

Assuming that μ_i is twice continuously differentiable at s^i, we obtain by Schwarz's Theorem that the cross-derivatives at s^i must be equal. This implies

$$(4.3) \qquad \forall k, k', \quad a^i_{ki} \frac{\partial q^i_k(s^i)}{\partial s^i_{k'i}} = \frac{\partial^2 \mu_i}{\partial s^i_{k'i} \partial s^i_{ki}}(s^i) = \frac{\partial^2 \mu_i}{\partial s^i_{ki} \partial s^i_{k'i}}(s^i) = a^i_{k'i} \frac{\partial q^i_{k'}(s^i)}{\partial s^i_{ki}};$$

$$(4.4) \qquad \forall k, k', \forall j \neq i, \quad a^i_{ki} \frac{\partial q^i_k(s^i)}{\partial s^i_{k'j}} = \frac{\partial^2 \mu_i}{\partial s^i_{k'j} \partial s^i_{ki}}(s^i) = \frac{\partial^2 \mu_i}{\partial s^i_{ki} \partial s^i_{k'j}}(s^i) = 0.$$

The mathematical idea behind the following impossibility results is to check whether efficient mechanisms yield conditional expected probability assignment functions that satisfy conditions (4.3) and (4.4).

Note that an efficient SCR is piecewise constant. Hence, for efficient mechanisms we obtain that the associated functions $\{q^i\}_{i=1}^n$ are everywhere continuously differentiable by recalling that the (convex) type spaces have a nonempty interior and a piecewise smooth boundary, and that for all i and all $s^i \in S^i$, $f_i(s^i) > 0$.

We first focus on the simpler condition (4.4).

[21] The integral can be defined on any path connecting t^i and s^i since Q^i is conservative. For example, we can choose a straight line: $\int_{t^i}^{s^i} Q^i(\tau^i)\, d\tau^i = \int_0^1 Q^i((1-\alpha)t^i + \alpha s^i) \cdot (s^i - t^i)\, d\alpha$.

[22] The Theorem implies a "Revenue Equivalence" result. The conditional expected payment of agent i in any incentive compatible mechanism is solely a function of the associated expected probability assignment, and of the expected utility of an arbitrary type. Any two incentive compatible mechanisms with the same probability assignment yield, up to a constant, the same conditional expected payments. The characterization of incentive compatibility is *not valid* if signals are not independent.

THEOREM 4.1: *Let* (p, x) *be an efficient DRM, and assume that the following are satisfied*: (1) *There exist* i, j, k *such that* $i \neq j$, $a^i_{ki} \neq 0$ *and* $a^i_{kj} \neq 0$. (2) *There exist*[23] *open neighborhoods* $\Theta^i \subset S^i$, $\Theta^{-i}_1, \Theta^{-i}_2 \subset S^{-i}$ *such that* $k \in \kappa^*(s^i, s^{-i})$ *for all* $(s^i, s^{-i}) \in \Theta^i \times \Theta^{-i}_1$ *and* $k \notin \kappa^*(s^i, s^{-i})$ *for all* $(s^i, s^{-i}) \in \Theta^i \times \Theta^{-i}_2$. *Then* (p, x) *cannot be incentive compatible*.

PROOF: Let $\{q^l\}_{l=1}^N$ be the conditional expected probability assignments associated with (p, x). Let $s^i \in \Theta^i$. By efficiency, we obtain

$$q^i_k(s^i) = \int_{\Delta_k(s^i)} f_{-i}(s^{-i})\, ds^{-i} \tag{4.5}$$

where

$$\Delta_k(s^i) = \{s^{-i} \mid k \in \kappa^*(s^i, s^{-i})\}. \tag{4.6}$$

By definition and by condition 2 in the statement of the Theorem, we obtain that $\Delta_k(s^i)$ is a nonempty, closed, proper subset of S^{-i}. Because $a^i_{kj} \neq 0$, we obtain that the area $\Delta_k(s^i)$ varies with s^i_{kj}, and hence $\partial q^i_k(s^i)/\partial s^i_{kj} \neq 0$ for all $s^i \in \Theta^i$.

Suppose now that (p, x) is incentive compatible. Since the expected equilibrium utility μ_i is twice differentiable almost everywhere, there exists $t^i \in \Theta^i$ where μ_i satisfies this requirement. Since $a^i_{ki} \neq 0$, equation (4.4) yields $\partial q^i_k(t^i)/\partial s^i_{kj} = 0$. This is a contradiction. *Q.E.D.*

So far we have assumed that s^i_{kj}, agent i's signal affecting the utility of agent j in alternative k, is one-dimensional. We next look at an example where this requirement is not satisfied. An impossibility result in such situations has been observed by Maskin (1992).

EXAMPLE 4.2: There are two agents $i = 1, 2$ and two alternatives $k = A, B$. Signals are two-dimensional, $\tau^i = (\tau^i_1, \tau^i_2)$, $i = 1, 2$. Valuations are given by:[24]

$$V^1_A(\tau^1, \tau^2) = \tau^1_1 + a(\tau^1_2 + \tau^2_2), \quad V^1_B(\tau^1, \tau^2) = 0,$$
$$V^2_B(\tau^1, \tau^2) = \tau^2_1 + a(\tau^1_2 + \tau^2_2), \quad V^2_A(\tau^1, \tau^2) = 0.$$

Consider the change of variables:

$$s^1 = (s^1_{A1}, s^1_{B2}) = (\tau^1_1 + a\tau^1_2, \tau^1_2), \quad s^2 = (s^2_{B2}, s^2_{A1}) = (\tau^2_1 + a\tau^2_2, \tau^2_2).$$

In the s^i type space we obtain

$$V^1_A(s^1, s^2) = s^1_{A1} + as^2_{A1}, \quad V^1_B(s^1, s^2) = 0,$$
$$V^2_B(s^1, s^2) = s^2_{B2} + as^1_{B2}, \quad V^2_A(s^1, s^2) = 0.$$

[23] This condition on the parameters of the social choice situation ensures that, for some types of agent i, the welfare-maximizing alternative is not a constant function of s^{-i}.

[24] Imagine an auction for an indivisible good where the components τ^i_1, $i = 1, 2$, are the private parts of the signals, while the components τ^i_2 are common parts.

