Stable Property Rights

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First Draft: March 2015
Current Draft: August 2016

Abstract

A collective choice model of an environment where a society must allocate but cannot legally enforce property rights over a single indivisible productive asset is studied. There are both production and pillage opportunities for coalitions. For strictly superadditive production technologies, we show that the set of efficient allocations is a vNM Stable Set. In contrast, while the stability imperatives of the Core support the efficient allocation of property rights, they do limit the possibilities for wealth distribution in the society. For a mild restriction on the state space and for a class of pillage technologies, the set of efficient allocations is also *Chwe*'s Largest Consistent Set.

JEL Classification: C71, C73, D71, D74
Keywords: property rights, allocation by force, political economy

"Throughout the history of mankind, it has been quite common that economic agents, individually or collectively, use power to seize control of assets held by others."

Piccione and Rubinstein (2007)

"All aspects of human life are responses ... to the interaction of two great life strategy options: on the one hand, production and exchange, on the other hand, appropriation and defense against appropriation."

Hirshleifer (1994)

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†I am indebted to James Jordan for the inception of the project and for continuous guidance and support. I also thank Kalyan Chatterjee and Vijay Krishna for valuable discussions.

1 Introduction

A collective choice model of an environment where a society must allocate but cannot legally enforce property rights over a single indivisible productive asset is studied. Property rights may be held by coalitions which may differ in terms of their power and their productivity with the asset. A state in the model describes which coalition owns the asset and how is the wealth produced from the asset distributed among all the players. Coalitions can move to costlessly pillage the asset from current owners if they are more powerful than the coalition of players who lose from such a move and the new owners are complicit in the pillage. Even in the absence of property rights enforcement, is an efficient allocation of property rights enforceable in this model ? This paper uses cooperative analysis to give an affirmative answer.

As Hirshleifer (1994) reminds us, the appropriative struggle for control of productive resources alongside the economic activity of production and exchange through market institutions provides the main storyline of human history. The acts of pillage can take many forms- war, theft, rent seeking activities like lobbying for licenses and monopoly privileges as well as coercive taxes/transfers of democratic governments. Pillage, therefore, is as much a feature of advanced societies as it appears to be of the primitive societies.

The classical efficiency of exchange in a free market economy is predicated on enforcement of property rights. The first welfare theorem postulates voluntary exchange as a premise for allocative efficiency of exchange. A strand of literature in law and economics starting with Calabresi and Melamed (1972) and Kaplow and Shavell (1996) look at the economic analysis of property and liability rules. Working in that tradition, Bar-Gill and Persico (2016) point to the possibility of achieving efficiency in an exchange model with a single durable asset even with a weak protection of property rights. The present paper is motivated by a similar question as to whether property rights protection is a necessary prerequisite for efficiency. We examine this question in a model with a single durable asset and where there are opportunities for production as well as pillage by coalitions.

Models of weak property rights either model pillage as costly investment in rent seeking activities (Murphy et al. (1991), Muthoo (2004), Hafer (2006)) or as costless (Piccione and Rubinstein (2007), Jordan (2006)). This paper is in the latter tradition of modeling pillage as costless. Our model has the same lack of commitment that characterizes dynamic political settings and has been emphasized in the recent literature (Jordan (2009), Acemoglu et al. (2012) etc.). A sufficiently powerful coalition may pillage the asset today from a less powerful coalition. However, members of the coalition cannot commit not to part ways and join another coalition tomorrow to appropriate the asset in hope of a better payoff. Enforceability thus depends on the distribution of political power.

All our results pertain to the class of strictly superadditive production technologies implying that an efficient state is one where the grand coalition holds the property rights and produces the greatest aggregate wealth. Our first result, Proposition 1, says that the set of efficient states is a vNM Stable Set. Thus stability imperatives of vNM Stable Set neither constrain the efficient allocation of property rights nor wealth distributions. In contrast, while the stability imperatives of the Core support the efficient allocation of property rights, they do limit the possibilities for wealth distribution in the society. Our second result, Proposition 2, characterizes the Core. At a wealth distribution in a Core state, every coalition whose total wealth falls short of its own productive potential cannot be more powerful than its complementary coalition. Both the Core and vNM Stable Set are defined with respect to the direct dominance relation. We next turn to see whether efficiency can be supported by expectations of rational farsighted players. Proposition 4 is our main result characterizing *Chwe*'s Largest Consistent Set (LCS). For a mild restriction on the state space and for a class of pillage technologies, the set of efficient states is the LCS. Thus for a class of pillage technologies, the farsighted stability imperatives of LCS also impose no constraints on either efficient property rights allocation or wealth distributions. We conclude that in our model which has absence of property rights enforcement, efficiency is supported by a range of stability concepts from cooperative theory namely vNM Stable Set, Core and LCS.

