The Majoritarian Social Welfare Relation: Some Results*

Mihir Bhattacharya† Nicolas Gravel‡

June 27, 2016

Abstract

For any given preference profile of voters, a *social welfare relation* is a complete binary relation over the set of alternatives. A *social welfare relation* is *distance-minimising* if it selects a binary relation that is 'closest' to the given voter profile according to the a notion of distance defined over the set of all possible preference orderings. We define a *quaternary relation* over the set of all preference orderings and assume that it satisfies certain 'weak' assumptions. We show that under these conditions there exists a distance function over which the *class* of *majoritarian social welfare relations* is *distance-minimising.* We also characterize the *majority binary relation* over which these conditions become necessary.

JEL classification: D71, D72

Keywords: Social Welfare Relation, Majority Relation, Distance Minimisation

PRELIMINARY AND INCOMPLETE. DO NOT QUOTE.

*The authors would like to thank Mathieu Faure, Antonin Macé, Ali Ihsan Ozkes, and (...)

†Postdoctoral researcher, GREQAM, AMSE, Aix-Marseille Université

‡Professor, GREQAM, AMSE, Aix-Marseille Université

1 Introduction

Social choice theory, from its conception, has been concerned with aggregation of voter preferences. The well-known Arrow's result states that no such aggregation is possible when certain 'appropriate' properties are imposed on the aggregating rule. A significant amount of literature has attempted to avoid these problems by either restricting the domain of preferences or by relaxing some of the axioms.[1] In this paper we take the latter approach by relaxing the assumption of transitivity of the social welfare relation.

We know from the literature that transitivity of the social welfare relation is a strong requirement and it is only guaranteed by imposing certain assumptions on individual preferences.[2] Transitivity of social welfare relations has been known to produce irrational outcomes. For example, Fishburn (1970) show that transitive social choice functions may contradict with the principles of unanimity for some vote profiles.

We already know social welfare relations may violate transitivity even though individual preferences are transitive.[3] This implies that requiring the social welfare relation to be transitive is much stronger requirement than imposing it on individual preferences. We provide an example of a situation when relaxing this assumption may be useful.

Consider a setting where the central planner has not yet decided to implement a single outcome. In such a situation her only concern is to obtain a *complete* social binary relation from the individual preferences. The decision to produce a good is therefore, postponed for a later time. Moreover, in cases of tie, i.e., when one outcome needs to be implemented and no majority winner exists, a tie-breaking rule can be used to obtain a final outcome.

We impose an ordinal notion of 'similarity' over the set of all preference orderings. This is represented by a *quaternary* relation which is defined over the set of all binary relations over the set of alternatives. A *quaternary* relation compares pairs of binary relations and orders them according to their 'similarity' or 'dissimilarity'.

We impose certain 'natural' properties on the quaternary relation and show that there exist distance functions, defined over the set of all preference orderings, which represent the given quaternary relation. We use the literature on measurement theory (Krantz et al. (1990)) to show that this distance function satisfies certain additional properties.

We show that these properties are sufficient for the class of *majoritarian* social welfare relations to be the unique class of *distance-minimising* social welfare relations. We also show that if we relax the assumption of transitivity on the voter preferences then these properties are *necessary* and *sufficient* for specific class of majoritarian social welfare relations. We give a brief description of the model and discuss the literature and our contribution to this literature.

[1]See Arrow et al. (2010) for detailed exposition of the social choice literature on avoiding the negative results of Arrow.

[2]Sen and Pattanaik (1969) provide conditions on individual choice functions which guarantee transitivity of the social choice function.

[3]This is exemplified by the famous Condorcet's paradox.

1.1 Model and Results

Consider a model with finite number of voters and alternatives. Each voter submits a preference ordering over the set of alternatives. The social welfare relation outputs a complete, but not necessarily transitive, binary relation.

We define a quaternary relation Q over the set of all binary relations over the set of alternatives as follows. We write $(W_1W_2)Q(W_3W_4)$ if and only if W_1 and W_2 are 'more similar' to each other than W_3 and W_4. Alternatively this can also be defined in terms of 'dissimilarity' (see Krantz et al. (1989) for a complete theory of measurement).

We impose the following properties over the quaternary relation- (i) *linear* (ii) *symmetric* (iii) *closest at identity* (iv) *equivalence of identical pairs* (v) *Q-betweenness* and (vi) *collinearity*. All the properties except Q-betweenness are standard in the literature on measurement theory. We briefly explain Q-betweenness.

We use the notion of *betweenness* of preferences to define *Q-betweenness*. A profile W_2 is between W_1 and W_3 if it contains all the common ordered pairs in W_1 and W_3 but not those which are in neither of them. For example, suppose the set of alternatives is $\{x, y, z\}$. Let $W_1 = \{(x, y), (y, x), (y, z), (z, x)\}$, $W_2 = \{(x, y), (y, z), (z, x)\}$, $W_3 = \{(x, y), (y, z), (x, z)\}$ and $W_4 = \{(x, y), (y, z), (z, y), (z, x)\}$ be four complete binary relations over the set of alternatives. Then W_2 is between W_1 and W_3 but W_4 is not between W_1 and W_3 since it consists of the pair (z, y) which is in neither W_1 nor W_3 even though it contains the common pairs (x, y) and (y, z) between W_1 and W_3. This has become a standard notion of betweenness in the social choice literature due to its properties.[4]

Suppose W_1 is *between* W_2 and W_3. Then Q-betweenness requires that W_1 and W_2 be strictly more similar to each other according to Q than W_1 and W_3 are to each other. Similarly, it also requires that W_2 and W_3 are strictly more similar to each other according to Q than W_1 is to W_3.

This property allows us to "embed" a notion of betweenness that is appropriate in this social choice setting. Moreover, the standard notion of imposing *collinearity* as in Krantz et al. (1990) is too weak and does not guarantee the *additivity* property of the distance metric that we require for our main result. We describe the *majoritarian* social welfare relation below.

Suppose for a given profile at least a majority of voters weakly prefer x over y. A social welfare relation is *majoritarian* if x is socially weakly preferred to y under this rule for the given profile. We use the following notion for our result.

