Path-Monotonicity and Truthful Implementation

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Abstract

We study the role of monotonicity in the characterization of truthfully implementable allocation rules when types are multi-dimensional, agents have quasi-linear preferences, and valuations for outcomes are either differentiable or convex in types. We define a stronger version of monotonicity, called *path-monotonicity*, and show that it characterizes in combination with an integrability condition truthfully implementable allocation rules in two cases: (1) valuations are differentiable and type spaces are path-connected; (2) valuations are convex and type spaces are convex. We analyze conditions under which monotonicity is equivalent to path-monotonicity. We show that a property closely related to the single-crossing property ensures this equivalence. In particular, the property holds when valuations for outcomes are linear functions of types. Next we show that for simply connected type spaces implementability of the allocation rule is equivalent to path-monotonicity plus implementability in some neighborhood of each type. This result is used to show that on simply connected type spaces implementable allocation rules with a finite range are completely characterized by path-monotonicity, and thus by monotonicity in cases where path-monotonicity and monotonicity are equivalent.

Keywords: Mechanism design, multi-dimensional types, truthful implementation, monotonicity.

1 Introduction

The goal of mechanism design is to design game forms that motivate agents with private information to choose equilibrium strategies that lead to an implementation of a desired social choice function. In this paper we assume that agents have preferences in terms of monetary valuations for outcomes and that the mechanism designer can use payments to orchestrate agent behavior. Preferences for each outcome are given by publicly known functions of private information of each player, called *valuation functions* and *types*, respectively. Types may be a single number (one-dimensional) or vectors of numbers (multi-dimensional). Agents are assumed to have quasi-linear utilities over outcomes and payments. Whenever the revelation principle holds, the question of implementability then reduces to the existence of a payment rule such that truthful reports of agent types becomes an equilibrium, in other words, *lying does not pay.* A social choice function is then viewed as an *allocation rule* that selects an outcome for each combination of reported types. An allocation rule is called *(truthfully) implementable*, if such payments exist.

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In this paper we study conditions on the structure of type spaces of agents and the structure of valuation functions that allow for easily verifiable properties to characterize truthfully implementable allocation rules. In particular, our aim is to have payment free, local characterizations, because with such a characterization implementability can be verified without the need to construct payments, and by understanding the influence of "small" deviations from the truth.

In single-item auctions where types are valuations for a single good and outcomes can be interpreted as probabilities to get the good, such a property is monotonicity: the probability to get the good has to be non-decreasing in the reported type (see, e.g. Myerson [14]). In multi-dimensional settings (a generalization of) monotonicity is still necessary for implementability, but often not sufficient. For example, in a combinatorial auction, type vectors represent values for each bundle of goods, and outcomes may be given by a probability to win each of the bundles. An allocation rule f that assigns probability vector $f(t)$ to report t, would be monotone if for any two types s, t we have $(f(s) - f(t)) \cdot (s - t) \geq 0$. It is well known that there exist non-implementable f that satisfy monotonicity (see Müller, Perea and Wolf [12] for an example). The goal is then to identify conditions that, if added to monotonicity, guarantee implementability. A well-known condition of this type is the existence and path-independence of path integrals of a particular vector field (Jehiel and Moldovanu [8]). As such a condition is difficult to verify—a claim that will become clear in Section 3—it is helpful to understand when the latter is not needed, in other words, when monotonicity alone implies implementability. A quite general result of this type is due to Saks and Yu [17], generalizing results in more restricted settings due to Jehiel, Moldovanu and Stacchetti [9], Gui et al. [6] and Bikhchandani et al. [4]. Saks and Yu show that if types are from a convex set in $\mathbb{R}^d$, representing valuations for d different outcomes, then monotonicity is sufficient for implementability. Shorter proofs of this result where later given by Ashlagi, Braverman, Hassidim and Monderer [2] as well as by Vohra [18]. Archer and Kleinberg [1] gave yet an alternative proof based on techniques that are of interest by their own. They show first that it is essentially sufficient for being implementable that an allocation rule is monotone and *locally implementable* in some arbitrary small neighborhood of each type t, even for infinite sets of outcomes. Second they show that for finite sets of outcomes monotonicity implies local implementability.

Gui et al. [6], Saks and Yu [17], Ashlagi et al. [2], Vohra [18] and Archer and Kleinberg [1] make use of a network interpretation of incentive constraints, which has been proposed in Gui et al. [6], and was already implicitly used in Rochet [15]. The network approach defines a network whose nodes are the types, and whose arcs between any two nodes are given weights in a way that lets potentials in the network coincide with truthful payments. Such payments exist whenever the network does not have a cycle of negative length, while monotonicity only guarantees that cycles of two arcs—forward and backward arcs between two types—have non-negative length. While monotonicity has thus a straightforward interpretation in terms of networks, path-integrals of the vector fields used in analytical characterizations of implementability (Jehiel et al. [8, 9]) and revenue equivalence (Krishna and Maenner [10], Milgrom and Segal [11]) seemed more difficult to grasp. Archer and Kleinberg [1] closed this gap for linear valuations by defining arc lengths differently, and linking lengths of paths and cycles in the network explicitly to path integrals of the aforementioned vector field.

All previous literature assumes in one or the other way that valuations for outcomes are linear functions of types. In many applications of mechanism design this is a very restrictive assumption. For example, a type of an agent might be parameters defining the cost structure of a firm, and outcomes might be various contracts. Valuations for outcomes could in such a case be given by a complex, non-linear function of the type. Theoretically, we could still think of a type as a vector of values for each possible outcome. In a revelation mechanism this would mean that agents have to report their value for each possible contract. Furthermore the non-linearity of the valuation function could turn a convex set of parameters into a non-convex, and maybe even disconnected set of type vectors. It is therefore desirable that characterizations of implementability are available in terms of compact representations of parameters, and thus in the same generality as theorems on revenue equivalence have been made available in Krishna and Maenner [10], Milgrom and Segal [11], Heydenreich et al. [7] and Chung and Olszewski [5].

In this paper we provide such a general characterization of implementability. We study the role of monotonicity in two very general settings. The first setting requires that types come from a connected subset in $\mathbb{R}^d$ and valuations for each outcome are differentiable functions of the type. The set of outcomes can be arbitrary. This setting has been studied with respect to revenue equivalence by Milgrom and Segal [11]. In the second setting we require that types come from a convex subset in $\mathbb{R}^d$, and valuations for each outcome are convex functions of the type. The set of outcomes is again arbitrary. In terms of revenue equivalence this setting has been studied by Krishna and Maenner [10]. For both settings we clarify the additional conditions that have to be satisfied in order to make a monotone rule implementable. However, we have to introduce a stronger version of monotonicity, which we call path-monotonicity. It makes use of the derivative—let us assume it exists, details on this issue are explained later—of the valuation function of the chosen allocation: say allocation a is chosen at type t, we consider the valuation for a as a function of the type, and take the derivative at t. Doing this for every type defines a vector field. Path-monotonicity requires that the change in value for the outcome a, if the agent would be of type s rather than of type t, must not be larger than the path-integral of this vector field along any path from t to s.

