Strategy-Proof Location of Two Public Bads in an Interval

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Abstract

We consider the decision of placing two public bads in a region, modelled by a line segment, based on a collective decision of a committee in charge. Committee members have single-dipped preferences determined lexicographically, by the distance to the nearer and the other public bad (lexmin preferences). A (decision) rule takes a profile of reported preferences as input and assigns the locations of the two public bads. All rules satisfying strategy-proofness and Pareto optimality are characterised. These rules pick only boundary locations.

Keywords: public bads, single-dipped preferences, strategy-proofness

JEL Classification: D71

1 Introduction

We consider the problem of locating two similar noxious facilities in a region. For example, consider the problem of locating two windmill parks along the coast line of a country. Here the final locations are thought of as the outcome of some rule representing the collective decision under plausible situations of the committee in charge of the problem. In particular, we investigate the implication of Pareto optimality and strategy-proofness for such a rule on locating two such facilities in a given region.

The region is modelled by a unit line segment and the plausible situations of the committee by combinations of preferences over location pairs. The later combinations are also known as profiles. A rule selects two points from the interval for every reported profile. We assume that each committee member (agent) has a single-dipped lexmin preference over all possible pairs of location. Each agent has a unique point (his dip) on the line segment as the worst location for any of the bads for him. Preference between two different pairs of location is determined by the distance from his dip to the nearer public bad and, in case of a tie, by the distance to the other public bad.

In this situation, a decision rule will take the preferences of all the agents as input, and give a pair of locations as output. We assume that there are finitely many agents. In this paper we define the class of all decision rules

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that simultaneously satisfy two properties. Strategy-proofness - which ensures that for every agent, truth telling is a weakly dominant strategy, and Pareto optimality - which says that given a decision about the locations of the bads, improving the locations for one agent would result in worsening them for other agents.

Under these properties, we show that either bads cannot be located at an interior point of the interval. We characterise all rules satisfying the two mentioned conditions by a pair of monotone voting rules between the boundary points of the region. The voting rules are not independent of each other. This class contains a variety of rules ranging from dictatorial voting to voting by majorities. Lastly we weaken Pareto optimality to unanimity and provide an example of a rule that selects inner points for the location of both the bads.

Our results can be seen as positive results compared to the seminal impossibility theorem of Gibbard (1973) and Satterthwaite (1975) which says that if there are three or more alternatives, then it is impossible to find a non-dictatorial decision rule which is also strategy proof and Pareto optimal. One way out from this impossibility result is to consider restricted preference domains. Here a restricted domain related to single-dipped preferences is considered. Peremans and Storcken (1999) have shown the equivalence between individual and group strategy-proofness on domains of single-dipped preferences. Manjunath (2014) has characterised the class of all non-dictatorial, strategy-proof and Pareto optimal decision rules when preferences are single-dipped on an interval. Barberà, Berga and Moreno (2012) have characterised the class of all non-dictatorial, group strategy-proof and Pareto optimal decision rules when preferences are single-dipped on a line.

But there are impossibility results in this domain as well. Öztürk et al. (2013, 2014) have shown that there does not exist a non-dictatorial decision rule that is strategy-proof and Pareto optimal when preferences are single-dipped on a disk, and on some, but not all, convex polytopes in the plane. Chatterjee et al. (2016) have extended these results to decision rules on a sphere, when preferences are single-dipped or, equivalently in this case, single-peaked.

All these results are about strategy-proof location of one public bad. As far as we know, the present paper is the first one to consider the location of two public bads in a region. Lahiri et al. (2016) has considered the problem of joint decision of placing public bads in each of two neighboring countries, modelled by two adjacent line segments. In that paper, they considered two different specification of single-dipped preferences, myopic extension and lexmin extension. Under myopic extensions, they characterised the class of rules satisfying strategy-proofness, country-wise Pareto optimality, non-corruptibility and far away condition. Finally, they compare this class with the class of rules satisfying strategy-proofness and country-wise Pareto optimality under lexmin extensions. The rules in the present paper bear similarities to the rules in the last papers. There is also literature adopting a mechanism design approach to the location of public bads, that is, including monetary side payments: e.g., recently, Lescop (2007) and Sakai (2012), but we are not aware of results in this area addressing more than one noxious facilities. On the other hand, the problem of locating two public goods on an interval has been considered previously. Ehlers (2002) considered lexicographic extension of single-peaked preference over pairs of locations where, in order to compare two pairs of locations, an agent first compares the best locations from each pair, and if they are same then he compares the worst locations. Ehlers (2002) characterises the class of rules that satisfies Pareto-optimality and replacement-domination.

This paper is organized as follows. Section 2 introduces the model and some preliminary results. Section 3 shows that internal locations are excluded, and Section 4 provides the characterisation of all rules satisfying our conditions. Section 5 concludes.

2 Model

The objective is to locate two bads in the unit interval $A = [0, 1]$. The set of possible alternatives therefore equals $\mathcal{A} = \{(\alpha, \beta) \in A \times A : \alpha \leq \beta\}$. This location is considered to be a collective decision of a non-empty and finite set of agents, say N, with cardinality $n \in \mathbb{N}$.

