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Operators for the adjudication of conflicting claims

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Abstract

We consider the problem of allocating some amount of an infinitely divisible and homogeneous resource among agents having claims on this resource that cannot be jointly honored. A "rule" associates with each such problem a feasible division. Our goal is to uncover the structure of the space of rules. For that purpose, we study "operators" on the space, that is, mappings that associate to each rule another one. Duality, claims truncation, and attribution of minimal rights are the operators we consider. We first establish a number of results linking them. Then, we determine which properties of rules are preserved under each of these operators, and which are not.

JEL classification: C79; D63; D74

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1. Introduction

We address the problem of allocating an endowment of an infinitely divisible and homogeneous resource among agents having claims on this resource that cannot be jointly honored. A primary example is when the liquidation value of a bankrupt firm has to be allocated among its creditors. A "division rule" is a function that associates with each situation of this kind, which we call a "claims problem," a division of the endowment. We call this division an "awards vector." It is interpreted as the choice that a judge or arbitrator could make. In the search for the most

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desirable rules, the literature,[1] initiated by O'Neill [12], has proceeded on several fronts, much recent progress having been made on the axiomatic front.

We will consider the issue from a higher perspective than is standard however, and examine the space of rules itself, our goal being to contribute to the understanding of its structure. When surveying the literature, one is struck by the richness of the inventory of rules that have been proposed. Such richness is intriguing but also confusing, and one feels the need to organize this inventory in some fashion. Several approaches can be taken for that purpose. The first approach simply consists in searching for resemblances between rules, in the formulas or algorithms defining them, and in the geometry of their graphs. Rules can be usefully organized in families exploiting these resemblances. The parametric family introduced by Young [25], as well as certain families defined by Thomson [21], collect a number of important rules that can be described in a common way. The identification of these families allows us to relate rules to one another, and also to understand what is unique to each of them. A second approach is to organize rules in classes by referring to the properties they share. Axiomatic analysis is the principal methodology here. Of course, these two approaches are related. The fact that a general formula can be written down to gather rules among which one has recognized patterns will underlie why all members of the family share certain properties.

The approach we follow here is based on a third way of relating rules. It exploits and generalizes a phenomenon one quickly notices, namely that one can often pass from one rule to another by means of a simple algebraic or geometric operation. Let us define an "operator" on the space of rules as a mapping that associates with each rule another one. We propose to undertake a systematic study of such objects. We consider three of them. First is a duality operator. When looking at a claims problem, two perspectives can be taken: we can think of the issue as dividing what is available; or, as dividing the deficit (the difference between the sum of the claims and the endowment). Let S be a rule. The rule associated with S by the duality operator, its "dual," treats what is available in the same way as S treats the deficit. The second operator associates with S the rule defined for each problem by first truncating claims at the endowment and then applying S to the problem so revised. The rule associated with S by the third operator calculates the awards vector for each problem in two steps: first, each claimant is attributed the difference between the endowment and the sum of the claims of the other agents (or 0 if this difference is negative); this difference is an obvious minimum to which he is entitled; second, S is applied to allocate what remains, the part that is truly contested, claims being adjusted down by the "minimal rights" of the first step.

We uncover a number of links between the three operators. Obviously the duality operator composed with itself is the identity; also, the claims truncation operator composed with itself is equivalent to itself; somewhat less obvious is that a similar statement holds for the attribution of minimal rights operator. We then show that if two rules are dual, then the version of one obtained by subjecting it to the attribution of minimal rights operator is dual to the version of the other obtained by subjecting it to the claims truncation operator. Next, we study the composition of the claims truncation and attribution of minimal rights operators (a composition on which a rule suggested by Curiel et al. [5] is based). We show that the order in which they are composed does not matter: the rule that results is independent of the order. Second, in the two-claimant case, starting from any two rules satisfying the basic property that agents with equal claims should receive equal amounts, subjecting them to the composition of the two operators always

[1] For surveys, see Thomson [22,24].

produces the same rule. Third, this rule is not just any rule, but it is one that has been central in the literature. We refer to it as "concede-and-divide" because it emerges from the following natural two-step scenario: each claimant first concedes to the other the difference between the endowment and his own claim (or 0 if this difference is negative); what remains, the part that we described earlier as being truly contested, is divided equally (Aumann and Maschler [1]; the name is proposed by Thomson [22]).

Given a property that a rule may have, a natural question is whether the property is also satisfied by the rule obtained by subjecting it to a certain operator. The fact that a property is preserved under an operator is an interesting and very useful feature it may have. We show that, of the properties that have been frequently discussed in the literature, most are preserved under the duality operator, but our main results concerning this operator pertain to two basic monotonicity properties, which somewhat surprisingly, are not. One is "claims monotonicity": if an agent's claim increases, his award should be at least as large as it was initially. The other is "population monotonicity": upon the arrival of additional claimants, the award to each claimant initially present should be at most as large as it was initially.

Next, we turn to the claims truncation and attribution of minimal rights operators. These operators tend to be more disruptive, but they are disruptive in "symmetric" ways. We also study their composition and find that the central property of "self-duality"—invariance under the duality operator—which is preserved by neither operator, is preserved under their composition.

Our results have a number of benefits. First, as was our goal, they allow us to structure the existing inventory of rules available to solve claims problems, and to help ensure that no important rule has been missed. The structural relations between the operators we uncover also allow us to provide easy proofs that certain properties hold for particular rules (examples are the properties established by Curiel et al. [5] for the rule they define), and they should also be useful in identifying which properties each newly constructed rule may or may not satisfy. Finally, the operators—the duality operator is particularly useful in this regard—allow us to derive new characterizations from existing ones. (For an earlier example of such a derivation, see Herrero and Villar [8].) Altogether, they should help clarify the literature and keep it organized as it develops further.

2. Model

There is a finite set of **claimants**, N. Each agent $i \in N$ has a **claim** $c_i \in \mathbb{R}_+$ over an **endowment** $E \in \mathbb{R}_+$. The endowment is insufficient to honor all of the claims. Altogether, a **claims problem** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_N c_i \geqslant E$.[2] Let $\boldsymbol{\mathcal{C}^N}$ denote the class of all claims problems. An **awards vector of (c, E)** is a point of $\mathbb{R}^N_+$ bounded above by c and whose coordinates add up to E, a condition we call "efficiency." Let $X(c, E)$ be the set of awards vectors of (c, E). A **rule** is a function defined on $\mathcal{C}^N$ that associates with each $(c, E) \in \mathcal{C}^N$ an awards vector of (c, E). Let S be our generic notation for rules. For the two-claimant case, a rule can be conveniently described in a two-dimensional space, for each claims vector, by means of the path followed by the awards vector as the endowment increases from 0 to the sum of the claims. We refer to this path as the **path of awards of the rule for this claims vector**. We denote by $p(S, c)$ the path of awards of S for c.

[2] By the notation $\mathbb{R}^N$ we mean the Cartesian product of $|N|$ copies of $\mathbb{R}$ indexed by the members of N. Vector inequalities: $x \geqslant y$, $x \geqslant y$, and $x > y$.

We also consider a variable-population version of the model. There is a population of "potential" claimants, either $\mathbb{N}$, the set of natural numbers, or some subset of it. However, only a finite number of claimants are present at any given time. Let $\mathcal{N}$ be the class of finite subsets of the set of potential claimants. To specify a claims problem, we first choose $N \in \mathcal{N}$, then $(c, E) \in \mathcal{C}^N$. A rule is a function defined over $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$, which associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, an awards vector of (c, E).

3. Operators

Next, we define our operators, the duality operator, $\boldsymbol{O^d}$, the claims truncation operator, $\boldsymbol{O^t}$, and the attribution of minimal rights operator, $\boldsymbol{O^m}$. Then, we illustrate them by means of examples. Given any rule S, the rule obtained by subjecting it to operator O^p, for $p = d, t$, or m, is denoted S^p.

1. Duality. The dual of a rule S treats what is available for division in the same way as S treats what is missing. Formally, given $(c, E) \in \mathcal{C}^N$, we replace E by $\sum c_i - E$; we use S to divide this difference, and then subtract the result from c. The idea is suggested by Aumann and Maschler [1], who provide motivation for it, as well as note passages in the Talmud to support their thesis that its seed was already there:

Dual of $\boldsymbol{S}$, $\boldsymbol{S^d}$: For each $(c, E) \in \mathcal{C}^N$, $S^d(c, E) \equiv c - S(c, \sum c_i - E)$.

It is easy to check that the pair $(c, \sum c_i - E)$ is a well-defined problem and that S^d is a well-defined rule. The operator O^d has a convenient geometric interpretation: for each $c \in \mathbb{R}^N_+$, $p(S, c)$ and $p(S^d, c)$ are symmetric of each other with respect to $\frac{c}{2}$. Also, since $(S^d)^d = S$, we can speak of rules being "dual." Examples of dual rules are the **constrained equal awards rule**, ***CEA***, which equates the amounts received by all claimants subject to no one receiving more than his claim, and the **constrained equal losses rule**, ***CEL***, which equates the losses experienced by all claimants subject to no one receiving a negative amount: formally, $CEA(c, E) \equiv (\min\{c_i, \alpha\})_{i \in N}$, and $CEL(c, E) \equiv (\max\{0, c_i - \alpha\})_{i \in N}$, where in each case, $\alpha \in \mathbb{R}_+$ is chosen so as to achieve efficiency. Fig. 1a illustrates the definitions, and this duality, for $|N| = 2$.[3]

A rule is **self-dual** if it treats the problem of dividing what is available symmetrically to the problem of dividing what is missing (Aumann and Maschler [1]).[4] To say that a rule is *self-dual* is to say that it is invariant under O^d. For such a rule S, and for each $c \in \mathbb{R}^N_+$, $p(S, c)$ is symmetric with respect to $\frac{c}{2}$. A number of rules are *self-dual*. An obvious example is the **proportional rule**, ***P***, which chooses awards proportional to claims. However, other important rules share this property. One of them is the **Talmud rule**, ***Tal***, (Aumann and Maschler [1]), which can be described as a hybrid of CEA and CEL: if $E \leqslant \frac{\sum c_i}{2}$, $Tal(c, E) = CEA(\frac{c}{2}, E)$ and otherwise, $Tal(c, E) = \frac{c}{2} + CEL(\frac{c}{2}, E - \frac{\sum c_i}{2})$. Another is the **random arrival rule**, ***RA*** (O'Neill [12]; see Thomson, [24], for a proof), which assigns to each claimant the expected value of what he would

[3] We write the formal definitions of rules for the fixed-population case. To obtain their variable-population versions, it suffices to add a universal quantification over N.