Let d be a distance metric defined over the set of binary relations. A social welfare relation is *distance-minimising* if it minimizes the sum of the distances between itself and the preference orderings of the voters. We briefly state our main result.

Suppose the quaternary relation Q satisfies assumptions (i)-(vi). Then there exists a dis-

[4]There are numerous works which deal with the same notion of betweenness. See, for example, Nehring and Puppe (2007), Can and Storcken (2013) and Lainé et al. (2016).

tance function d which is a numerical representation of Q such that the class of majoritarian social welfare relations is the unique class of distance-minimising. This implies that *any* majoritarian social welfare relation will minimize the sum of the distances (corresponding to d) between itself and the voter preferences when compared to any other non-majoritarian social welfare relation.

We show that the properties satisfied by the distance metric d which makes this result possible are weaker than the ones used in the literature. More specifically, the additivity of the distance function guarantees our result. We show that Kemeny's notion of distance belongs to this broad class of 'additive' distance metrics.

Later, in the discussion of our mail result we provide an example of a distance metric which satisfies additivity and is not Kemeny. Therefore, the set of distance metrics over which the class of majoritarian social welfare relations is distance-minimising is large.

Our second result characterizes the specific class of majoritarian social welfare relation that is distance-minimising over the same distance metric that was used for our main result. Moreover, this distance metric satisfies properties which are necessary and sufficient to guarantee that this class of social welfare relations will be distance-minimising. We call the social welfare relations which belong to this class as majority binary relation.

It is necessary to relax the assumption of transitivity of voter preferences to obtain the properties of the distance metric. We describe this rule below.

The majority binary relation always strictly prefers x over y if either (i) a strict majority strictly prefers x over y or (ii) a strict minority (i.e. strictly less than half the number of voters) prefer (weakly or otherwise) y over x.

If half the voters strictly prefer x over y and the other half strictly prefers y over x then a tie-breaking rule is applied. The tie-breaking rule is such that any of the three possible orderings can be chosen x and y: (i) x strictly preferred over y (ii) y strictly preferred over x and (iii) x and y indifferent. We assume for completeness that it chooses each with equal probability but this is not important for our result. However, the fact that *each* of the three orderings is possible under the tie-breaking rule is important for our second result.

1.2 Literature Review

A large amount of work in the literature on social choice theory has dealt rationalizing different social aggregators through the use of a distance notions defined over the set of possible preferences.[5]

It is well known in the literature on *median rules* that the majoritarian social welfare

[5] A part of this literature deals with choosing aggregating rules which select outcomes or rankings closest to the profile which would have unanimously selected the prescribed outcome (see Lerer and Nitzan (1985)) and Andjiga et al. (2014)).

Other works like Bossert and Storcken (1992), Nehring and Puppe (2002) and Nehring and Puppe (2007) study the existence of 'suitable' strategy-proof social welfare functions in median spaces.

relation maximises the sum of pairwise agreements between the voters' preferences and itself (Monjardet (2005)).

The social welfare relation which satisfies the above condition is also known as a *median rule* or Kemeny's rule (Kemeny (1959), Kemeny (1972)). These rules pick the 'median' in *some* metric space depending on the corresponding notion of distance. However, in most of these papers Kemeny's notion of distance is used for computing median rules.[6]

As noted in the literature, it is easy to verify that Condorcet's majority relation and Kemeny's median procedure are the same (Monjardet (2005)). This implies that the majority binary relation is the unique distance minimising rule under the Kemeny's notion of distance. In other words, when the social welfare relation is transitive, then the majority binary relation is the unique representative distance-minimising social welfare relation. This is a direct application of Demange (2012).

However, all of these works assume Kemeny's notion of distance when measuring distances between preference orderings. Even though Kemeny's notion is 'suitable' for social choice theoretic settings, our result implies that for a broader class of distance functions the *majoritarian social welfare relations* are *distance-minimising*. And, as noted above, Kemeny's notion of distance is a special case of these *additive* distance notions.

We aim to add to the literature in social choice theory which deals with the relaxation of transitivity of the social welfare relation. A prominent result in this literature is that of May (1952) which states that when there are two alternatives the majority rule is the only rule that is *decisive*, *egalitarian*, *neutral* and *positively responsive*. The result immediately extends to the case of multiple alternatives if the social welfare relation satisfies *independence of irrelevant alternatives*.

Other works include Dasgupta and Maskin (2008) which shows that the majority rule is the 'most robust' voting rule for 'any' domain. More specifically, they show that it satisfies *pareto*, *anonymity*, *neutrality*, *independence of irrelevant alternatives* and *decisiveness*.

Our paper adds to the literature on majoritarian social choice relations and provides additional results in favour of these rules. This work also aims to contribute to the literature on median rules and distance-based analysis in the social choice literature.

The paper is organised as follows. Section 2 describes the model. Section 3 states the main result and the proof while section 3.1 provides a discussion of its implications. Section 4 provides a characterization while section 5 concludes.

[6] The Kemeny distance is defined as follows: $d(W_1, W_2) = |W_1 \backslash W_2| + |W_2 \backslash W_1|$ for all W_1, W_2. This has become a widely used notion of distance in the social choice literature. See Young and Levenglick (1978) for an application of Kemeny's distance to the problem of Condorcet consistency.

2 The Model

2.1 Notation

The set of voters is $\mathbf{N}$. Let X be the finite set of alternatives. A *binary relation* W on X is a subset of $X \times X$ such that (i) $(x, y) \in W$ if and only if x is weakly preferred to y (ii) $(x, y) \in W$ and $(y, x) \in W$ if and only if x and y are indifferent and (iii) $(x, y) \in W$ and $(y, x) \notin W$ if and only if x is strictly preferred to y. Alternatively, we will write xWy when x is weakly preferred to y, xW_Py when x is strictly preferred to y, and xW_Iy when x and y are indifferent according to W. A binary relation W is *complete* if either $(x, y) \in W$ or (y, x) or both for any $x, y \in X$. Let the set of all complete binary relations be $\mathcal{W}$. In this paper we will only consider complete binary relations and we will use W to denote a generic element of $\mathcal{W}$. A binary relation $S \in \mathcal{W}$ is *strict* if $[(x, y) \in S] \Leftrightarrow [(y, x) \notin S]$ for all $x, y \in X$. Let $\mathcal{S}$ be the set of all strict binary relations.

