A Theory of Endogenous Coalition Formation in Group Contests *

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Abstract

Agents compete for a reward by forming coalitions through a sequential process. The probability that a coalition wins the reward depends on the relative size of the coalition in the economy. The winning coalition equally divides the reward among all its members. Agents strategically form coalitions to maximise the expected value of their individual reward share. We extend the three axioms for contest success functions (formalized by Skaperdas(1996)) to coalition structures and show that the coalition structure at the Stationary Perfect Equilibrium contains at least two coalitions. One coalition necessarily consists of at least a majority of the agents and the size of that coalition is characterised. Further, we derive conditions under which the remaining agents form a single coalition. As examples to this theory, we use the two common functions used in this literature: ratio and difference functions. We plot the size of the majority coalition against the number of agents in the economy discuss the differences between the two.

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1 Introduction

Consider a contest where competing agents benefit by forming coalitions with their rivals in self interest. The synergy created by forming a coalition increases the probability of their coalition winning the reward. There are plenty of examples such as formation of alliances during wars, political parties to win elections, cartels of firms to lobby in the government, etc. In all these applications agents not just invest staggering amount of resources, but also form coalitions to enhance their chance to win the scarce reward. This paper contributes significantly to literature, first, by extending the axiomatizing of the contest success function by Skaperdas (1996) to partitions of agents (or coalition structures). The second, and the major contribution, is that we show that the stationary perfect equilibrium consists of coalition structure with at least two coalitions. One of those coalitions consists of at least a majority of agents in the economy, but not the grand coalition, and characterise its size for any general contest success function satisfying the axioms.

This problem can be formulated as the endogenous coalition formation in non-cooperative terms by Hart and Kurz (1983). The individual payoff to agents in their model depends not only on the members of their own coalition, but also on the coalitions formed by agents outside their own coalition. Thus, agents form coalitions with their rivals, accounting for the actions of other agents. The time line for the game is the following: Formation of coalitions - Choose efforts - Compete - Outcome The process of coalition formation is modelled sequentially formed as in the game Δ introduced by Hart and Kurz(1983) where agents announce coalitions from the strategy space $S_i = \{C \subset N, i \in C\}$. The coalitions are formed by all players who have announced the same coalition, whether or not the formation of the coalition has been approved unanimously by all its members. As noted by Bloch (2003) solution concepts such as Strong Nash or Coalition Proof Nash Equilibrium are appropriate for coalition formation in non-cooperative setting. However, Hart and Kurz (1984) show that such models may not admit a Strong Nash Equilibrium. For this reason, we cannot make any predictions in the general model using SNE as the solution concept. Therefore, I use a sequential coalition formation mechanism as defined by Bloch(1993) and Stationary

Perfect Equilibrium as the solution concept.

To get a sense of the model, consider the rent-seeking game introduced by Tullock (1967). In this model, agents invest resources to compete for a reward of fixed value R. In a group contest, agents form groups and agree on a sharing rule to allocate the reward when a group wins the contest. The probability that agent i wins the prize increases with an increase in the resources invested by his group and decreases with an increase in resources invested by other groups. Consider the simplest sharing rule – equal sharing where every group member has an equal probability of getting the prize. The formation of a group induces two opposing effects on an agent's utility: on the one hand, it increases her probability of winning the contest, on the other hand, it reduces the expected value of the prize if the group wins the contest. The balance between these two effects shapes the incentives to form groups, or secede from the universal agreement.

We extend the three axioms introduced by Skaperdas (1996) to the coalition structures. The three axioms are (1) increasing amount of resource invested by coalition C_k increases the probability that it wins (2) If coalition C_l increases the resource invested, it decreases the winning probability of C_k (3) Coalitions investing equal resource have equal chance of winning. Our main result is that at the Stationary Proof Equilibrium, the coalition structure consists of at least two coalitions. Thus, the grand coalition is not formed. One coalition consists of at least a majority of agents in the economy. The size of this coalition is that which maximises the individual expected payoff to its members. Further, we state a sufficient condition for the remaining members to form a coalition.