Each agent $i \in N$ has a so called *lexmin* preference $R_{z(i)}$ over $\mathcal{A}$, characterised by its dip $z(i) \in [0, 1]$ as follows. For alternatives (a_1, b_1), $(a_2, b_2) \in \mathcal{A}$, (a_1, b_1) is at least as good as (a_2, b_2) at $R_{z(i)}$, with the usual notation $(a_1, b_1)R_{z(i)}(a_2, b_2)$, if

$$\min\{|a_1 - z(i)|, |b_1 - z(i)|\} > \min\{|a_2 - z(i)|, |b_2 - z(i)|\}, \text{ or if}$$

$$\begin{aligned}\min\{|a_1 - z(i)|, |b_1 - z(i)|\} &= \min\{|a_2 - z(i)|, |b_2 - z(i)|\} \text{ and} \\ \max\{|a_1 - z(i)|, |b_1 - z(i)|\} &\geq \max\{|a_2 - z(i)|, |b_2 - z(i)|\}\end{aligned}$$

As a lexmin preference is completely determined by its dip, we identify these with their dips and denoted by $z(i)$ instead of $R_{z(i)}$. As usual, $P_{z(i)}$ denotes the strict or asymmetric part of $R_{z(i)}$ and $I_{z(i)}$ denotes the symmetric part.

A preference profile z assigns to each agent i in N a lexmin preference $z(i)$ over $\mathcal{A}$. The set of all preference profiles is denoted by $\mathcal{R}$.

For a profile z and a non-empty set $S \subseteq N$, let z_S denote the restriction of z to S; i.e. $z_S = (z(i))_{i \in S}$. For $i \in N$, profile z' is an i-deviation of z if $z_{N \setminus \{i\}} = z'_{N \setminus \{i\}}$. Define the restriction of a preference $z(i)$ to a subset $\mathcal{C}$ of $\mathcal{A}$ by $z(i)|_{\mathcal{C}} = (\mathcal{C} \times \mathcal{C}) \cap z(i)$. Further, define the restriction of a profile z to $\mathcal{C}$ component wise; i.e., $z|_{\mathcal{C}} = (z(i)|_{\mathcal{C}})_{i \in N}$. For $a, b \in \mathcal{A}$, we denote $z|_{\{a,b\}}$ and $z(i)|_{\{a,b\}}$ also by $z|_{a,b}$ and $z(i)|_{a,b}$ respectively. For $a \in A$ and $S \subseteq N$, let $(a^S, z_{N \setminus S})$ denotes the profile, say z', where for all $i \in N \setminus S$ we have $z'(i) = z(i)$ and for all $i \in S$ we have $z'(i) = a$.

A *rule* f assigns to each preference profile z an alternative $f(z) = (\alpha^f(z), \beta^f(z)) \in \mathcal{A}$ such that for all $z \in \mathcal{R}$, $\alpha^f(z) \leq \beta^f(z)$. For $x, y \in \mathbb{R}$, $\mu(x, y) = \frac{x+y}{2}$ denotes the midpoint of x and y. For any profile $z \in \mathcal{R}$, in case there is no confusion we write $\mu(z)$ instead of $\mu(\alpha^f(z), \beta^f(z))$.

We consider the following properties for a rule f.

Strategy-Proofness f is strategy-proof if for any $i \in N$ and any $z \in \mathcal{R}$ and any i-deviation z' of z, we have $f(z)R_{z(i)}f(z')$.

Remark 1. We define a rule f to be intermediate strategy-proof if for any coalition $S \subseteq N$ and for any profile $z \in \mathcal{R}$ and any S-deviation z' of z such that $z(i) = z(j)$ for all $i, j \in S$, we have $f(z)R_{z(i)}f(z')$ for all $i \in S$. Note that strategy-proofness and intermediate strategy-proofness are equivalent in our setting.

Strategy-proofness says that truth-telling is a weakly dominant strategy.

Pareto optimality Rule f is Pareto optimal if for every profile z there does not exist an $a \in \mathcal{A}$ such that $aR_{z(i)}f(z)$ for all $i \in N$ with $aP_{z(j)}f(z)$ for at least one $j \in N$.

Monotonicity f is monotone if $f(z) = f(z')$ for all $z, z' \in \mathcal{R}$ such that for all agents $i \in N$:

- $z'(i) \leq z(i) \leq \alpha^f(z)$ or
- $0 = \alpha^f(z) \leq z(i) \leq z'(i) < \mu(z)$ or
- $\mu(z) < z'(i) \leq z(i) \leq \beta^f(z) = 1$ or
- $\beta^f(z) \leq z(i) \leq z'(i)$.

Monotonicity is a familiar consequence in the presence of strategy-proofness: in this case it says, roughly, that if the preference of an agent changes such that the chosen pair becomes better when evaluated according to the new preference, then it remains to be chosen. As an aside, it can be shown that this monotonicity condition is weaker than what Maskin Monotonicity would demand in this framework.