A binary relation $R \in \mathcal{W}$ is an *ordering* if it is *reflexive*, *complete* and *transitive*.[7] Let $\mathcal{R}$ be the domain of all orders. An ordering is strict $P \in \mathcal{W}$ if it also a strict binary relation. Let $\mathcal{P}$ be the set of all strict orderings. Therefore, $\mathcal{P} \subseteq \mathcal{R} \subseteq \mathcal{S} \subseteq \mathcal{W}$.

For any $W_1, W_2, W_3 \in \mathcal{W}$ we say that W_2 is *between* W_1 and W_3 if and only if $(W_1 \cap W_3) \subseteq W_2 \subseteq (W_1 \cup W_3)$. In other words, W_2 contains all the ordered pairs common to both W_1 and W_3 but not those which are in neither of them. This notion of betweenness is natural in this setting and is commonly used in the social choice literature.[8]

Let $\mathcal{B}(W_1, W_2) = \{W' \in \mathcal{W} | W' \text{ is between } W_1 \text{ and } W_2\}$ and let $b(W_1, W_2) = |\mathcal{B}(W_1, W_2)|$ denote the number of profiles between W_1 and W_2.

Let $W_1 \backslash W_2$ denote the ordered pairs in W_1 which are not in W_2. Two binary relations $W_1, W_2 \in \mathcal{W}$ are *adjacent* if $|W_1 \backslash W_2| + |W_2 \backslash W_1| = 1$ i.e. they only differ by one ordered pair.

A *path* between W and W' is collection of profiles $\rho = (W_0, W_1, \ldots, W_q)$ where (i) $W_0 = W$ and $W_q = W'$ (ii) $|W \backslash W'| + |W' \backslash W| = q$ and (iii) W_j and W_{j+1} are adjacent for all $j \in \{0, \ldots, q-1\}$. Note that there may be multiple paths between any two profiles W_1 and W_2.

Let N denote a non-empty subset of $\mathbf{N}$ consisting of n voters. A *social welfare relation* $F : \cup_{N \in \mathbf{N}} \mathcal{R}^n \to \mathcal{W}$ is a binary relation over the set of alternatives. Therefore, for any non-empty subset of voters $N \subseteq \mathbf{N}$ and vote profile $\pi \in \mathcal{R}^n$ the social welfare relation outputs a binary relation $F(\pi)$.[9] We denote by $\mathcal{F}$ the set of all social welfare functions.

We impose the following ordinal notion of 'similarity' between pairs of binary relations. A *quaternary* relation Q on $\mathcal{W}$ is a subset of $\mathcal{W}^2 \times \mathcal{W}^2$. We write $(W_1W_2, W_3W_4) \in Q$ or

[7]A binary relation R is complete if either aRb or bRa or both, reflexive if aRa for all a and transitive if aRb and bRc implies aRc for all a, b, c.

[8]For other applications of *betweenness* in the social choice literature see Nehring and Puppe (2007), Can and Storcken (2013) and Lainé et al. (2016).

[9]Our main result does not depend on this variable population definition of the *social welfare relation*.

$(W_1W_2)Q(W_3W_4)$ if and only if W_1 and W_2 are 'more similar' to each other than the pair W_3 and W_4.[10] Let Q_P and Q_I denote the strict or asymmetric and the indifferent component of Q respectively defined in the usual way. We impose the following properties on Q.

(i) Linear: Q is reflexive, complete and transitive.

(ii) Symmetric: For all $W_1, W_2 \in \mathcal{W}$, $(W_1, W_2)Q(W_2, W_1)$.

(iii) Closest at identity: For all distinct $W_1, W_2 \in \mathcal{W}$, $(W_1, W_1)Q_P(W_2, W_1)$.

(iv) Equivalence of identical pairs: For all $W_1, W_2 \in \mathcal{W}$, $(W_1, W_2)Q(W_2, W_2)$.

(v) Q-betweenness: For any $W_1, W_2, W_3 \in \mathcal{W}$, if $W_2 \in \mathcal{B}(W_1, W_3)$ then $(W_1, W_2)Q_P(W_1, W_3)$ and $(W_2, W_3)Q_P(W_1, W_3)$.

This property is different from the more general notion of betweenness used in the theory of measurement. However, in the social choice setting this is a natural assumption since it states that when a binary relation W_2 is *between* W_1 and W_3. Moroever, *Q-betweenness* along with the *collinearity* (defined below) implies that profiles which have more ordered pairs in common between them are closer to each other terms of Q.

(vi) Collinearity: Let $W_1, W_2, W_3 \in \mathcal{W}$ distinct be such that $W_2 \in \mathcal{B}(W_1, W_3)$. For all distinct $W_1', W_2', W_3' \in \mathcal{W}$ such that $W_1' \in \mathcal{B}(W_2', W_3')$,

1. If $(W_1, W_2)Q(W_1', W_2')$ and $(W_1', W_3')Q(W_1, W_3)$, then $(W_2, W_3)Q(W_2', W_3')$.
2. If either of the preceding antecedent inequalities is strict, then the consequent inequality is strict as well.

The ternary relation $(W_1, W_2, W_3\rangle$ holds if W_1, W_2, W_3 satisfy collinearity.[11] We write $\langle W_1, W_2, W_3\rangle$ if both $(W_1, W_2, W_3\rangle$ and $\langle W_1, W_2, W_3)$ hold.

By *respects betweenness* and *collinearity*, $[W_2 \in \mathcal{B}(W_1, W_3)] \Leftrightarrow [\langle W_1, W_2, W_3\rangle]$. The structure $\langle \mathcal{W}, Q\rangle$ is a *proximity structure* iff Q satisfies properties (i)-(iv). In the literature on measurement theory it has been shown that a quaternary relation can be numerically represented by a distance function (Krantz et al. (1990)). We define a distance function below.

Definition 1 (Distance) A function $d : \mathcal{W} \times \mathcal{W} \to \mathbb{R}_+$ is a distance function if it satisfies the following properties,

(i) Non-negativity: $d(W_1, W_2) \geq 0$ for all $W_1, W_2 \in \mathcal{W}$.