In the rent-seeking literature, the issue of group and alliance formation has received some attention since the early 80's (See Tullock (1980), Katz, Nitzan and Rosenberg (1991), Nitzan (1991), and the survey by Sandler (1993).) The early literature treated groups and alliances as exogenous, and did not consider incentives to form groups in contests. Baik and Shogren (1995), Baik and Lee (1997) and Baik and Lee (2001) obtain partial results on group formation in rent seeking models with linear costs. They consider a three-stage model, where players form groups, decide on a sharing rule, and then choose noncooperatively the resources they spend on con‡ict. Baik and Shogren (1995) analyze a situation where a single group faces isolated players, Baik and Lee (1997) consider competition between two groups and Baik and Lee (2001) analyze a general model with an arbitrary number of groups. In all three models, through examples, it appears that the group formation model leads to the formation of groups containing approximately one half of the players. This fits in with our main result in the paper.

In section 2 we describe the sequential coalition formation. Section 3 builds our model and Section 4 contains the analysis for the results. Section 5 uses two standard contest success functions in literature and contrasts the size of the majority coalition with the size of the economy.

2 Sequential Coalition Formation

The coalition formation game is played in two stages: first agents form coalitions and then they receive their payoffs. This model of coalition formation is formed sequentially based on the process given in Bloch(1993). The coalition formation mechanism is based on Rubinstein (1982)'s alternating offers bargaining game and its extensions to n players by Selten (1981) and Chatterjee et al. (1993) that is modelled for non-cooperative coalition formation game with a fixed sharing rule; identical to the model here. The process is as follows.

A randomly chosen agent makes the coalition proposal $s_i = \{C \in N, i \in C\}$. Prospective members of C play strategies from the set $\{Y, N\}$. If any prospective member rejects the proposal, the first member to reject it is chosen as the initiator in the next round and he must make a counteroffer and propose a coalition C' to which he belongs. If all prospective members of C accepts the proposal, the coalition is formed. A random agent, among the remaining agents in $N \backslash C$, is chosen as the initiator. This process continues till no initiator is left. Once a coalition has been formed, the game is only played among the remaining players. The horizon for this game is infinite. There is no discount of payoff, but in case of infinite play I assume all agents to get zero payoff. The outcome of the sequential coalition formation game is a partition of the set of agents into disjoint coalitions, called a coalition structure. A stationary perfect Equilibrium (SPE) of the coalition formation game is a strategy profile

$\mathbf{S} = \{s_1, s_2..s_n\}$ such that (1) for every agent i, the strategy s_i is a stationary strategy and (2) for every agent i after every history at which i moves s_i is a best response to the strategies of the other players s_{-i}.

3 The Model

Consider a set N composed of n ex-ante identical agents who compete for a reward $R > n$. A *coalition structure* $\pi = \{C_1, C_2..., C_K\}$ is a partition of N into a collection of disjoint coalitions indexed by k. Let $|C_k|$ denote the cardinality of the coalition C_k. Without loss of generality, within any coalition structure, we order coalitions in ascending order of sizes: $|C_k| < |C_{k+1}|$. Once a coalition is formed, each agent $i \in C_k$ chooses to invest an amount $y_i \in [0, Y]$ of the resource in order that his coalition wins the contest. The total resources invested by coalition C_k is $Y_k = \sum_{i=1}^{k} y_i$. The time horizon of the game is as follows: in the first period agents endogenously form coalitions. They choose amount of resource to invest for winning in the second period. The outcome of the contest is decided in the third period. Only one of the K competing coalitions wins the reward. The reward is split equally among all members of the coalition.

The contest success function (CSF) maps every vector of resources invested by all coalitions $\mathbf{Y} = \{Y_1, Y_2..., Y_K\}$ into a vector $p(\mathbf{Y}) = \{p_k(\mathbf{Y})\}_{k=1}^{K}$ of coalitional winning probabilities. $p_k(\mathbf{Y})$ is interpreted as an agent's prior that coalition C_k wins, given that the resources invested are $\mathbf{Y}$. All agents have identical priors. The resource invested, y_i, is private information.The payoff to agent $i \in C_k$ is:

$$v_i(|C_k|, \mathbf{Y}) = \frac{p_k(\mathbf{Y})R}{|C_k|} - y_i$$

We assume that more the resource a coalition invests, the greater is its winning probability. This assumption is quite intuitive. We show that the problem of moral hazard does not arise when the reward is sufficiently large, $R > n$.