Lemma 1. *Let $f : \mathcal{R} \to \mathcal{A}$ satisfy strategy-proofness. Then f satisfies monotonicity.*

Proof. It is sufficient to prove monotonicity for an i-deviation $z' \in \mathcal{R}$ of $z \in \mathcal{R}$ for an agent $i \in N$, under the following two cases.

Case 1 : $z'(i) < z(i) \leq \alpha^f(z)$

In this case, we have the following situations

$\alpha^f(z) < \alpha^f(z')$: This is a violation of strategy-proofness, as agent i can manipulate from z to z'.

$\alpha^f(z) > \alpha^f(z')$: In this case, we have $\alpha^f(z') \leq z'(i) - r$, where $r = \alpha^f(z) - z'(i)$, otherwise i manipulates from $z'(i)$ to $z(i)$. In turn this implies $|z(i) - \alpha^f(z')| > |z(i) - \alpha^f(z)|$, so we must have $\alpha^f(z) = \beta^f(z')$, otherwise i manipulates from $z(i)$ to $z'(i)$. Now $\beta^f(z) \leq z'(i) + (z'(i) - \alpha^f(z'))$, otherwise i manipulates from $z'(i)$ to $z(i)$; and $\beta^f(z) \geq z(i) + (z(i) - \alpha^f(z'))$, otherwise i manipulates from $z(i)$ to $z'(i)$. These two inequalities combined, however, contradict the assumption that $z(i) > z'(i)$.

The only remaining possibility is $\alpha^f(z) = \alpha^f(z')$, and by strategy-proofness this implies $\beta^f(z) = \beta^f(z')$.

Case 2 : $0 = \alpha^f(z) \leq z(i) < z'(i) < \mu(z)$

In this case, strategy-proofness for the deviation from z to z' implies that either $\alpha^f(z')$ or $\beta^f(z')$ is in the interval $[0, 2z(i)]$. Also strategy-proofness for the deviation from z' to z implies that neither $\alpha^f(z')$ nor $\beta^f(z')$ is in the interval $(0, 2z'(i))$. As $z(i) < z'(i) < \mu(z)$, we have $2z(i) < 2z'(i) < \beta^f(z)$. So it follows that $[0, 2z(i)] \backslash (0, 2z'(i)) = \{0\}$. So suppose that $\beta^f(z') = 0 = \alpha^f(z)$. As $z'(i) < \mu(z)$, we have $f(z)P_{z'(i)}f(z')$, which is a violation of strategy-proofness for the deviation from z' to z. So it follows that $\alpha^f(z') = 0$ and by strategy-proofness this implies $\beta^f(z') = \beta^f(z)$.

This concludes the proof of Lemma 1. $\square$

3 No internal solution

In this section, let f be a strategy-proof and Pareto optimal rule. We show that f cannot assign an internal location in A, i.e. the alternative chosen at any profile is a corner point of $\mathcal{A}$. Formally,

Theorem 1. *Let f is a strategy-proof and Pareto optimal rule. Then for any profile $z \in \mathcal{R}$, $f(z) \in \{(1,1),(0,1),(0,0)\}$.*

We prove this Theorem by the following three lemmas.
The first lemma shows that if one of the two bads is located at the extreme end of A, then the other one cannot be located at an interior point of A.

Lemma 2. *For a profile z, $\alpha^f(z) = 0$ implies $\beta^f(z) \in \{0,1\}$ and $\beta^f(z) = 1$ implies $\alpha^f(z) \in \{0,1\}$.*

Proof. Due to symmetry, it is sufficient to prove that $\alpha^f(z) = 0$ implies $\beta^f(z) \in \{0,1\}$. So suppose $\alpha^f(z) = 0$ but to the contrary $\beta^f(z) \in (0,1)$. Consider coalitions $S = \{i \in N : z(i) < \mu(z)\}$ and $T = \{i \in N : z(i) \geq \beta^f(z)\}$. Pareto optimality of f implies that both S and T are non-empty.
Let $a = \max_{i \in S(z)} z(i)$. In view of Lemma 1, we may assume that $z = (\mu(a,\mu(z))^S, 1^T, z_{N\setminus(S\cup T)})$.
Now consider three profiles z^1, z^2 and z^3 as follows.

	S	T	$N\setminus(S\cup T)$
z	$\mu(a,\mu(z))$	1	$z(i)$
z^1	$\mu(z)$	1	$z(i)$
z^2	$\mu(a,\mu(z))$	$\mu(\beta^f(z),1)$	$z(i)$
z^3	$\mu(z)$	$\mu(\beta^f(z),1)$	$z(i)$

The first line of this table shows the profile z. The second line shows the profile z^1, which is an $S-$deviation from z, where $z^1(i) = \mu(z)$ for all $i \in S$. Similarly, profiles z^2 and z^3 are defined by the proceeding lines.