(ii) Identity of indiscernibles: $d(W_1, W_2) = 0$ if and only if $W_1 = W_2$.

[10] Alternatively, Q can also be defined as a notion of 'dissimilarity'. See Krantz et al. (1971), Krantz et al. (1990) and Krantz et al. (1989) for a comprehensive theory of measurement.

[11] Most of the terminology is adopted from Krantz et al. (1990).

(iii) Symmetry: $d(W_1, W_2) = d(W_2, W_1)$ for all $W_1, W_2 \in \mathcal{W}$.

(iv) Triangle Inequality: $d(W_1, W_3) \leq d(W_1, W_2) + d(W_2, W_3)$ for all $W_1, W_2, W_3 \in \mathcal{W}$.

For any given profile $\pi \in \mathcal{W}$ let $n(x, y) = |\{i|(x, y) \in W_i\}|$ be the number of voters who weakly prefer x over y. Similarly define $n(y, x)$. Our main result characterizes the distance metric over which the following class of majority binary relations are the distance minimising. We define this below.

Definition **2 (Distance-minimising)** A social welfare relation $\mathcal{F}$ is *distance-minimising* with respect to a distance function d if for every profile $\pi = (W_1, \ldots, W_n) \in \mathcal{W}^n$,

$$F \in \underset{F' \in \mathcal{F}}{\operatorname{argmin}} \sum_{i=1}^{n} d(W_i, F'(\pi)) \qquad \forall F \in \mathcal{F}.$$

Therefore, a social welfare relation is distance-minimising if it picks the 'closest' aggregate preference relation according to the distance function d when compared to any element outside this class.

These rules are called "median" rules by Kemeny (1959) and characterized in Young and Levenglick (1978) and Can and Storcken (2013).[12] We can define this for a *class* of social welfare functions as follows.

A *class* of social welfare relations $\mathcal{F}^c$ is *distance-minimising* with respect to a distance function d if for every profile $\pi = (W_1, \ldots, W_n) \in \mathcal{W}^n$,

$$F \in \underset{F' \in \mathcal{F} \setminus \mathcal{F}^c}{\operatorname{argmin}} \sum_{i=1}^{n} d(W_i, F'(\pi)) \qquad \forall F \in \mathcal{F}^c.$$

Therefore, a class of social welfare relation is distance-minimising if any element of the class picks the 'closest' aggregate preference relation according to the distance function d when compared to any element outside this class.

It is well-known that majority outcomes over pairwise decisions form a class of distance-minimising social welfare relations (Monjardet (2008)). However, most of the literature on median rules only consider the Kemeny notion of distance to define. It is well known that if the distance is Kemeny then the majority binary relation will be the median rule for any odd profile (see Demange (2012)). We define *majoritarian* social welfare relations below.

Definition **3 (Majoritarian)** A social welfare relation F^M is a *majoritarian* if for all $\pi \in \mathcal{R}^n$ for all $x, y \in X$

$$\left[n(x, y) \geq \frac{n}{2}\right] \Rightarrow \left[(x, y) \in F^M(\pi)\right].$$

[12]Rules which minimize the square of sum of distances between the social welfare relation and the corresponding vote profile are called *mean* rules.

Let $\mathcal{F}^M$ denote the class of all majoritarian social welfare relations. A majority binary relation always weakly prefers x over y if atleast a weak majority of voters weakly prefer x over y.

3 Main Result

Theorem 1 Suppose Q, the quaternary relation on $\mathcal{W} \times \mathcal{W}$, satisfies properties (i)-(vi). Then there exists a distance function d which is the numerical representation of Q over which the *class* of *majoritarian* social welfare relations is *distance-minimising*.

In other words, Theorem 1 states that given a quaternary relation that satisfies the mentioned properties any majoritarian social welfare relation will be 'closer' to the preference profile when compared to any non-majoritarian social welfare relation with respect to *any* d which is a numerical representation of Q. Moreover, as we will show later, the class of distance functions over which majoritarian social welfare relations are distance-minimising contains the Kemeny distance.

Proof: Suppose Q is a quaternary relation on $\mathcal{W}$ which satisfies properties (i)-(vi). We first show the existence of a distance function d which numerically represents Q. We then show that a majoritarian binary relation is distance-minimising with respect to d.

By definition, $\langle \mathcal{W}, Q \rangle$ is a proximity structure. As argued above, $\langle W_1 W_2 W_3 \rangle \Leftrightarrow W_1 \in \mathcal{B}(W_2, W_3)$ for all $W_1, W_2, W_3 \in \mathcal{W}$. We use the results of Krantz et al. (1990) to prove our claim. We appropriately modify their definition of *segmentally additive* proximity structure as follows.

Definition 4 A proximity structure $\langle \mathcal{W}, Q \rangle$ with a ternary relation $\langle\rangle$ defined in terms of Q is segmentally additive iff the following condition holds for all $W_1, W_2, W_3, W_5 \in \mathcal{W}$, If $W_1 \neq W_2$, then there exist $W_0', \ldots, W_n' \in \mathcal{W}$ such that $W_0' = W_1$, $W_n' = W_2$ and $(W_{i-1}', W_i') Q (W_1, W_2)$.

Therefore, $\langle \mathcal{W}, Q \rangle$ is a *segmentally additive* with the ternary relation $\langle\rangle$ as defined above. The definition in Krantz et al. (1990) includes an added condition which states that if $(W_1, W_2) Q (W_3, W_5)$ then there exits $W_4 \in \mathcal{W}$ such that $(W_1, W_2) Q_I (W_3, W_4)$. With our additional notion of betweenness above their following result can be proved without this additional assumption.

Proposition 1 (Krantz et al. (1990)) Suppose $\langle \mathcal{W}, Q \rangle$ is a segmentally additive proximity structure. Then there exists a real-valued function $d : \mathcal{W} \times \mathcal{W} \to \mathbb{R}_+$ such that for any $W_1, W_2, W_3, W_4 \in \mathcal{W}$,

1. $\langle \mathcal{W}, d \rangle$ is a metric space.

2. $(W_1, W_2)Q(W_3, W_4)$ iff $d(W_1, W_2) \leq d(W_3, W_4)$.