Proposition 1: If winning probability of a coalition, $p_k(\mathbf{Y}$, is increasing in the amount of resource invested Y_k: $\frac{\partial p_k(\mathbf{Y})}{\partial Y_k} > 0$ and the reward is sufficiently large, $R > n$, then it is optimal for every agent to invest $y_i = Y$.

Proof. The payoff[1] to agent $i \in C_k$ is:

$$v_i(|C_k|, \mathbf{Y}) = \frac{p_k(\mathbf{Y})R}{|C_k|} - y_i \text{ where } C_k \in \pi$$

If the marginal increase in agent i's utility is positive, the agent will want to put in the maximum possible effort.

$$\begin{aligned} \frac{\partial v_i}{\partial y_i} &= \frac{\partial p_k(\mathbf{Y})}{\partial y_i} \frac{R}{|C_k|} - 1 \\ &= \frac{\partial p_k(\mathbf{Y})}{\partial Y_k} \frac{R}{|C_k|} - 1 \end{aligned}$$

The maximum value of $|C_k| = n$ (grand coalition). Thus, when $R > n$ the marginal increase is the agents utility is always positive: $\frac{\partial v_i}{\partial y_i} > 0$

Thus, it is optimal for every agent in C_k to invest all resources: $y_i = Y$. □

Thus, agents have no incentive to invest less than Y and eliminate the problem of moral hazard: incentive exists for agents to invest lesser than the amount committed ex-ante. This also speaks about the efficiency of the sharing rule. It induces the agents to invest full resources. If the resources are interpreted as efforts, it is a desirable outcome from the perspective of the principal who is offering the reward for the contest.

Under the assumptions of increasing winning probabilities and sufficiently large rewards, the resource invested is proportional to coalition size: $Y_k = |C_k|Y$. Thus, an increase (decrease) in the invested resources is equivalent to increasing (decreasing) coalition size. Normalising $Y = 1$, we have the winning probability as a function of the coalition sizes.

$$p_k(\pi) = f(|C_k|, |C_{-k}|)$$

[1]Throughout the paper we use payoff and expected payoff equivalently

where $|C_{-k}|$ is the sizes of all coalitions in π except C_k.

Although, the arguments of $f(.)$ are discrete, it is a continuous and differentiable function. Also, $\sum_{k=1}^{K} f(|C_k|, |C_{-k}|) = 1$ follows as it is the winning probability. An example for this function is

$$f(|C_k|, |C_{-k}|) = \frac{|C_k|^\alpha}{\sum_{k=1}^{K} |C_k|^\alpha}$$

As every agent invests all resources, the only way a coalition can increase its winning probability is by increasing the size of their coalition. The transfer of one or more members from one coalition to another must imply that while the winning probability of one coalition increases, it must also decrease the winning probability of at least one coalition. To formalise this concept consider the coalition structure $\pi = \{C_1, C_2 ..., C_K\}$. Choose any coalition $C_l \neq C_k$ and keep the coalitions sizes of all coalitions, except C_k and C_l, constant such that $m = \sum_{i=1}^{K-2} |C_i|$. Writing the coalition size of C_l in terms of C_k:

$$|C_l| = (n - m) - |C_k|$$

Restating the assumption of increasing winning probability with resources in terms of coalition size.