Consider the deviation from z to z^1. Strategy-proofness for this deviation implies that $\alpha^f(z^1) = 0$ and $\beta^f(z^1) \in \{0, \beta^f(z)\}$. So $f(z^1) \in \{(0,0), f(z)\}$. As $(0,0)I_{z^1(i)}f(z)$ for all $i \in S$ and $(0,0)P_{z^1(k)}f(z)$ for all $k \in N\setminus S$, Pareto optimality implies that $f(z^1) = (0,0)$.
As $\mu(\beta^f(z),1) > \frac{1}{2}$, strategy-proofness for the deviation from z^1 to z^3 implies that $f(z^3) = (0,0)$.
On the other hand, consider the deviation from z to z^2. Strategy-proofness implies that $f(z^2) \in \{f(z),(0,1)\}$. As $(0,1)I_{z^2(i)}f(z)$ for all $i \in T$ and $(0,1)P_{z^2(k)}f(z)$ for all $i \in N\setminus T$, Pareto optimality implies that $f(z^2) = (0,1)$.
Next consider the deviation from z^2 to z^3. As $\mu(a,\mu(z)) < \mu(z) < \frac{1}{2}$, strategy-proofness for this deviation implies that $f(z^3) = f(z^2) = (0,1)$, which contradicts the fact that $f(z^3) = (0,0)$. □

The second lemma shows that both bads cannot be located at a common inner point.

Lemma 3. *For the profile z, if $\alpha^f(z) = \beta^f(z) = c$, then $c \in \{0,1\}$.*

Proof. Suppose $\alpha^f(z) = \beta^f(z) = c$ but to the contrary $c \in (0,1)$. Define a coalitions $S = \{i \in N : z(i) \leq c\}$. Note that, Pareto optimality implies that S is a non-trivial subset of N. In view of Lemma 1, we may assume that $z = (0^S, 1^{N\backslash S})$. Now consider the three profiles z^1, z^2 and z^3 as follows.

	S	$N\backslash S$
z	0	1
z^1	$\mu(0,c)$	1
z^2	0	$\mu(c,1)$
z^3	$\mu(0,c)$	$\mu(c,1)$

Consider the deviation from z to z^1. Strategy-proofness for this deviation implies that $\alpha^f(z^1) \in \{0, c\}$. Then Lemma 2 and strategy-proofness implies that $f(z^1) \in \{(0,0), (0,1), f(z)\}$. Note that $(0,0)I_{z^1(i)}f(z)$ for all $i \in S$ and $(0,0)P_{z^1(k)}f(z)$ for all $k \in N\backslash S$. Also $(\mu(c,1), \mu(c,1))P_{z^1(i)}(0,1)$ for all $i \in N$. So Pareto optimality implies that $f(z^1) = (0,0)$. As $\mu(c,1) > \frac{1}{2}$, strategy-proofness for the deviation from z^1 to z^3 implies that $f(z^3) = (0,0)$.

Similarly, we can derive from $f(z) = (c,c)$, that $f(z^2) = (1,1)$, and therewith $f(z^3) = (1,1)$, which contradicts the fact that $f(z_3) = (0,0)$. □

The third lemma shows that the two bads cannot be both in the interior of A.

Lemma 4. *For the profile z, $(\alpha^f(z), \beta^f(z)) \notin (0,1) \times [\alpha^f(z), 1)$.*

Proof. Due to Lemma 3, it is sufficient to show that $(\alpha^f(z), \beta^f(z)) \notin (0,1) \times (\alpha^f(z), 1)$. First we show that in such a case we may assume that $\{i \in N : \alpha^f(z) < z(i) < \beta^f(z)\} = \emptyset$.

Claim 1. *Suppose $0 < \alpha^f(z) < \beta^f(z) < 1$ and there exists $\emptyset \neq M \subsetneq N$ such that $\alpha^f(z) < z(i) < \beta^f(z)$ for all $i \in M$. Then there exists $z' \in \mathcal{R}$ such that $0 < \alpha^f(z') < \beta^f(z') < 1$ and $\{i \in N : \alpha^f(z') < z'(i) < \beta^f(z')\} = \emptyset$.*

Proof of claim 1. Define coalition $T = \{i \in N : z(i) \geq \beta^f(z)\}$. Note that Pareto optimality implies that T is a non-trivial subset of N. Using Lemma 1, we may assume that $z = (0^{N\backslash(M\cup T)}, z_M, 1^T)$. Let agent $i \in M$. Suppose $z(i) \leq \frac{1}{2}$. Consider the $i-$deviation z^1 of z, where $z^1(i) = 0$. Strategy-proofness for the deviations between z and z^1 imply that $\alpha^f(z^1) \in [\alpha^f(z), \beta^f(z)]$.
As $0 < \alpha^f(z) < \beta^f(z) < 1$, so Lemma 2 implies that neither $\alpha^f(z^1) \in \{0,1\}$ nor $\beta^f(z^1) \in \{0,1\}$. So it follows that $0 < \alpha^f(z^1) \leq \beta^f(z^1) < 1$. Then Lemma 3 implies $0 < \alpha^f(z^1) < \beta^f(z^1) < 1$. If $z(i) > \frac{1}{2}$, then consider the $i-$deviation z^1 of z, where $z^1(i) = 1$. By a similar argument we have $0 < \alpha^f(z^1) < \beta^f(z^1) < 1$. Define $M^1 = \{i \in N : \alpha^f(z^1) < z^1(i) < \beta^f(z^1)\}$. Now if $M^1 = \emptyset$, then this concludes the proof of Claim 1. Otherwise repeat the procedure above with M^1 in the role of M and construct M^2. As $|M| < n$, there exists a finite $k \in \mathbb{N}$ such that $M^k = \emptyset$. This concludes the proof of Claim 1.