3. $\langle W_1, W_2, W_3\rangle$ iff $d(W_1, W_2) + d(W_2, W_3) = d(W_1, W_3)$.

4. If d' is another metric on $\mathcal{W}$ satisfying the above conditions, then there exists $\alpha > 0$ such that $d' = \alpha d$.

We refer the reader to Krantz et al. (1990) for the proof of this result. Their proof can be modified easily using our definition of segmentally additive proximity structures. This is due to fact that our notion of 'betweenness' is a special case of their notion of betweenness as given by collinearity in the description of the model above. Therefore, we omit the proof of this result and proceed to proving our claim.

We now show that given any majoritarian social welfare relation F^M and for all $\pi \in \mathcal{R}^n$,

$$\sum_{i=1}^{n} d(R_i, F^M(\pi)) \leq \sum_{i=1}^{n} d(R_i, F'(\pi)) \quad \forall F' \notin \mathcal{F}^M.$$

We prove this by contradiction. Suppose F^M is a majoritarian social welfare relation and for some $F' \notin \mathcal{F}^M$ and $\pi \in \mathcal{R}^n$, $\sum_{i=1}^{n} d(R_i, F^M(\pi)) > \sum_{i=1}^{n} d(R_i, F'(\pi))$.

We fix the profile π and for simplicity write F instead of $F(\pi)$ for the remaining part of the proof. We construct a sequence of social welfare relations $(F^0, F^1, \ldots, F^q)$ such that $F^0 = F'$, $F^q = F^M$ and the following holds,

$$\sum_{i=1}^{n} d(R_i, F^{j+1}) \leq \sum_{i=1}^{n} d(R_i, F^j) \;\; \forall \;\; j \in \{0, q-1\}.$$

Let $F^0 = F'$. Since $F'(\pi) \neq F^M(\pi)$ there exists a pair $x, y \in X$ such that either (i) $(x, y) \in F^M$ and $(x, y) \notin F'$ or (ii) $(x, y) \notin F^M$ and $(x, y) \in F'$. We construct F^1 as follows.

- If condition (i) above holds let $(x, y) \in F^1$
- If condition (ii) holds let $(x, y) \notin F^1$.
- For all other pairs let $F^1|_{R\setminus(x,y)} = F^0|_{R\setminus(x,y)}$.

Therefore, $(F^0 \cap F^M) \subset F^1 \subset (F^0 \cup F^M) \Rightarrow F^1 \in \mathcal{B}(F^0, F^M)$. Either $F^1 = F^M$ in which case the sequence ends. If $F^1 \neq F^M$, then by the same arguments as above we can construct $F^2 \in \mathcal{B}(F^1, F^M)$. Therefore, by repeating these steps we have a sequence $(F^0, F^1 \ldots, F^q)$ such that $F^0 = F'$, $F^q = F^M$ and $\langle F^j F^{j+1} F^{j+2}\rangle$ for $q \in \{0, 1, \ldots, q-2\}$. We first show that, $\sum_{i=1}^{n} d(R_i, F^1) \leq \sum_{i=1}^{n} d(R_i, F^0)$.

We introduce the notion of *adjacency* to prove this. Two profiles W_1, W_2 are *adjacent* if there is no profile $W_3 \notin \{W_1, W_2\}$ such that $W_3 \in \mathcal{B}(W_1, W_2)$. It is easy to check that in the sequence $(F^0, F^1, \ldots, F^q)$ every pair F^j, F^{j+1} for $q \in \{0, 1, \ldots q-1\}$ are adjacent.

LEMMA **1** Suppose W_1 and W_2 are *adjacent*. Then for any $W_3 \in \mathcal{W}$ such that $W_3 \notin \{W_1, W_2\}$, either $\langle W_1W_2W_3\rangle$ or $\langle W_3W_1W_2\rangle$.[13]

Proof: Since W_1 and W_2 are adjacent there exists one and *only* one pair $x, y \in X$ such that either (i) $(x, y) \in W_1$ and $(x, y) \notin W_2$ or (ii) $(x, y) \notin W_1$ and $(x, y) \in W_2$. W.l.o.g. suppose (i) holds. Then either (a) $\langle W_1W_2W_3\rangle$ if $(x, y) \notin W_3$ or (b) $\langle W_3W_2W_1\rangle$ if $(x, y) \in W_3$. This proves the lemma.

Let $A_1 = \{i|b(R_i, F^0) \geq b(R_i, F^1)\}$ and $A_2 = \{i|b(R_i, F^1) \geq b(R_i, F^0)\}$. By construction of F^0 and F^1, $|R_i \cap F^0| \leq |R_i \cap F^1|$ for at least a majority of voters. Therefore, by Lemma 1, either $\langle R_iF^1F^0\rangle$ for a majority of voters and $\langle R_iF^0F^1\rangle$ for the remaining voters. Therefore, $|A_1| \geq |A_2|$.

Since $F^1 \in \mathcal{B}(R_i, F^0)$ for all $i \in A_1$, by Lemma 1 either $\langle R_iF^1F^0\rangle$ or $\langle F^0F^1R_i\rangle$ for all $i \in A_1$. Therefore, by statement (3) of Proposition 1,

$$d(R_i, F^0) = d(R_i, F^1) + d(F^1, F^0) \text{ for all } i \in A_1.$$

Therfore, by taking the sum over A_1,

$$\sum_{i\in A_1} d(R_i, F^0) = \sum_{i\in A_1} d(R_i, F^1) + \sum_{i\in A_1} d(F^1, F^0).$$

Since $F^0 \in \mathcal{B}(R_i, F^1)$ for all $i \in A_2$, we can use the same arguments as those used above and sum over A_2,

$$\sum_{i\in A_2} d(R_i, F^1) = \sum_{i\in A_2} d(R_i, F^0) + \sum_{i\in A_2} d(F^1, F^0).$$

Since $|A_1| \geq |A_2|$,

$$\sum_{i\in A_2} d(F^1, F^0) \leq \sum_{i\in A_1} d(F^1, F^0).$$

$$\text{i.e.} \quad \sum_{i\in A_2} d(R_i, F^1) - \sum_{i\in A_2} d(R_i, F^0) \leq \sum_{i\in A_1} d(R_i, F^0) - \sum_{i\in A_1} d(R_i, F^1).$$

Therefore, by manipulating the terms in the above inequality appropriately,

$$\sum_{i=1}^{n} d(R_i, F^1) \leq \sum_{i=1}^{n} d(R_i, F^0).$$

Using similar arguments as those used above,

$$\sum_{i=1}^{n} d(R_i, F^2) \leq \sum_{i=1}^{n} d(R_i, F^1).$$

[13]Note that the ternary relation is *symmetric* i.e. $\langle W_1W_2W_3\rangle = \langle W_3W_2W_1\rangle$ for all $W_1, W_2, W_3 \in \mathcal{W}$.