Assumption 1: The winning probability of a coalition C_k increases with an increase in its size $|C_k|$.

$$\frac{\partial p_k(\pi)}{\partial |C_k|} = \frac{\partial f(|C_k|, (n-m) - |C_k|)}{\partial |C_k|} > 0$$

where $0 \leq |C_k| \leq n - m$for all $m \in [K-2, n-2]$ and $C_l \in \pi, C_l \neq C_k$

This assumption means that transferring members from C_l to C_k increases the winning probability of C_k. It also implies that the winning probability of C_l decreases. Further, by allowing $|C_k|$ to take boundary values of 0 and $n-m$, we can compare the winning probability across coalition structures with unequal number of coalitions. For example, consider a three element coalition structure $\pi = \{2, 3, 6\}$ where $2, 3$ and 6 are the coalition sizes of C_1, C_2 and C_3 respectively. Henceforth, we use this shorthand for examples. Transfer all members of C_1 to C_2 to get a two element coalition structure $\pi' = \{5, 6\}$. By Assumption 1, the winning probability of C_1' is greater than C_1.

Note that Assumption 1 does not allow a complete comparison of winning probabilities among all coalition stuctures; $\pi = \{3, 3\}$ and $\pi' = \{1, 1, 4\}$ cannot be compared. As this incompleteness does not interfere with our results, we ignore this issue.

We deviate slightly from our story to define convexity of $f(.)$ using the terms above.

Definition 1:The function $f(.)$ is convex in $|C_k|$ if

$$\frac{\partial^2 f(|C_k|, (n-m) - |C_k|)}{\partial |C_k|^2} > 0$$

Similarly, $f(.)$ is concave if $\frac{\partial^2 f}{\partial |C_k|^2} < 0$ and linear if $\frac{\partial^2 f}{\partial |C_k|^2} = 0$. Such a definition is required as the arguments of $f(.)$ are dependent on each other. Thus, one argument cannot be changed while keeping all the rest unchanged.

Going back to the example where we compare $\pi = \{2, 3, 6\}$ and $\pi' = \{5, 6\}$, the effect of such a transfer on the winning probability of C_3 is not specified. We assume that such a transfer reduces the winning probability of C_3'. Thus, we assume negative spillovers in our model. To formalise this, let C_x and C_y be any coalitions in $\pi = \{C_1, C_2 ..., C_K\}$ such that $|C_x| \geq |C_y|$. Keeping all other coalitions constant, $\sum\limits_{i \neq x,y} |C_i| = m$. Thus, $C_y = (n-m) - C_x$.

Assumption 2: If agents are transferred from coalition C_y to C_x, where $|C_x| \geq |C_y|$, winning probability of coalition C_k decreases (increases) with an increase (decrease) in its size of C_x.

$$\frac{\partial p_k(\pi)}{\partial |C_x|} = \frac{\partial f(|C_k|, |C_x|, (n-m) - |C_x|)}{\partial |C_x|} < 0$$

where $0 \leq |C_k| \leq n - m$ for all $m \in [K-2, n-2], C_x, C_y \in \pi, C_x \neq C_y \neq C_k$

The intuition behind this is that, for an increase in resource invested by other coalitions, keeping the resource invested by C_k unchanged, the winning probability of C_k decreases. Note that it is essential that members of a C_y are transferred to a coalition, C_x, of equal or greater size. Such a requirement ensures that transfers that lead to equivalent coalition structures are excluded. For example, consider $\pi = \{3, 4, 5\}$. Transfer an agent from C_3 to C_2. As agents are identical, the coalition structure after the transfer remains the same. The requirement $|C_y| \leq |C_x|$ ensures that such a transfer does not change the winning probability of $|C_1|$.

Lastly, we assume that coalitions investing equal resources have equal winning probabilities.

Assumption 3: If $|C_k| = |C_l|$ where $C_k, C_l \in \pi$, then it must be that $p_k(\pi) = p_l(\pi)$.

It follows that in a coalition structure $\pi = \{C_1, C_2..., C_K\}$ if $|C_1| = |C_2|... = |C_K|$, then $p_1(\pi) = p_2(\pi)... = p_K(\pi) = \frac{1}{K}$. Such symmetric coalition structures are frequently used as reference points while proving the results ahead.

All information, including the assumptions, are common knowledge. However, the nature of function $f(.)$: the extent of increase or decrease the winning probabilities, is unknown to the agents. In the next section, we prove our main results.