[4] Aumann and Maschler [1] note a number of passages in the Talmud where the idea that the two perspectives should be equivalent is implicit.

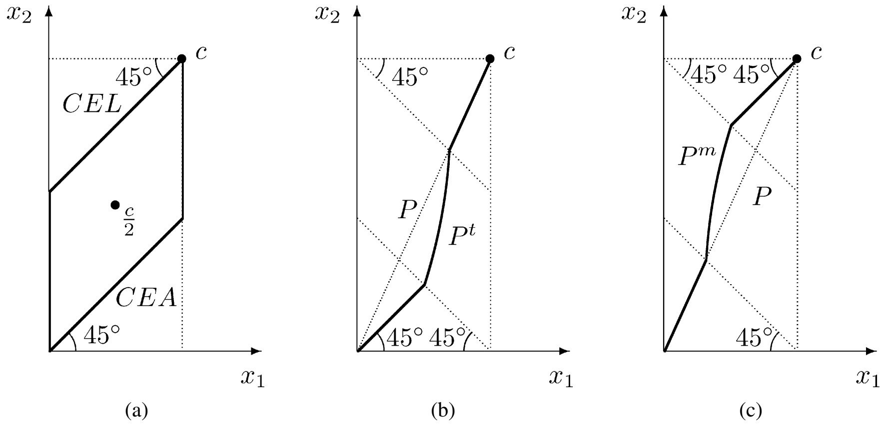

Fig. 1. Illustrating the three operators. (a) The operator O^d applied to *CEA* produces *CEL*: for each $c \in \mathbb{R}^N_+$, $p(CEA, c)$ and $p(CEL, c)$ are symmetric of each other with respect to $\frac{c}{2}$. (b) The operator O^t applied to P. (c) The operator O^m applied to P.

obtain on a first-come first-serve basis, assuming that all orders of arrival of claimants occur with equal probabilities.[5]

2. Claims truncation. The second operator truncates claims: given $(c, E) \in \mathcal{C}^N$, each claim that is greater than E is replaced by E. The operator O^t is critical for the study of claims problems as "games with transferable utility" (O'Neill [12]). Indeed, if a rule is such that for each problem, the awards vector it recommends is the payoff vector chosen by a solution to TU games for the game associated with the problem in the manner first suggested by O'Neill [12][6], then it is invariant under O^t (Curiel et al. [5]). Formally, for each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, let $t_i(c, E) \equiv \min\{c_i, E\}$ denote **agent *i*'s truncated claim at the endowment**, and $t(c, E) \equiv (t_i(c, E))_{i \in N}$ the vector of truncated claims. Fig. 1b illustrates O^t applied to P for $|N| = 2$.

Rule *S* operated from truncated claims, S^t: For each $(c, E) \in \mathcal{C}^N$, $S^t(c, E) \equiv S(t(c, E), E)$.

The inequality between $\sum c_i$ and E is not reversed by the truncation: after carrying it out, we still have a well-defined claims problem.

[5] For references to the relevant ancient literature, see O'Neill [12], Aumann and Maschler [1], Young [25], and Dagan [6]. Both *CEA* and *CEL* are discussed by Maimonides. Proportionality is explicitly advocated by Aristotle as the basis for "just" distribution. The Talmud rule is defined by Aumann and Maschler [1] to rationalize numerical examples given in the Talmud. We should also mention the "minimal overlap rule," *MO* (O'Neill [12]; Chun and Thomson [4]), which calculates awards by arranging claims over the endowment so as to minimize in a lexicographic way the extent to which they conflict, and then dividing each unit equally among all agents claiming it. Remarkably, *RA*, *MO*, and *Tal* all coincide for $|N| = 2$; moreover, they coincide with "concede-and-divide," defined below. For further discussion of these relationships, see Thomson [22]. We will see below that many other rules share this feature.

[6] Given $(c, E) \in \mathcal{C}^N$, and $S \subseteq N$, the "worth of S" is defined to be $\max\{E - \sum_{i \in N \setminus S} c_i, 0\}$. "Correspondences" between rules and solutions to coalitional games have proved to be very useful tools in the literature on the problem of claims resolution.

If a rule is invariant under O^t, it is **invariant under claims truncation**: for each $(c, E) \in \mathcal{C}^N$, one can equivalently calculate the awards vector (i) directly, or (ii) after truncating claims at E (Dagan [6], Herrero and Villar [9]).

3. Attribution of minimal rights. Given $(c, E) \in \mathcal{C}^N$ and $i \in N$, it is natural to think of the difference $E - \sum_{N\setminus\{i\}} c_j$ (or 0 if this difference is negative), as a minimal amount that agent i can reasonably expect. There should be no dispute about this payment. Given any rule S, a version of it can be defined by first attributing to each claimant his minimal amount; then after adjusting all claims down by these "first-round awards," applying S to divide the remainder. This remainder is what is truly disputed. Formally, for each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, let $m_i(c, E) \equiv \max\{E - \sum_{N\setminus\{i\}} c_j, 0\}$ denote **claimant *i*'s minimal right** and $m(c, E) \equiv (m_i(c, E))_{i\in N}$ the vector of these rights. Fig. 1c illustrates the operator O^m applied to P for $|N| = 2$.

Rule *S* operated from minimal rights, S^m: For each $(c, E) \in \mathcal{C}^N$, $S^m(c, E) \equiv m(c, E) + S(c - m(c, E), E - \sum m_i(c, E))$.

Since $E - \sum m_i(c, E) \geqslant 0$ (Curiel et al. [5]), and $\sum(c_i - m_i(c, E)) \geqslant E - \sum m_i(c, E)$, here too, at the second round, we obtain a well-defined claims problem.

If a rule is invariant under O^m, it satisfies **minimal rights first**: for each problem, one can equivalently calculate the awards vector (i) directly, or (ii) in two steps, first attributing to each claimant his "minimal right," and after adjusting down each agent's claim by his minimal right, dividing what remains (Curiel et al. [5]).

4. Relating the operators

Given any rule S, the rule obtained by subjecting it to the operator O^p and then to the operator $O^{p'}$ is denoted $S^{p'\circ p}$.[7] It is obvious that for each rule S, we have $S^{d\circ d} = S$ and $S^{t\circ t} = S^t$. Also, $S^{m\circ m} = S^m$. To prove this, let $(c, E) \in \mathcal{C}^N$. We need to show that, in the problem obtained from (c, E) by attributing minimal rights, namely $(c - m(c, E), E - \sum m_j(c, E))$, minimal rights are all 0. Let $i \in N$, and note that claimant i's minimal right in this revised problem is $\max\{E - \sum m_j(c, E) - \sum_{N\setminus\{i\}}(c_j - m_j(c, E)), 0\}$. After canceling out terms, we obtain $\max\{E - \sum_{N\setminus\{i\}} c_j - m_i(c, E), 0\}$, which is easily seen to be equal to 0, by using the definition of $m_i(c, E)$.

For $|N| = 2$, straightforward calculations reveal that *CEA* subjected to O^m and *CEL* subjected to O^t are dual. Indeed, they both coincide with **concede-and-divide**, ***CD***. This rule is defined only for $|N| = 2$ but it is very important because a large number of ways of looking at the issue of adjudicating conflicting claims lead to it.[8] Formally, setting $N \equiv \{i, j\}$, $CD(c, E) \equiv (\max\{E - c_j, 0\} + \frac{E - \sum_N \max\{E - c_k, 0\}}{2}, \max\{E - c_i, 0\} + \frac{E - \sum_N \max\{E - c_k, 0\}}{2})$. This duality result

[7] We find this notation a little easier in formulas than $O^{p'} \circ O^p(S)$.

[8] The following scenario, which provided the reason for the name we chose for the rule, is one of them (Aumann and Maschler [1]): agent i, by claiming c_i, is implicitly conceding to claimant j the difference $E - c_i$, or 0 if this difference is negative, namely $\max\{E - c_i, 0\}$. Similarly, by claiming c_j, agent j can be understood as conceding $\max\{E - c_j, 0\}$ to agent i. Let us first attribute to each claimant the amount conceded to him by the other (this can be done because the sum of these concessions is at most as large as the endowment), and in a second step, let us divide the remainder, the "contested part," equally (no agent ends up with more than his claim).

between CEA^m and CEL^t is not an accident. It is a consequence of the following theorem, which holds for any number of claimants:

Theorem 1. *Let S and R be two dual rules. Then S^m and R^t are dual too.*

Proof. We need to show that for each $(c, E) \in \mathcal{C}^N$, $S^m(c, E) = c - R^t(c, \sum_N c_i - E)$, or equivalently that

$$m(c,E) + S\Bigl(c - m(c,E), E - \sum_N m_i(c,E)\Bigr) = c - R\Bigl(t\Bigl(c, \sum_N c_i - E\Bigr), \sum_N c_i - E\Bigr). \qquad (*)$$

Since S is dual to R,

$$S\Bigl(c - m(c,E), E - \sum_N m_i(c,E)\Bigr) = c - m(c,E) - R\Bigl(c - m(c,E), \sum_N \bigl(c_i - m_i(c,E)\bigr) - \Bigl(E - \sum_N m_i(c,E)\Bigr)\Bigr),$$

and substituting in $(*)$, we obtain

$$R\Bigl(c - m(c,E), \sum_N c_i - E\Bigr) = R\Bigl(t\Bigl(c, \sum_N c_i - E\Bigr), \sum_N c_i - E\Bigr).$$

We prove this equality by showing that for each $i \in N$, $c_i - m_i(c, E) = t(c, \sum_N c_i - E)$, equivalently, that

$$c_i - \max\Bigl\{E - \sum_{N\setminus\{i\}} c_j, 0\Bigr\} = \min\Bigl\{c_i, \sum_N c_j - E\Bigr\}. \qquad (**)$$

If $E \leqslant \sum_{N\setminus\{i\}} c_j$, then $\max\{E - \sum_{N\setminus\{i\}} c_j, 0\} = 0$ and the left-hand side of $(**)$ is c_i; the right-hand side is also c_i. If $\sum_{N\setminus\{i\}} c_j < E$, the left-hand side of $(**)$ is $c_i - E + \sum_{N\setminus\{i\}} c_j = -E + \sum_N c_j$, and so is the right-hand side. □

We give two other illustrations of Theorem 1 for $|N| = 2$. First, an implication of this theorem is that if a rule S is such that S^m is *self-dual* and $R \equiv S^d$, then $R^t = S^m$. This is what occurs for **Piniles' rule**, ***Pin***, which is defined, for each $(c, E) \in \mathcal{C}^N$, as follows: if $E \leqslant \frac{\sum c_i}{2}$, $Pin(c, E) = CEA(\frac{c}{2}, E)$ and otherwise $Pin(c, E) = \frac{c}{2} + CEA(\frac{c}{2}, E - \frac{\sum c_i}{2})$.[9] The rule is represented for $|N| = 2$ in Fig. 2b and its dual in Fig. 2e. It is easy to calculate that for $|N| = 2$, $Pin^m = CD$ (Fig. 2a). Since CD is *self-dual*, Theorem 1 implies that $Pin^{t\circ d} = CD$, as is also easily verified (Fig. 2d).