Hence, agent 1 has a signal s^1_{B2} that does not affect her utility (in particular it does not affect her utility in alternative A), but affects the utility of agent 2 in alternative B. In incentive compatible mechanisms we obtain by condition (4.4) that agent 1's interim expected probability for alternative A cannot depend on s^1_{B2}, while s^1_{B2} clearly matters for the determination of ex-post efficiency. Hence, incentive-compatible, efficient mechanisms do not exist.

The example[25] can be extended to the case where $V^1_A(\tau^1,\tau^2) = \tau^1_1 + a\tau^1_2 + b\tau^2_2$ and $V^2_B(\tau^1,\tau^2) = \tau^2_1 + a\tau^2_2 + b\tau^1_2$. Even when the dependence of an agent's valuation on the signal of another agent is very small (i.e., b is very close to zero), efficiency cannot be attained.

In Theorem 4.1 and Example 4.2, the intuition behind the impossibility results is that a (one-dimensional) payment associated to each alternative is not sufficient to elicit multi-dimensional information whose varoius components are all important for efficiency considerations.

The natural next step is to inquire the existence of efficient, incentive compatible mechanisms in a framework where, intuitively, K payments—one for each possible alternative—should suffice to elicit the entire information: consider then K-dimensional type-spaces,[26] where s^i_k is agent i's *one-dimensional* piece of information affecting (possibly in different ways) the utility of *all* agents in alternative k, i.e., each V^j_k is a function of $(s^1_k, s^2_k, \ldots, s^N_k)$. In this setup, the remaining question is whether the conditional expected probability assignment functions generated by efficient mechanisms satisfy condition (4.3).[27]

Recall that we have derived conditions (4.3) and (4.4) for signals of dimension $K \times N$. For each K-dimensional signal $\tilde{t}^i$, define $\tilde{\mu}_i(\tilde{t}^i) \equiv \mu_i(t^i)$ and $\tilde{q}^i_k(\tilde{t}^i) \equiv q^i_k(t^i)$, where t^i is the $K \times N$-dimensional signal such that $t^i_{kj} = \tilde{t}^i_k$ for all k, j. Assuming that μ_i is differentiable at t^i, we obtain by conditions (4.3) and (4.4) that

$$\forall k, \quad \frac{\partial \tilde{\mu}_i}{\partial \tilde{t}^i_k}(\tilde{t}^i) = \sum_{j=1}^{N} \frac{\partial \mu_i}{\partial t^i_{kj}}(t^i) = a^i_{ki} q^i_k(t^i) = a^i_{ki} \tilde{q}^i_k(\tilde{t}^i).$$

The equality of cross-derivatives implies that

$$(4.7) \qquad a^i_{ki} \frac{\partial \tilde{q}^i_k(\tilde{t}^i)}{\partial \tilde{t}^i_{k'}} = a^i_{k'i} \frac{\partial \tilde{q}^i_{k'}(\tilde{t}^i)}{\partial \tilde{t}^i_k}.$$

In order to simplify notation, we drop from now on the "tilde" and denote by $s^i = (s^i_1, \ldots, s^i_K)$ a K-dimensional signal of agent i, yielding expected probability assignments $\{q^i_k\}^K_{k=1}$, and equilibrium expected utility μ_i.

[25] Compte and Jehiel (1998) look at related examples in order to study the value of competition in standard auctions.

[26] We assume below that the respective spaces $\{S^j\}^N_{j=1}$ and densities $\{f_j\}^N_{j=1}$ satisfy all conditions imposed in Section 2 (relative to $\Re^K$).

[27] An affirmative answer would imply an affirmative answer also for frameworks where $\forall i, j$, $i \neq j$, $\forall k$, s^i_{kj} is a linear function of the signals $s^i_{k'i}$, $k' = 1, \ldots, K$, and where each $s^i_{k'i}$ is one-dimensional. The situation treated in the text corresponds to the case where $\forall i, j$, $i \neq j$, $\forall k$, $s^i_{kj} = s^i_{ki}$.

THEOREM 4.3: *Assume that an efficient, incentive compatible DRM exists, and assume that there exist an agent i and a pair of distinct alternatives k, k' such that*: (1) $a^i_{k'i} \neq 0$; (2) *there exist*[28] *open neighborhoods* $\Theta^i \subset S^i$, Θ_1^{-i}, $\Theta_2^{-i} \subset S^{-i}$ *such that* $\Theta_1^{-i} \cap \Theta_2^{-i} \neq \emptyset$, *and such that* $k \in \kappa^*(s^i, s^{-i})$ *for all* $(s^i, s^{-i}) \in \Theta^i \times \Theta_1^{-i}$ *and* $k' \in \kappa^*(s^i, s^{-i})$ *for all* $(s^i, s^{-i}) \in \Theta^i \times \Theta_2^{-i}$. *Then it must be the case that*

$$\frac{a^i_{ki}}{a^i_{k'i}} = \frac{\sum_{j=1}^N a^i_{kj}}{\sum_{j=1}^N a^i_{k'j}}. \tag{4.8}$$

PROOF: See Appendix.[29]

Condition 4.8 is a congruence requirement between private and social rates of information substitution (see the examples below for more intuition about these terms). The implied algebraic relations among parameters cannot be generically satisfied.[30] Note that condition (4.8) is trivially satisfied in two interesting cases: the private values case where $\forall i, j,\ i \neq j, \forall k,\ a^i_{kj} = 0$, and the case where $\forall i, j, k,\ a^i_{kj} = a^i_{ki}$. In the next two examples we provide the intuition for Theorem 4.3. The first example (which was sketched in the Introduction) is very simple since only one agent receives a private signal.[31]

EXAMPLE 4.4: There are two agents $i = 1, 2$ and three alternatives $k = A, B, C$. Suppose that only agent 1 receives a signal, denoted by $s = (s_A, s_B, s_C)$.[32]

A mechanism can be defined here by a triple of transfers to agent 1, $x = (x_A, x_B, x_C)$. Given x, agent 1 chooses an alternative $k \in \text{argmax}_{k'}(V^1_{k'}(s) + x_{k'})$. In contrast, an efficient rule chooses an alternative $k \in \text{argmax}_{k'}(V^1_{k'}(s) + V^2_{k'}(s))$.

Consider an incentive compatible, efficient mechanism,[33] i.e., a mechanism where x is such that the two choice rules coincide. Keeping fixed the signal affecting payoffs in alternative C, consider two types $s^* = (s^*_A, s^*_B, s^*_C)$ and

[28] This condition on the parameters of the social choice situation ensures that, for some types of agent i, the welfare maximizing alternative switches from k to k' as s^{-i} varies. For $s^i \in \Theta^i$ we obtain, in particular, that $\sum_{j=1}^N a^i_{k'j} \neq 0$ and $\sum_{j=1}^N a^i_{kj} \neq 0$.