The expectations that make efficient states stable when players are farsighted rely on punishing acts of pillage. As such, they differ from the expectations embodying a vNM Stable Set. To see this, consider a 3-player symmetric production environment given by $f(N) = 1.2$, $f(S) = 1$ if $|S| = 2$ and $f(S) = 0$ if $|S| = 2$. The power of a coalition at a state is its total wealth at that state. Let $z^0 = (N, (0.2, 0.4, 0.6))$ and $z^1 = (\{1, 3\}, (0.3, 0, 0.7))$. A deviation by coalition $\{1, 3\}$ from the efficient state z^0 to the inefficient state z^1 cannot be deterred by the expectation of a move back to an efficient state. So the standard markovian expectations in Anesi (2010) that support the vNM Stable Set cannot stabilize the efficient state z^0 as they leave no scope for punishing either player 1 or player 3. This is primarily because "winning coalitions" are endogenous and may depend on the particular move under consideration. So even though there is no "winning coalition" for a direct move from z^0 to say $z'' = (N, (0.35, 0.1, 0.75))$, coalition $\{1, 3\}$ may enforce a move from z^0 to z^1 in anticipation of a subsequent move to the efficient state z'', thus eventually profiting from the move. Nevertheless, we can construct an expectation that makes z^0 stable against a deviation to z^1. Let $z^2 = (\{2, 3\}, (0, 0.2, 0.8))$ and $z^3 = (N, (0.1, 0.2, 0.9))$. Player 1 will block the deviation from z^0 to z^1 as the path z^2, z^3 that is expected subsequent to the deviation punishes player 1 in its final conclusion.

There is a growing literature on resource allocation in environments with weak or no protection of property rights. Jordan (2006) invents single good *pillage games* as a setting for study of power and studies the core and stable sets of these games. Rowat (2009) adds a second good to single good *pillage games*'s single good pillage games, thus introducing opportunities for exchange. Consequently, gains from trade may also motivate pillage in their setting. Moreover, vNM stability may preclude efficiency. Jordan (2009) further develops *production pilllage games* in which in addition to pillage opportunities, each player has an individual production technology which she uses to produce consumption from the share of asset that she owns. Our model differs from these papers in that unlike them we allow for coalitional ownership and coalitional production. Moreover, we have a single indivisible asset while in Jordan (2009), players own varying quantities of the asset. Our main result on supporting efficient states as LCS is similar in spirit to Jordan (2009) who supports efficient states in his model via his own solution concept of 'Legitimate Set' that relies on punishing acts of pillage.

2 Model

2.1 Environment

There is a single indivisible and durable asset. There is a set $N = \{1, \ldots, n\}$ of n agents who live forever. The asset may be owned by a coalition of agents. Various groups of agents may produce varying amount of wealth with the asset which is described by a function f that for a coalition S gives the wealth $f(S)$ that it can generate, in units of lifetime payoffs, if S had the ownership of the asset.

There is a set of states $\mathbb{Z}$ that encode information about property rights and wealth distribution in the society. A state $z \in \mathbb{Z}$ specifies a coalition S that jointly owns the asset and a distribution $u = (u_1, \ldots, u_n) \in \mathbb{R}^n_+$ of lifetime payoff $f(S)$ that S gets from the asset.

$$\mathbb{Z} = \{(S, u) \in 2^N \setminus \phi \times \mathbb{R}^n_+ : \sum_{i \in N} u_i = f(S)\}$$

Agent i's payoff in state $z = (T, w)$ is $u_i(z) = w_i$ i.e. his share of the wealth distribution at that state. We will refer to the owning coalition at the state z as $o(z)$.

There is absence of property rights enforcement which makes any prevailing state vulnerable to pillage by another group. The technology of pillage is specified by the construct of a power function as defined by Jordan (2006) that satisfies the following properties

Definition 1. (Power Function). A power function is a function $\pi : 2^N \times \mathbb{Z} \to \mathbb{R}$ satisfying

(p1). the power of a coalition does not decrease as the coalition adds new members i.e.

$$\forall z \in \mathbb{Z}, \quad S \subset T \implies \pi\big[T; z\big] \geq \pi\big[S; z\big]$$

(p2). the power of a coalition does not decrease if there is no member of the coalition whose wealth decreases

$$\forall z, z' \in \mathbb{Z} \text{ st } o(z) = o(z') \quad \big(\forall i \in S,\, u_i(z') \geq u_i(z)\big) \implies \big(\pi\big[S; z'\big] \geq \pi\big[S; z\big]\big)$$

(p3). the power of a coalition increases if the wealth of every member of the coalition increases

$$\forall z, z' \in \mathbb{Z} \text{ st } o(z) = o(z') \quad \big(\forall i \in S,\, u_i(z') > u_i(z)\big) \implies \big(\pi\big[S; z'\big] > \pi\big[S; z\big]\big)$$

That the notion of power function specifies the technology of pillage is reflected in the following definition of feasible moves.