As a final illustration of Theorem 1, once again we consider Pin for $|N| = 2$, but this time we subject it to O^t. The resulting rule is shown in Fig. 2c. The rule obtained by subjecting

[9] Piniles' [16] rule is an only partially successful attempt to explain the recommendations made in the Talmud for the numerical examples given there. On the other hand, the rule obtained from *Pin* by subjecting it to O^t is not *self-dual*. We return to this example to illustrate a later theorem.

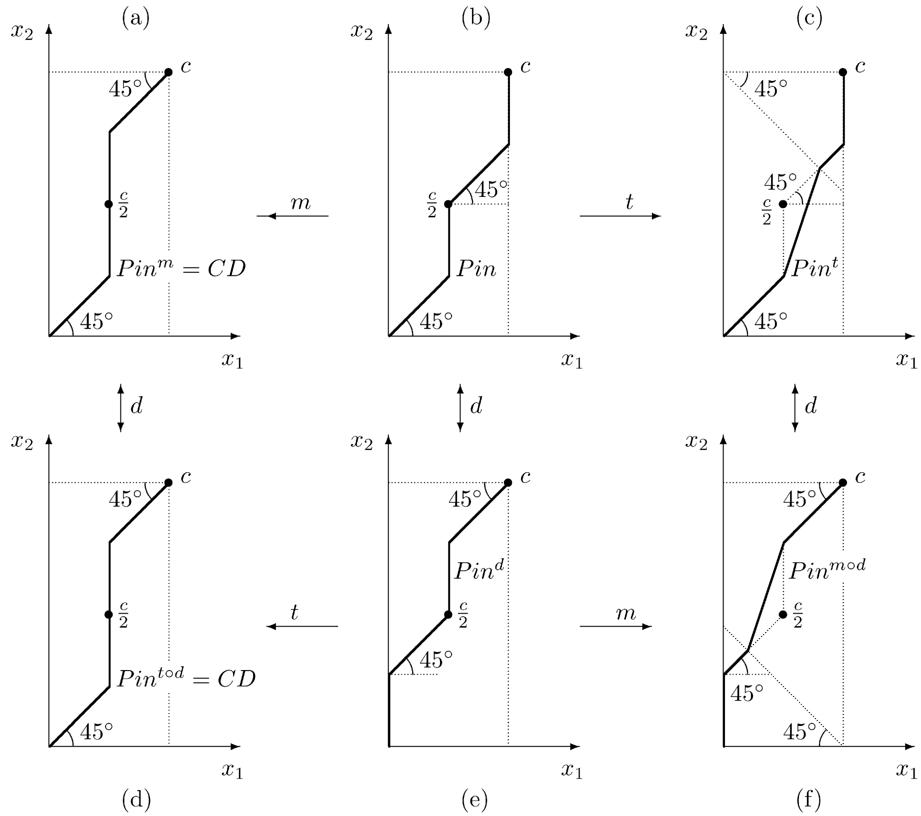

Fig. 2. Illustrating Theorem 1. (a) Piniles' rule subjected to O^m. (b) Piniles' rule. (c) Piniles' rule subjected to O^t. (d) The dual of Piniles' rule subjected to O^t. (e) The dual of Piniles' rule. (f) The dual of Piniles' rule subjected to O^m. The dual of each rule in the top row is the rule below it.

Pin^d to O^m is shown in Fig. 2f. For each $c \in \mathbb{R}^N_+$, the symmetry of $p(Pin^t, c)$ and $p(Pin^{m \circ d}, c)$ announced by Theorem 1 can be verified on panels (c) and (f).

When a rule is subjected to both O^t and O^m, the question arises whether the order in which these operators are applied matters. It is an important question since neither order appears more compelling than the other. Fortunately, the answer is no. We give the proof of this invariance first for $|N| = 2$, as it is very transparent, and also because then, not only is the resulting rule independent of the order, but it is also independent of which rule is taken as a starting point, provided the rule assigns equal awards to agents with equal claims. This is the property of **equal treatment of equals**: for each $(c, E) \in \mathcal{C}^N$ and each pair $\{i, j\} \subseteq N$, if $c_i = c_j$, then $S_i(c, E) = S_j(c, E)$. Moreover, the end-result is CD. This feature of CD is in fact one of the reasons why we feel that this rule is so important.

A preliminary observation is worth making. Consider a rule S satisfying *equal treatment of equals*. Then, for each problem in which the endowment is at most as large as the smallest claim, the rule obtained by subjecting S to O^t chooses equal division. Also, for each problem in which the endowment is at least as large as the sum of the $n - 1$ largest claims, the rule obtained by subjecting S to O^m imposes equal losses on all claimants.

Theorem 2. *For $|N| = 2$. For each rule S satisfying equal treatment of equals, $S^{tom} = S^{mot} = CD$.*

Proof. We assume, without loss of generality, that $c_1 \leqslant c_2$.

Case 1. $E \leqslant c_1$. The amount conceded to each claimant (also his minimal right) is 0. First-round awards are all 0, and no adjustment of claims is needed. Truncation of claims at E yields revised claims both equal to E. By *equal treatment of equals*, equal division prevails.

Case 2. $c_1 < E \leqslant c_2$. Claimant 1 concedes to claimant 2 the amount $E - c_1$, and claimant 2 concedes nothing to claimant 1. Claims are adjusted down to c_1 and $c_2 - (E - c_1)$. After these first-round awards, what remains to divide is c_1. Truncating claims at c_1 yields new claims both equal to c_1. In the second round, by *equal treatment of equals*, each claimant receives half of the remaining endowment, namely $\frac{c_1}{2}$. Altogether, claimant 1 receives $\frac{c_1}{2}$ and claimant 2 what is left.

Case 3. $c_2 < E$. The amounts conceded are $E - c_2$ and $E - c_1$. Claims are adjusted down to $c_1 - (E - c_2)$ and $c_2 - (E - c_1)$, and the amount that remains is $E - \sum(E - c_i)$. After truncation at this revised endowment, claims are equal (and in fact, equal to the revised endowment $c_1 + c_2 - E$). Then, in the second round, by *equal treatment of equals*, equal division of the remaining endowment prevails.

It is easy to see that the awards made in each of the three cases are those specified by *CD*, and that reversing the order in which the two operators are composed also yields *CD*. □

If in Theorem 2, *equal treatment of equals* is dropped, order independence still holds but now a family of rules is obtained (defined in Hokari and Thomson [10]). If the rule is such that multiplying all data of a problem by some positive number results in a problem whose awards vector is obtained from the awards vector of the original problem by the same multiplication (see below for a more formal statement of this property of "homogeneity"), a one-parameter subfamily is obtained. In the general n-claimant case, we lose uniqueness also, but not order independence.

Theorem 3. *For each rule S, we have $S^{mot} = S^{tom}$.*

Proof. The proof is in three steps. Let $(c, E) \in \mathcal{C}^N$ be given.

Step 1. $m(c, E) = m(t(c, E), E)$.

Let $i \in N$. If there is $j \in N\backslash\{i\}$ such that $c_j \geqslant E$, then $m_i(c, E) = 0$. Also, $t_j(c, E) = E$, so $m_i(t(c, E), E) = 0$. Thus, $m_i(c, E) = m_i(t(c, E), E)$. If for each $j \in N\backslash\{i\}$, $c_j < E$, then for each $j \in N\backslash\{i\}$, $t_j(c, E) = c_j$, and thus, $m_i(t(c, E), E) \equiv \max\{E - \sum_{N\backslash\{i\}} t_j(c, E), 0\} = \max\{E - \sum_{N\backslash\{i\}} c_j, 0\} \equiv m_i(c, E)$.

Step 2. $t(c - m(c, E), E - \sum m_i(c, E)) = t(c, E) - m(t(c, E), E)$.

By Step 1, we only need to show that for each $i \in N$,

$$t_i\left(c - m(c, E), E - \sum m_k(c, E)\right) = t_i(c, E) - m_i(c, E). \tag{†}$$

Using the definitions of $t(\cdot,\cdot)$ and $m(\cdot,\cdot)$, (†) reads:

$$\min\left\{c_i - \max\left\{E - \sum_{N\setminus\{i\}} c_j, 0\right\}, E - \sum_{h\in N} \max\left\{E - \sum_{N\setminus\{h\}} c_j, 0\right\}\right\}$$
$$= \min\{c_i, E\} - \max\left\{E - \sum_{N\setminus\{i\}} c_j, 0\right\}.$$

Adding $\max\{E - \sum_{N\setminus\{i\}} c_j, 0\}$ to both sides, we have to prove that

$$\min\left\{c_i, E - \sum_{h\in N\setminus\{i\}} \max\left\{E - \sum_{N\setminus\{h\}} c_j, 0\right\}\right\} = \min\{c_i, E\}. \tag{$*$}$$

If, for each $h \in N \setminus \{i\}$, $E - \sum_{N\setminus\{h\}} c_j \leqslant 0$, the desired conclusion follows directly. Otherwise, there is $h^* \in N \setminus \{i\}$ such that $E - \sum_{N\setminus\{h^*\}} c_j > 0$. Then, obviously $c_i < E$ and

$$\begin{aligned}
&E - \sum_{h\in N\setminus\{i\}} \max\left\{E - \sum_{N\setminus\{h\}} c_j, 0\right\} \\
&= E - \left(E - \sum_{N\setminus\{h^*\}} c_j\right) - \sum_{h\in N\setminus\{i,h^*\}} \max\left\{E - \sum_{N\setminus\{h\}} c_j, 0\right\} \\
&= c_i + \sum_{j\in N\setminus\{i,h^*\}} c_j - \sum_{h\in N\setminus\{i,h^*\}} \max\left\{E - \sum_{N\setminus\{h\}} c_j, 0\right\} \\
&= c_i + \sum_{h\in N\setminus\{i,h^*\}} \min\left\{\sum_{N} c_j - E, c_h\right\} \\
&\geqslant c_i.
\end{aligned}$$

Hence, both left- and right-hand sides of ($*$) are equal to c_i.