Now suppose $0 < \alpha^f(z) < \beta^f(z) < 1$. In view of Claim 1 and Lemma 1, we can assume that there is a coalition S, such that $z = (0^S, 1^{N\backslash S})$. Pareto optimality implies that S is a non -trivial coalition. But now $(\mu(z), \mu(z))$ Pareto dominates $f(z)$. This contradiction proves Lemma 4. □

Proof of Theorem 1. Follows from Lemmas 2, 3 and 4. □

4 Characterisation

In this section, we characterise the class of rules satisfying strategy-proofness and Pareto optimality. Note that because of Theorem 1, the range of any such rule is a subset of $\mathcal{B} = \{00, 01, 11\}$, where 00 denotes $(0,0)$ and so on. Restricted to $\mathcal{B}$, we have the following preferences.

Dips	Preferences
$z(i) < 0.5$	$11P_{z(i)}01P_{z(i)}00$
$z(i) = 0.5$	$11I_{z(i)}01I_{z(i)}00$
$z(i) > 0.5$	$00P_{z(i)}01P_{z(i)}11$

Note that if $z(i) < 0.5$, then 11 is the unique top ranked alternative of $R_{z(i)}$. Similarly, if $z(i) > 0.5$, then 00 is the unique top ranked alternative of $R_{z(i)}$. On the other hand if $z(i) = 0.5$, then the top ranked alternatives of $R_{z(i)}$ is the set $\mathcal{B}$. This brings us to the following lemma.

Lemma 5. *Let f be a strategy-proof and Pareto optimal rule. Then $f(z) = f(z')$ for any $z, z' \in \mathcal{R}$ such that $z|_{\mathcal{B}} = z'|_{\mathcal{B}}$.*

Proof. Follows from Theorem 1 and strategy-proofness. □

Next we introduce a strong monotonicity property of a rule f as follows.

Strong Monotonicity Rule f is strongly monotone if $\alpha^f(z) \geq \alpha^f(z')$ and $\beta^f(z) \geq \beta^f(z')$ for all $i \in N$ and for all profiles $z, z' \in \mathcal{R}$ such that $z(i) \leq z'(i)$.

Next lemma shows an implication of strategy-proofness and Pareto optimality.

Lemma 6. *Suppose f be a strategy-proof and Pareto optimal rule. Then f is strongly monotone.*

Proof. As f is strategy-proof and Pareto optimal, Theorem 1 implies that $f(z) \in \mathcal{B}$ for any $z \in \mathcal{R}$. It is sufficient to show that $\alpha^f(z) \geq \alpha^f(z')$ and $\beta^f(z) \geq \beta^f(z')$, where $z' \in \mathcal{R}$ is an $i-$deviation of $z \in \mathcal{R}$ such that $z(i) < z'(i)$. Now consider the following cases.

$z(i) < \frac{1}{2}$: In this case, we have $11P_{z(i)}01P_{z(i)}00$. Hence strategy-proofness for the deviation from z to z' implies that $\alpha^f(z) \geq \alpha^f(z')$ and $\beta^f(z) \geq \beta^f(z')$.

$z'(i) > \frac{1}{2}$: In this case, we have $00P_{z'(i)}01P_{z'(i)}11$. Hence strategy-proofness for the deviation from z' to z implies that $\alpha^f(z) \geq \alpha^f(z')$ and $\beta^f(z) \geq \beta^f(z')$.

This concludes the proof of Lemma 6. □

Next lemma shows the opposite direction.

Lemma 7. *Suppose f be a strongly monotone rule. Also suppose that $f(z) \in \mathcal{B}$ for all $z \in \mathcal{R}$. Then f is strategy-proof.*

Proof. Suppose $z' \in \mathcal{R}$ be an $i-$deviation of some $z \in \mathcal{R}$. Without loss of generality, assume that $z(i) < z'(i)$. Now we consider the following cases.

If $z(i) = \frac{1}{2}$, hence $11 I_{z(i)} 01 I_{z(i)} 00$, strategy-proofness follows as i is indifferent between all the elements in $\mathcal{B}$.

If $z(i) > \frac{1}{2}$, then strategy-proofness follows because $f(z) = f(z')$ by Lemma 5.