Definition 2. (Feasible Moves). A move from state z to state $z' = (T, w)$ is feasible for a coalition S , written $z \to_S z'$ if

1. the new owners i.e. players in T support the move i.e. $T \subset S$. This is needed as coalition T is supposed to produce the wealth from the asset at state z'.

2. S is more powerful at state z than the coalition of players who lose from the move

$$\pi\big[S; z\big] > \pi\big[\{i \in N : u_i(z') < u_i(z)\}; z\big]$$

The environment $\langle \mathbb{Z}, \pi, (\to_S)_{S\subset N}, (u_i)_{i\in N}\rangle$ may be seen as a cooperative game.

3 Cooperative Analysis

Individual preferences represented by u_i, feasibility relations $\to_T$ and power function $\pi[.; z]$ induce a direct dominance relation $<$ on the set of states $\mathbb{Z}$.

Definition 3. (Direct Dominance). $z < z'$ (z' directly dominates z) if

1. there exists S such that $z \to_S z'$; and

2. Players in S do not lose from the move and some player in S gains from the move i.e.

$$S \subset \{i \in N : u_i(z') \geq u_i(z)\} \text{ and } S \cap \{i \in N : u_i(z') > u_i(z)\} \neq \emptyset$$

Let $<(z)$ be the set of states that dominate z. The Core and vNM Stable Set are two of the earliest solution concepts from cooperative game theory based on the direct dominance relation.

Definition 4. (Core). A set of states $\mathbb{V} \subset \mathbb{Z}$ is the Core of $\langle N, \mathbb{Z}, \pi, (u_i)_{i\in N}, \mathscr{W}\rangle$ if no state in $\mathbb{V}$ is dominated by another state in $\mathbb{Z}$.

In other words, Core is the set of undominated states in $\mathbb{Z}$. We now define a stable set for an abstract binary relation on the state space.

Definition 5. (Stable Set). Given a binary irreflexive relation $\lhd$ on the state space $\mathbb{Z}$, a set of states $\mathbb{V} \subset \mathbb{Z}$ is a stable set of $\langle \mathbb{Z}, \lhd\rangle$ if the following two conditions hold:

1. Internal Stability (IS). No state in $\mathbb{V}$ is dominated by another state in $\mathbb{V}$.

$$z \in \mathbb{V} \implies \lhd(z) \subset \mathbb{Z} \setminus \mathbb{V}$$

2. External Stability (ES). Every state not in $\mathbb{V}$ is dominated by another state in $\mathbb{V}$.

$$z' \in \mathbb{Z} \setminus \mathbb{V} \implies \exists z \in \mathbb{V} : z \in \lhd (z')$$

Definition 6. (vNM Stable Set). A set $\mathbb{Y} \subset \mathbb{Z}$ is vNM stable set if $\mathbb{Y}$ is the stable set of $\langle \mathbb{Z}, < \rangle$.

The first result below shows that the set of all efficient states can be supported as the vNM stable set of the game. In other words, the stability imperatives in vNM stability concept do not constrain efficiency in any way.

Proposition 1. If the coalitional production function $f(.)$ is strictly superadditive, then the set of efficient states $\mathbb{E}$ is a vNM Stable Set.

Proof. No efficient state a is dominated by another efficient state b because in moving from a to b, some player i must be getting less than in a. However the move is feasible only for coalition N making i's cooperation necessary. i can, therefore, block the move. Thus the set of efficient states is internally stable.

Every inefficient state a with total wealth produced $f(S)$ (for some coalition S that holds the property rights over the asset at state a) is dominated by an efficient state b with total wealth produced $f(N)$ in which the extra wealth produced $f(N) - f(S) > 0$ is equally shared with everyone in N i.e. $b = \big(N, (u_i(a) + \frac{f(N)-f(S)}{N})_{i \in N}\big)$. Thus the set of efficient states is externally stable. Q.E.D.

The above result also implies that the direct dominance relation $<$ is not a tournament on the state space. The next result says that not all efficient states can be supported as the Core. Thus the stability imperatives in the concept of Core do not constrain the efficient allocation of property rights but do limit the possibilities for wealth distribution in the society.

Proposition 2. Suppose the coalitional production function $f(.)$ is strictly superadditive. Let

$$\mathbb{Y} = \{z \in \mathbb{E} : \forall S \subset N \sum_{i \in S} u_i(z) \geq f(S) \text{ or } \pi[S; z] \leq \pi[N \setminus S; z]\}$$

Then $\mathbb{Y}$ is the Core of $\langle \mathbb{Z}, < \rangle$.