Step 3. Conclusion. Using Step 2 and Step 1 in turn, we obtain,

$$\begin{aligned}
S^{t\circ m}(c, E) &\equiv m(c, E) + S\left(t\left(c - m(c, E), E - \sum m_i(c, E)\right), E - \sum m_i(c, E)\right) \\
&= m(c, E) + S\left(t(c, E) - m\big(t(c, E), E\big), E - \sum m_i(c, E)\right) \\
&= m\big(t(c, E), E\big) + S\left(t(c, E) - m\big(t(c, E), E\big), E - \sum m_i\big(t(c, E), E\big)\right) \\
&\equiv S^{m\circ t}(c, E). \qquad \square
\end{aligned}$$

What would happen if the operators O^t and O^m were reapplied? The answer is: nothing. We have already noted that once minimal rights are attributed, claims adjusted down by the minimal rights, and the endowment adjusted down by the sum of the minimal rights, the minimal rights in the problem that results are all 0. In other words, the minimal rights in $(c - m(c, E), E - \sum m_j(c, E))$ are all 0. But consider the problem obtained from the above by truncating claims at the endowment $E - \sum m_j(c, E)$. In this new problem, namely

$$(c', E') \equiv \left(t\left(c - m(c, E), E - \sum m_j(c, E)\right), E - \sum m_j(c, E)\right),$$

we assert that minimal rights are still all 0. Formally:

Proposition 1. *For each problem, consider the problem obtained from it by attributing minimal rights, revising claims down by these minimal rights and the endowment down by the sum of the minimal rights. Then, after claims truncation, minimal rights are all* 0.

Proof. Let $(c, E) \in \mathcal{C}^N$. We need to show that for each $i \in N$, $m_i(c', E') \equiv \max\{E' - \sum_{N\backslash\{i\}} c'_j, 0\} = \max\{E' - \sum_{N\backslash\{i\}} t_j(c - m(c, E), E'), 0\} = 0$. Replacing E' by its value, this is equivalent to showing that

$$E - \sum m_j(c, E) \leqslant \sum_{N\backslash\{i\}} t_j\left(c - m(c, E), E - \sum m_k(c, E)\right),$$

and using the equality established in the proof of Theorem 3 ((†) written for claimant j),

$$t_j\left(c - m(c, E), E - \sum m_k(c, E)\right) = t_j(c, E) - m_j(c, E),$$

showing that $E - m_i(c, E) \leqslant \sum_{N\backslash\{i\}} t_j(c, E)$, and equivalently that

$$E - \sum_{N\backslash\{i\}} t_j(c, E) \leqslant m_i(c, E). \qquad (*)$$

To prove $(*)$, we distinguish two cases. (i) If there is $j \in N\backslash\{i\}$ such that $c_j \geqslant E$, then the left-hand side of $(*)$ is at most 0, whereas the right-hand side is 0. The desired inequality holds. (ii) Otherwise, the left-hand side of $(*)$ is equal to $E - \sum_{N\backslash\{i\}} c_j$ and the right-hand side is the maximum of that same expression and 0. Here too, the desired inequality holds. □

Thanks to Theorem 3, we conclude that parallel statements can be made when O^t and O^m are applied in reverse order.

5. Preservation of properties under operators

In this section we undertake a systematic investigation of which properties are preserved under the operators defined in the previous section. The properties we consider have a straightforward interpretation, and to save space we refer readers to earlier literature for motivation and formal definitions. For the same reason, we do not consider properties that have been less frequently discussed.[10] We apologize for the enumeration, which nevertheless has the advantage of gathering all the material we need. Formal definitions can be found in Thomson [22]. The proofs are available from the authors upon request.

Order preservation (Aumann and Maschler [1]): if agent i's claim is at least as large as agent j's claim, his award should be at least as large as agent j's award; also, his loss should be at least as large as agent j's loss; **group order preservation** (Thomson [20]; Chambers and Thomson [2]) is a parallel statement for groups of agents, obtained by comparing the aggregate

[10] Additional results are listed in Thomson [24].

claims of two groups, which pertains to the aggregate amounts awarded to them and the aggregate losses imposed on them; **anonymity**: any "renaming" of claimants should be accompanied by a parallel reassignment of awards; **homogeneity**: if claims and endowment are multiplied by the same positive number, so should all awards; **continuity**, which is self-explanatory; **claims monotonicity**: if an agent's claim increases, his award should be at least as large as it was initially; **resource monotonicity**: if the endowment increases, each claimant's award should be at least as large as it was initially.[11] **No advantageous transfer** (Moulin [13]; Ju et al. [11]): no group of claimants should receive more in the aggregate by redistributing their claims among themselves. Two "composition" properties follow. If the endowment decreases from some initial value, this decrease can be dealt with in either one of two ways: (i) by canceling the initial division and recalculating the awards for the final endowment; (ii) by taking the awards calculated on the basis of the initial endowment as claims in dividing the final endowment. **Composition down** (Moulin [14]) says that (i) and (ii) should result in the same awards vector. Now, suppose that instead, the endowment increases from some initial value. Here too, we can handle this increase in either one of two ways: (i) by canceling the initial division and recalculating the awards for the final endowment; (ii) by letting claimants keep their initial awards, revising their claims down by these awards, and reapplying the rule to divide the incremental amount (the difference between the final and initial endowments). **Composition up** (Young [25]) says that (i) and (ii) should give the same awards vector. **Population monotonicity** (Thomson [17,18]): if new claimants arrive, the award to each of the claimants initially present should be at most as large as it was initially;[12] **consistency** (Young [25]): if some claimants leave with their awards and the problem of dividing among the remaining claimants what is left is considered, these claimants should receive the same awards as initially; **converse consistency** (see Chun [3] and Thomson [23], for discussions of the property in this context): suppose that an awards vector x is such that its restriction to each two-claimant group is chosen for the problem of dividing between them the sum of their components of x; then, x should be chosen.[13]

A property is preserved under an operator if whenever a rule S satisfies it, so does the rule obtained by subjecting S to the operator. In the following pages, we discuss which properties are preserved under our operators, and which are not. The results are summarized in Table 1. Appendix A contains the proofs of those results that are more difficult.

1. Duality operator. The properties that are preserved under O^d are numerous. **Two properties are dual** if whenever a rule satisfies one of them, its dual satisfies the other. A simple example of a pair of dual properties are the two parts of *order preservation*. This is most easily seen for $|N| = 2$, thanks to the convenient geometric interpretation of *self-duality*. Let $N \equiv \{1, 2\}$ and $c \in \mathbb{R}^N_+$ be such that $c_1 \leqslant c_2$, say. Then, $p(S, c)$ lies above the 45° line (the first part of *order preservation*) if and only if $p(S^d, c)$ lies below the line of slope 1 passing through c (the second part of *order preservation*). **A property is self-dual** if it is preserved under O^d.

Two basic monotonicity properties are not preserved under O^d (the proofs of these facts are in Appendix A), and we state their duals. First is *claims monotonicity*. Its dual says that if an agent's claim and the endowment increase by the same amount γ, this claimant's award

[11] For the "inequality conditions," a "strict" version is obtained by requiring that the conclusion should be strict if the inequality appearing in the hypothesis is strict.

[12] For a survey of the literature on *population monotonicity*, see Thomson [19].

[13] For a survey of the literature on *consistency* and its *converse*, see Thomson [23].

Table 1
Showing which properties are preserved under the operators

Prop\operators	Duality	Truncation	Min rights
Equal treat of equals	+	+	+
Order pres	+	+	+
Anonymity	+	+	+
Group order pres	+	− (P)	− (P)
Continuity	+	+	+
Claims mon	− (Prop 3)	+	− (Prop 4)
Resource mon	+	− (*CEL*) (Prop 6)	− (*CEA*)
Homogeneity	+	+	+
Claims trunc inv	− (*CEA*)	+	+
Min rights first	− (*CEL*)	+	+
Comp down	− (ES^u) (Prop 2)	− (P)	− (P)
Comp up	− (ES^u) (Prop 2)	− (P)	− (P)
Self-duality	+	− (P)	− (P)
No adv trans	+	− (P)	− (P)
Pop mon	− (Prop 5)	+	− (*CEA*)
Consistency	+	− (P)	− (P)
Conv cons	+	− (P)	− (P)

In each cell for which a negative result holds, we indicate in parenthesis a rule allowing to prove the assertion. For instance, the notation (P) at the intersection of the row labeled "*group order preservation*" and of the column labeled "truncation" means that P satisfies the property but that P^t does not.

should not increase by more than γ.[14] We include the derivation, to show how one performs this operation. It is obtained by simply replacing every occurrence of E by $\sum c_i - E$ and every occurrence of $S(c, E)$ by $c - S(c, \sum c_i - E)$. Indeed, let $i \in N$, and note that $S_i(c_i + \gamma, c_{-i}, E) \geqslant S_i(c, E)$ is equivalent to $c_i + \gamma - S_i^d(c_i + \gamma, c_{-i}, \sum c_j + \gamma - E) \geqslant c_i - S_i^d(c, \sum c_j - E)$.[15] After canceling out c_i from both sides of this inequality and replacing $\sum c_j - E$ by E', we obtain $\gamma \geqslant S_i^d(c_i + \gamma, c_{-i}, E' + \gamma) - S_i^d(c, E')$, as announced.