If $z(i) < \frac{1}{2}$, hence $11 P_{z(i)} 01 P_{z(i)} 00$, strategy-proofness follows as by monotonicity $\alpha(z') \leq \alpha(z)$

This concludes the proof of Lemma 7. □

Now we define a class of rules as follows. Each rule is characterised by two families of pairs of coalitions $\mathcal{W}_\alpha, \mathcal{W}_\beta \in 2^N \times 2^N$. The first coalition in such a pair can be thought of as the collection of agents for whom 11 is the unique top ranked alternative. The second coalition in such a pair can be thought of as the collection of agents who are indifferent among all alternatives in $\mathcal{B}$. $\mathcal{W}_\alpha$ can be thought of as the collection of such pairs who are decisive to locate $\alpha^f()$ at 1. Similarly $\mathcal{W}_\beta$ can be thought of as the collection of such pairs who are decisive to locate $\beta^f()$ at 1. We define two such families $\mathcal{W}_\alpha$, $\mathcal{W}_\beta$ as decisive if they satisfy the following properties.

Inclusion : $\mathcal{W}_\alpha \subseteq \mathcal{W}_\beta$.
This follows from the fact that $\alpha^f() \leq \beta^f()$.

Monotonicity : For $(S, T) \in \mathcal{W}_\alpha(\mathcal{W}_\beta)$, we have $(S', T') \in \mathcal{W}_\alpha(\mathcal{W}_\beta)$ whenever $S \subseteq S'$ and $S \cup T \subseteq S' \cup T'$.
This is a direct translation of the strong monotonicity property introduced previously in this section.

Boundary condition : $(X, N \backslash X) \in \mathcal{W}_\alpha$ for all non-empty $X \subseteq N$, and $(\emptyset, Y) \notin \mathcal{W}_\beta$ for all $Y \subsetneq N$.
This is an implication of Pareto optimality, restricted to the boundary points.

Non-compromising at maximal conflict : $T \neq \emptyset$ for all $(S, T) \in \mathcal{W}_\beta \backslash \mathcal{W}_\alpha$.
This property ensures that 01 cannot be selected if there are no agents indifferent among all the alternatives in $\mathcal{B}$. The fact that this holds for any strategy-proof and Pareto optimal rule can be seen as follows. Suppose f be a strategy-proof and Pareto optimal rule, but to the contrary $f(z) = 01$ for some $z \in \mathcal{R}$ with $z(i) \neq 0.5$ for all $i \in N$. This implies that in the profile z, there are no agents indifferent among all the alternatives in $\mathcal{B}$. Now consider another profile $z^\star \in \mathcal{R}$ as follows. $z^\star(i) = 0$ if $z(i) < 0.5$ and $z^\star(i) = 1$ if $z(i) > 0.5$. Note that $z|_\mathcal{B} = z^\star|_\mathcal{B}$. As f is strategy-proof and Pareto optimal, Lemma 5 implies that $f(z) = f(z^\star) = 01$. This contradicts Pareto optimality as $(\alpha, \alpha) P_{z^\star(i)} (0, 1)$ for all $i \in N$ and for any $\alpha \in (0, 1)$.

Based on these two families, we define a rule $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}$ as follows. For any profile $z \in \mathcal{R}$, define

$S(z) = \{i \in N : z(i) < \frac{1}{2}\}$.

$T(z) = \{i \in N : z(i) = \frac{1}{2}\}$.

$$f^{\mathcal{W}_\alpha \mathcal{W}_\beta}(z) = \begin{cases} 00 & \text{if } (S(z), T(z)) \notin \mathcal{W}_\beta \\ 01 & \text{if } (S(z), T(z)) \in \mathcal{W}_\beta \backslash \mathcal{W}_\alpha \\ 11 & \text{if } (S(z), T(z)) \in \mathcal{W}_\alpha \end{cases}$$

This brings us to our final theorem.

Theorem 2. *Let $f : \mathcal{R} \longrightarrow \mathcal{A}$ be a rule. Then f is strategy-proof and Pareto optimal if and only if there exist two families of pairs of coalitions $\mathcal{W}_\alpha$ and $\mathcal{W}_\beta$, which are decisive; such that $f(z) = f^{\mathcal{W}_\alpha \mathcal{W}_\beta}(z)$ for all $z \in \mathcal{R}$.*

We prove this theorem with the help of the following two lemmas. The first lemma shows the only if direction of Theorem 2. For any rule f, define

$$\mathcal{W}_\alpha^f = \left\{ (S, T) \in 2^N \times 2^N : \exists\ z \in \mathcal{R} \text{ with } \begin{array}{c} \alpha^f(z) = 1 \text{ and} \\ S = S(z) \text{ and } T = T(z) \end{array} \right\}.$$

$$\mathcal{W}_\beta^f = \left\{ (S, T) \in 2^N \times 2^N : \exists\ z \in \mathcal{R} \text{ with } \begin{array}{c} \beta^f(z) = 1 \text{ and} \\ S = S(z) \text{ and } T = T(z) \end{array} \right\}.$$

This brings us to the following lemma.