Proof. Suppose $z \in \mathbb{Y}$ and $z < z' = (T, w)$ through coalition D. Let $S = \{i \in N : u_i(z') \geq u_i(z)\}$. Then $D \subset S$ and $\pi[S; z] \geq \pi[D; z] > \pi[N \setminus S; z]$ where the first inequality follows from property (p1) of the power function and the second from the definition of $<$. Since $z \in \mathbb{Y}$, this implies $\sum_{i \in S} u_i(z) \geq f(S)$, and

$$f(T) = \sum_{i \in N} u_i(z') \geq \sum_{i \in S} u_i(z') > \sum_{i \in S} u_i(z) \geq f(S)$$

where the strict inequality is because some player in coalition D must necessarily gain in the move from z to z' and $D \subset S$. For $T = S$, the contradiction is obvious. For $T \subsetneq S$, the contradiction is to the strict superadditivity of f. This completes the proof that all states in $\mathbb{Y}$ are Core states. To complete the proof that $\mathbb{Y}$ is the Core, consider a state $z \notin \mathbb{Y}$. There is a coalition $S \subset N$ such that $\sum_{i \in S} u_i(z) < f(S)$ and $\pi[S; z] > \pi[N \setminus S; z]$. Define a state $z'' = (S, w)$ such that for every player i in S, $u_i(z'') = w_i = u_i(z) + \frac{f(S) - \sum_{i \in S} u_i(z)}{|S|}$. Then $z < z''$. So z cannot be a Core state. Q.E.D.

We present an example with specific production and pillage technologies to get a sense of what the Core looks like.

Example. There are three players, $N = \{1, 2, 3\}$. There is a symmetric production environment and power of a coalition is simply its total wealth.

$$f(S) = \begin{cases} F & \text{if } |S| = 3 \\ G & \text{if } |S| = 2 \,, \\ 0 & \text{if } |S| = 1 \end{cases} \qquad F > G$$

$$\pi[S; z] = \textstyle\sum_{i \in S} u_i(z)$$

In this example, the Core is given by

$$\mathbb{Y} = \Big\{ \Big(N, \big(w_i, w_j, w_k\big)\Big) : 0 \le w_i \le w_j < w_k;\ w_k \ge G;\ w_i + w_j + w_k = F; \quad i, j, k \in \{1, 2, 3\} \Big\}$$

The vNM stable set suffers from a conceptual flaw that it does not take farsightedness of players into account. The next definition of dominance takes into account the farsightedness of players and adapts Chwe (1994)'s definition of indirect dominance and consistent set to our environment.

Definition 7. (Farsighted Dominance). z^0 is farsightedtly dominated by z^m, or, $z^0 \ll z^m$ if there exists a path $z^0 = (T^0, w^0), z^1 = (T^1, w^1), \ldots, z^m = (T^m, w^m)$ and coalitions $S^1, S^2, \ldots, S^m$ such that for every $i = 1, \ldots, m$,

1. $T^i \subset S^i$;

2. S^i is more powerful at state z^{i-1} than the coalition of players who lose from the move in its final conclusion under the path

$$\pi\big[S^i; z^{i-1}\big] > \pi\big[\{i \in N : u_i(z^m) < u_i(z^{i-1})\}; z^{i-1}\big]$$

3. Players in S^i do not lose from the move and some player in S^i gains from the move i.e.

$$S^i \subset \{i \in N : u_i(z^m) \geq u_i(z^{i-1})\} \text{ and } S^i \cap \{i \in N : u_i(z^m) > u_i(z^{i-1})\} \neq \emptyset$$

Definition 8. (*Chwe*'s Consistent Set). A set $\mathbb{Y} \subset \mathbb{Z}$ is a *Chwe*'s Consistent Set if $z^0 \in \mathbb{Y}$ if and only if $\forall z^1 = (T^1, w^1)$ and $S^1 \subset N$ such that $T^1 \subset S^1$, $\exists z^m \in \mathbb{Y}$, where $z^1 = z^m$ or $z^1 \ll z^m$ and a player $i \in S^1$ for whom $u_i(z^m) < u_i(z^0)$.

Definition 9. (*Chwe*'s Largest Consistent Set). A set $\mathbb{Y} \subset \mathbb{Z}$ is *Chwe*'s Largest Consistent Set if $\mathbb{Y}$ is a Consistent Set that includes any other Consistent Set.

Proposition 3. Suppose the coalitional production function $f(.)$ is strictly superadditive. Then the set of efficient states, $\mathbb{E}$, is externally stable with respect to the farsighted dominance relation, $\ll$. However, there are pillage technologies for which $\mathbb{E}$ is not internally stable with respect to $\ll$.

Proof. External stability of $\mathbb{E}$ with respect to $\ll$ follows because $<\subset\ll$ and $\mathbb{E}$ is externally stable with respect to $<$ by Proposition 1. For the second assertion, consider an example. There are three players, $N = \{1, 2, 3\}$. There is a symmetric production environment given by

$$f(S) = \begin{cases} 1.2 & \text{if } |S| = 3 \\ 1 & \text{if } |S| = 2 \\ 0 & \text{if } |S| = 1 \end{cases}$$

The power $\pi[S; z]$ of a coalition S at the state z is the total wealth of S i.e. $\pi[S; z] = \sum_{i \in S} u_i(z)$. This notion of power depends on the prevailing state.

$$z = (N, (0.2, 0.4, 0.6)) \quad z' = (\{1, 3\}, (0.3, 0, 0.7)) \quad z'' = (N, (0.35, 0.1, 0.75))$$

It may be checked that $z \ll z''$ via the path z, z', z''. Q.E.D.