Therefore,

$$\sum_{i=1}^{n} d(R_i, F^{j+1}) \leq \sum_{i=1}^{n} d(R_i, F^{j}) \text{ for all } j \in \{0, q-1\}.$$

By construction,

$$\sum_{i=1}^{n} d(R_i, F^{M}) \leq \sum_{i=1}^{n} d(R_i, F').$$

Note that the choice of the majoritarian social welfare relation F^M, π and F' was arbitrary. Therefore, for any $F^M \in \mathcal{F}^M$ and for all $\pi \in \mathcal{R}^n$,

$$\sum_{i=1}^{n} d(R_i, F^{M}(\pi)) \leq \sum_{i=1}^{n} d(R_i, F'(\pi)) \quad \forall F' \notin \mathcal{F}^M.$$

□

3.1 Discussion

The variable population definition of the *social welfare relation* is not required for Theorem 1. The result also holds for any fixed number of voters. In the next section, we will characterize the *majority binary relation* using the variable population definition of the *social welfare relation* which is *distance-minimising* over the set of *additive* distance metrics.

Property 3 in Proposition 1 is critical for our result. This property has been called *additivity* in the literature on measurement theory. In the next section we will show that for a specific *majoritarian* social welfare relation *additivity* of the distance functions is necessary and sufficient for this *majoritarian binary relation* to be *distance-minimising*.

These notions of distances are broader than Kemeny's. It can be shown that the distance metric characterized by the quaternary relation Q need not be *neutral* à la Can and Storcken (2013). We provide an example of a metric which is not Kemeny but satisfies additivity. Therefore, the *majoritarian social welfare relation* is *distance-minimising* according to this distance.

Example 1 We define a distance metric d^α as follows. Let $\delta : X \times X \to (0, 1)$ be such that (i) $\delta(j, j') = \delta(j', j)$ for all $j, j' \in X$. Then,

$$d^\alpha(W_1, W_2) = \sum_{\substack{j,j' \in X \\ j \neq j \\ (j,j') \in (W_1 \setminus W_2) \\ (j,j') \in (W_2 \setminus W_1)}} \delta(j, j').$$

We can show that d satisfies all the properties of a metric as in Definition 1.

- Non-negativity: This holds by definition of δ.

- Identity of indiscernibles: For any $W_1 = W_2$ we have $(W_1 \backslash W_2) = (W_2 \backslash W_1) = \phi$. Therefore, $d^\alpha(W_1, W_1) = 0$.

- Symmetry: Since $(W_1 \backslash W_2) \cup (W_2 \backslash W_1) = (W_1 \backslash W_2) \cup (W_2 \backslash W_1)$ for all $W_1, W_2 \in \mathcal{W}$, we have $d^\alpha(W_1, W_2) = d^\alpha(W_2, W_1)$.

- Triangle inequality: This is true by the property that for any $\delta_1, \delta_2, \delta_3 \in (0, 1)$ we have $\delta_1 + \delta_2 \geq \delta_3$.

Moreover, we show that it satisfies *additivity*. For any W_1, W_2, W_3 such that $W_2 \in \mathcal{B}(W_1, W_3)$ we show that $d^\alpha(W_1, W_2) + d^\alpha(W_2, W_3) = d^\alpha(W_1, W_3)$. We show this in steps. From the definition of the distance metric,

$$d^\alpha(W_1, W_2) + d^\alpha(W_2, W_3) = \sum_{\substack{j,j' \in X \\ j \neq j \\ (j,j') \in (W_1 \backslash W_2) \\ (j,j') \in (W_2 \backslash W_1)}} \delta(j, j') + \sum_{\substack{j,j' \in X \\ j \neq j \\ (j,j') \in (W_2 \backslash W_3) \\ (j,j') \in (W_3 \backslash W_2)}} \delta(j, j')$$

Since $(W \backslash W') \cup (W' \backslash W) = (W \cup W') \backslash (W \cap W')$ for all W, W',

$$d^\alpha(W_1, W_2) + d^\alpha(W_2, W_3) = \sum_{\substack{j,j' \in X \\ j \neq j \\ (j,j') \in (W_1 \cup W_2) \backslash (W_1 \cap W_2)}} \delta(j, j') + \sum_{\substack{j,j' \in X \\ j \neq j \\ (j,j') \in (W_2 \cup W_3) \backslash (W_2 \cap W_3)}} \delta(j, j')$$

By definition of betweenness of profiles and the associativity of $\cup$ and $\cap$,

$$\big((W_1 \cup W_2) \backslash (W_1 \cap W_2)\big) \bigcup \big((W_2 \cup W_3) \backslash (W_2 \cap W_3)\big) = (W_1 \cup W_3) \backslash (W_1 \cap W_3).$$

Therefore,

$$\sum_{\substack{j,j' \in X \\ j \neq j \\ (j,j') \in (W_1 \cup W_2) \backslash (W_1 \cap W_2)}} \delta(j, j') + \sum_{\substack{j,j' \in X \\ j \neq j \\ (j,j') \in (W_2 \cup W_3) \backslash (W_2 \cap W_3)}} \delta(j, j') = \sum_{\substack{j,j' \in X \\ j \neq j \\ (j,j') \in (W_1 \cup W_3) \backslash (W_1 \cap W_3)}} \delta(j, j')$$

$$\Rightarrow d^\alpha(W_1, W_2) + d^\alpha(W_2, W_3) = \sum_{\substack{j,j' \in X \\ j \neq j \\ (j,j') \in (W_1 \cup W_3) \backslash (W_1 \cap W_3)}} \delta(j, j') = d^\alpha(W_1, W_3).$$

Therefore, $d^\alpha(W_1, W_2) + d^\alpha(W_2, W_3) = d^\alpha(W_1, W_3)$. Therefore, d satisfies *additivity*.