[29] The Theorem has also a converse: If condition (4.8) is satisfied for all relevant pairs of alternatives, and if an efficient SCR p yields for each agent i a monotone vector field Q^i, then there exist payment schedules x such that (p, x) is incentive compatible.

[30] I.e., the set of parameters satisfying the condition is closed and has Lebesgue-measure zero.

[31] This implies that the conditional expected probability assignment function is deterministic and coincides with the social choice rule.

[32] For ease of notation we omit here the superscript 1.

[33] Note that setting $x_{k'} = V^2_{k'}(s)$ is not incentive compatible.

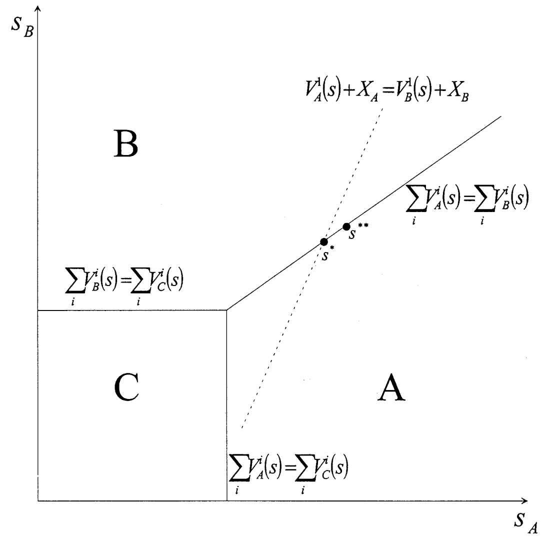

FIGURE 1

$s^{**} = (s_A^{**}, s_B^{**}, s_C^*)$ such that:

$$(4.9) \qquad \sum_{i=1}^{2} V_A^i(s^*) = \sum_{i=1}^{2} V_B^i(s^*) > \sum_{i=1}^{2} V_C^i(s^*),$$

$$\sum_{i=1}^{2} V_A^i(s^{**}) = \sum_{i=1}^{2} V_B^i(s^{**}) > \sum_{i=1}^{2} V_C^i(s^{**}).$$

Together with the continuity of the valuation functions, efficiency and incentive compatibility imply that[34]

$$(4.10) \qquad V_A^1(s^*) + x_A = V_B^1(s^*) + x_B,$$

$$V_A^1(s^{**}) + x_A = V_B^1(s^{**}) + x_B.$$

The above equalities yield

$$(4.11) \qquad V_A^1(s^{**}) - V_B^1(s^{**}) = V_A^1(s^*) - V_B^1(s^*).$$

Equations (4.9) and (4.11) show that, as we move in the (s_A, s_B) subspace from s^* to s^{**} along the curve defined by $\sum_{i=1}^2 V_A^i(s) = \sum_{i=1}^2 V_B^i(s)$ we must also keep the difference $V_A^1(s) - V_B^1(s)$ constant (and equal to $V_A^1(s^*) - V_B^1(s^*)$). But it is obvious that the locus in the (s_A, s_B) subspace where this difference is constant

[34] To see this, consider types $s' = s^* + \varepsilon e_A$, and $s'' = s^* - \varepsilon e_A$ where $\varepsilon > 0$ is small and where $e_A = (1,0,0)$. At s' efficiency requires that alternative A is chosen, so that, by incentive compatibility, $V_B^1(s') + x_B \le V_A^1(s') + x_A$. At s'' efficiency requires that alternative B be chosen so that $V_A^1(s') + x_A \le V_B^1(s') + x_B$. The assertion follows by letting ε go to zero in the two inequalities above. The argument for s^{**} is analogous.

need not coincide with the locus defined by the society's indifference between alternatives k and k' (i.e., the curve from s^* to s^{**}). In particular, for the two curves to coincide around s^* it is necessary that

$$\frac{\frac{\partial V_A^1}{\partial s_A}(s^*)}{\frac{\partial V_B^1}{\partial s_B}(s^*)} = \frac{\frac{\partial}{\partial s_A}(\sum_{i=1}^2 V_A^i(s^*))}{\frac{\partial}{\partial s_B}(\sum_{i=1}^2 V_B^i(s^*))}. \tag{4.12}$$

The next example shows how the above intuition generalizes to the more complex case where several agents are privately informed.

EXAMPLE 4.5: There are two agents $i=1,2$ and two alternatives $k=A,B$. Signals are two-dimensional, $s^i=(s_A^i,s_B^i)$, $i=1,2$. For $i=1,2$ let $-i$ denote the agent other than i. Valuations are given by

$$V_k^i(s^i,s^{-i}) = a_{ki}^i s_k^i + a_{ki}^{-i} s_k^{-i} \qquad (i=1,2;\ k=A,B).$$

Assume that an efficient, incentive compatible DRM exists, and denote it by (p,x). Let q_k^i denote i's conditional expected probability that the mechanism chooses alternative k.

We will first show that, as a consequence of equation (4.7), incentive compatible mechanisms must yield the same vector of conditional expected probability assignments for types of agent i, $i=1,2$, lying on lines with slope a_{Ai}^i/a_{Bi}^i. We next show that efficient mechanisms yield the same vector of conditional expected probability assignments for types lying on lines with slope $(a_{Ai}^i + a_{A-i}^i)/(a_{Bi}^i + a_{B-i}^i)$. Hence, incentive compatibility can be consistent with efficiency only if these two slopes are equal.