The example presented in the proof of Proposition 3 also helps to illustrate a general feature of the model. A deviation by coalition $\{1, 3\}$ from the state $z \in \mathbb{E}$ to state $z' \in \mathbb{Z} \setminus \mathbb{E}$ cannot be deterred by the expectation of a move back to $\mathbb{E}$. So the standard markovian expectations that support the vNM Stable Set do not work in the model. This is primarily because "winning coalitions" are endogenous and may depend on the particular move under consideration. So even though there is no "winning coalition" for a direct move from z to

z", coalition $\{1,3\}$ may enforce a move from z to z' in anticipation of a subsequent move to the efficient state z", thus eventually profiting from the move.

We use the same example to argue that even though the path z, z', z" displayed in the proof of Proposition 3 destabilizes the efficient state z, we can construct an expectation that makes z stable against a deviation to z'. For ease of notation, lets relabel z as z^0 and z' as z^1. We will display states z^2 and z^3 such that the path z^0, z^1, z^2, z^3 makes z^0 stable against a deviation to z^1.

$$\begin{aligned} z^0 &= (N, (0.2, 0.4, 0.6)) \\ z^1 &= (\{1,3\}, (0.3, 0, 0.7)) \\ z^2 &= (\{2,3\}, (0, 0.2, 0.8)) \\ z^3 &= (N, (0.1, 0.2, 0.9)) \end{aligned}$$

Player 1 will block the deviation from z^0 to z^1 as the path z^2, z^3 that is expected subsequent to the deviation punishes player 1 in its final conclusion.

We next argue that not all pillage technologies support the construction of expectations that serve as punishment for some player who is complicit in the initial deviation. For this, take a 3-player environment and suppose the strictly superadditive production function is specified as

$$f(S) = \begin{cases} 1.2 & \text{if } |S| = 3 \\ 0.85 & \text{if } |S| = \{1,2\} \text{ or } \{1,3\} \\ 0.1 & \text{if } |S| = \{2,3\} \\ 0.8 & \text{if } |S| = \{1\} \\ 0 & \text{if } |S| = \{2\} \text{ or } \{3\} \end{cases}$$

The power $\pi[S;z]$ of a coalition S at the state z is the total wealth of S i.e. $\pi[S;z] = \sum_{i \in S} u_i(z)$. Let $z^0 = (N, (0.7, 0.3, 0.2))$ and $z^1 = (\{1\}, (0.8, 0, 0))$. This deviation cannot be deterred by an expectation of a move back to an efficient state. Also observe that it is impossible to construct an expectation that punishes player 1. To see this, suppose we could construct a path embodying such an expectation. The coalition, say S^2, that gets this path rolling from z^1 must be more powerful than the coalition of players who eventually lose from the move. This implies that the coalition S^2 must be more powerful than player 1 which is impossible given the pillage technology.

In order to be able to construct an expectation punishing player 1, we need to make the players other than player 1 more powerful. One power function that enables us to do that specifies the power $\pi[S;z]$ of a coalition S at the state z as $\pi[S;z] = v\,|S| + \sum_{i \in S} u_i(z)$. Suppose in the above example $v > 0.8$. Then $\pi[\{2,3\}; z^1] > \pi[\{1\}; z^1]$. Letting $z^2 = (\{2,3\}, (0, 0.05, 0.05))$ and $z^3 = (N, (0.2, 0.5, 0.5))$, we now have the desired expectations. Player 1 will not deviate from z^0 to z^1 as the path z^2, z^3 that is expected subsequent to the deviation punishes player 1 in its final conclusion.

In the next result, we restrict the state space to allow for only those states at which every player who holds property rights over the asset enjoys a positive payoff. Let

$$\mathbb{Z}_{++} = \{z = (S, u) \in \mathbb{Z} : \forall i \in S,\ u_i > 0\}$$
$$\mathbb{E}_{++} = \mathbb{E} \cap \mathbb{Z}_{++}$$

We show that with this mild restriction on state space, the set of efficient states has good farsighted stability properties for some pillage technologies. Formally, the set of efficient states is a *Chwe*'s Largest Consistent Set that is externally stable with respect to $\ll$. The pillage technology in Proposition 4 is specified by a power function that depends both on coalitional size and coalitional wealth. The parameter v specifies the degree of substitutability of size and wealth for attaining the same power. [1]

The logic of the proof of Proposition 4 is as follows. In the first step, we show that for every deviation by coalition S^1 from an efficient state z^0 to an inefficient state z^1 where say coalition T^1 holds the property rights, there is a feasible opportunity for pillage at z^1 that leaves some player $i \in T^1$ losing her property rights. The second step shows that $\mathbb{E}_{++}$ must be a *Chwe's* Consistent Set and involves the construction of a two-step path z^2, z^3 from the deviation z^1. The first move to z^2 leads to player i losing her property rights; the second move ends up in some efficient state z^3 and serves as a punishment for i with respect to z^0. The third step shows that $\mathbb{E}_{++}$ must be a Largest Consistent Set as the stability imperatives preclude the inclusion of any inefficient state.