The conditions specified on Q and the resulting properties on the distance function are not necessary for the class of *majoritarian social welfare relations* to be *distance-minimsing*. This is due to the fact that the class of *majoritarian social welfare relations* is large. Therefore, to obtain necessary conditions a complete binary relation has to be specified.

In the next section we show for a specific *majoritarian social welfare relation* the conditions (i)-(vi) on Q and the resulting properties on d are both necessary and sufficient for this *majoritarian social welfare relation* to be *distance-minimising*. We use the variable population definition of the social welfare relation to obtain these necessary conditions.[14]

4 Characterization

In this section we characterize a specific majority binary relation which is the unique distance-minimising social welfare function over a class of distance metrics which satisfy *additivity*.

In the next result we show that part (3) of Proposition 1 i.e., *additivity* over triples on a path is necessary (and of course, sufficient) for the following *majoritarian* social welfare relation. This allows us to obtain a quaternary relation from the corresponding distance function d.

We use the following notation for our next definition. Let $n_P(x,y) = |\{i|xP_iy\}|$ and $n_P(y,x) = |\{i|yP_ix\}|$. We define a tie-breaking rule $\tau : X \times X \Rightarrow X \times X$ such (i) $\tau(x,y) = \tau(y,x)$ and (ii) $\tau(x,y) \subseteq \{(x,y),(y,x)\}$. Let $\mathcal{T}$ be the set of all tie-breakers.

Therefore, a tie-breaking relation for every pair of alternatives picks a binary relation over that pair of alternatives. Note that for every pair of alternatives x, y the tie-breaking rule either chooses $\{(x,y)\}, \{(y,x)\}$ or $\{(x,y),(y,x)\}$. In the following definition we will assume that either of these may be chosen.

Definition **5 (Majority binary relation)** A *social welfare relation* F^* is a majority binary relation if for all $\pi \in \mathcal{R}^n$ and every pair of alternatives $(x,y) \in X \times X$ there exists a tie-breaking rule $\tau : X \times X \Rightarrow X \times X$ such that,

(i) $[n_P(x,y) > \frac{N}{2}] \Rightarrow [xF^*_Py]$.

(ii) $[n_P(y,x) < \frac{N}{2}] \Rightarrow [xF^*y]$.

(iii) If $n_P(x,y) = n_P(y,x) = \frac{N}{2}$ then $F^*(\pi)|_{xy} = \tau(x,y)$.

(iv) If none of the above (i)-(iii) hold then $(x,y) \notin F^*$.

We denote this *class* of majority binary relations as $\mathcal{F}^*$. It is easy to verify that the above rule is complete. We say that there is a *tie* between x and y at a given profile π if part (iii) of the above definition holds i.e. $n_P(x,y) = n_P(y,x) = \frac{N}{2}$. In such a case, the tie-breaking rule comes into effect and any of the three possible binary relations may be chosen by the *majority binary relation*.[15] We show this in the following example.

Example **2** The set of voters is $N = \{1,2,3,4\}$ and the set of alternatives is $X = \{x,y\}$. Consider the profile π shown in Table 1 below.

[14] More specifically, to obtain necessary conditions we need the *social welfare relation* to be defined over every *triple* of voter preferences.

[15] Of course, as the number of pairwise ties increase, the number of possible majority binary relations increases threefold. For a profile with ties over m pairs the number of profiles in the set of all majority binary relations is 3^m.

Table 1: Preference profile $\pi \in \mathcal{R}^4$

1	2	3	4
x	x	yz	y
y	yz	x	x
z			z

By Definition 5 there are three equally likely outcomes. Denote these by $F_1^*(\pi)$, $F_2^*(\pi)$ and $F_3^*(\pi)$. These are listed below.

- $F_1^*(\pi) = \{(x,y),(y,z),(x,z)\}$ if $\tau(x,y) = \{(x,y)\}$.
- $F_2^*(\pi) = \{(x,y),(y,x),(y,z),(x,z)\}$ if $\tau(x,y) = \{(x,y),(y,x)\}$.
- $F_3^*(\pi) = \{(y,x),(y,z),(x,z)\}$ if $\tau(x,y) = \{(y,x)\}$.

Let $\mathcal{F}^*(\pi)$ denote the set of possible binary relations for the profile π. This set is singleton when there are no ties. Therefore, in the above example $\mathcal{F}^*(\pi) = \{F_1^*(\pi), F_2^*(\pi), F_3^*(\pi)\}$.

THEOREM **2** The *class* of *majority binary relation* is *distance-minimising* with respect to d if and only if Q satisfies the conditions (i)-(vi) and d is a numerical representation of Q.

Proof: To prove sufficiency we can use the same arguments as those used in the proof of Theorem 1 to prove this. By Lemma 1, for any $F' \neq F^*$ we can construct a sequence of functions $F^0, \dots F^q$ such that $F^0 = F'$ and $F^q = F^*$ and $\sum_{i=1}^n d(R_i, F^{j+1}) \leq \sum_{i=1}^n d(R_i, F^j)$ for all $j \in \{0, q-1\}$.

To prove necessity we need to extend the domain of the majority rule from $\mathcal{R}$ to $\mathcal{W}$. Since the majority binary relation only considers pairwise orderings Definition 5 is extended to the domain $\mathcal{W}$.

Suppose the majority binary relation is distance-minimising with respect to some distance d. We show that d satisfies *additivity* (property (iii) in Proposition 1). As a result, we can obtain a quaternary relation on $\mathcal{W}$ such that $(W_1, W_2)Q(W_3, W_4) \Leftrightarrow d(W_1, W_2) \leq d(W_3, W_4)$. We show this in steps.

Since d is a distance function by assumption $\langle \mathcal{W}, d \rangle$ is a metric space. Therefore, part (i) of Proposition 1 is satisfied. We show that d is *additive* over triples such that one of them is *between* the other two.