Second is *population monotonicity*. Its dual says that if new claimants arrive and the endowment increases by an amount equal to the sum of their claims, then the award to none of the claimants initially present should decrease.

2. Claims truncation and minimal rights operators. Many properties are preserved under O^t. Having at hand such a list, the concept of duality of properties, together with the following theorem, allows us to easily determine which properties are preserved under O^m. The only properties whose case cannot be settled by invoking these theorems are *claims monotonicity* and *population monotonicity*, and direct proofs are needed (see Appendix A).

Theorem 4. *A property is preserved under O^t if and only if its dual is preserved under O^m.*

Proof. Let A be a property that is preserved under O^t, A^d its dual, and let S be a rule satisfying A^d. We need to show that S^m satisfies A^d. Since A is dual to A^d, S^d satisfies A. Since A is

[14] This property is independently formulated by Moulin [15] for a discrete version of the model of claims resolution.

[15] The notation c_{-i} designates the vector c from which the ith coordinate has been deleted and (c_i', c_{-i}) the vector c in which the ith coordinate has been replaced by c_i'.

preserved under O^t, $S^{t \circ d}$ satisfies A. Since A^d is dual to A, $S^{d \circ t \circ d}$ satisfies A^d. We will show that $S^{d \circ t \circ d} = S^m$.

Recall that Theorem 1 asserts that if R is the dual of S, then R^t is the dual of S^m. Thus, $R^{d \circ t} = S^m$. Since $R = S^d$, then $S^{d \circ t \circ d} = S^m$.

We have therefore shown that $S^{d \circ t \circ d} = S^m$, and since $S^{d \circ t \circ d}$ was assumed to satisfy A^d, so does S^m. This completes the proof of the theorem in one direction.

We omit the "dual" proof for the other direction. □

Theorem 4 suggests an additional definition: **two operators are dual** if whenever a property is preserved under the first one, the dual property is preserved under the second one. Theorem 4 says that O^t and O^m are dual.

3. Composition of the claims truncation and attribution of minimal rights operators. The next theorem says that the composition of O^t and O^m preserves duality of rules (according to Theorem 3, the operators can be composed in either order):

Theorem 5. *If two rules S and R are dual, then so are S^t and R^m.*

Proof. Let S and R be a pair of dual rules. By Theorem 1, S^t and R^m are dual too. Applying Theorem 1 to this second pair, we deduce that $S^{m \circ t}$ and $R^{t \circ m}$ are dual. By Theorem 3, $R^{t \circ m} = R^{m \circ t}$. Thus, $S^{m \circ t}$ and $R^{m \circ t}$ are dual (and of course, so are $S^{t \circ m}$ and $R^{t \circ m}$). □

If a property is preserved under both O^t and O^m separately, then clearly it is preserved under $O^{t \circ m}$. However, a property may be preserved under neither O^t nor O^m and yet be preserved under $O^{t \circ m}$. An example is *self-duality*, for which we obtain the following result, a corollary of Theorem 5.

Corollary 1. *If a rule S is self-dual, so is $S^{m \circ t}$.*

Curiel et al. [5] start from P and define a new rule by first attributing minimal rights and then truncating claims. They show that their rule is *self-dual* by invoking a game-theoretic argument. Since P is *self-dual*, this conclusion can be obtained by applying Corollary 1. Typically however, when a property is preserved under neither O^t nor O^m, it is not recovered under their composition.

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Appendix A

In this appendix, we provide the proofs of selected results presented in Table 1. We use the following additional notation. Given $x^1, x^2, \ldots, x^k \in \mathbb{R}^N$, seg$[x^1, x^2]$ denotes the segment connecting them and seg$]x^1, x^2] \equiv$ seg$[x^1, x^2] \setminus \{x^1\}$; also bro.seg$[x^1, x^2, \ldots, x^k] \equiv$ seg$[x^1, x^2] \cup$ seg$[x^2, x^3] \cup \cdots \cup$ seg$[x^{k-1}, x^k]$.

Proposition 2. *The following properties are not preserved under the duality operator but they come in dual pairs*: *invariance under claims truncation and minimal rights first*; *composition down and composition up.*

For the proof of the second part of this proposition, we use the following "equal sacrifice rule" (Young [26]; Moulin [14]). Let $u:\mathbb{R}\to\mathbb{R}$ be the function defined by $u(x)\equiv\frac{-1}{x}$ and ES^u the rule that selects for each $(c,E)\in\mathcal{C}^N$ the vector $x\in X(c,E)$ such that for each pair $\{i,j\}\subseteq N$, $u(c_i)-u(x_i)=u(c_j)-u(x_j)$. (This is equivalent to setting $x_i=\frac{c_i}{1+\beta c_i}$, where $\beta\in\mathbb{R}_+$ is chosen so as to achieve efficiency.)

Proof. The duality between *invariance under claims truncation* and *minimal rights first* is proved by Herrero [7] (Dagan [6] proves a related result).

Minimal rights first: *CEL* can be used to make the point. One can also appeal to the example used to prove that *invariance under claims truncation* is not preserved and to the fact that this property and *minimal rights first* are dual properties.

Composition down: We assert first that ES^u satisfies *composition down*. To see this, let $(c,E)\in\mathcal{C}^N$ be given and $E'<E$. Let $x\equiv ES^u(c,E)$, $x'\equiv ES^u(c,E')$, and $y\equiv ES^u(x,E')$. We show that $x'=y$. Let $i\in N$. Let $\beta_x\in\mathbb{R}_+$ be such that $\sum\frac{c_i}{1+\beta_x c_i}=E$. Let $\beta_{x'}$ and β_y be similarly defined. By definition of ES^u, $x_i=\frac{c_i}{1+\beta_x c_i}$ and $y_i=\frac{x_i}{1+\beta_y x_i}$. Thus, $y_i=\frac{c_i}{1+(\beta_y+\beta_x)c_i}$. Since $\sum y_i=\sum x'_i=E'$ and $\beta_{x'}$ is uniquely determined, $\beta_{x'}=\beta_y+\beta_x$. Thus, $x'_i=y_i$, as announced.

Next, we assert that $(ES^u)^d$ violates *composition down*. Let $N\equiv\{1,2\}$, $(c,E)\in\mathcal{C}^N$ be defined by $(c,E)\equiv(1,3;\frac{68}{21})$, and $E'=\frac{11}{4}$. Then, $(ES^u)^d(c,E)=(\frac{2}{3},\frac{18}{7})$ and $(ES^u)^d(c,E')=(\frac{1}{2},\frac{9}{4})$. Let $c'\equiv(\frac{2}{3},\frac{18}{7})$ and $x\equiv(ES^u)^d(c',E')$. We claim that $x\neq(\frac{1}{2},\frac{9}{4})$. Suppose that $x=(\frac{1}{2},\frac{9}{4})$. Since $x\equiv c'-(ES^u)(c',\sum c'_i-E')$, then $(ES^u)(c',\sum c'_i-E')=(\frac{1}{6},\frac{9}{28})$. Let $\beta\in\mathbb{R}_+$ be such that $\sum(ES^u)_i(c',\sum c'_i-E')=\sum\frac{c'_i}{1+\beta c'_i}=\sum c'_i-E'$. We obtain $\beta=\frac{9}{2}$ but also $\frac{49}{18}$. This is impossible since β is uniquely determined.

The duality between the two *composition* properties is proved by Moulin [14]. □

Our next result concerns *claims monotonicity*, a property that is satisfied by every rule encountered in the literature.[16] Unfortunately we have:

Proposition 3. *Claims monotonicity is not preserved under* O^d.

The proof is by means of an example. It is of interest that the example is *anonymous*, *order-preserving*, *homogeneous*, and *resource monotonic* (and therefore *resource continuous*; it is in fact fully *continuous*, that is, jointly continuous with respect to the claims and the endowment). This shows that these properties do not help preserve *claims monotonicity*.

Proof. We define a rule S on $\mathcal{C}^N$, where $N\equiv\{1,2\}$. The rule is depicted in Figs. 3 and 4. We show that S is *claims monotonic* whereas S^d is not.

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[16] Here too, few of the standard rules satisfy the stronger requirement that an agent whose claim increases should receive more, unless $E=0$ of course (equality is not permitted any more). The rule P is a rare example that does. However, it is easy to construct rules that do. Most "parametric rules" (Young [25]) do.

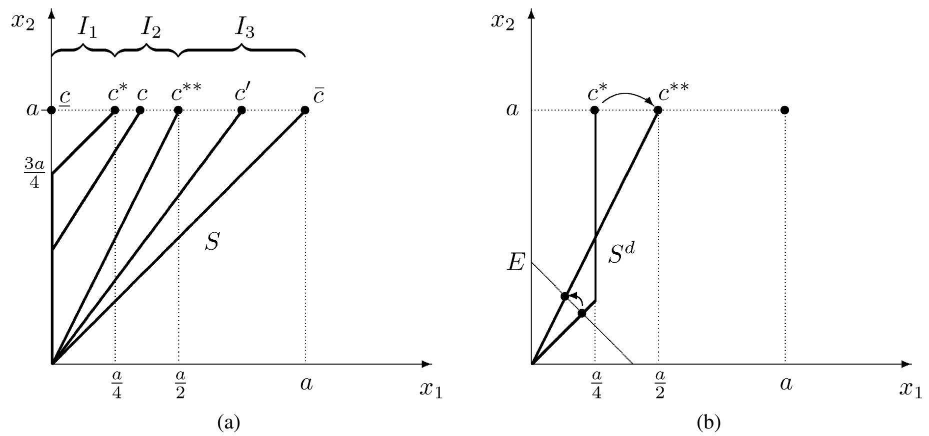

Fig. 3. Claims-monotonicity is not preserved under O^d (Proposition 3). (a) Sample paths of awards of the rule S defined in the proof. (b) Paths $p(S^d, c^*)$ and $p(S^d, c^{**})$. If the endowment is E, as agent 1's claim increases from $c_1^* = \frac{a}{4}$ to $c_1^{**} = \frac{a}{2}$, he receives less (follow the arrows).