Next we discuss the relation between monotonicity and path-monotonicity. Generally, the latter implies the first, but not the other way round. In special cases however, which can be easily explained in terms of the network approach, both notions coincide. A property, called increasing differences, guarantees this network structure. Increasing differences holds when a single-crossing property is present in the structure of the preferences. However, the single-crossing property is not as generally applicable as it requires to take derivatives of valuation functions with respect to outcomes.

Next we generalize the concept of local implementability to both our settings. We get results of the flavor that an allocation rule on a simply connected type space is implementable if and only if it is (path-)monotone and locally implementable. Such characterizations allow us to generalize the Theorem by Saks and Yu: whenever A is finite, and T is simply connected, then (path-)monotonicity is sufficient for implementability.

Our results are stated and proven in terms of single agent models. This is w.l.o.g. for the type of theorems we are aiming for. In case of several agents, they have an interpretation that depends on the equilibrium concept used. An allocation rule will be dominant strategy incentive compatible, if the conditions of the various theorems hold for each agent and each type report of all other agents. An allocation rule will be ex-post incentive compatible, if the conditions hold for each agent and each truthful report of all other agents. Ex-post incentive compatibility in interdependent value models is covered as well, but valuation functions of the agent under consideration change now with the type of other agents. An allocation rule will be Bayes-Nash implementable, if for each agent the conditions hold for the randomized allocation rule that is induced by the distribution of reports of the other agents when they are truth-telling. However, it will have to be checked whether the required properties of the valuation functions are robust with respect to randomization. Generally, settings with finite set of outcomes will turn into settings with infinitely many outcomes if randomization is added.

Organization. Section 2 defines our setting and introduces necessary notation. We prove the main characterization theorems for arbitrary outcome sets and differentiable or convex valuation functions in Section 3. Section 4 identifies properties under which path-monotonicity and monotonicity are equivalent. In Section 5 we show that local implementability and path-monotonicity imply implementability if the type space is simply connected. Section 6 uses this result to prove generalizations of the Theorem of Saks and Yu. Parts of the results on convex valuations appeared earlier in Berger, Müller, and Naeemi [3].

2 Definitions and Setting

Let $f : T \to A$ be an allocation rule from a set of types T to a set of outcomes A. The valuation for an outcome $a \in A$ of a certain type $t \in T$ is defined by the value $v(a,t)$ given by the function $v : A \times T \to \mathbb{R}$.

Definition 1. *A* (direct) mechanism *is a pair* (f,p) *of an allocation rule* f *and a payment function* $p : T \to \mathbb{R}$*. The mechanism is called* truthful *or* incentive compatible *if for all* $s,t \in T$*:*

$$v(f(s),s) + p(s) \geq v(f(t),s) + p(t), \tag{1}$$

i.e. a player of type s *maximizes his utility when he reports* s*. The allocation rule* f *is called* implementable *if there exists such a payment function* p *that makes the mechanism* (f,p) *truthful.*

Our goal is to identify properties of allocation rules f that are necessary and sufficient to guarantee the existence of a p that satisfies (1).

Rochet [15] identified a property called *cyclical monotonicity*, which was later related to node potentials in *type graphs* by Gui et al. [6]. Here, and further on, a graph consists of a set of *nodes* and a set of (directed) *arcs* between pairs of nodes. Given an allocation rule f the set of nodes of the type graph T_f is equal to T. Every pair of types $s,t \in T$ is connected by arcs from s to t and from t to s.

Gui et al. [6] define the arc length from s to t as[3]

$$l_p(s,t) \quad := \quad v(f(s),s) - v(f(t),s)$$

Archer and Kleinberg [1] proposed alternative arc lengths $l_u(s,t)$, defined as[4]:

$$l_u(s,t) \quad := \quad v(f(t),t) - v(f(t),s).$$

We call $l_p(s,t)$ and $l_u(s,t)$ the *p-length* and *u-length*, respectively. A path from node s to node t in T_f, or (s,t)-path for short, is defined as $P = (s = s_0, s_1, ..., s_k = t)$ such that $s_i \in T$ for $i = 0, ..., k$. The u-length of P is defined as

$$length_u(P) = \sum_{i=0}^{k-1} l_u(s_i, s_{i+1}.$$

Similarly, $length_p(P)$ is defined with respect to p-length. A cycle is a path with $s = t$. For any t, we regard the path from t to t without any arcs as a (t,t)-path and define its length to be 0. Let $P(s,t)$ be the set of all (s,t)-paths. The distance from s to t is defined as

$$dist_u(s,t) = \inf_{P \in P(s,t)} length_u(P).$$

[3] In Gui et al. [6] this would be the length of the arc from t to s, but at some point it is more convenient to do it the other way round.

[4] Archer and Kleinberg [1] use subtitle s instead of u in $l_u(s,t)$

It is obvious that if T_f does not contain a negative cycle then $dist_u(s,t)$ is finite.

We can now define cyclical monotonicity and monotonicity for allocation rules. Monotonicity is weaker, but will play a central role in our characterization efforts. Monotonicity of the type graph generalizes the role of ordinary monotonicity of the allocation rule used in Myerson [14] to characterize incentive compatible single item auctions.

Definition 2. *An allocation rule $f : T \to A$ is called* monotone*, if for all $s,t \in T$ it holds that $l_u(s,t) + l_u(t,s) \geq 0$. f is called* cyclically monotone*, if for all cycles C, $length_u(C) \geq 0$.*

We observe that p-length and u-length of any cycle in T_f are the same, as for all $s,t \in T$ we have $l_p(s,t) = l_u(s,t) + v(f(s),s) - v(f(t),t)$.

Therefore monotonicity and cyclical monotonicity could have been defined in terms of p-length as well.

A node potential in a (type) graph is a function π from T to $\mathbb{R}$ such that for all $s,t \in T$ we have $\pi(t) \leq \pi(s) + l_x(s,t)$, where subscript x indicates that we might choose either p or u lengths.

Rochet provided the following characterization of implementable allocation rule.

Theorem 1 (Rochet [15])**.** *An allocation rule $f : T \to A$ is implementable if and only if it is cyclically monotone.*

The simple proof employs the fact that, due to our choice of arc lengths, node potentials in the type graph with respect to p-length coincide with payment rules that implement the allocation rule. Node potentials exist if and only if the type graph does not have a negative cycle. From a node potential π with respect to u-lengths we get payments by setting $p(t) = \pi(t) - v(f(t),t)$, thus node potentials with respect to u-length provide utilities of truthful reports, given a truthful payment rule p.