Let $W_1, W_2, W_3 \in \mathcal{W}$ such that $W_2 \in \mathcal{B}(W_1, W_3)$. We first show that $F(W_1, W_3) = W_2$. As a result we show that $d(W_1, W_2) + d(W_2, W_3) = d(W_1, W_3)$. Fix any arbitrary pair of alternatives $x, y \in X$. For the profile $\pi = (W_1, W_3)$ one of the four possibilities in Definition 5 will hold. We prove the following claim.

Claim: Suppose $\pi = (W_1, W_3)$. Then (i)$[(x,y) \in F^*] \Rightarrow [(x,y) \in W_2]$ and (ii) $[(x,y) \notin F^*] \Rightarrow [(x,y) \notin W_2]$.

This implies that $F^*(W_1, W_3) = W_2$. We prove this in parts. We slightly abuse notation and write $xW_i^P y$ when x is strictly preferred to y in W_i for any $i \in N$.

- Suppose part (i) of Definition 5 holds i.e. $n_P(x, y) > \frac{N}{2}$. By definition $xF_P^* y$. By betweenness of preferences $xW_2^P y$.
- Similar arguments can be made to show that if part (ii) of Definition 5 holds then xF^*y and $xW_2 y$.
- Suppose part (iii) of Definition 5 holds. Then either $(x, y) \in W_2$ or $(x, y) \notin W_2$. By definition, there exists a tie-breaking rule $\tau \in \mathcal{T}$ such that $\tau(x, y) = W_2|_{xy}$. Therefore, $W_2 \in \mathcal{F}^*(W_1, W_3)$.
- Suppose none of the parts (i)-(iii) of the definition hold. Since conditions (i) and (ii) do not hold, $yW_1^P x$ and $yW_3^P x$. By the definition of betweenness of preferences, $[W_2 \in \mathcal{B}(W_1, W_3)] \Rightarrow [W_2 \subseteq W_1 \cup W_3]$. Therefore, $(x, y) \notin W_2$.

Therefore, for all parts (i)-(iv) of Definition 5, $[(x, y) \in F^*] \Rightarrow [(x, y) \in W_2]$ and $[(x, y) \notin F^*] \Rightarrow [(x, y) \notin W_2]$ for some tie-breaking function $\tau \in \mathcal{T}$. This implies that $W_2 \in \mathcal{F}^*(W_1, W_3)$. Since the *majority binary relation* is *distance-minimising*,

$$d(W_1, W_2) + d(W_3, W_2) \leq d(W_1, W_1) + d(W_3, W_1)$$

By *closest at identity* of Q and the *triangle inequality* of d,

$$d(W_1, W_2) + d(W_2, W_3) = d(W_1, W_3).$$

□

Therefore, for a specific class of majoritarian social welfare function the *additivity* of the distance metric is both necessary and sufficient to make this the unique distance minimising class of social welfare relations.

5 Conclusion

We showed that for a large class of distance functions, which are representations of ordinal measures, the class of majoritarian social welfare relations are uniquely distance-minimising. We characterized the specific class of majoritarian social welfare relations for which this class of distance measures are necessary *and* sufficient.

References

Andjiga, N. G., A. Y. Mekuko, and I. Moyouwou (2014): "Metric rationalization of social welfare functions," *Mathematical Social Sciences*, 72, 14–23.

Arrow, K. J., A. Sen, and K. Suzumura (2010): *Handbook of Social Choice & Welfare*, vol. 2, Elsevier.

Bossert, W. and T. Storcken (1992): "Strategy-proofness of social welfare functions: the use of the Kemeny distance between preference orderings," *Social Choice and Welfare*, 9, 345–360.

Can, B. and A. J. A. Storcken (2013): *A re-characterization of the Kemeny distance*, Maastricht University School of Business and Economics, Graduate School of Business and Economics (GSBE).

Dasgupta, P. and E. Maskin (2008): "On the robustness of majority rule," *Journal of the European Economic Association*, 6, 949–973.

Demange, G. (2012): "Majority relation and median representative ordering," *SERIEs*, 3, 95–109.

Fishburn, P. C. (1970): "The irrationality of transitivity in social choice," *Behavioral Science*, 15, 119–123.

Kemeny, J. G. (1959): "Mathematics without numbers," *Daedalus*, 88, 577–591.

Kemeny, J. G. J. G. (1972): "Mathematical models in the social sciences," Tech. rep.

Krantz, D. H., P. Suppes, R. D. Luce, and A. Tversky (1971): *Foundations of measurement: vol. I.: Additive and polynomial representations*, vol. 1, Academic Press, Inc.

——— (1989): *Foundations of Measurement, vol. II: Geometrical, threshold, and probabilistic representation*, Academic Press, Inc.

——— (1990): *Foundations of measurement, vol. III: Representation, axiomatization, and invariance*, Academic Press, Inc.

Lainé, J., A. I. Ozkes, and R. Sanver (2016): "Hyper-stable social welfare functions," *Social Choice and Welfare*, 46, 157–182.

Lerer, E. and S. Nitzan (1985): "Some general results on the metric rationalization for social decision rules," *Journal of Economic Theory*, 37, 191–201.

May, K. O. (1952): "A set of independent necessary and sufficient conditions for simple majority decision," *Econometrica: Journal of the Econometric Society*, 680–684.

Monjardet, B. (2005): "Social choice theory and the "Centre de Mathèmatique Sociale": some historical notes," *Social choice and Welfare*, 25, 433–456.

——— (2008): "Mathèmatique Sociale" and Mathematics. A case study: Condorcet's effect and medians," *Electronic Journal for History of Probability and Statistics*, 4, 1–26.

Nehring, K. and C. Puppe (2002): "Strategy-proof social choice on single-peaked domains: possibility, impossibility and the space between," *Unpublished manuscript, Department of Economics, University of California at Davis.*

——— (2007): "The structure of strategy-proof social choiceŮPart I: General characterization and possibility results on median spaces," *Journal of Economic Theory*, 135, 269–305.

Sen, A. and P. K. Pattanaik (1969): "Necessary and sufficient conditions for rational choice under majority decision," *Journal of Economic Theory*, 1, 178–202.

Young, H. P. and A. Levenglick (1978): "A consistent extension of Condorcet's election principle," *SIAM Journal on Applied Mathematics*, 35, 285–300.