4 Results

From Proposition 1 we have $y_i = Y$ for every $i \in N$. Thus, the payoff to agent $i \in C_k$ is

$$v_i(|C_k|, \pi) = \frac{p_k(\pi)R}{|C_k|} - Y$$

To simplify our analysis, we use a linear transformation of the function above:

$$u_i(|C_k|, \pi) = u(av_i(.) + b) = \frac{p_k(\pi)}{|C_k|} \quad \text{where} \quad a = \frac{1}{R}, b = \frac{Y}{R}$$

A coalition's worth is defined as the sum of payoffs to all members of a coalition. The coalition worth for $C_k \in \pi$ is

$$U(|C_k|, \pi) = \sum_{i \in C_k} u_i(|C_k|, \pi) = p_k(\pi)$$

Thus, the winning probability of a coalition can also be interpreted as its worth. The convexity (and concavity) of the winning probability function, $f(.)$, defined earlier in this section have an interesting interpretation here.

Proposition 2: If the nature of $f(.)$ is
(1) Linear in C_k, the individual payoff is $u_i(|C_k|, \pi) = \frac{1}{n}$ for all $i \in N$
(2) Concave in C_k, the individual payoff is $u_i(|C_k|, \pi) > \frac{1}{n}$ for all $i \in C_k$
(3) Convex in C_k, the individual payoff is $u_i(|C_k|, \pi) < \frac{1}{n}$ for all $i \in C_k$

Proof. First, we prove the result for the case where $f(.)$ is linear. Using that result, the other two can be proved easily.
Let $f(.)$ be linear in $|C_k|$
Let $f(|C_k|, |C_{-k}|) = a|C_k|$ where a is a non-zero constant. Summing the winning probabilities across all coalitions

$$\sum_{k=1}^{K} f(|C_k|, |C_{-k}|) = a \sum_{k=1}^{K} |C_k| = 1$$

$$a = \frac{1}{\sum\limits_{k=1}^{K} |C_k|} = \frac{1}{n}$$

Thus, the coalition worth is $U(|C_k|, \pi) = \frac{|C_k|}{n}$ and the individual payoff is $u_i(|C_k|, \pi) = \frac{1}{n}$. Note that this will be the payoff to all agents, irrespective of the coalition to which they belong. □

A linear function $f(.)$ implies that adding more members to C_k proportionally increases its worth such that the individual payoff remains unchanged. Adding members increases the worth more than proportional if $f(.)$ is convex and less than proportional if concave. There is some cost of cooperation while forming coalitions, that is considered negligible here. Thus, if $f(.)$ is linear, agents do not form any coalitions, as forming coalitions provides them with no increase in payoffs.

Next, we show that rational agents cannot have a prior that the $f(.)$ is either strictly convex or strictly concave over the entire interval of coalition sizes.

Proposition 3: For a K element coalition structure $\pi = \{C_1, C_2..., C_K\}$, if agents have rational priors, then it cannot be that $\frac{\partial f(|C_k|,(n-m)-|C_k|)}{\partial |C_k|} > 0$ or $\frac{\partial f(|C_k|,(n-m)-|C_k|)}{\partial |C_k|} < 0$ for all $0 < |C_k| < n-m$ and $m \in [K-2, n-2]$.

For all K partitioned coalition structures, it is not possible that a coalition $U(|C_k|, \pi)$ is concave or convex for all values of $|C_k|$. The intuition for this proposition is that when $U(|C_k|, \pi)$ is concave for, say, $c_1 \geq |C_k| < c_2$, every members payoff is greater than $\frac{1}{n}$. As it is a constant sum game, there must be at least one other coalition, say C_l, whose members receive less than $\frac{1}{n}$, hence their coalition worth is convex. The interval of C_l will be different from C_k, say $d_1 \geq C_l \geq d_2$. Now as agents are identical, when $c_1 \geq |C_l| < c_2$ and $d_1 \geq |C_k| < d_2$ the coalition worth of C_l will be concave and C_k will be convex. Hence, our result.

For the rest of the paper consider a two element coalition structure $\pi = \{C_1, C_2\}$ where $|C_2| = n - |C_1|$. There are two reasons for analysing this coalition structure. First, it can be represented in a two dimensional graph, hence explaining the analysis is easier. Second, is these results can then be extended for K element coalition structures.