Consider the special case when A is finite, and $T \subseteq \mathbb{R}^A$, with the interpretation $v(a,t) = t_a$ for all $a \in A, t \in T$, and T convex. Saks and Yu [17] show for this setting that an allocation rule f is monotone *if and only if* it is cyclically monotone. Ashlagi et al. [2] and Archer and Kleinberg [1] give alternative proofs (and extend the result slightly by allowing outcomes to be lotteries over A). Both proofs can be extended to arbitrary T and $v(a,t)$ being any linear function in t. If $T \subseteq \mathbb{R}$ and convex, it is easy to see that monotonicity implies cyclical monotonicity if the following property of the type graph holds [12].

Definition 3. *Let T be convex. We say that the type graph satisfies* decomposition monotonicity *if for all s,t and x, where x is a convex combination of s and t, it holds $l_u(s,t) \geq l_u(s,x) + l_u(x,t)$*

Decomposition monotonicity holds in particular if valuations are linear in types, as it is implied by monotoncity. For general valuations and/or multi-dimensional settings neither decomposition monotonicity is implied by monotonicity, nor would it be sufficient in combination with monotonicity for implementability. In the remainder of the paper we will strengthen the notion of monotonicity and identify conditions under which the Saks and Yu theorem *can* be generalized to settings with non-linear valuations. To do so we will establish a link between the network approach to implementability (and revenue equivalence) and the traditional, analytical approach as developed in Jehiel, Moldovanu, and Stacchetti [9], Jehiel and Moldovanu [8], Krishna and Maenner [10] and Milgrom and Segal [11]. In the analytical approach, it has been observed that implementable rules give raise to a conservative vector field, that is path-integrals of the vector field exist and are equal to 0 on closed paths. Our theorems relate path integrals to u-lengths of arcs. The following definitions will be needed.[5]

[5]Mathematical definitions and theorems used in this paper but not stated can be found in Royden [16].

A *path*[6] σ is a mapping $\sigma : [0,1] \to T$. A path $\sigma \in C^\infty$ is called *smooth*.[7] For simplicity of notation we drop the adjective *smooth* throughout the paper. We write $t \in \sigma$ if $t \in \sigma([0,1])$. A path is said to go from s to t if $\sigma(0) = s$ and $\sigma(1) = t$. Where relevant this is indicated by σ_{st}. T is called *path-connected* if any two types $s, t \in T$ are connected by a path. Our theorems will frequently refer to the following assumption:

Assumption 1. *The type space $T \subseteq \mathbb{R}^d$ ($d \geq 1$) is path connected.*

A *closed path* is a path such that $s = t$. Observe that closed paths can be parameterized by mappings $\sigma : S^1 \to T$, where S^1 is the unit cycle in $\mathbb{R}^2$.

We denote by $\sigma'(\lambda)$ the vector of derivatives of σ at λ. A vector field is a mapping $g : T \to \mathbb{R}^d$. Given a smooth path σ, the *path integral* of g along σ, if it exists, is defined as:

$$\int_\sigma g \cdot d\sigma = \int_0^1 g(\sigma(\lambda)) \cdot \sigma'(\lambda) d\lambda.$$

We denote by $L_{s,t} := \{s + \lambda(t - s) : \lambda \in [0,1]\}$ the line segment between two types $s, t \in T$, by $\blacktriangle_{s_1,s_2,s_3}$ the convex hull of $s_1, s_2, s_3 \in T$, all three distinct and by $\triangle_{s_1,s_2,s_3}$ the path describing the boundary of $\blacktriangle_{s_1,s_2,s_3}$, i.e $L_{s_1,s_2} \cup L_{s_2,s_3} \cup L_{s_3,s_1}$, with direction $s_1 \to s_2 \to s_3 \to s_1$.

A line segment $L_{s,t} \subseteq T$ is the image of the smooth path $\sigma(\lambda) = s + \lambda(t - s)$. We denote the path integral of a vector field g along $L_{s,t}$ by

$$\int_{L_{s,t}} g \cdot d\sigma = \int_{L_{s,t}} g(s + \lambda(t - s)) \cdot (t - s) d\lambda.$$

Consistently, we define

$$\int_{\triangle_{s_1,s_2,s_3}} \Psi \cdot d\sigma = \int_{L_{s_1,s_2}} \Psi \cdot d\sigma + \int_{L_{s_2,s_3}} \Psi \cdot d\sigma + \int_{L_{s_3,s_1}} \Psi \cdot d\sigma.$$

A continuous vector field Ψ is conservative if there exists a differentiable function $F : T \to \mathbb{R}^d$ such that $\Psi = \nabla F$. Equivalently, the path-integral of Ψ along every smooth path σ exists, and for all $s, t \in T$, path integrals for all paths from s to t are equal. Since we do not want to bother whether vectors fields as defined later are continuous, our theorems will be stated in terms of existence of path-integrals and their value on closed paths.

In each of the two settings studied in the following sections we will derive a specific vector field Ψ from an allocation rule f and valuations v. Given this Ψ the following definition provides the key property of implementable allocation rules.

Definition 4. *Given T, v and f, and vector field Ψ (to be specified later). f is called* path-monotone *if path-integrals of Ψ exist and for all s, t and all paths σ from s to t:*

$$l_u(s,t) \geq \int_\sigma \Psi \cdot d\sigma.$$

3 Characterizing Incentive Compatibility

In this section we fully characterize implementable allocation rules for two settings. First, we study the case where valuations for outcomes are differentiable functions of types. Second, we consider the case where valuations for outcomes are convex functions of types.

[6] The term *path* refers in this paper as well to paths in type graphs as to smooth paths in $\mathbb{R}^d$. The meaning will always be clear from the context.

[7] It will be sufficient throughout the paper that σ is a concatenation of finitely many paths all contained $\in C^1$, however we decided to refer always to smooth path in order to simplify notation.

3.1 Differentiable Valuations

We need here the following additional assumptions.

Assumption 2. *For any $a \in A$, $v(a,t) : T \to \mathbb{R}$ is differentiable in t.*

Assumption 3. *The function $\boldsymbol{r} : T \to \mathbb{R}$, $\boldsymbol{r}(t) \to \sup_{a \in A} \|\nabla_t v(a,t)\|$ is bounded on T.*

By assumption 2 the allocation rule f defines for all $t \in T$ a differentiable function $v(f(t),.) : T \to \mathbb{R}$, $s \mapsto v(f(t),s)$. By taking the gradient of this function at $s = t$ we get a vector field Ψ, more formally:

$$\begin{aligned} \Psi : T &\to \mathbb{R}^d \\ \Psi : t &\mapsto \nabla_s v(f(t),s)_{|s=t} \end{aligned}$$

The following theorem tells us that implementable rules are characterized by properties of Ψ. That the conditions in Theorem 2 are necessary for implementability can also be shown using a result by Milgrom and Segal [11], who show that under Assumptions 2 and 3, incentive compatible payments of implementable allocation rules satisfy (5). For the sake of presentation, we give a direct proof of that part of the theorem below. The main contribution of Theorem 2 is however to identify path-monotonicity as crucial ingredient in order to prove an equivalence.