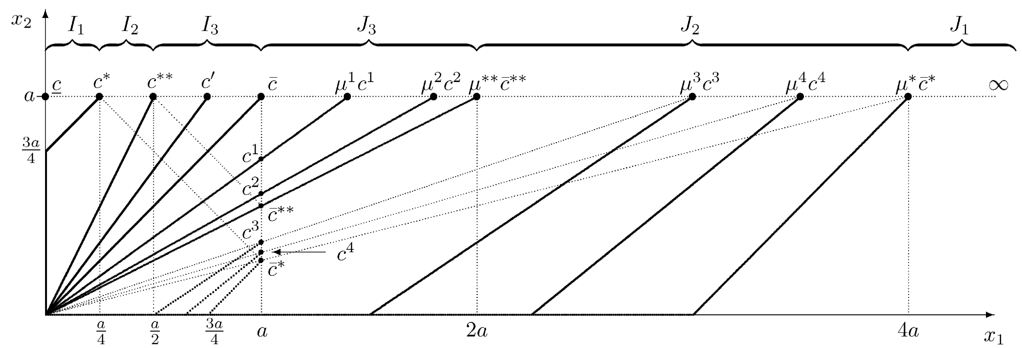

Fig. 4. The rule S of Proposition 3 is claims monotonic. Sample paths $p(S, c)$ for $c \equiv (c_1, a)$ when $c_1 \in [0, \infty[$. For each $c \in \text{seg}[(a, 0), \bar{c}]$, where $\bar{c} \equiv (a, a)$, $p(S, c)$ is obtained by symmetry from the path for the symmetric image of c with respect to the 45° line. The paths for two critical claims vectors, $\bar{c}^*$ and $\bar{c}^{**}$, the symmetric images of c^* and c^{**}, are represented. For each $c \in J_3 \cup J_2 \cup J_1$, $p(S, c)$ is obtained by homothetic expansion of the path for the homothetic image of c that belongs to $\text{seg}[(a, 0), \bar{c}]$. For each endowment, as agent 1's claim increases, he receives at least as much as he did initially.

Let $a > 0$, $\underline{c} \equiv (0, a)$, $c^* \equiv (\frac{a}{4}, a)$, $c^{**} \equiv (\frac{a}{2}, a)$, and $\bar{c} \equiv (a, a)$. We first specify $p(S, c)$ for each $c \in \text{seg}[\underline{c}, \bar{c}]$. We then choose $p(S, c)$ for each $c \in \text{seg}[(a, 0), \bar{c}]$ as the symmetric image with respect to the 45° line of $p(S, (c_2, c_1))$. Finally, we choose $p(S, c)$ for each other $c \in \mathbb{R}_+^N$ by first calculating μ such that $\mu c \in \text{bro.seg}[(a, 0), \bar{c}, (0, a)]$ and subjecting $p(S, \mu c)$ to a homothetic transformation of ratio $\frac{1}{\mu}$. This construction guarantees that S is *anonymous* and *homogeneous*.

For each $c \in I_1 \equiv \text{seg}[\underline{c}, c^*]$ (Fig. 3a illustrates I_1 and I_2 and I_3 defined below), $p(S, c) = p(CEL, c)$. For each $c \in I_2 \equiv \text{seg}]c^*, c^{**}]$, $p(S, c)$ is piecewise linear in two pieces: given $0 \leqslant \lambda \leqslant 1$, $p(S, \lambda c^* + (1-\lambda)c^{**}) = \text{bro.seg}[(0, 0), \lambda(0, \frac{3}{4}a), \lambda c^* + (1-\lambda)c^{**}]$. (Note that for $\lambda = 0$, the path is that of P.) For each $c \in I_3 \equiv \text{seg}]c^{**}, \bar{c}]$, $p(S, c) = p(P, c)$.

Fig. 4 illustrates that when agent 2's claim is fixed at a, and as agent 1's claim increases from 0 to ∞, agent 1's award does not decrease. The *claims monotonicity* of S is a consequence of this fact and of its being *anonymous* and *homogeneous*. The figure indicates some paths of S for $c \in \mathbb{R}^N_+$ such that $c_2 = a$. We show that these paths never cross. For each $c_1 \in [a, \infty[$, there is a claims vector on seg$[(a,0), \bar{c}]$ that is proportional to $c \equiv (c_1, a)$. We call $\mu \geqslant 1$ the expansion factor required to pass from (c_1, a) to the former, using the same superscript to keep track of this pairing, $\bar{c}^*$ and $\mu^*\bar{c}^*$ being an example of a pair so defined.

1. For each $c \in J_3 \equiv$ seg$]\bar{c}, \mu^{**}\bar{c}^{**}]$ (Fig. 4), where $\bar{c}^{**}$ is the symmetric image of c^{**}, $p(S,c) = p(P,c)$ (examples are $p(S, \mu^1 c^1)$ and $p(S, \mu^2 c^2)$).
2. For each $c \in J_2 \equiv$ seg$]\mu^{**}\bar{c}^{**}, \mu^*\bar{c}^*]$, $p(S,c)$ is obtained by a homothetic expansion of the path for the reduced image of c that belongs to seg$[(a,0), \bar{c}]$. For example, consider two points in J_2, such as $\mu^3 c^3$ and $\mu^4 c^4$ in the figure, where $\mu^4 c^4$ is to the right of $\mu^3 c^3$. Then, the paths for these points are obtained by homothetic expansions of the paths for c^3 and c^4, with c^4 below c^3. The slope of the oblique segment in $p(S, c^4)$ is greater than the slope of the oblique segment in $p(S, c^3)$. Therefore the same statement can be made about the slopes of the oblique segments in $p(S, \mu^4 c^4)$ and $p(S, \mu^3 c^3)$, which imply that they do not cross.
3. Finally, for each $c \in J_1 \equiv \{(c_1, a)\colon\ c_1 \in]4a, \infty[\}$ (J_1 is the open half-line $\{\mu^*\bar{c}^* + t(1,0)\colon\ t > 0\}$ in the figure), $p(S,c)$ consists of a horizontal segment from the origin and a segment of slope 1.

The fact that S^d violates *claims monotonicity* can be seen by considering $p(S^d, c^*)$ and $p(S^d, c^{**})$. These paths are obtained by symmetry of $p(S, c^*)$ and $p(S, c^{**})$. Inspection of Fig. 3b reveals that the paths cross: in fact, for each endowment in the interval $]0, \frac{3}{4}a[$, agent 1 loses as his claim increases from $c_1^* = \frac{a}{4}$ to $c_1^{**} = \frac{a}{2}$, agent 2's claim being kept fixed at a. □

The strengthening of *claims monotonicity* obtained by requiring that if an agent's claim increases, he should receive more, is not preserved under the duality operator either. To see this, it suffices to modify the example used to prove Proposition 3. Informally, for each $c \in I_1 \cup I_2$, replace the vertical segment of $p(S,c)$ by a very steep segment whose slope varies continuously and monotonically between ∞ and 2 as c varies in $I_1 \cup I_2$ from $\underline{c}$ to c^{**}.

Proposition 4. *Claims monotonicity is not preserved under* O^m.

We prove this result by exhibiting a rule that is *claims monotonic*, but the rule obtained from it by applying O^m is not. The rule is also *order preserving*, *anonymous*, *resource monotonic*, and *continuous*.

Proof. The proof is by means of an example of a rule S defined on $\mathcal{C}^N$ where $N \equiv \{1,2\}$. It is depicted in Fig. 5a.

Step 1. Construction of S. We first consider $c \in \mathbb{R}^N_+$ with $c_1 \leqslant c_2$. If $c_2 \leqslant 2$, then $p(S,c) = p(P,c)$. If $c_2 \geqslant 4$, then $p(S,c) = p(CEL,c)$. If $2 < c_2 < 4$ (the shaded region), then $p(S,c)$ is a linear combination of $p(P,c)$ and $p(CEL,c)$. The construction uses an arbitrary continuous and monotone function $g\colon [0,1] \to [0,1]$ such that for each $t \in [0,1]$, $g(t) \leqslant t$, and $g(0) = 0$, $g(\frac{1}{2}) = \frac{1}{4}$, and $g(1) = 1$. Now, let $k(c) \equiv g(\frac{c_2-2}{2})(c_2 - c_1)$, and $p(S,c) \equiv$ bro.seg$[(0,0), (0, k(c)), c]$.

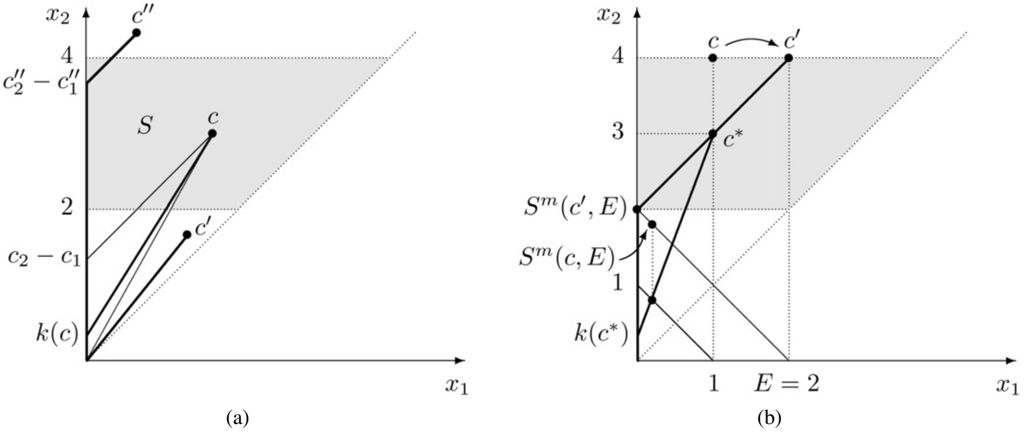

Fig. 5. Claims monotonicity is not preserved under the attribution of minimal rights operator (Proposition 4). (a) Sample paths of awards of the rule S defined in the proof. The path for c in the shaded region is a linear combination of $p(P, c)$ and $p(CEL, c)$. (b) Paths $p(S^m, c)$ and $p(S^m, c')$. If $E = 2$, as agent 1's claim increases from 1 to 2, he receives less.