We know that

$$\forall i, \forall s^i, \quad q_A^i(s^i) + q_B^i(s^i) = 1. \tag{4.13}$$

Consider agent 1. Equation (4.7) yields

$$a_{A1}^1 \frac{\partial q_A^1(s^1)}{\partial s_B^1} = a_{B1}^1 \frac{\partial q_B^1(s^1)}{\partial s_A^1}. \tag{4.14}$$

By taking the derivative with respect to s_A^1 in identity (4.13), we get

$$\frac{\partial q_B^1(s^1)}{\partial s_A^1} = -\frac{\partial q_A^1(s^1)}{\partial s_A^1}.$$

By equation (4.14), we get

$$a_{A1}^1 \frac{\partial q_A^1(s^1)}{\partial s_B^1} + a_{B1}^1 \frac{\partial q_A^1(s^1)}{\partial s_A^1} = 0. \tag{4.15}$$

Fix now $t^1=(t^1_A,t^1_B)\in\Theta^1$ (see statement of Theorem 4.3) and consider a line in the type space of agent 1 having the form $s^1=s^1(r)=(t^1_A+r,t^1_B+(a^1_{A1}/a^1_{B1})r)$. By equation (4.15) we have:

$$\frac{dq^1_A\left(t^1_A+r,t^1_B+\frac{a^1_{A1}}{a^1_{B1}}r\right)}{dr}=\frac{\partial q^1_A(s^1)}{\partial s^1_A}+\frac{a^1_{A1}}{a^1_{B1}}\frac{\partial q^1_A(s^1)}{\partial s^1_B}=0. \tag{4.16}$$

Hence, in incentive compatible mechanisms the function q^1_A is constant along lines having the form $(t^1_A+r,t^1_B+(a^1_{A1}/a^1_{B1})r)$. By equation (4.13) the same is true for q^1_B.

We now turn to the consequences of efficiency. Alternative A is chosen by an efficient DRM at reports (s^1,s^2) if and only if

$$\sum_{i=1}^{2}\sum_{j=1}^{2}a^j_{Ai}s^j_A\ge\sum_{i=1}^{2}\sum_{j=1}^{2}a^j_{Bi}s^j_B.$$

This is equivalent to

$$(a^1_{A1}+a^1_{A2})s^1_A-(a^1_{B1}+a^1_{B2})s^1_B\ge(a^2_{B1}+a^2_{B2})s^2_B-(a^2_{A1}+a^2_{A2})s^2_A. \tag{4.17}$$

Efficiency implies that

$$q^1_A(s^1)=\int_{\Delta(s^1)}f_2(s^2)\,ds^2$$

where $\Delta(s^1)=\{s^2$ such that condition (4.17) is satisfied$\}$. By definition and by condition 2 in the statement of Theorem 4.3 we obtain that $\Delta(s^1)$ is a nonempty, closed, proper subset of S^2.

Consider a line in agent 1's type space having the form

$$s^1=s^1(r)=\left(t^1_A+r,t^1_B+\frac{a^1_{A1}+a^1_{A2}}{a^1_{B1}+a^1_{B2}}r\right).$$

For any two signals θ^1,τ^1, on this line, we have $\Delta(\theta^1)=\Delta(\tau^1)$. Therefore $q^1_A(s^1(r))$ does not depend on r. Taking the derivative with respect to r, and multiplying by $(a^1_{B1}+a^1_{B2})\neq 0$, this yields

$$(a^1_{B1}+a^1_{B2})\frac{\partial q^1_A(s^1)}{\partial s^1_A}+(a^1_{A1}+a^1_{A2})\frac{\partial q^1_A(s^1)}{\partial s^1_B}=0. \tag{4.18}$$

Equations (4.16) and (4.18) yield together

$$\frac{a^1_{A1}}{a^1_{B1}}=\frac{a^1_{A1}+a^1_{A2}}{a^1_{B1}+a^1_{B2}}. \tag{4.19}$$

The same logic yields an analogous condition for $i=2$.

Several remarks regarding Theorem 4.3 follow.

REMARK 1: The impossibility of incentive compatible, efficient mechanisms is a general phenomenon, and it is not confined to our linear setting. Indeed, recall that the *only* crucial property of valuation functions used in Example 4.4 is continuity.[35] Linearity (which implies that marginal valuations are constant) was used to get the simple global formula (4.8). Without linearity, one obtains congruence conditions that locally equate average private and social rates of substitution (where the average is taken over the area in which two alternatives are equally efficient).

REMARK 2: Theorem 4.3 applies as stated to the case where the dimension of signal spaces coincides with the number of alternatives $K \geq 2$. But the argument in Example 4.4 clarifies that the same type of result holds whenever, for at least one agent, the dimension of the signal space is greater than one. In particular, we obtain impossibility results for auctions of several heterogeneous objects, where the dimension of signal spaces is greater than 1, but usually smaller than the number of alternatives (which equals the number of possible partitions of the set of auctioned objects).

REMARK 3: Dasgupta and Maskin (2000) suggest that the difficulties appearing in multi-dimensional models can be circumvented by performing a reduction to a one-dimensional model for which efficient, incentive compatible mechanisms can be constructed under less restrictive assumptions (see next Section). The efficient mechanism for the reduced model is then constrained efficient (i.e., second-best) for the original multi-dimensional model. Dimension reductions are indeed readily available in two cases: (i) If a variable $\hat{s}^i_{kj}$, $j \neq i$, moves independently of $(\hat{s}^i_{k'i})_{k'}$, condition (4.4) shows that incentive compatible mechanisms cannot condition on it. Hence, such variables can be eliminated without affecting the maximum efficiency performance obtainable by incentive compatible mechanisms. (ii) In Example 4.5 there were only two social alternatives A and B, and we have shown that incentive compatible mechanisms have the property that the conditional expected probability assignment vector field q^i is constant along lines in the type space S^i with the slope a^i_{Ai}/a^i_{Bi}. A parameterization of this family of parallel lines yields a one-dimensional type space[36] for which an efficient, incentive compatible mechanism can be constructed.[37] This mechanism is necessarily second-best for the original model where first-best efficiency was impossible.

[35] Differentiability was used there only to quantify the condition relating the two loci.

[36] Instead of reporting a type (s^i_A, s^i_B), agent i reports, say, the intercept of a line with slope a^i_{Ai}/a^i_{Bi}.

[37] Similar reductions can be performed in models where there are possibly more than two alternatives, but each agent perceives only two outcomes as payoff relevant. For example, in an auction for one unit of an indivisible good without allocative externalities, an agent cares only about "winning" or "losing."

We wish to stress here that this insight does not hold anymore if at least one agent perceives more than two payoff relevant alternatives.[38] Recall Example 4.4 where we had to keep one coordinate constant while operating on the other two. For each pair of alternatives we obtain a one-dimensional family of lines as above, but, since there are at least three different pairs, it is not a-priori clear how to *consistently* combine the pair-wise information in order to reduce the dimension of the signal spaces.