Proposition 4. Suppose $n \geq 3$, the state space is $\mathbb{Z}_{++}$ and the following conditions hold:
(1) the production function $f(.)$ is strictly superadditive;
(2) the pillage technology as specified by the power function $\pi[.; z]$ is given by

$$\forall z \in \mathbb{Z}, \quad \pi[S; z] = v\,|S| + \sum_{i \in S} u_i(z) \text{ where } v > \max_{k \in N} \frac{f(\{k\})}{n-2}$$

Then the set of efficient states, $\mathbb{E}_{++}$, is a *Chwe*'s Largest Consistent Set of $\langle \mathbb{Z}_{++}, \ll \rangle$ that is externally stable with respect to $\ll$.

Proof. External stability of $\mathbb{E}_{++}$ with respect to $\ll$ follows because $< \subset \ll$ and $\mathbb{E}_{++}$ is externally stable with respect to $<$.

Step 1. We show that

$$\forall z = (S, u) \in \mathbb{Z}_{++} \setminus \mathbb{E}_{++}, \quad \exists\, T \neq N \text{ such that } S \setminus T \neq \emptyset,\ \pi[T; z] > \pi[N \setminus T; z] \text{ and } f(T) > 0$$

Without loss of generality let $u_1 \geq u_2 \geq \ldots \geq u_n$. Choose the smallest $m \in \{1, \ldots, n\}$ such that

$$\pi[\{1, \ldots, m\}; z] > \pi[\{m+1, \ldots, n\}; z]$$
$$\text{i.e. } vm + \sum_{i=1}^{m} u_i > v(n-m) + \sum_{i=m+1}^{n} u_i$$

[1] If coalitional size reduces by 1 player, then an increase in coalitional wealth by v units would keep the power unchanged.

Such an $m < n$ exists as the above inequality holds for $m = n-1$. To see this, suppose first $u_n > 0$. Then $u_{n-2} \geq u_{n-1} \geq u_n > 0$. This implies $u_{n-2} + u_{n-1} > u_n$ which further implies $v(n-2) + \sum_{i=1}^{n-1} u_i > u_n$. Suppose next $u_n = 0$. This implies $\sum_{i=1}^{n-1} u_i = f(S) > 0 = u_n$ which further implies $v(n-2) + \sum_{i=1}^{n-1} u_i > u_n$.

Let $U = \{1, \ldots, m\}$. Suppose $S \setminus U \neq \emptyset$. If $m \geq 2$, then set $T = U$. Else, if $m = 1$ i.e. $U = \{1\}$, then let $k \in S \setminus U$ and set $T = N \setminus \{k\}$ which implies $k \in S \setminus T$.

$$
\begin{aligned}
\pi[T; z] &= v(n-1) + \sum_{i \in N \setminus \{k\}} u_i \\
&> v + u_1 && \text{as } 1 \in N \setminus \{k\} \text{ and } n \geq 3 \\
&> v(n-1) + \sum_{m=2}^{n} u_m && \text{as } U = \{1\} \\
&\geq v + u_k = \pi[N \setminus T; z]
\end{aligned}
$$

Suppose $S \setminus U = \emptyset$. Let k be the biggest player index such that $k \in S$. There are two cases. Suppose first $k \geq 2$. Then set $T = N \setminus \{k\}$ so that $k \in S \setminus T$. In this case $u_{k-1} \geq u_k$ which implies $\sum_{i \in N \setminus \{k\}} u_i \geq u_k$ so that

$$
\pi[T; z] = v(n-1) + \sum_{i \in N \setminus \{k\}} u_i > v + u_k = \pi[N \setminus T; z]
$$

as $n \geq 3$. Suppose next $k = 1$ so that $S = \{k\}$. Then set $T = N \setminus \{k\}$ so that $k \in S \setminus T$. In this case

$$
\begin{aligned}
\pi[T; z] &= v(n-1) + \sum_{i \in N \setminus \{k\}} u_i \\
&= v(n-1) + f(S) - u_k && \text{as } z = (S, u) \in \mathbb{Z}_{++} \\
&> v + 2f(S) - u_k && \text{by definition of } v \\
&\geq v + 2u_k - u_k && \text{as } z = (S, u) \in \mathbb{Z}_{++} \\
&= v + u_k = \pi[S; z] = \pi[N \setminus T; z]
\end{aligned}
$$

Since f is strictly superadditive and $|T| \geq 2$, we have $f(T) > 0$.