We begin by plotting the coalition worth $U(|C_k|, \pi)$ against its coalition size $|C_k|$. The graph in *Figure 1* is a box plot where

(1) The bottom horizontal axis represents $|C_1|$ where its range is $0 < |C_1| < n$ from left to right.

(2) The top horizontal axis represents the coalition size $|C_2| = n - |C_1|$ where its range is $0 < |C_2| < n$ from right to left.

(3) The left vertical axis represents the coalition worth $U(|C_1|, \pi)$ of $|C_1|$ where its range is

$0 < U(|C_1|, \pi) < 1$ from bottom to top.
(4) The right vertical axis represents the coalition worth $U(|C_2|, \pi) = 1 - U(|C_1|, \pi)$ of $|C_2|$ where its range is $0 < U(|C_1|, \pi) < 1$ from bottom to top.

The red curve is the coalition worth of C_1: $f(|C_1|, \pi)$ and blue curve is that of C_2: $f(|C_2|, \pi)$. The intersection of the red and blue curves, point O, is a symmetric coalition stucture: $|C_1| = |C_2| = \frac{n}{2}$. Note that line XX' is the plot of the coalition worth against its size when $f(.)$ is linear.

In *Lemma 1*, we prove that the nature of the curves is as shown in *Figure 1*, that is, the coalition worth of $|C_1|$ is concave when its size is above $\frac{n}{2}$ and convex below $\frac{n}{2}$. As C_2 is the only other coalition, $U(|C_2|, \pi) = 1 - U(|C_1|, \pi)$ must be convex for the interval the coalition worth of C_1 is concave and vice versa (as proved in Proposition 3). This lemma is a foundation for *Theorem 1*.

Lemma 1: In a two element partition $\pi = \{C_1, C_2\}$, the coalition worth for C_k where $k = 1, 2$ is always a concave for $\frac{n}{2} \leq |C_k| \leq n$ and convex for $0 \leq |C_k| \leq \frac{n}{2}$.

Proof. We prove it through the contrapositive approach:
(1) Assume the coalition worth to be linear.
A sequential coalition formation process, allows the formation of the following coalition structures where $c_1 < \frac{n}{2}$:

$$\begin{aligned}
\pi &= \{c_1, \frac{n}{2}, \frac{n}{2} - c_1\} \\
\pi' &= \{c_1, n - c_1\} \\
\pi'' &= \{\frac{n}{2}, \frac{n}{2}\} \\
\pi''' &= \{c_1 + \frac{n}{2}, \frac{n}{2} - c_1\}
\end{aligned}$$

Except the first coalition structure, the individual payoff to every agent in π', π'' and π'''