Theorem 2. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \to \mathbb{R}$ satisfy Assumptions 1-3. Then, $f : T \to A$ is implementable if and only if path integrals of Ψ exist and*

(a) f is path-monotone,

(b) for every closed path σ

$$\int_\sigma \Psi \cdot d\sigma = 0.$$

Proof. $\Rightarrow$
Implementability implies existence of payments p such that:

$$v(f(s),s) + p(s) \geq v(f(t),s) + p(t) \quad \forall s,t \in T. \tag{2}$$

Fix $s,t \in T$ and a path σ from s to t. Define $M : [0,1] \to \mathbb{R}$ as:

$$M(\lambda) = \sup_{x \in \sigma} \{v\left(f(x), \sigma(\lambda)\right) + p(x)\}. \tag{3}$$

Since p implements f, the supremum is finite and

$$\sigma(\lambda) \in \arg\max_{x \in \sigma} \{v\left(f(x), \sigma(\lambda)\right) + p(x)\}, \tag{4}$$

i.e. the player maximizes his utility with declaring $\sigma(\lambda)$, given his true type is $\sigma(\lambda)$. Using (3), for any $\alpha_1, \alpha_2 \in [0,1]$ we have:

$$\begin{aligned} |M(\alpha_2) - M(\alpha_1)| &\leq \sup_{x \in \sigma} |v\left(f(x), \sigma(\alpha_2)\right) - v\left(f(x), \sigma(\alpha_1)\right)| \\ &= \sup_{x \in \sigma} |\int_{\alpha_1}^{\alpha_2} \frac{\partial}{\partial \lambda} v(f(x), \sigma(\lambda)) d\lambda| \\ &= \sup_{x \in \sigma} |\int_{\alpha_1}^{\alpha_2} \nabla_s v(f(x),s)_{|s=\sigma(\lambda)} \cdot \sigma'(\lambda) d\lambda| \\ &\leq |\alpha_2 - \alpha_1| \sup_{\lambda} \|\sigma'(\lambda)\| \sup_{x \in \sigma} \|\nabla_s v(f(x),s)_{|s=\sigma(\lambda)}\|. \end{aligned}$$

This implies that M is Lipschitz continuous and therefore differentiable almost everywhere[8] and

$$M(1) - M(0) = \int_0^1 M'(\lambda) d\lambda.$$

By the Envelope Theorem (we make use of the version in Milgrom and Segal [11]) it follows from (4) that for every $\lambda \in (0, 1)$:

$$\begin{aligned} M'(\lambda) &= \frac{\partial}{\partial \mu}(v(f(\sigma(\lambda)), \mu) + p(\sigma(\lambda)))_{|\mu=\sigma(\lambda)} \\ &= \frac{\partial}{\partial \mu} v(f(\sigma(\lambda)), \mu)_{|\mu=\sigma(\lambda)} \cdot \sigma'(\lambda) \\ &= \Psi(\sigma(\lambda)) \cdot \sigma'(\lambda). \end{aligned}$$

So, Ψ is integrable along the path σ and

$$\begin{aligned} \int_\sigma \Psi \cdot d\sigma &= M(1) - M(0) \\ &= v(f(t), t) + p(t) - v(f(s), s) - p(s). \end{aligned} \tag{5}$$

It follows that path integrals of Ψ on closed paths are equal to 0. Furthermore,

$$\begin{aligned} l_u(s, t) &= v(f(t), t) - v(f(t), s) \\ &= v(f(t), t) - v(f(t), s) + v(f(s), s) - v(f(t), s) \\ &\geq v(f(t), t) - v(f(t), s) + p(t) - p(s) \\ &= \int_\sigma \Psi \cdot d\sigma, \end{aligned}$$

where the inequality follows from (2). Hence f is path-monotone.

$\Leftarrow$)
Fix $x \in T$. For every $w \in T$ define the payments as:

$$p(w) = \int_{\sigma_{x,w}} \Psi \cdot d\sigma - v(f(w), w),$$

where $\sigma_{x,w}$ is an arbitrary path from x to w. Now for every $s, t \in T$ we have:

$$p(t) - p(s) = \int_{\sigma_{x,w}} \Psi \cdot d\sigma - v(f(t), t) - \int_{\sigma_{x,s}} \Psi \cdot d\sigma + v(f(s), s)$$

Since path-integrals of Ψ on closed paths are equal to 0 and $l_u(s, t) \geq \int_{\sigma_{s,t}} \Psi \cdot d\sigma$ we get

$$p(t) - p(s) \quad \leq \quad v(f(s), s) - v(f(t), s).$$

which means f is implementable. □

[8] In fact, every Lipschitz continuous function is absolutely continuous and therefore it is differentiable almost everywhere

Example 1 and Example 2 in the appendix show that condition (a) and (b) in Theorem 2 do not imply each other.

Suppose payments p and q both implement f. Fix $s \in T$ and let $k = q(s) - p(s)$. According to (5), for any $t \in T$ we have:

$$\begin{aligned} q(t) &= \int_{\sigma_{s,t}} \Psi \cdot d\sigma - v(f(t),t) + v(f(s),s) + q(s) \\ &= \int_{\sigma_{s,t}} \Psi \cdot d\sigma - v(f(t),t) + v(f(s),s) + p(s) + k \\ &= p(t) + k. \end{aligned}$$

This shows

Corollary 1. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \to \mathbb{R}$ satisfy Assumptions 1-3. If f is implementable, then any two payments that implement f differ by at most a constant.*

We conclude by observing that for convex T we may replace in Theorem 2 general path integrals by path integrals on line segments.

Corollary 2. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \to \mathbb{R}$ satisfy Assumptions 2 and 3 and let T be convex. Then, $f : T \to A$ is implementable if and only if for any $s, t \in T$, $\int_{L_{s,t}} \Psi \cdot d\sigma$ exists and the following hold:*

(*a*) *for any $s, t \in T$:*

$$l_u(s,t) \geq \int_{L_{s,t}} \Psi \cdot d\sigma,$$

(*b*) *for all $s_1, s_2, s_3 \in T$, all three distinct:*

$$\int_{\Delta_{s_1,s_2,s_3}} \Psi \cdot d\sigma = 0.$$

3.2 Convex Valuations

In this section we study the case where valuations for outcomes are convex functions of the type. We also assume T to be convex.