5. ONE-DIMENSIONAL SIGNALS

We now assume that agents have one-dimensional signals. Agent i's payoff in alternative k is given by

$$V_k^i(s^i, s^{-i}) = \sum_{j=1}^{N} a_{ki}^j s^j$$

where $s^j \in [\underline{s}^j, \bar{s}^j]$ denotes the one-dimensional signal of agent j. Signals need not be independently distributed, and the result below does not depend on the signals' distribution functions.

In order to avoid a tedious case differentiation, we assume that, for each agent i, there are no alternatives k, k', $k' \neq k$, such that $a_{ki}^i = a_{k'i}^i$. Our result will rely on the following assumption:

$$\text{(5.1)} \qquad \forall i, \forall k, k', \quad a_{ki}^i > a_{k'i}^i \Rightarrow \sum_{j=1}^{N} a_{kj}^i > \sum_{j=1}^{N} a_{k'j}^i.$$

Consider first the impact of i's signal on i's payoff in each alternative, and order the alternatives such that the alternative where the impact is highest appears first, the alternative where the impact is second highest appears second, and so on.... Consider then the impact of i's signal on social welfare in each alternative, and order the alternatives as above. Condition (5.1) (to which we refer below as the weak congruence condition) requires that the two orders coincide. By rewriting this condition as

$$\forall i, \forall k, k', \quad \frac{a_{ki}^i}{a_{k'i}^i} > 1 \Rightarrow \frac{\sum_{j=1}^{N} a_{kj}^i}{\sum_{j=1}^{N} a_{k'j}^i} > 1,$$

we note a certain (formal) analogy with condition (4.8), but also the gained slack in the one-dimensional framework. This slack (i.e., required inequalities instead of equalities) allows the condition to be satisfied for an open set of parameters' values.

[38] This is the general case in auctions of several heterogenous objects or in auctions of one object with allocative externalities.

THEOREM 5.1: *Assume that the weak congruence condition* (5.1) *is satisfied. Then there exists an efficient, Bayesian incentive compatible mechanism. Moreover, the associated transfers do not depend on the distribution of signals.*[39]

PROOF: See Appendix.

The "Clarke-Groves-Vickrey logic" behind the efficient, incentive compatible revelation mechanism constructed in the proof of Theorem 5.1 works for any one-dimensional framework with quasi-linear utility. The needed conditions on marginal valuations are generally more complex than condition (5.1) (see also Dasgupta and Maskin (2000)). Since the construction does not depend on distributional assumptions (it works, in particular, for one agent or for several agents with independently distributed types), the results in the previous Section show that this logic cannot be extended to multi-dimensional frameworks where agents have interdependent valuations.

6. CONCLUSIONS

We have shown that efficient, Bayesian incentive compatible mechanisms can exist only if a congruence condition relating private and social rates of information substitution is satisfied. If signals are multi-dimensional, the congruence condition is determined by an integrability constraint, and it can be satisfied only in nongeneric cases. If signals are one-dimensional, the congruence condition reduces to a monotonicity constraint and it can be generically satisfied.

The impossibility results in the multi-dimensional case suggest a quest for a second-best (or constrained efficient) mechanism. It is straightforward to construct second-best mechanisms if the inefficiency is purely due to the fact that some signals must have a zero marginal effect on incentive compatible expected probability assignments. It is then possible to reduce the dimension of the model (without loss of efficiency) by eliminating such variables. If, after performing these reductions, it is still the case that the payoff-relevant information depends in a nontrivial way on the chosen alternative (as it is the case, say, in a general multi-object auction), we are left in a framework covered by Theorem 4.3 and further dimension reductions become endogenous. The construction of a second-best mechanism is then equivalent to the difficult problem of finding a monotone and conservative vector field that maximizes the (expected) welfare functional.[40] This will be the subject of future work.

[39] Technically, this result is not a special case of Crémer and McLean (1985) or Dasgupta and Maskin (2000) because they study multi-object auctions (without allocative externalities), while we study a general social choice problem. Dasgupta and Maskin's mechanism is more complex since it also elicits reports about valuation functions, which, in their model, are not known to the designer. This allows them to construct a mechanism whose rules do not depend on valuation functions.

[40] Jehiel, Moldovanu, and Stacchetti (1999) discuss the mathematically related question of revenue maximization in a multi-dimensional private values model. The constraint on cross-derivatives boils down to a certain partial differential equation. For some special cases, the equation is an ordinary one, and examples can be analytically computed.

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APPENDIX

PROOF OF THEOREM 3.1: (a) Assume first that a DRM (p,x) satisfies the conditions in the Theorem. Choose any agent i. We must show that $\forall s^i, t^i$, $U_i(s^i,s^i) - U_i(t^i,s^i) \geq 0$. We obtain the following chain of equalities:

$$\begin{aligned} U_i(s^i,s^i) - U_i(t^i,s^i) &= \mu_i(s^i) - \mu_i(t^i) - Q^i(t^i)\cdot(s^i - t^i) \\ &= \int_{t^i}^{s^i} Q^i(\tau^i)\cdot d\tau^i - Q^i(t^i)\cdot(s^i-t^i) \\ &= \int_0^1 [Q^i((1-\alpha)t^i + \alpha s^i) - Q^i(t^i)]\cdot(s^i - t^i)\, d\alpha. \end{aligned}$$

The first equality follows by equation (2.1) and by the definition of μ_i. The second equality follows by assumption. The last equality follows by choosing to integrate on the straight line connecting t^i and s^i.

The condition $\forall s^i, t^i$, $U_i(s^i,s^i) - U_i(t^i,s^i) \geq 0$ is therefore equivalent to the condition

$$\forall s^i, t^i, \quad \int_0^1 [Q^i((1-\alpha)t^i + \alpha s^i) - Q^i(t^i)]\cdot(s^i-t^i)\, d\alpha \geq 0.$$

It is enough to show that the integrand is nonnegative for any α, $0 \leq \alpha \leq 1$. For $\alpha = 0$, the claim is obvious. Assume that $\alpha > 0$. Noting that $s^i - t^i = (1/\alpha)((1-\alpha)t^i + \alpha s^i - t^i)$, we obtain

$$\begin{aligned} &[Q^i((1-\alpha)t^i + \alpha s^i) - Q^i(t^i)]\cdot(s^i - t^i) \\ &\quad = \frac{1}{\alpha}[Q^i((1-\alpha)t^i + \alpha s^i) - Q^i(t^i)]((1-\alpha)t^i + \alpha s^i - t^i) \geq 0. \end{aligned}$$

The last inequality follows from the monotonicity of Q^i.