Step 2. In this step, we show $\mathbb{E}_{++}$, is a *Chwe*'s Consistent Set. Observe that a deviation to another efficient state from an efficient state is not feasible as some player must lose from the move and that player will block the deviation. Fix $z^0 \in \mathbb{E}_{++}$ and a deviation by coalition S^1 to $z^1 = (T^1, w^1) \in \mathbb{Z}_{++} \setminus \mathbb{E}_{++}$. We will define $z^2 = (T^2, w^2) \in \mathbb{Z}_{++} \setminus \mathbb{E}_{++}$ and $z^3 = (N, w^3) \in \mathbb{E}_{++}$ such that $z^1 \ll z^3$ but there exists a player $i \in S^1$ such that $u_i(z^3) < u_i(z^0)$ so that i blocks the initial deviation to z^1.

Since $z^1 \in \mathbb{Z}_{++} \setminus \mathbb{E}_{++}$, by *Step 1*, $\exists T^2 \neq N$ such that $T^1 \setminus T^2 \neq \emptyset, \pi[T^2; z^1] > \pi[N \setminus T^2; z^1]$ and $f(T^2) > 0$. Let $i \in T^1 \setminus T^2 \subset S^1$. Then since $z^0 \in \mathbb{E}_{++}$, we have $u_i(z^0) > 0$.

Construction of w^2.
Step a. Define $\tilde{w}^2 \in \mathbb{R}^n_+$ by

$$\begin{aligned}
\forall j \in T^2 \cup \{i\} \quad & \tilde{w}^2_j = \epsilon \in \Big(0, \min\ \big\{u_i(z^0), (w^1_j)_{j \in T^2 : w^1_j > 0}, \frac{f(T^2)}{1+|T^2|}\big\}\Big) \\
\forall j \in N \setminus (T^2 \cup \{i\}) \quad & \tilde{w}^2_j = 0 \\
\text{so that the excess } & \tilde{e} = f(T^2) - \sum_{j \in N} \tilde{w}^2_j > 0
\end{aligned}$$

Step b. Consider the optimization problem

$$\begin{aligned}
& \max_{(\eta_j)_{j \in T^2 : w^1_j > 0}} \sum_{j \in T^2 : w^1_j > 0} \eta_j \\
\text{subject to} \quad & \forall j \in T^2 : w^1_j > 0 \quad \tilde{w}^2_j + \eta_j \le w^1_j \\
& \sum_{j \in T^2 : w^1_j > 0} \eta_j \le \tilde{e} \\
& \forall j \in T^2 : w^1_j > 0 \quad \eta_j \ge 0
\end{aligned}$$

Abusing notation, let $(\eta_j)_{j \in T^2 : w^1_j > 0}$ be the solution to this optimization problem. Define $\hat{w}^2 \in \mathbb{R}^n_+$ by

$$\begin{aligned}
\forall j \in T^2 : w^1_j > 0 \quad & \hat{w}^2_j = \tilde{w}^2_j + \eta_j \\
\forall j \in N \setminus \{k \in T^2 : w^1_k > 0\} \quad & \hat{w}^2_j = \tilde{w}^2_j \\
\text{Recompute the excess } & \hat{e} = f(T^2) - \sum_{j \in N} \hat{w}^2_j \ge 0
\end{aligned}$$

Step c. If $\hat{e} = 0$, then $w^2 := \hat{w}^2$. If $\hat{e} > 0$, then

$$\begin{aligned}
\forall j \in T^2, \quad & w^2_j := \hat{w}^2_j + \frac{\hat{e}}{|T^2|} \\
\forall j \in N \setminus T^2, \quad & w^2_j := \hat{w}^2_j \\
\text{At this point the excess } & e = f(T^2) - \sum_{j \in N} w^2_j = 0
\end{aligned}$$

so that $z^2 = (T^2, w^2) \in \mathbb{Z}_{++}$.

Construction of w^3. Let $\tilde{h} = f(N) - f(T^2) > 0$ by strict superadditivity of f. There are two cases.

Case a. If $\hat{e} = 0$ in *Step b* in the construction of w^2, then consider the optimization problem

$$\begin{aligned}
&\max_{(\gamma_j)_{j \in T^2 : w_j^1 > 0}} \sum_{j \in T^2 : w_j^1 > 0} \gamma_j \\
\text{subject to} \quad &\forall j \in T^2 : w_j^1 > 0 \quad \hat{w}_j^2 + \gamma_j \leq w_j^1 \\
&\sum_{j \in T^2 : w_j^1 > 0} \gamma_j \leq \tilde{h} \\
&\forall j \in T^2 : w_j^1 > 0 \quad \gamma_j \geq 0
\end{aligned}$$