Assumption 4. *For any $a \in A$, $v(a,t) : T \to \mathbb{R}$ is convex in t.*

Assumption 5. *The type space $T \subseteq \mathbb{R}^d$ is a convex set.*

Recall that a vector $\nabla \in \mathbb{R}^d$ is a subgradient of a function $h : \mathbb{R}^d \to \mathbb{R}$ at t if $h(s) \geq h(t) + \nabla \cdot (s - t)$ for all $s \in T$. For every $t \in T$ allocation rule f defines a convex function $v(f(t),.) : T \to \mathbb{R}$, $s \mapsto v(f(t),s)$. We make the following technical assumption:

Assumption 6. *For every $t \in T$ and $a \in A$ the set of subgradients of $v(a,.)$ at t is nonempty.*

As convex functions have a subgradient at each point in the interior of T, this assumption restricts the set of valuations for which our results hold only in terms of their behavior on the boundary of T.

We can now define a vector field $\Psi : T \to \mathbb{R}^d$ by selecting for each $t \in T$ an element from the subgradient of $v(f(t),.)$ at t. Any such vector field satisfies for all $s, t \in T$

$$v(f(t),s) \geq v(f(t),t) + \Psi(t) \cdot (s - t). \tag{6}$$

The main result of this section is the following analogue of Corollary 2.

Theorem 3. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \to \mathbb{R}$ satisfy Assumptions 4, 5 and 6. Then, $f : T \to A$ is implementable if and only if for any $s, t \in T$, $\int_{L_{s,t}} \Psi \cdot d\sigma$ exists and the following hold:*

(a) for any $s, t \in T$:

$$l_u(s,t) \geq \int_{L_{s,t}} \Psi \cdot d\sigma$$

(b) for all $s_1, s_2, s_3 \in T$, all three distinct:

$$\int_{\Delta_{s_1,s_2,s_3}} \Psi \cdot d\sigma = 0$$

In order to proof Theorem 3 we start with a lemma that relates monotonicity to the existence of path integrals on line segments, which is of interest by its own.[9]

Lemma 1. *Let $s, t \in T$ and assume that $f : T \to A$ is monotone. Then the following hold for all $s, t \in T$:*

(a) $\Psi(s) \cdot (t - s) \leq l_u(s,t) \leq \Psi(t) \cdot (t - s)$,

(b) the path integral of Ψ on $L_{s,t}$ exists.

Proof. The first property follows immediately from monotonicity and the definitions of $l_u(s,t)$ and Ψ.

For the second property define $g : [0,1] \to \mathbb{R}$ by $g(\lambda) = \Psi(s + \lambda(t - s)) \cdot (t - s)$ and let $0 \leq \lambda_1 < \lambda_2 \leq 1$, $r_1 = s + \lambda_1(t - s)$ and $r_2 = s + \lambda_2(t - s)$. Then, by using monotonicity and Property 1, we get that

$$\begin{aligned} 0 &\leq l_u(r_1, r_2) + l_u(r_2, r_1) \\ &\leq \Psi(r_2) \cdot (r_2 - r_1) + \Psi(r_1) \cdot (r_1 - r_2) \\ &= (\lambda_2 - \lambda_1)(g(\lambda_2) - g(\lambda_1)), \end{aligned}$$

i.e. $g(\lambda_1) \leq g(\lambda_2)$. Since g is non-decreasing, g is integrable on $[0,1]$ and

$$\begin{aligned} \int_0^1 g(\lambda) d\lambda &= \int_0^1 \Psi(s + \lambda(t - s)) \cdot (t - s) d\lambda \\ &= \int_{L_{s,t}} \Psi \cdot d\sigma. \end{aligned}$$

Thus the path integral of Ψ along line segment $L_{s,t}$ exists. □

The following lemma establishes the relation between the path integral of Ψ and u-lengths of paths in the type graph of f, again for the case of monotone f.

Lemma 2. *Let $s, t \in T$ and assume that $f : T \to A$ is monotone. For every $n \geq 1$ we let $S_n = \sum_{i=0}^{n-1} l_u(r_i^n, r_{i+1}^n)$, where $r_k^n := s + \frac{k}{n}(t - s)$ for $0 \leq k \leq n$. Then*

$$\lim_{n \to \infty} S_n = \int_{L_{s,t}} \Psi \cdot d\sigma,$$

$$dist_u(s,t) \leq \int_{L_{s,t}} \Psi \cdot d\sigma.$$

[9] Archer and Kleinberg [1] make even in the case of linear valuations functions the assumption that the allocation rule is locally path integrable in order to get this property. Our lemma shows that this is not necessary.

Proof. Fix $n \geq 1$. According to Lemma 1 we have that for $0 \leq i \leq n-1$

$$\Psi(r_i^n) \cdot (r_{i+1}^n - r_i^n) \leq l_s(r_i^n, r_{i+1}^n) \leq \Psi(r_{i+1}^n) \cdot (r_{i+1}^n - r_i^n).$$

If we sum up the inequalities we get that

$$\sum_{i=0}^{n-1} \Psi(r_i^n) \cdot (r_{i+1}^n - r_i^n) \leq S_n \leq \sum_{i=0}^{n-1} \Psi(r_{i+1}^n) \cdot (r_{i+1}^n - r_i^n).$$

For every $n \in \mathbb{N}$ we define $L_n := \sum_{i=0}^{n-1} \Psi(r_i^n) \cdot (r_{i+1}^n - r_i^n)$ and $U_n := \sum_{i=0}^{n-1} \Psi(r_{i+1}^n) \cdot (r_{i+1}^n - r_i^n)$. Since Ψ is path-integrable on the path $L_{s,t}$ we have that

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = \int_{L_{s,t}} \Psi \cdot d\sigma.$$

Furthermore, since $L_n \leq S_n \leq U_n$, we conclude that

$$\lim_{n\to\infty} S_n = \int_{L_{s,t}} \Psi \cdot d\sigma.$$

The second property follows from the fact that:

$$dist_u(s,t) \leq S_n, \quad \forall n.$$

□

We are now ready to prove Theorem 3.