(b) For the converse, assume that the DRM (p,x) is incentive compatible. This implies that $\mu_i(s^i) = U_i(s^i,s^i) = \max_{t^i} U_i(t^i,s^i)$. The function μ_i is the supremum of a collection of affine functions and it must be convex. Convex functions are twice differentiable almost everywhere.[41] The convexity of μ_i implies the monotonicity of the subdifferential map $\partial\mu_i$. At all points where μ_i is differentiable (i.e., a.e.) the subdifferential $\partial\mu_i$ consists of a unique point, the gradient $\nabla\mu_i$. Hence, the function $\nabla\mu_i$ is a.e. well-defined, monotone, and differentiable. Assuming that μ_i is differentiable at s^i we obtain by expression (2.1) and by the Envelope Theorem that

$$\forall k, \quad \frac{\partial\mu_i}{\partial s^i_{ki}}(s^i) = \frac{\partial U_i}{\partial s^i_{ki}}(t^i,s^i)|_{t^i=s^i} = a^i_{ki} q^i_k(s^i), \tag{7.1}$$

$$\forall k, \forall j \neq i, \quad \frac{\partial\mu_i}{\partial s^i_{kj}}(s^i) = \frac{\partial U_i}{\partial s^i_{kj}}(t^i,s^i)|_{t^i=s^i} = 0. \tag{7.2}$$

[41] This and all following properties of convex functions are listed in the classical text of Rockafellar (1997).

Hence, we obtain $\nabla\mu_i(s^i) = Q^i(s^i)$ whenever the gradient is well-defined (a.e.). The integral representation is immediately obtained from the Fundamental Theorem of Calculus if μ_i is everywhere differentiable. Otherwise, the result follows by noting that a convex function is (up to a constant) uniquely determined by its subdifferential (see Rockafellar (1997, Theorem 24.9)), and that it can be recovered (up to a constant) by integrating any measurable selection from its subdifferential map (see Krishna and Maenner (1999)).

Proof of Theorem 4.3: Let (p, x) be an efficient, incentive compatible DRM, and let $(q_k^i)_{k=1}^K$ be the associated vector field of conditional expected probabilities for agent i. Since (p, x) is incentive compatible, the associated indirect utility function μ_i is twice-differentiable a.e. Since (p, x) is efficient, the associated functions $(q_k^i)_{k=1}^K$ are continuously differentiable everywhere.

We consider below signals $s^i \in \Theta^i$. By equation (4.7) we obtain for almost all s^i:

$$\forall k, k', \quad a_{ki}^i \frac{\partial q_k^i(s^i)}{\partial s_{k'}^i} = a_{k'i}^i \frac{\partial q_{k'}^i(s^i)}{\partial s_k^i}. \tag{7.3}$$

Since p is efficient, we obtain

$$q_k^i(s^i) = \int_{\Delta_k(s^i)} f_{-i}(s^{-i})\, ds^{-i} \tag{7.4}$$

where

$$\Delta_k(s^i) = \{s^{-i} \mid k \in \kappa^*(s^i, s^{-i})\}. \tag{7.5}$$

An analogous expression holds for $q_{k'}^i(s^i)$. For a fixed s^i define now the locus in S^{-i} where alternatives k and k' achieve the same highest utility:

$$\Omega_{k,k'}(s^i) = \{s^{-i} \mid \{k, k'\} \subset \kappa^*(s^i, s^{-i})\}. \tag{7.6}$$

Condition 2 in the statement of the Theorem ensures that $\Omega_{k,k'}(s^i) \neq \emptyset$. We now want to calculate the derivative $\partial q_k^i(s^i)/\partial s_{k'}^i$ using expressions (7.4), (7.5), and (7.6): a marginal variation of $s_{k'}^i$ affects $q_k^i(s^i)$ only by marginally shifting the boundary $\Omega_{k,k'}(s^i) \subset \Delta_k(s^i)$ where k and k' are equally efficient. Hence $\partial q_k^i(s^i)/\partial s_{k'}^i$ is equal to an integral over the boundary multiplied by the marginal shift, which is proportional to $(\sum_{g=1}^N a_{k'g}^i)$, the constant coefficient of $s_{k'}^i$ in the equation defining $\Omega_{k,k'}(s^i)$.[42]

To make this observation precise, define:

$$\begin{aligned} z_0 &= \sum_{j\neq i} \sum_{g=1}^N a_{kg}^j s_k^j - \sum_{j\neq i} \sum_{g=1}^N a_{k'g}^j s_{k'}^j, \\ c &= -\left(\sum_{g=1}^N a_{kg}^i\right) s_k^i + \left(\sum_{g=1}^N a_{k'g}^i\right) s_{k'}^i. \end{aligned} \tag{7.7}$$

Note that

$$\begin{aligned} \Delta_k(s^i) &= \left\{ s^{-i} \,\middle|\, z_0 \geq c \wedge \sum_{j=1}^N \sum_{g=1}^N a_{kg}^j s_k^j \geq \sum_{j=1}^N \sum_{g=1}^N a_{k''g}^j s_{k''}^j \text{ for } k'' \neq k' \right\}, \\ \Omega_{k,k'}(s^i) &= \Delta_k(s^i) \cap \{s^{-i} \mid z_0 = c\}. \end{aligned} \tag{7.8}$$

[42] This is the generalization to several dimensions of a standard one-dimensional result: define $H(y) = \int_{d(y)}^{c(y)} g(x)\,dx$ where g is continuous and where c, d are continuously differentiable. By the Fundamental Theorem of Calculus (FTC), $H'(y) = g(c(y))c'(y) - g(d(y))d'(y)$. A general proof of the multi-dimensional analog uses a multi-dimensional version of the FTC, called the *Divergence Theorem* (see Lang (1973)).

Consider an affine, bijective change of variables in the space S^{-i}, where z_0 is one of the new variables, and where $\mathbf{z}$ denotes the set of the remaining variables (with differentiable element $d\mathbf{z}$).[43] Such a bijective change of variables exists because z_0 is not identically equal to zero (recall that $\sum_{j=1}^N a_{k'j}^i \neq 0$ and $\sum_{j=1}^N a_{kj}^i \neq 0$). To fix ideas, suppose that the coefficients are such that for all alternatives k'' there exists an agent $j(k'') \neq i$, such that $a_{k''j(k'')}^{j(k'')} \neq 0$. Consider the mapping G: $\{s_{k''}^j\}_{j \neq i, k''} \to \{z_{k''}^j\}_{j \neq i, k''}$ where:

(i) for $k'' \neq k$, $j = j(k'')$, $z_{k''}^{j(k'')} = \sum_{j \neq i} \sum_{g=1}^N a_{kg}^j s_k^j - \sum_{j \neq i} \sum_{g=1}^N a_{k''g}^k s_{k''}^j$ (observe that $z_{k'}^{j(k')} = z_0$);

(ii) for all (j, k'') such that $k'' = k$ or $j \neq j(k'')$, $z_{k''}^j = s_{k''}^j$.