We then choose $p(S, c)$ for each $c \in \mathbb{R}^N_+$ with $c_1 > c_2$ as the symmetric image with respect to the 45° line of $p(S, (c_2, c_1))$. This guarantees that S is *anonymous*.

Step 2. S is *claims monotonic*. Since S is *anonymous*, it is enough to examine the rule in the region $\{c \in \mathbb{R}^N_+\colon\ c_1 \leqslant c_2\}$. First, let $c_2 > 0$ and let $c_1, c'_1 \in [0, c_2]$ be such that $c'_1 < c_1$. Let $c'_2 \equiv c_2$ and $c' \equiv (c'_1, c'_2)$. There are three subcases. If $c_2 \leqslant 2$, then $p(S, c) = p(P, c)$ and $p(S, c') = p(P, c')$, and since P satisfies *claims monotonicity*, we are done. If $c_2 \geqslant 4$, then $p(S, c) = p(CEL, c)$ and $p(S, c') = p(CEL, c')$, and since CEL satisfies *claims monotonicity*, we are done. If $2 < c_2 < 4$, then $p(S, c) = \text{bro.seg}[(0, 0), (0, k(c)), c]$. Also, $p(S, c') = \text{bro.seg}[(0, 0), (0, k(c')), c']$. The conclusion follows from the fact that $c_2 - c_1 < c'_2 - c'_1$, and since $c'_2 = c_2$, $g(\frac{c_2-2}{2}) = g(\frac{c'_2-2}{2})$, so that altogether $k(c) \equiv g(\frac{c_2-2}{2})(c_2 - c_1) < g(\frac{c'_2-2}{2})(c'_2 - c'_1) \equiv k(c')$.

Next, let $c_1 > 0$ and let $c_2, c'_2 \in [c_1, \infty[$ be such that $c_2 < c'_2$. Let $c'_1 \equiv c_1$ and $c' \equiv (c'_1, c'_2)$. We have $p(S, c) = \text{bro.seg}[(0, 0), (0, k(c)), c]$; also, $p(S, c') = \text{bro.seg}[(0, 0), (0, k(c')), c']$. The conclusion follows from the fact that since $c'_2 > c_2$ and $c'_1 = c_1$, then $c_2 - c_1 < c'_2 - c'_1$, and since g is increasing, $g(\frac{c_2-2}{2}) \leqslant g(\frac{c'_2-2}{2})$, so that altogether $k(c) \equiv g(\frac{c_2-2}{2})(c_2 - c_1) < g(\frac{c'_2-2}{2})(c'_2 - c'_1) \equiv k(c')$.

Step 3. S^m is not *claims monotonic* (Fig. 5b). To see this, let $c \equiv (1, 4)$, $c' \equiv (2, 4)$, and $E = 2$. Note that $m(c, E) = (0, 1)$ and $m(c', E) = (0, 0)$. We have $S^m(c, E) = m(c, E) + S(c - m(c, E), E - \sum m_i(c, E))$. Let $c^* \equiv c - m(c, E)$. To calculate the second term in this sum, we note that $c^* = (1, 3)$ and $E - \sum m_i(c, E) = 1$. Then, $p(S, c^*)$ is $\text{seg}[(0, 0), k(c^*), c^*]$, where $k(c^*) \equiv g(\frac{c^*_2-2}{2})(c^*_2 - c^*_1)$. Since $g(\frac{c^*_2-2}{2}) = g(\frac{1}{2}) = \frac{1}{4}$, we have $k(c^*) = \frac{1}{4}(c^*_2 - c^*_1) = \frac{1}{2} < 1 = E - \sum m_i(c, E)$. So, $S_1(c^*, E - \sum m_i(c, E)) > 0$. This implies $S^m_1(c, E) > 0$. Also, $S^m_1(c', E) = S_1(c', E) = CEL_1(c', E) = 0$. Thus, as agent 1's claim increases from $c_1 = 1$ to $c'_1 = 2$, he receives less, in violation of *claims monotonicity*. □

Next, we turn to *population monotonicity* for which a negative result also holds. We prove this fact by exhibiting a rule S that is *anonymous*, *homogeneous*, *resource monotonic*, and *population monotonic*, but S^d is not *population monotonic*. (Since *resource monotonicity* implies *resource continuity,* S is also *resource continuous*.)

Proposition 5. *Population monotonicity is not preserved under* O^d.

Proof. The proof is by means of an example of a rule S defined on $\bigcup_{N'\subseteq N} \mathcal{C}^{N'}$ where $N \equiv \{1, 2, 3\}$. Sample paths of awards of S are plotted in Figs. 6a and 6b. We show that S is *population monotonic* but S^d is not.

Step 1. Construction of S. On the subdomain of two-claimant problems, $S \equiv P$. Let Q be the unit cube in $\mathbb{R}^N_+$, and for each $t \in \{1, 2, 3\}$, let F_t be the face of Q consisting of all $c \in \mathbb{R}^N_+$ such that $c_t = 1$. Given $c \equiv (c_1, 1, c_3) \neq (1, 1, 1)$, a typical claims vector in F_2, let L be the line passing through c and $e \equiv (1, 1, 1)$. Also, let $x \equiv L \cap \text{seg}[(\frac{2}{3}, 1, 1), (1, 1, \frac{2}{3})]$ and $y \equiv L \cap \text{seg}[(0, 1, 1), (1, 1, 0)]$ (we use the notation L', x', y' for the corresponding objects calculated for the claims vector c'). If $0 < c_1 + c_3 \leqslant 1$, then $p(S, c) = p(P, c)$ (c in Fig. 6a). If $1 < c_1 + c_3 \leqslant 1 + \frac{2}{3}$, then $p(S, c) = \text{bro.seg}[(0, 0, 0), y, c]$ (c' in Fig. 6a). If $1 + \frac{2}{3} < c_1 + c_3 \leqslant 2$, then $p(S, c)$ is piecewise linear in two pieces defined as follows: let $0 \leqslant \lambda \leqslant 1$ be such that $c = \lambda x + (1 - \lambda)e$. Then $p(S, c) = \text{seg}[(0, 0, 0), d, c]$ where $d \equiv \lambda y + (1 - \lambda)\frac{2}{3}e$ (Fig. 6b). Finally, if $c = e$, then $p(S, c) = p(P, c)$. We deduce $p(S, c)$ for each $c \in F_1$ by symmetry with respect to the plane of equation $x_1 = x_2$ of $p(S, c')$ where c' is the symmetric image of c with respect to that plane; similarly we deduce $p(S, c)$ for each $c \in F_3$ by symmetry with respect to the plane of equation $x_2 = x_3$ of $p(S, c')$ where c' is the symmetric image of c with respect to that plane. If c is not in any of the faces F_1, F_2, and F_3, let $\mu \in \mathbb{R}_+$ be such that μc does belong to such a face. Then, $p(S, c)$ is obtained from $p(S, \mu c)$ by the homothetic transformation of ratio $\frac{1}{\mu}$. This construction guarantees that S is *anonymous* and *homogeneous.*

Step 2. S is *population monotonic*. Let $E > 0$ and $c \equiv (c_1, 1, c_3)$ be an arbitrary point in F_2. We distinguish three cases.

Case 1. $0 < c_1 + c_3 \leqslant 1$. Then, $S(c, E) \equiv P(c, E)$. Since $S(c_{N'}, E) \equiv P(c_{N'}, E)$ for each N' with $|N'| = 2$ and P is *population monotonic*, the *population-monotonicity* inequalities hold.

Case 2. $1 < c_1 + c_3 \leqslant 1 + \frac{2}{3}$. We imagine the departure of each agent in turn (Fig. 6c).

Subcase 2.1. Claimant 1 leaves. We have to compare $z \equiv S(c, E)$ and $z' \equiv S(c_{\{2,3\}}, E)$. We assume that $E \leqslant 1 + c_3$ since otherwise there is nothing to check. Since $y = L \cap \text{seg}[(0, 1, 1), (1, 1, 0)]$, then $y_1 + 1 + y_3 = 2$. Note that y belongs to the simplex in the plane of equation $\sum v_i = 2$. Thus $z = (\frac{y_1 E}{2}, \frac{E}{2}, \frac{y_3 E}{2})$. Also $z' = (\frac{E}{1+c_3}, \frac{c_3 E}{1+c_3})$. Then $z'_3 - z_3 = \frac{c_3 E}{1+c_3} - \frac{y_3 E}{2}$. Since $c_3 \geqslant y_3$ and $1 + c_3 \leqslant 2$, then $z'_3 - z_3 \geqslant 0$.

Also $z'_2 - z_2 = \frac{E}{1+c_3} - \frac{E}{2}$. Since $1 + c_3 \leqslant 2$, then $z'_2 - z_2 \geqslant 0$.

Subcase 2.2. Claimant 2 leaves. We have to compare $z \equiv S(c, E)$ and $z' \equiv S(c_{\{1,3\}}, E)$. We assume that $E \leqslant c_1 + c_3$ since otherwise there is nothing to check. Since $y = L \cap \text{seg}[(0, 1, 1), (1, 1, 0)]$, as already calculated, $y_1 + 1 + y_3 = 2$. Thus $z = (\frac{y_1 E}{2}, \frac{E}{2}, \frac{y_3 E}{2})$. Also $z' = (\frac{c_1 E}{c_1+c_3}, \frac{c_3 E}{c_1+c_3})$. Thus $z'_1 - z_1 = \frac{c_1 E}{c_1+c_3} - \frac{y_1 E}{2}$. Since $c_1 \geqslant y_1$ and $c_1 + c_3 \leqslant 2$, then $z'_1 - z_1 \geqslant 0$.