Proof. $\Rightarrow$
As f is implementable it is monotone. From Lemma 1 it follows that for any $s, t \in T$, the path integral of Ψ exists and

$$dist_u(s,t) \leq \int_{L_{s,t}} \Psi \cdot d\sigma.$$

As $dist_u(s,t) < \int_{L_{s,t}} \Psi \cdot d\sigma$ would imply $dist_u(t,s) + dist_u(s,t) < 0$, which would contradict cyclical monotonicity, we get:

$$dist_u(s,t) = \int_{L_{s,t}} \Psi \cdot d\sigma,$$

Since $l_u(s,t) \geq dist_u(s,t)$, we conclude (a).
To show *(b)* observe that

$$\int_{\Delta_{s_1,s_2,s_3}} \Psi \cdot d\sigma \quad = \quad dist_u(s_1,s_2) + dist_u(s_2,s_3) + dist_u(s_3,s_1) \geq 0$$

By the same reasoning

$$-\int_{\Delta_{s_1,s_2,s_3}} \Psi \cdot d\sigma \geq 0.$$

This proves (b).
$\Leftarrow$) is proven in the same way as in Theorem 2. □

Example 2 and Example 3 in the Appendix show that (a) and (b) do not imply each other. In Heydenreich et al. [7] it is shown that for any implementable rule f revenue equivalence holds if and only if $dist_p(s,t) = -dist_p(t,s)$ in T_f. By the relation between p-lengths and s-lengths, the same characterization can be stated in terms of distances with respect to u-lengths. From the proof of Theorem 3, we know that:

$$dist_u(s,t) + dist_u(t,s) = 0.$$

Thus we get as a corollary a revenue equivalence result by Krishna and Maenner.

Corollary 3. (Krishna and Maenner [10]) *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \to \mathbb{R}$ satisfy Assumptions 4, 5 and 6. If f is implementable, then any two payments that implement f differ by at most a constant.*

This result of Krishna and Maenner provides actually as well a means of proving $\Rightarrow$ of Theorem 3, however in a less elementary way.

4 Monotonicity, Path-Monotonicity and Linear Valuations

We assume in this section that we are either in the setting of Section 3.1 with the additional assumption that T is convex, or in the setting of Section 3.2. In either setting we assume that path integrals of Ψ exist.

Obviously path-monotonicity implies that f is monotone, but monotonicity does not always imply path-monotonicity (see Example 3 in the appendix). This raises the question when it does, in other words, when can we replace condition (a) in Theorems 2 and 3 and Corollary 2 by monotonicity? The key will be decomposition monotonicity.

We recall first that under the assumption of decomposition monotonicity, all monotone allocation rules are implementable when restricted to line segments.

Lemma 3. *Let $T \subseteq \mathbb{R}^n$, A a set of outcomes, $v : T \times A \to \mathbb{R}$, and $f : T \to \mathbb{R}$ be decomposition monotone. Let $s, t \in T$. Then f is implementable on $L_{s,t}$ if and only if f is monotone on $L_{s,t}$.*

The proof is very straightforward. It uses the simple observation that the length of any cycle with nodes in $L_{s,t}$ can be lower-bounded by the sum of lengths of two-cycles. The latter is non-negative if f is monotone.

Now assume that we are in the setting of Corollary 2 or Theorem 3, and f is decomposition monotone. By Lemma 3 we know that a monotone f is implementable on lines. By Corollary 2 and Theorem 3, respectively, we get that f must be path-monotone when restricted to such lines. This proves:

Lemma 4. *Let T, v and f be such that Assumption 5 and either Assumptions 2 and 3 or Assumptions 4 and 6 hold, and let f be decomposition monotone. Then f is monotone if and only if it is path-monotone.*

This yields the following versions of Corollary 2 and Theorem 3.

Theorem 4. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \to \mathbb{R}$ and $f : T \to A$ satisfy Assumptions 2, 3 and 5, and assume $f : T \to A$ is monotone and decomposition monotone. Then path integrals of Ψ exist. Furthermore f is implementable if and only if for all $s_1, s_2, s_3 \in T$, all three distinct*

$$\int_{\Delta_{s_1,s_2,s_3}} \Psi \cdot d\sigma = 0.$$

Theorem 5. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \to \mathbb{R}$ and $f : T \to A$ satisfy Assumptions 4, 5 and 6, and assume $f : T \to A$ is monotone and decomposition monotone. Then for any $s, t \in T$, $\int_{L_{s,t}} \Psi \cdot d\sigma$ exists. Furthermore f is implementable if and only if for all $s_1, s_2, s_3 \in T$, all three distinct*

$$\int_{\Delta_{s_1,s_2,s_3}} \Psi \cdot d\sigma = 0.$$

Decomposition monotonicity puts a condition on f. It is illustrative to understand which conditions on T, v, and A let this condition hold for any monotone f.

Definition 5. *Let T be convex, A a set of outcomes, and $v : T \times A \to \mathbb{R}$. We say that the* increasing difference property *holds if for all $s, t \in T$, $x \in L_{s,t}$, $a, b \in A$, we have that $v(b, t) - v(a, t) \geq v(b, x) - v(a, x)$ implies $v(b, x) - v(a, x) \geq v(b, s) - v(a, s)$).*

We note that the increasing difference property is implied by the well-known single-crossing property, though the definition of the latter requires assumptions on differentiability which we do not need to make here.

A simple verification shows that the following lemma holds:

Lemma 5. *Let T be convex, A a set of outcomes, and $v : T \times A \to \mathbb{R}$ such that the increasing difference property holds. Then every monotone f is decomposition monotone.*

Let us finally consider the case where $v(a, t) : T \to \mathbb{R}$ is linear in t and T is convex. Obviously this is a special case of differentiable valuations and convex valuations. Furthermore, it is easy to see that the increasing difference property holds in such a setting and Theorem 5 applies.

Corollary 4. *Let T be convex. Assume that for every fixed $a \in A$ the function $v(a, .) : T \to \mathbb{R}$ is linear. Let $f : T \to A$ be monotone. Then for any $s, t \in T$, $\int_{L_{s,t}} \Psi \cdot d\sigma$ exists. Furthermore, f is implementable if and only if for all $s_1, s_2, s_3 \in T$, all three distinct*

$$\int_{\Delta_{s_1,s_2,s_3}} \Psi \cdot d\sigma = 0.$$

Theorems in the spirit of Corollary 4 have been obtained by Jehiel and Moldovanu [8], Müller et al. [12], and Archer and Kleinberg [1]. Wolf [19] provided an alternative proof of Theorem 5.

5 Local Implementability

Motivated by results in Archer and Kleinberg [1] for the case of valuations that are linear in the type, we introduce in this section the notion of *local implementability.* We want to understand, when local implementability implies global implementability. This understanding will be crucial to show that in the case of finite A (path-) monotonicity alone implies implementability. We are aware that we use terminology from topology and differential geometry quite freely. It is beyond the scope of this paper to work out the subtle differences between, e.g., connectedness of a set T with respect to smooth paths and connectedness with respect to continuous paths. We believe however, that in most cases of economic relevance both notions will be equivalent. Having this said we use the following definition.