Let G^{-1} be the inverse of G, and let $|\Delta_{G^{-1}}| \neq 0$ denote the absolute value of the Jacobian determinant associated with G^{-1}. Note that $|\Delta_{G^{-1}}|$ is a constant (i.e., it does not depend on $(z_0, \mathbf{z})$) because G is affine. We obtain now:

$$\begin{aligned}
(7.9)\qquad \frac{\partial q_k^i(s^i)}{\partial s_{k'}^i} &= \frac{\partial}{\partial s_{k'}^i}\left(\int_{\Delta_k(s^i)} f_{-i}(s^{-i})\, ds^{-i}\right) \\
&= \frac{\partial}{\partial s_{k'}^i}\left[|\Delta_{G^{-1}}| \int_{G(\Delta_k(s^i))} f_{-i}(G^{-1}(z_0, \mathbf{z}))\, d\mathbf{z}\, dz_0\right] \\
&= -\frac{\partial c}{\partial s_{k'}^i} |\Delta_{G^{-1}}| \int_{G(\Omega_{k,k'}(s^i))} f_{-i}(G^{-1}(c, \mathbf{z}))\, d\mathbf{z} \\
&= -\left(\sum_{g=1}^N a_{k'g}^i\right) |\Delta_{G^{-1}}| \int_{G(\Omega_{k,k'}(s^i))} f_{-i}(G^{-1}(c, \mathbf{z}))\, d\mathbf{z}.
\end{aligned}$$

The first equality in (7.9) follows by the definition of $q_k^i(s^i)$; the second equality follows by the multi-dimensional *change of variables formula* (see Lang (1973)); the third follows by expressions (7.8) and by the argument following expression (7.6); the last equality follows by the definition of c in (7.7).

Using the same change of variables as above, the term $\partial q_{k'}^i(s^i)/\partial s_k^i$ is analogously computed:[44]

$$\begin{aligned}
(7.10)\qquad \frac{\partial q_{k'}^i(s^i)}{\partial s_k^i} &= \frac{\partial c}{d s_k^i} |\Delta_{G^{-1}}| \int_{G(\Omega_{k,k'}(s^i))} f_{-i}(G^{-1}(c, \mathbf{z}))\, d\mathbf{z} \\
&= -\left(\sum_{g=1}^N a_{kg}^i\right) |\Delta_{G^{-1}}| \int_{G(\Omega_{k,k'}(s^i))} f_{-i}(G^{-1}(c, \mathbf{z}))\, d\mathbf{z}.
\end{aligned}$$

Combining equations (7.9) and (7.10), we obtain that

$$(7.11)\qquad \frac{\partial q_k^i(s^i)}{\partial s_{k'}^i}\left(\sum_{g=1}^N a_{kg}^i\right) = \frac{\partial q_{k'}^i(s^i)}{\partial s_k^i}\left(\sum_{g=1}^N a_{k'g}^i\right).$$

Equations (7.3) and (7.11) yield together the wished result. *Q.E.D.*

Proof of Theorem 5.1: Since all a_{ki}^i are assumed to be different, we can rename the alternatives so that the sequence $(a_{ki}^i)_k$ is strictly increasing, i.e. $a_{(k+1)i}^i > a_{ki}^i$ for $k = 1, \ldots, K-1$. Condition (5.1) implies then that the sequence $\{\sum_{j=1}^N a_{kj}^i\}_k$ is also strictly increasing.

[43] If $\mathbf{z} = (z_1, \ldots, z_m)$, then $d\mathbf{z} = dz_1 dz_2 \cdots dz_m$. The purpose of the change of variables is to concentrate the entire dependence on s_k^i and $s_{k'}^i$ in a single dimension, z_0. This allows us to use the one-dimensional argument of the previous footnote in the derivation of expression (7.9) below. The choice of variables $\mathbf{z}$ is entirely arbitrary as long as they are well defined (we need to take care about possible zero coefficients).

[44] Note that the area in $\Delta_{k'}(s^i)$ where marginal variations of s_k^i are relevant is also $\Omega_{k,k'}(s^i)$. The resulting expressions in (7.9) and (7.10) contain the same integrand over the same boundary, but differ in terms of orientations (since the respective outward normal vectors have opposite signs) and shifts (since the variables $s_{k'}^i$ and s_k^i appear with different coefficients in the definition of c).

We construct an efficient, incentive compatible, DRM. For any reported signals the mechanism chooses an efficient alternative given those reports. To specify transfers, we proceed as follows. For fixed reports s^{-i}, denote by $k^*(t^i)$ a selection out of $\kappa^*(t^i, s^{-i})$ as a function of i's report t^i, i.e.

$$k^*(t^i) \in \arg\max_k \sum_{j=1}^{N} V_k^j(t^i, s^{-i}).$$

Because the sequence $\{\sum_{j=1}^{N} a_{kj}^i\}_k$ is strictly increasing, we can define for every vector s^{-i}, a *nondecreasing* sequence of agent i's signals $\{\bar{s}^{i,k}(s^{-i})\}_k$ with the property that, for any $t^i \in (\bar{s}^{i,k}(s^{-i}), \bar{s}^{i,k+1}(s^{-i}))$, the efficient alternative is $k^*(t^i) = k$.