Again, let $(\gamma_j)_{j \in T^2 : w_j^1 > 0}$ be the solution to this optimization problem. Define $\hat{w}^3 \in \mathbb{R}_+^n$ by

$$\begin{aligned}
\forall j \in T^2 : w_j^1 > 0 \quad & \hat{w}_j^3 = w_j^2 + \gamma_j \\
\forall j \in N \setminus \{k \in T^2 : w_k^1 > 0\} \quad & \hat{w}_j^3 = w_j^2 \\
\text{Compute the excess } \hat{h} = f(N) - \sum_{j \in N} \hat{w}_j^3 \geq 0
\end{aligned}$$

If $\hat{h} = 0$, then $w^3 := \hat{w}^3$. If $\hat{h} > 0$, then choose $\hat{\epsilon} \in (0, \hat{h})$ and define

$$\begin{aligned}
\forall j \in T^2, \quad & w_j^3 := \hat{w}_j^3 + \frac{\hat{h} - \hat{\epsilon}}{|T^2|} \\
\forall j \in N \setminus (T^2 \cup \{i\}), \quad & w_j^3 := \hat{w}_j^3 + \frac{\hat{\epsilon}}{|N \setminus (T^2 \cup \{i\})|} \\
& w_i^3 := \hat{w}_i^3 \\
\text{At this point the excess } h = f(N) - \sum_{j \in N} w_j^3 = 0
\end{aligned}$$

so that $z^3 = (N, w^3) \in \mathbb{Z}_{++}$.

Case b. If $\hat{e} > 0$ in *Step b* in the construction of w^2, then $\forall j \in T^2 : w_j^1 > 0, w_j^2 \geq w_j^1$. So choose $\tilde{\epsilon} \in (0, \tilde{h})$ and define

$$\begin{aligned}
\forall j \in T^2, \quad & w_j^3 := w_j^2 + \frac{\tilde{h} - \tilde{\epsilon}}{|T^2|} \\
\forall j \in N \setminus (T^2 \cup \{i\}), \quad & w_j^3 := w_j^2 + \frac{\tilde{\epsilon}}{|N \setminus (T^2 \cup \{i\})|} \\
& w_i^3 := w_i^2 \\
\text{At this point the excess } h = f(N) - \sum_{j \in N} w_j^3 = 0
\end{aligned}$$

so that $z^3 = (N, w^3) \in \mathbb{Z}_{++}$.

This construction implies that for every $j \in T^2$, $w_j^3 \geq \max\,(w_j^2, w_j^1)$ and for every $j \in N \setminus T^2$, $w_j^3 \geq w_j^2$. This further implies that $z^1 \lll z^3$ but $u_i(z^3) := w_i^3 < u_i(z^0)$ so that i blocks the initial deviation to z^1.

Step 3. In this step, we show $\mathbb{E}_{++}$, is *Chwe*'s Largest Consistent Set (LCS). Suppose by way of contradiction that $z = (S, u(z)) \in \mathbb{Z}_{++} \setminus \mathbb{E}_{++} \cap$ LCS. Define $z' = (N, (u_i(z) + \frac{f(N)-f(S)}{n})_{i \in N})$. Then $z' \in \mathbb{E}_{++} \subset$ LCS. Moreover $z < z'$ and since $\{z, z'\} \subset$ LCS , a deviation from z to z' cannot be deterred. But this contradicts that z is stable by virtue of being in LCS. This proves that $\mathbb{E}_{++}$ is LCS. Q.E.D.

4 Concluding Remarks

The possibility of efficient allocation of property rights in a model of asset exchange where coalitions have both production and pillage opportunities is studied. The model reflects the feature of human societies that any cooperative production happens in the shadow of conflict. This conflict arises from distribution of wealth primarily because any such distribution has implications for power. More powerful coalitions may coercively take control of the asset from less powerful ones. We show that efficient allocation of property rights is supported as a stable allocation using a range of cooperative stability notions like vNM Stable Set, Core and Largest Consistent Set (LCS). The LCS is widely seen as one of the most permissive notions of stability. As such, its conclusions for what is unstable is robust. Our main result, Proposition 4, says that for a class of production and pillage technologies, the set of efficient states is the LCS. The LCS expectations that make the set of efficient states stable rely on punishing the acts of pillage feasible at any efficient state, thereby disciplining any initial coalition contemplating pillage. Proposition 4 may also be interpreted as saying that for a class of environments, property rights enforcement is not necessary for achieving efficiency.

We work with strictly superadditive production technologies. The pillage technologies that support the punishments in our main result are ones that provide that in determining power of a coalition, an increase in coalitional size can compensate for at least some reduction in wealth.

The advantages of a cooperative analysis is that it allows us to deduce conclusions without writing detailed institutional structures. But the treatment of expectations is ad-hoc. A noncooperative analysis would involve writing a dynamic bargaining game in a style similar to Konishi and Ray (2003) providing an institution for generating those expectations. A noncooperative analysis is the subject matter of future work.

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