Lemma 8. *Suppose $f : \mathcal{R} \longrightarrow \mathcal{A}$ be a strategy-proof and Pareto optimal rule. Then $\mathcal{W}_\alpha^f$ and $\mathcal{W}_\beta^f$ are decisive.*

Proof. As f is strategy-proof and Pareto optimal, $\mathcal{W}_\alpha^f$ and $\mathcal{W}_\beta^f$ are well defined sets. Next we show that $\mathcal{W}_\alpha^f$ and $\mathcal{W}_\beta^f$ are decisive. Note that the inclusion property follows from $\alpha^f(z) \leq \beta^f(z)$. As f is strategy-proof and Pareto optimal, Lemma 6 implies that f is strongly monotone. This in turn implies the monotonicity property. Also the fact that $\mathcal{W}_\alpha^f$ and $\mathcal{W}_\beta^f$ are non-compromising at maximal conflict and satisfies the boundary condition follows directly from Pareto optimality of f. □

The next lemma shows the if direction of Theorem 2.

Lemma 9. *Suppose $\mathcal{W}_\alpha \subset 2^N \times 2^N$ and $\mathcal{W}_\beta \subset 2^N \times 2^N$ be two families of pairs of coalitions which are decisive. Then $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}$ is strategy-proof and Pareto optimal.*

Proof. Note that because of the inclusion property, $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}$ is a well defined function. Next we show that $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}$ is strategy-proof.

Proof of strategy-proofness. Note that $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}(z) \in \mathcal{B}$ for all $z \in \mathcal{R}$. Then in view of Lemma 7, it is sufficient to show that $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}$ is strongly monotone. Consider $z, z' \in \mathcal{R}$ such that $z(i) \leq z'(i)$ for all $i \in N$. This implies that either $S(z) \subseteq S(z')$ or $S(z) \cup T(z) \subseteq S(z') \cup T(z')$. Then monotonicity of $\mathcal{W}_\alpha$ and $\mathcal{W}_\beta$ yields that $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}$ is strongly monotone.

This shows that $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}$ is strategy-proof. Next we show that $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}$ is Pareto optimal.

Proof of Pareto optimality. Note that $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}$ is Pareto optimal if for any $z \in \mathcal{R}$; either there is an agent $j \in N$ for whom $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}(z)$ is the unique top ranked alternative at $R_{z(j)}$; or $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}(z)$ is one of the top ranked alternatives of all agents. As $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}(z) \in \mathcal{B}$ for all $z \in \mathcal{R}$, we distinguish the following cases.

Case I : $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}(z) = 11$

We are done if there exists an agent $j \in N$ such that $z(j) < 0.5$. If $z(i) \geq 0.5$ for all $i \in N$, by the boundary condition it follows that $z(i) = 0.5$ for all $i \in N$. But then 11 is among the top ranked alternatives of all agents and the alternative is Pareto optimal at this profile.

Case II : $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}(z) = 00$

Similar to the case I.

Case III : $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}(z) = 01$

In this case, the non-compromising at maximal conflict property implies that $T(z) \neq \emptyset$. Note that $\mathcal{B}$ is the set of all top ranked alternatives for all agents in $T(z)$. So 01 cannot be improved by any alternatives in $\mathcal{A} \backslash \mathcal{B}$. Now suppose, for contradiction that 11 is weakly better than 01 for all agents and 11 is strictly better than 01 for atleast one agent $j \in N$. This implies that $S(z) \cup T(z) = N$ and $S(z) \neq \emptyset$. Then the boundary condition implies that $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}(z) = 11$, which contradicts our assumption that $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}(z) = 01$. Similarly suppose, for contradiction that 00 is weakly better than 01 for all agents and 00 is strictly better than 01 for atleast one agent $j \in N$. This implies that $S(z) = \emptyset$ and $T(z) \neq N$. Then the boundary condition implies that $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}(z) = 00$, which contradicts our assumption that $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}(z) = 01$. So we can conclude that 01 is a Pareto optimal outcome at this profile.

Combining these cases yields that $f^{\mathcal{W}_\alpha \mathcal{W}_\beta}$ is Pareto optimal and concludes the proof of Lemma 9. □

Proof of Theorem 2. In view of Lemmas 8 and 9, it is sufficient to show that $f(z) = f^{\mathcal{W}^f_\alpha \mathcal{W}^f_\beta}(z)$ for all $z \in \mathcal{R}$.

Note that $\alpha^f(z) = 1 (\beta^f(z) = 1) \Leftrightarrow (S(z), T(z)) \in \mathcal{W}^f_\alpha$ $(\mathcal{W}^f_\beta)$. This shows that $f(z) = f^{\mathcal{W}^f_\alpha \mathcal{W}^f_\beta}(z)$ for all $z \in \mathcal{R}$ and concludes the proof of Theorem 2. □

5 Conclusion

In this section, first we provide an example of a non-dictatorial rule that belongs to the class described in Section 4.

Example 1. Define two families of pairs of coalitions ($\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$) as follows.

$$\mathcal{V}_\alpha = \left\{ (S,T) \in 2^N \times 2^N : \begin{array}{l} \text{either } S \cup T = N \text{ and } S \neq \emptyset \\ \text{or } |S| > \frac{3n}{4} \text{ and } T \neq \emptyset \end{array} \right\}.$$

$$\mathcal{V}_\beta = \mathcal{V}_\alpha \cup \left\{ (S,T) \in 2^N \times 2^N : |S| > \tfrac{n}{4} \text{ and } T \neq \emptyset \right\}.$$

Note that the two families $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$ are decisive. So Theorem 2 implies that $f^{\mathcal{V}_\alpha \mathcal{V}_\beta}$ is strategy-proof and Pareto optimal.