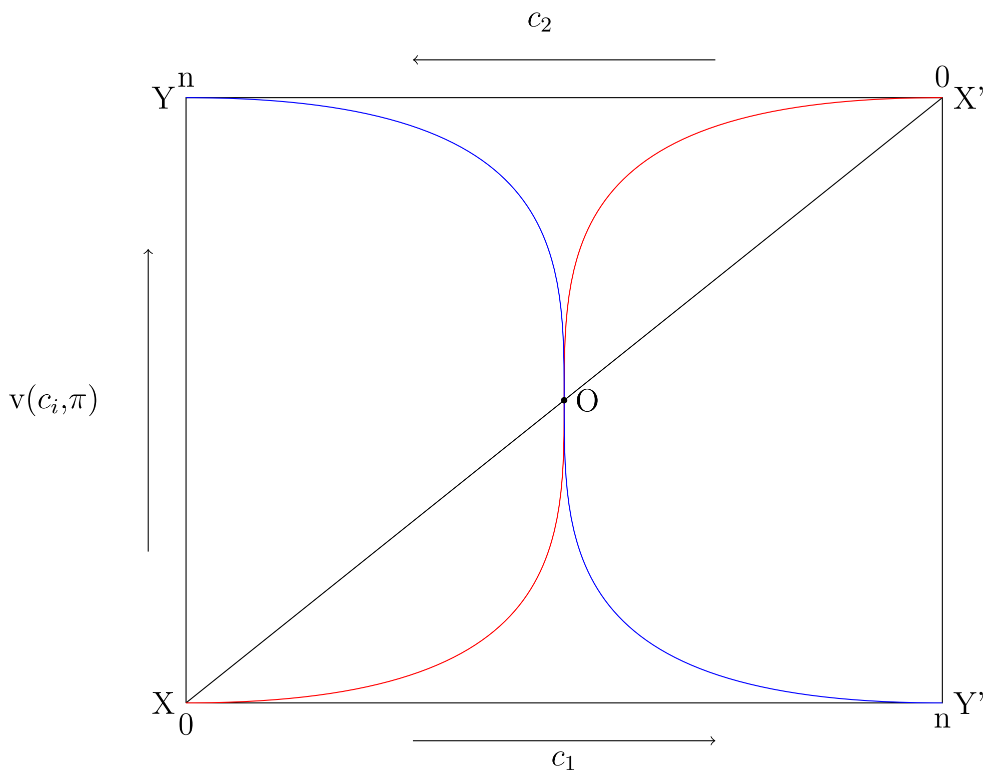

Figure 1: Concave Coalition worth beyond $c_1 = \frac{n}{2}$

is $\frac{1}{n}$ as the coalition worth is linear with coalition size (proposition 2):

$$u_i(|C_k|, \pi') = u_i(|C_k|, \pi'') = u_i(|C_k|, \pi''') = \frac{1}{n} \text{ where } k = \{1, 2\}$$

By assumption 2, we have

$$\begin{aligned} u_i(|C_1|, \pi) &> u_j(|C_1'|, \pi') = \frac{1}{n} \\ u_i(|C_2|, \pi) &> u_j(|C_1''|, \pi'') = \frac{1}{n} \\ u_i(|C_3|, \pi) &> u_j(|C_2'''|, \pi''') = \frac{1}{n} \end{aligned}$$

Thus,

$$\begin{aligned} u|C_1|u_i(|C_1|, \pi) + |C_2|u_i(|C_2|, \pi) + |C_3|u_i(|C_3|, \pi) > 1 \\ U(|C_1|, \pi) + U(|C_2|, \pi) + U(|C_3|, \pi) > 1 \end{aligned}$$

However, this cannot be possible as $\sum_{k=1}^{K} U(|C_k|, \pi) = \sum_{k=1}^{K} p_k(\pi) = 1$ Hence, the coalition worth cannot be linear.

(2) Assume the coalition worth to be convex when the size of C_1 is less than $\frac{n}{2}$ and concave when greater than $\frac{n}{2}$.
The analysis for this case is similar to the one above. Let $c_1 < \frac{n}{2}$ and

$$\begin{aligned} \pi &= \{c_1, \frac{n}{2}, \frac{n}{2} - c_1\} \\ \pi' &= \{c_1, n - c_1\} \\ \pi'' &= \{\frac{n}{2}, \frac{n}{2}\} \\ \pi''' &= \{c_1 + \frac{n}{2}, \frac{n}{2} - c_1\} \end{aligned}$$

As $U(|C_1|, \pi)$ is concave for $0 < |C_1| < \frac{n}{2}$, $u_i(|C_1'|, \pi') > \frac{1}{n}$. Similarly, $u_i(|C_2'''|, \pi''') > \frac{1}{n}$. As π'' is a symmetric coalition strucutre, $u_i(|C_1''|, \pi'') = u_i(|C_2''|, \pi'') = \frac{1}{n}$.

As before, by assumption 2, we have

$$\begin{aligned} u_i(|C_1|, \pi) &> u_j(|C_1'|, \pi') > \frac{1}{n} \\ u_i(|C_2|, \pi) &> u_j(|C_1''|, \pi'') = \frac{1}{n} \\ u_i(|C_3|, \pi) &> u_j(|C_2'''|, \pi''') > \frac{1}{n} \end{aligned}$$

Thus,

$$\begin{aligned} u|C_1|u_i(|C_1|, \pi) + |C_2|u_i(|C_2|, \pi) + |C_3|u_i(|C_3|, \pi) > 1 \\ U(|C_1|, \pi) + U(|C_2|, \pi) + U(|C_3|, \pi) > 1 \end{aligned}$$

However, this cannot be possible as $\sum_{k=1}^{K} U(|C_k|, \pi) = \sum_{k=1}^{K} p_k(\pi) = 1$ Hence, the coalition worth of C_1 cannot be concave when the size of C_1 is less than $\frac{n}{2}$ and convex when greater than $\frac{n}{2}$ in a two element coalition structure $\pi = \{C_1, C_2\}$.