For each vector s^{-i} we inductively define a sequence of transfers, $\{\bar{x}_i^k(s^{-i})\}_k$, as follows: $\bar{x}_i^1(s^{-i}) \in \Re$ is an arbitrary constant, and for all k, $1 < k \leq K-1$,

$$\bar{x}_i^{k+1}(s^{-i}) - \bar{x}_i^k(s^{-i}) = \sum_{j, j \neq i} [V_{k+1}^j(\bar{s}^{i,k+1}(s^{-i}), s^{-i}) - V_k^j(\bar{s}^{i,k+1}(s^{-i}), s^{-i})]. \tag{7.12}$$

If the vector of reports is (t^i, s^{-i}), then i's transfer is defined to be $x_i^*(t^i, s^{-i}) = \bar{x}_i^{k^*(t^i)}(s^{-i})$.[45]

The logic underlying the specification of the transfers is as follows. Fix a vector of reports s^{-i}. Suppose that both intervals $(\bar{s}^{i,k}(s^{-i}), \bar{s}^{i,k+1}(s^{-i}))$ and $(\bar{s}^{i,k+1}(s^{-i}), \bar{s}^{i,k+2}(s^{-i}))$ are nonempty. For s^i slightly above $\bar{s}^{i,k+1}(s^{-i})$ the only efficient alternative is $k+1$. For s^i slightly below $\bar{s}^{i,k+1}(s^{-i})$ the only efficient alternative is k. At $s^i = \bar{s}^{i,k+1}(s^{-i})$ both alternatives are efficient. The transfers are adjusted so that, given s^{-i}, agent i with type $\bar{s}^{i,k+1}(s^{-i})$ is made indifferent between alternative k with transfer $\bar{x}_i^k(s^{-i})$ and alternative $k+1$ with transfer $\bar{x}_i^{k+1}(s^{-i})$.

We now show that it is optimal for agent i to report truthfully if all other agents report truthfully. Fix s^{-i} the (truthfully) reported signal of all agents other than i. In order to have a more transparent notation, we omit below the dependence of $\bar{s}^{i,k}$ and $\bar{x}_i^k$ on the fixed s^{-i}.

Suppose without loss of generality that agent i's true type s^i lies in $[\bar{s}^{i,k}, \bar{s}^{i,k+1})$. If agent i truthfully reports $t^i = s^i$, his payoff is

$$R_i(s^i, s^{-i}) = V_k^i(s^i, s^{-i}) + \bar{x}_i^k.$$

For any report $t^i \in [\bar{s}^{i,k}, \bar{s}^{i,k+1})$, agent i gets the same payoff. Suppose that agent i makes a report $t^i \in [\bar{s}^{i,k+r}, \bar{s}^{i,k+r+1})$ with $r > 0$. This nontruthful report yields for agent i a payoff of

$$R_i(t^i, s^{-i}) = V_{k+r}^i(s^i, s^{-i}) + \bar{x}_i^{k+r}.$$

Noting that $\bar{x}_i^{k+r} = \sum_{l=1}^r (\bar{x}_i^{k+l} - \bar{x}_i^{k+l-1}) + \bar{x}_i^k$ and using expression (7.12), we obtain

$$\begin{aligned} &R_i(s^i, s^{-i}) - R_i(t^i, s^{-i}) \\ &\quad = V_k^i(s^i, s^{-i}) - V_{k+r}^i(s^i, s^{-i}) \\ &\qquad - \sum_{l=1}^{r} \left[\sum_{j, j \neq i} (V_{k+l}^j(\bar{s}^{i,k+l}, s^{-i}) - V_{k+l-1}^j(\bar{s}^{i,k+l}, s^{-i})) \right]. \end{aligned}$$

By the definition of $\bar{s}^{i,k+l}$ (at which both alternatives $k+l-1$ and $k+l$ are efficient), we obtain

$$\begin{aligned} &\sum_{j, j \neq i} (V_{k+l}^j(\bar{s}^{i,k+l}, s^{-i}) - V_{k+l-1}^j(\bar{s}^{i,k+l}, s^{-i})) \\ &\quad = -(V_{k+l}^i(\bar{s}^{i,k+l}, s^{-i}) - V_{k+l-1}^i(\bar{s}^{i,k+1}, s^{-i})). \end{aligned}$$

[45] To avoid a cumbersome case differentiation, we have assumed that, given s^{-i}, the set $\{k^*(t^i)\}_{t^i \in S^i}$ includes the entire set of alternatives. If this is not the case, then some of the intervals $(\bar{s}^{i,k}(s^{-i}), \bar{s}^{i,k+1}(s^{-i}))$ may be empty. Transfers are then defined up to the arbitrary value of the transfer in the first nonempty interval. Furthermore, if a signal $\bar{s}^{i,k+1}(s^{-i})$ hits the upper bound of agent i's signal interval, then the transfer for all reports $t^i > \bar{s}^{i,k}(s^{-i})$ is set to be equal to $\bar{x}_i^k(s^{-i})$.

Finally, we obtain that

$$\begin{aligned}
&R_i(s^i, s^{-i}) - R_i(t^i, s^{-i}) \\
&\quad = V_k^i(s^i, s^{-i}) - V_k^i(\bar{s}^{i,k+1}, s^{-i}) \\
&\qquad + \sum_{l=1}^{r-1} (V_{k+l}^i(\bar{s}^{i,k+l}, s^{-i}) - V_{k+l}^i(\bar{s}^{i,k+l+1}, s^{-i})) \\
&\qquad + V_{k+r}^i(\bar{s}^{i,k+r}, s^{-i}) - V_{k+r}^i(s^i, s^{-i}) \\
&\quad = a_{ki}^i(s^i - \bar{s}^{i,k+1}) + \sum_{l=1}^{r-1} (a_{(k+l)i}^i(\bar{s}^{i,k+l} - \bar{s}^{i,k+l+1})) + a_{(k+r)i}^i(\bar{s}^{i,k+r} - s^i) \\
&\quad = \sum_{l=1}^{r} (a_{(k+l-1)i}^i - a_{(k+l)i}^i)(s^i - \bar{s}^{i,k+l}) \geq 0.
\end{aligned}$$

The last inequality follows because each of the terms in the sum is nonnegative.[46]

The proof for a report $t^i \in [\bar{s}^{i,k+r}, \bar{s}^{i,k+r+1})$ with $r < 0$ is completely analogous.

Note that the transfers defined above do not depend on the distribution of signals, and our mechanism implements the efficient social choice rule no matter how the signals of the various agents are distributed.[47] *Q.E.D.*

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[46] By the assumption on the sequence $\{a_{ki}^i\}_k$, we have $(a_{(k+l-1)i}^i - a_{(k+l)i}^i) < 0$; because s^i lies in $[\bar{s}^{i,k}, \bar{s}^{i,k+1})$, and because the sequence $\{\bar{s}^{i,k}\}_k$ is nondecreasing, we have $(s^i - \bar{s}^{i,k+l}) \leq 0$.

[47] Truth-telling constitutes here an *ex-post equilibrium*.

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