Theorem 2 characterises the class of rules in our model. Although, the class contain rules other than dictatorships, this is indeed limited. These rules do not select inner points for any profile of reported preferences. Now suppose we consider a weakening of Pareto optimality, namely unanimity, which says that if at a given profile there exists atleast one alternative which is common among the top ranked alternatives of all agents, then the rule should select one of those alternative at that profile. Then we can show that Theorem 2 does not hold any more. In our domain of preferences, strategy-proofness together with unanimity does not imply Pareto optimality. We show this by means of the following example.

Example 2. This rule is characterised by an alternative $(\alpha, \beta) \in (0,1) \times (0,1)$. For any profile $z \in \mathcal{R}$, define

$N_1(z) = \{i \in N : z(i) \leq \frac{1}{2}\}$.

$N_0(z) = \{i \in N : z(i) > \frac{1}{2}\}$.

$y_l(z) = \max\limits_{i \in N_1(z)} z(i)$.

$y_u(z) = \min\limits_{i \in N_0(z)} z(i)$.

$$h^{(\alpha,\beta)}(z) = \begin{cases} (1,1) & \text{if } z(i) \leq \frac{1}{2} \text{ for all } i \in N \\ (0,0) & \text{if } z(i) \geq \frac{1}{2} \text{ for all } i \in N \\ & \quad \text{with } z(j) > \frac{1}{2} \text{ for atleast one } j \in N \\ (\alpha, \beta) & \text{if } 2y_l(z) < \alpha \leq \beta < 2y_u(z) - 1 \\ (0,1) & \text{otherwise} \end{cases}$$

Note that this rule is unanimous. Next, we show that this rule satisfies strategy-proofness. Without loss of generality consider a profile z such that there exists an $i \in N$ with $z(i) < \frac{1}{2}$. Consider an $i-$deviation z' of z and assume that $h^{(\alpha,\beta)}(z) \neq h^{(\alpha,\beta)}(z')$. We have to show that $h^{(\alpha,\beta)}(z) R_{z(i)} h^{(\alpha,\beta)}(z')$. As $z(i) < \frac{1}{2}$, so $(1,1)$ is the unique top ranked alternative of agent i according to the preference $R_{z(i)}$. So, if $h^{(\alpha,\beta)}(z) = (1,1)$, then it follows that $h^{(\alpha,\beta)}(z) P_{z(i)} h^{(\alpha,\beta)}(z')$. So assume that $h^{(\alpha,\beta)}(z) \neq (1,1)$. As $z(i) < \frac{1}{2}$, we can conclude that $h^{(\alpha,\beta)}(z) \neq (0,0)$. Also as $h^{(\alpha,\beta)}(z) \neq (1,1)$, there exists an agent $j \in N \backslash \{i\}$ such that $z(j) > \frac{1}{2}$. Without loss of generality, assume that $y_l(z) = z(i)$ and $y_u(z) = z(j)$ As z' is an $i-$deviation of z, it follows that $h^{(\alpha,\beta)}(z') \neq (1,1)$. Now we consider the following cases.

$h^{(\alpha,\beta)}(z) = (\alpha, \beta)$:

In this case, $2y_l(z) < \alpha \leq \beta < 2y_u(z) - 1$. This implies $|\alpha - z(i)| > |0 - z(i)|$. So we can conclude that $(\alpha, \beta) P_{z(i)} (0,0)$ and $(\alpha, \beta) P_{z(i)} (0,1)$. So in this case we can conclude that $h^{(\alpha,\beta)}(z) P_{z(i)} h^{(\alpha,\beta)}(z')$.

$h^{(\alpha,\beta)}(z) = (0,1)$:

Note that if $h^{(\alpha,\beta)}(z') = (0,0)$ then we have $h^{(\alpha,\beta)}(z) P_{z(i)} h^{(\alpha,\beta)}(z')$. So we consider the following sub cases.

$\alpha \leq 2y_l(z) = 2z(i)$: In this situation, as $\beta < 1$, we have $(0,1) P_{z(i)} (\alpha, \beta)$.

$\alpha > 2y_l(z) = 2z(i)$: In this situation, as $h^{(\alpha,\beta)}(z) = (0,1)$, so we have $2y_l(z) - 1 = 2z(j) - 1 \leq \beta$. Then from the definition of $h^{(\alpha,\beta)}$ it follows that agent i cannot change the outcome to (α, β) by unilateral deviation.

From these sub cases, it follows that in this case we have $h^{(\alpha,\beta)}(z)P_{z(i)}h^{(\alpha,\beta)}(z')$.

Also note that if $z(i) = \frac{1}{2}$, then $h^{(\alpha,\beta)}(z) \neq (\alpha, \beta)$, and the remaining three alternatives are his top ranked alternatives. So we can conclude that this rule is strategy-proof. Evidently, this rule is not Pareto optimal.

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