Definition 6. *$T \subseteq \mathbb{R}^d$ is said to be* simply connected *if for every closed smooth path $\sigma : S_1 \to T$ there exists a smooth extension $\tau : D_2 \to T$ from the unit disk $D_2 \subseteq \mathbb{R}^2$ onto T such that $\tau_{|S_1} = \sigma$.*

Definition 7. *An allocation rule* $f : T \to A$ *is called* locally implementable *if for every* $t \in T$ *there exists an open neighborhood* $U(t)$ *such that* $f : T \cap U(t) \to f(U(t) \cap T)$ *is implementable.*

Theorem 6. *Let* $T \subseteq \mathbb{R}^d$ *and* $v : A \times T \to \mathbb{R}$ *satisfy Assumptions 1, 2 and 3. Furthermore, assume that* T *is simply connected. Then,* $f : T \to A$ *is implementable if and only if path integrals of* Ψ *exist and:*

1. f *is path-monotone,*
2. f *is locally implementable.*

Proof. Applying Theorem 2 proves $\Rightarrow$. In order to prove the other direction, it is, due to the same Theorem, sufficient to show that (2) implies

$$\int_{\sigma} \Psi \cdot d\sigma = 0.$$

for any closed path σ.

Fix $\sigma : S_1 \to T$, and consider an extension τ of σ to D_2. Since $\tau(D_2)$ is closed and bounded it is compact. Local implementability implies that for every $t \in \tau(D_2)$ there exists $\varepsilon(t)$ and an open neighborhood $U(t, \varepsilon(t))$ such that f is implementable on $U(t, \varepsilon(t)) \cap T$. For every $x \in D_2$ there exists a $\delta(x)$ such that $\tau(U(x, \delta(x)) \cap D_2) \subseteq U(\tau(x), \varepsilon(\tau(x)))$.

The Lebesgue Number Lemma (see, e.g. [13]) implies that there is a δ' such that every subset of D_2 of diameter less than δ' is contained in at least one of the sets $U(x, \delta(x))$, $x \in D_2$. Now partition D_2 by a grid of which each cell has diameter strictly smaller than δ'. This decomposes D_2 into cells (of which some have a part of S_1 as border) with the property that any border B of these cells is completely contained in $U(x, \delta(x))$ for some x. The path $\tau(B)$ is contained in the neighborhood $U(\tau(x), \varepsilon(\tau(x)))$. As f is implementable on this neighborhood the path integral of Ψ with respect to this path is equal to 0.

By construction the path integral of Ψ with respect to σ equals the sum of the path integrals on the border of the cells, as path integrals on lines originating from the grid cancel each other out. □

Note that every convex set T is simply connected. Using the same arguments as in the proof of Theorem 6 we get an analogue theorem for the setting from Section 3.2.

Theorem 7. *Let* $T \subseteq \mathbb{R}^d$ *and* $v : A \times T \to \mathbb{R}$ *satisfy Assumptions 4, 5, and 6. Then,* $f : T \to A$ *is implementable if and only if for any* $s, t \in T$ $\int_{L_{s,t}} \Psi \cdot d\sigma$ *exists and:*

1. *for any* $s, t \in T$*:*
$$l_u(s, t) \geq \int_{L_{s,t}} \Psi \cdot d\sigma.$$
2. f *is locally implementable.*

Obviously, there are similar variants of the characterization theorems in Section 4. We state only the following.

Theorem 8. *Let* T *be convex. Assume that for every fixed* $a \in A$ *the function* $v(a, .) : T \to \mathbb{R}$ *is linear. Then* f *is implementable if and only if it is monotone and locally implementable.*

Theorem 8 has previously been proven in Archer and Kleinberg [1]. However, their proof needed the assumption that path integrals on line segments exist. We have shown in Lemma 1 that this property follows from monotonicity, even in the case of convex valuation functions.

6 Finite Outcome Space

Saks and Yu [17] were the first to show that in case of finite A, convex T and valuations that are linear in types, monotonicity alone implies already implementability. Such a result simplifies significantly the identification of implementable rules. We show in this section that such simplification is not limited to the linear case, but works for all other settings studied in the previous sections. Key will be the following lemma, which has been proven for the case of linear valuations in Ashlagi et al. [2].

Lemma 6. *Let A be finite and $T \subseteq \mathbb{R}^d$ and $v : A \times T \to \mathbb{R}$ be continuous in t for fixed a. For all $a \in A$ let*

$$D_a := \overline{f^{-1}(a)}.$$

If $f : T \to A$ is monotone and $\bigcap_{a \in A} D_a \neq \varnothing$, then f is implementable.[10]

Proof. Let $\{s_1, \ldots, s_k\} \subseteq T$ for some $k \geq 3$ and $t \in \bigcap_{a \in A} D_a$. Fix $1 \leq i \leq k$. Since $t \in D_{f(s_{i+1})}$, there is a sequence $(t_j)_{j \in \mathbb{N}}$, such that $f(t_j) = f(s_{i+1})$ for every $j \in \mathbb{N}$ and $\lim_{j \to \infty} t_j = t$.[11] Note that

$$\begin{aligned} l_p(s_i, s_{i+1}) &= v(f(s_i), s_i) - v(f(s_{i+1}), s_i) = v(f(s_i), s_i) - v(f(t_j), s_i) \\ &\geq v(f(s_i), t_j) - v(f(t_j), t_j) = v(f(s_i), t_j) - v(f(s_{i+1}), t_j). \end{aligned}$$

By continuity of v in t we get

$$l_p(s_i, s_{i+1}) \geq v(f(s_i), t) - v(f(s_{i+1}), t).$$

If we sum up all inequalities ,we have:

$$\sum_{i=1}^{k} l_p(s_i, s_{i+1}) \geq \sum_{i=1}^{k} v(f(s_i), t) - v(f(s_{i+1}), t) = 0.$$

Invoking Theorem 1 completes the proof. □

Theorem 9. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \to \mathbb{R}$ satisfy Assumptions 1, 2 and 3. Furthermore let T be simply connected and A be finite. Then, $f : T \to A$ is implementable if and only if path integrals of Ψ exist and f is path-monotone.*

Proof. We prove that f is locally implementable, then according to Theorem 6 f is implementable. Fix $t \in T$. For all $a \in A$ let $\varepsilon_a(t) := \inf_{x \in D_a} \|x - t\|_2$ if $D_a \neq \emptyset$ and $\varepsilon_a(t) = \infty$ otherwise . Then,

$$t \in D_a \Leftrightarrow \varepsilon_a(t) = 0.$$

We show existence of a neighborhood $U(t)$ around t such that $t \in D_a$ for all $a \in f(U(t))$. Set $A(t) := \{a \in A : \varepsilon_a(t) = 0\}$. As $t \in D_{f(t)}$, we have that $A(t) \neq \emptyset$ and $t \in \bigcap_{a \in A(t)} D_a$. If $A(t) = A$ we let $U(t) = \mathbb{R}^d$, otherwise let

$$\varepsilon = \min\{\varepsilon_a(t) : a \in A \setminus A(t)\}.$$

Note that $\varepsilon > 0$. Define $U(t) = \{x \in \mathbb{R}^d : \|x - t\|_2 < \varepsilon\}$. Since $v(a, t)$ is continuous in t for all a, we can invoke Lemma 6 to prove that f is implementable on $U(t)$. In other words, f is locally implementable. □

[10] $\overline{X}$ denotes the topological closure of a set $X \subseteq \mathbb{R}^d$.