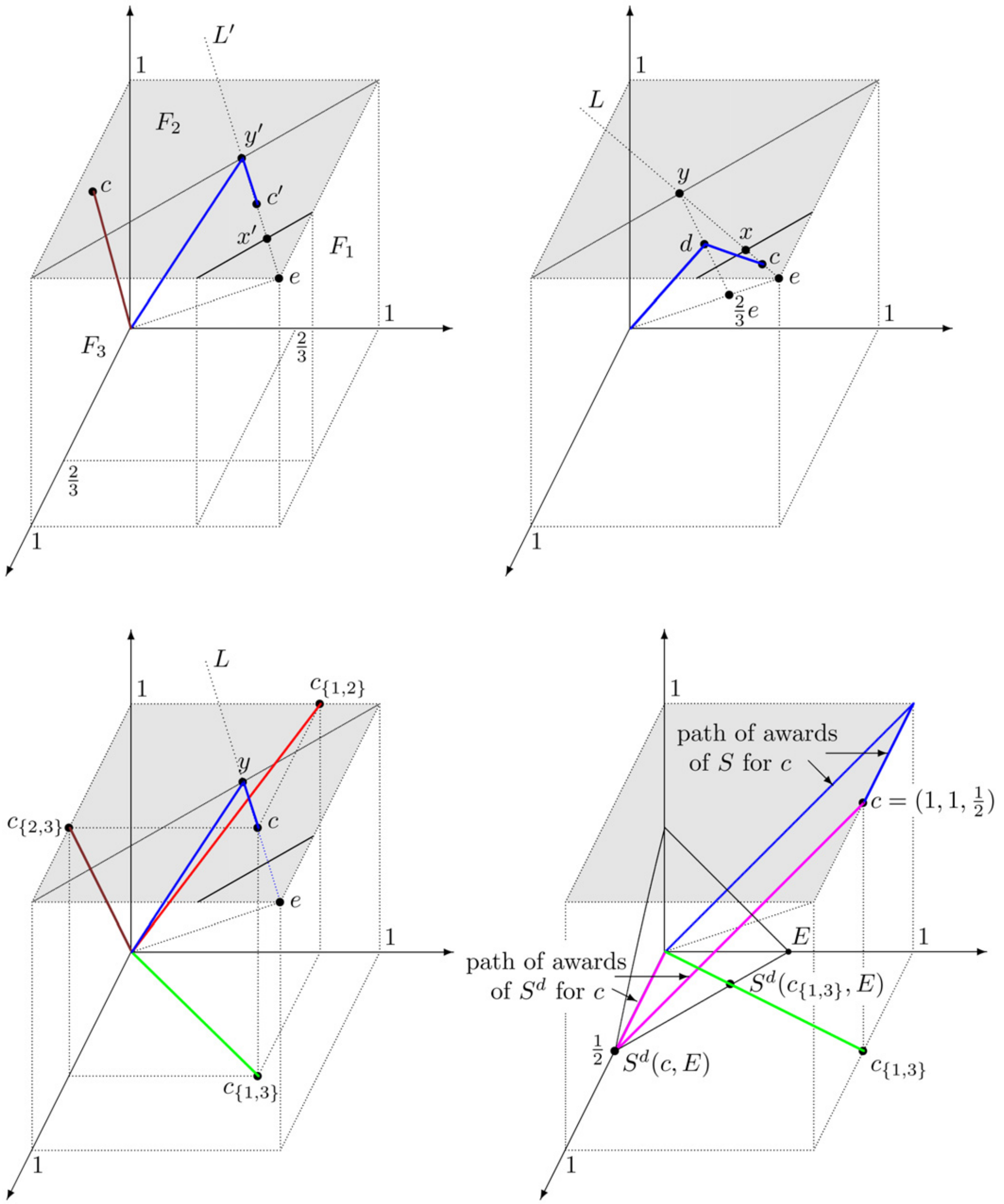

Fig. 6. Population monotonicity is not preserved under the duality operator (Proposition 5). (a) Construction of $p(S, c)$ for c such that $c_2 = 1$ and $0 \leqslant c_1 + c_3 \leqslant 1$, and for c' such that $c'_2 = 1$ and $1 < c'_1 + c'_3 \leqslant 1 + \frac{2}{3}$. (b) Construction of $p(S, c)$ for c such that $c_2 = 1$ and $1 + \frac{2}{3} < c_1 + c_3 \leqslant 2$. (c) The rule S is *population monotonic*. Given c in F_2, we determine $p(S, c)$ (it consists of two line segments), and the paths of awards of S for each of the projections of c onto the three two-dimensional coordinates subspaces (these paths are segments connecting the origin to these projections). Then, given E, we calculate the awards vectors selected by S for the resulting problems. (d) The rule S^d is not *population monotonic*. For $(c, E) \equiv (1, 1, \frac{1}{2}; \frac{1}{2})$, it selects $(0, 0, \frac{1}{2})$, but for the problem that results from the departure of claimant 2, it selects $(\frac{1}{3}, \frac{1}{6})$. Claimant 3 loses.

Also, $z'_3 - z_3 = \frac{c_3E}{c_1+c_3} - \frac{y_3E}{2}$. Since $c_3 \geqslant y_3$ and $c_1 + c_3 \leqslant 2$, then $z'_3 - z_3 \geqslant 0$.

Subcase 2.3. Claimant 3 leaves. We apply the same argument as in Subcase 2.1.

Case 3. $1 + \frac{2}{3} < c_1 + c_3 \leqslant 2$. Let λ be such that $c = \lambda x + (1-\lambda)e$ and $d \equiv \lambda y + (1-\lambda)\frac{2}{3}e$. Since $x \geqslant y$ and $e > \frac{2}{3}e$, then $c \geqslant d$.

Subcase 3.1. Claimant 1 leaves. We have to compare $z \equiv S(c, E)$ and $z' \equiv S(c_{\{2,3\}}, E)$. We assume that $E \leqslant 1 + c_3$ since otherwise there is nothing to check. Note that $d_1 + d_2 + d_3 = 2$ and $E \leqslant 1 + c_3 \leqslant d_1 + d_2 + d_3$, so $z = (\frac{d_1E}{2}, \frac{d_2E}{2}, \frac{d_3E}{2})$ and $z' = (\frac{E}{1+c_3}, \frac{c_3E}{1+c_3})$. Thus $z'_3 - z_3 = \frac{c_3E}{1+c_3} - \frac{d_3E}{2}$. Since $c_3 \geqslant d_3$ and $1 + c_3 \leqslant 2$, then $z'_3 - z_3 \geqslant 0$.

Also, $z'_2 - z_2 = \frac{E}{1+c_3} - \frac{d_2E}{2}$. Since $1 \geqslant d_2$ and $1 + c_3 \leqslant 2$, $z'_2 - z_2 \geqslant 0$.

Subcase 3.2. Claimant 2 leaves. We have to compare $z \equiv S(c, E)$ and $z' \equiv S(c_{\{1,3\}}, E)$. We assume that $E \leqslant c_1 + c_3$ since otherwise there is nothing to check. Note that $d_1 + d_2 + d_3 = 2$ and $E \leqslant c_1 + c_3 \leqslant d_1 + d_2 + d_3$, so $z = (\frac{d_1E}{2}, \frac{d_2E}{2}, \frac{d_3E}{2})$ and $z' = (\frac{c_1E}{c_1+c_3}, \frac{c_3E}{c_1+c_3})$. Thus $z'_1 - z_1 = \frac{c_1E}{c_1+c_3} - \frac{d_1E}{2}$. Since $c_1 \geqslant d_1$ and $c_1 + c_3 \leqslant 2$, then $z'_1 - z_1 \geqslant 0$.

Also, $z'_3 - z_3 = \frac{c_3E}{c_1+c_3} - \frac{d_3E}{2}$. Since $c_3 \geqslant d_3$ and $c_1 + c_3 \leqslant 2$, then $z'_3 - z_3 \geqslant 0$.

Subcase 3.3. Claimant 3 leaves. We apply the same argument as in Subcase 3.1.

Step 3. S^d is not *population monotonic* (Fig. 6d). Let $(c, E) \equiv (1, 1, \frac{1}{2}; \frac{1}{2})$. We have $p(S, c) =$ bro.seg$[(0,0,0), (1,1,0), (1,1,\frac{1}{2})]$. The path $p(S^d, c)$ is obtained from $p(S, c)$ by symmetry with respect to $\frac{c}{2}$. Thus, $p(S^d, c) =$ bro.seg$[(0,0,0), (0,0,\frac{1}{2}), (1,1,\frac{1}{2})]$. Then, $S^d(1, 1, \frac{1}{2}; \frac{1}{2}) = (0, 0, \frac{1}{2})$.

Let claimant 2 leave. Then $c_{\{1,3\}} \equiv (1, \frac{1}{2})$. By definition of S, $S(1, \frac{1}{2}; \frac{1}{2}) = P(1, \frac{1}{2}; \frac{1}{2}) = (\frac{1}{3}, \frac{1}{6})$. Since P is *self-dual*, $S^d(1, \frac{1}{2}; \frac{1}{2}) = (\frac{1}{3}, \frac{1}{6})$. Since claimant 3 receives less in the two-claimant problem than in the three-claimant problem, S^d violates *population monotonicity*. □

Proposition 6. *Resource monotonicity is not preserved under O^t.*

Proof. The rule *CEL* satisfies the property but CEL^t does not. To see this, let $N \equiv \{1, 2, 3\}$ and $(c, E) \in \mathcal{C}^N$ be defined by $(c, E) \equiv (10, 20, 30; 10)$. Then $CEL^t(c, E) = (\frac{10}{3}, \frac{10}{3}, \frac{10}{3})$. However, for $E' \equiv 20$, we obtain $CEL^t(c, E') = (0, 10, 10)$. Claimant 1 loses when the endowment increases from E to E'.

Since for $|N| = 2$, CEL^t coincides with *CD*, which is *resource monotonic*, this negative result can be proved by means of *CEL* only with an example involving at least three claimants. However, rules can be constructed to make the point that the property is not preserved under the claims truncation operator for $|N| = 2$. Any such rule has to fail *claims monotonicity*, a property that *CEL* satisfies. The proof is by means of an example S. Let $c \equiv (4, 7)$, $c' \equiv (4, 6)$, $p(S, c) \equiv$ bro.seg$[(0,0), (3.5, 3.5), c]$, and $p(S, c') \equiv$ bro.seg$[(0,0), (2,4), c']$. Both of these paths are monotone, and to obtain a *resource monotonic* rule, it suffices to choose $p(S, \tilde{c})$ to be a monotone path for any other $\tilde{c}$. Let $E = 7$ and $E' = 6$. Now, note that $S^t(c, E) = (3.5, 3.5)$ but $S^t(c, E') = S(t(c, E'), E') = S(c', E') = (2, 4)$. Claimant 2 gains when the endowment increases from E to E'. □

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