[11] Indices are taken modulo k.

Theorem 10. *Let $T \subseteq \mathbb{R}^d$ and $v : A \times T \to \mathbb{R}$ satisfy Assumptions 4, 5 and 6. Furthermore let A be finite. Then, $f : T \to A$ is implementable if and only if for any $s, t \in T$, $\int_{L_{s,t}} \Psi \cdot d\sigma$ exits and*

$$l_u(s,t) \geq \int_{L_{s,t}} \Psi \cdot d\sigma.$$

Proof. Analogues to the proof of Theorem 9. □

In light of the results from Section 4 we get the Theorem by Saks and Yu as a corollary.

Corollary 5. (Saks and Yu [17])
Let T be convex. Assume that for every fixed $a \in A$ the function $v(a,.) : T \to \mathbb{R}$ is linear. Then f is implementable if and only if it is monotone.

Using the Lebesque Number Lemma, one can easily see that under the additional assumption of decomposition monotonicity, monotonicity is implied by local monotonicity. Due to Theorem 8 it will then be sufficient to verify that monotonicity holds in some neighborhood of each type. Indeed, if this is the case, we get local implementability.

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Appendix

In this appendix we give various examples of valuations functions and allocation rules that demonstrate that natural generalizations of our theorems do not hold, i.e. we show that in the various settings none of the conditions that characterize implementability can be omitted.

Example 1. *This example shows that condition (b) in Theorem 2 does not imply path-monotonicity of an allocation rule in the case of differentiable valuations.*

Suppose $T = [0,1]$. Consider the following allocation rule:

$$f(t) = \begin{cases} a & 0 \leq t \leq \frac{1}{3} \\ b & \frac{1}{3} < t \leq \frac{2}{3} \\ c & \frac{2}{3} < t \leq 1. \end{cases}$$

and valuation functions: $v(a,t) = 3t$, $v(b,t) = -3t$ and $v(c,t) = 1 - 9t^2$. It is

$$\Psi(t) = \begin{cases} 3 & 0 \leq t \leq \frac{1}{3} \\ -3 & \frac{1}{3} < t \leq \frac{2}{3} \\ -18t & \frac{2}{3} < t \leq 1. \end{cases}$$

While path integrals of Ψ on closed paths are equal to 0, the rule f is not path-monotone as

$$-9 = l_u(0,1) < \int_{L_{0,1}} \Psi \cdot d\sigma = -5.$$

Example 2. *This example shows that path-monotonicity does not imply condition (b) in Theorems 2 and 3 in the case of either differentiable or convex valuations.*

Let $s = (1, 0, 0)$, $t = (0, 1, 0)$ and $u = (0, 0, 1)$ and let $T \subseteq \mathbb{R}^3$ be the convex hull of $\{s, t, u\}$. For any $\mathbf{t} \in T$, $\mathrm{f}(\mathbf{t}) = \mathbf{t}$ and the valuation functions are given by:

$$v(\mathbf{a}, \mathbf{t}) = (a_1 + a_2)t_1 + (a_2 + a_3)t_2 + (a_1 + a_3)t_3,$$

where a_i and t_i are the i-th component of the vector a and t, respectively. The valuations are linear and thus also differentiable and convex. We now have that

$$\Psi(\mathbf{t}) = (t_1 + t_2, t_2 + t_3, t_1 + t_3).$$

While f is path-monotone, the integral of Ψ over $\triangle_{s,t,u}$ is not equal to zero:

$$\begin{aligned}\int_{\triangle_{s,t,u}} \Psi \cdot d\sigma &= \int_{L_{s,t}} \Psi \cdot d\sigma + \int_{L_{t,u}} \Psi \cdot d\sigma + \int_{L_{u,s}} \Psi \cdot d\sigma \\ &= \frac{-1}{2} + \frac{-1}{2} + \frac{-1}{2} \\ &= \frac{-3}{2}.\end{aligned}$$

Example 3. *The following example shows that condition (b) in Theorem 3 does not imply path-monotonicity of an allocation rule for convex valuations. It shows as well that the result of Saks and Yu [17] cannot be extended to convex valuations without adding the condition of decomposition monotonicity. The given allocation rule with a finite outcome space is monotone, but not path-monotone and not implementable.*

Suppose $T = [0, 1]$. Consider the following allocation rule:

$$f(t) = \begin{cases} a & 0 \le t \le \frac{1}{3} \\ b & \frac{1}{3} < t \le \frac{2}{3} \\ c & \frac{2}{3} < t \le 1. \end{cases}$$

Define the valuation functions by

$$v(a, t) = \begin{cases} 0 & t \le \frac{2}{3} \\ 3t - 2 & t > \frac{2}{3}, \end{cases}$$

$v(b, t) = 3t$ and

$$v(c, t) = \begin{cases} 2 - 3t & t \le \frac{1}{3} \\ 3t & t > \frac{1}{3}. \end{cases}$$

Then we have that

$$\Psi(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{3} \\ 3 & \frac{1}{3} < t \le 1. \end{cases}$$

While condition (b) in Theorem 3 is satisfied, f is not path-monotone:

$$1 = l_u(0, 1) < \int_{L_{0,1}} \Psi \cdot d\sigma = 2.$$

Example 4. *Our final example shows that decomposition monotonicity is not a necessary condition for an allocation rule to be implementable in the case of convex valuations. The rule given below is implementable and hence monotone, but not decomposition monotone.*

Suppose $T = [0, 1]$. Consider the following allocation rule:

$$f(t) = \begin{cases} a & 0 \leq t \leq \frac{1}{8} \\ b & \frac{1}{8} < t \leq \frac{3}{8} \\ c & \frac{3}{8} < t \leq \frac{5}{8} \\ b & \frac{5}{8} < t \leq \frac{7}{8} \\ a & \frac{7}{8} < t \leq 1. \end{cases}$$

We define the corresponding valuation functions: $v(a, t) = \mid 2t - 1 \mid$,

$$v(b, t) = \begin{cases} 1 - t & t \leq \frac{1}{2} \\ t & t > \frac{1}{2} \end{cases}$$

and $v(c, t) = 1$. It is easy to check that the function is monotone and that it does not satisfy decomposition monotonicity, but that it is implementable.

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