Conducting such an analysis for the case when C_1 is convex when the size of C_1 is less than $\frac{n}{2}$ and concave when greater than $\frac{n}{2}$ shows that such an impossibility does not exist. As the analysis is almost identical to the ones above, we leave it to the interested reader to verify it.

□

The intuition for this lemma is that when the coalition worth is linear in coalition size (figure 2) or concave for size less and concave for size more than $'\frac{n}{2}$ (figure 3), for certain coalition structures, either assumption 2 is violated or the winning probabilities sum up to greater than one. As rational agents are not supposed to violate our assumptions, we prove that the such functions for coalition worth cannot exist. Using this result we show in theorem 1 that the Stationary Perfect Equilibrium always contains a coalition that consists, at least, of a majority. We also characterise the size of the coalition.

Theorem 1: The Stationary Perfect Equilibrium is a $K + 1$ element coalition structure

$\pi = \{C_M, C_1 ..., C_K\}$ where $\frac{n}{2} < |C_M| < n$ is the solution to $\frac{df(|C_k|, n-|C_k|)}{d|C_k|} = \frac{f(|C_k|, n-|C_k|)}{|C_k|}$ and $K \geq 1$.

Proof. Assume that in the first step of the sequential process of coalition formation, the coalition of size $|C_M|$ is formed. In an SPE, the coalition formed at every step is a best response. The payoff to a coalition is minimum when all remaining agents form a coalition. Therefore, the size of C_M, at which individual payoff is maximised, is calculated when the remaining agents form C_1 of size $n - |C_M|$.

From *Proposition 1*, we know that $u_i(|C_M|, \pi) > \frac{1}{n}$ when concave, $u_i(|C_M|, \pi) < \frac{1}{n}$ when convex and $u_i(|C_M|, \pi) = \frac{1}{n}$ when linear or $|C_M| = \frac{n}{2}$. As π consists of only two coalitions, from lemma 1 we know that the coalition worth of C_M is concave for $|C_M| > \frac{n}{2}$ and convex for $|C_M| < \frac{n}{2}$. Therefore, it is always the case that $|C_M| > \frac{n}{2}$ as its members will receive the maximum payoff that way. Agents will not form the grand coalition because $u_i(|C_M|, \pi) = \frac{1}{n}$ when $|C_M| = n$. Therefore, $\frac{n}{2} < |C_M| < n$. The value of the individual payoff to agents in C_M is maximised when $\frac{\partial u_i(|C_M|, \pi)}{\partial |C_M|}$=0.

$$\begin{aligned}
\frac{du_i(|C_M|, \pi)}{d|C_M}| &= \frac{d\frac{U(|C_M|, \pi)}{|C_M|}}{d|C_M|} \\
&= \frac{\frac{df(|C_k|, n-|C_k|)}{d|C_k|} - \frac{f(|C_k|, n-|C_k|)}{|C_k|}}{|C_k|^2} \\
&= 0
\end{aligned}$$

□

Theorem 1 shows that the equilibrium coalition structure always contains one coalition of at least a of majority members, $c*$, but never forms the grand coalition. However, nothing can be said about the remaining $n - c*$ members in the economy. They may form multiple coalitions among themselves depending on the estimated winning probability function. In the corollary below we state the condition under which the remaining members form a single coalition.

Corollary 1: The SPE is a coalition structure consisting of two coalitions $\pi = \{c_1, c_2\}$ of size $c_1 = c*$,$c_2 = n - c*$ where $c*$ is the solution of $\frac{df(c_1,n-c_1)}{dc_1} = \frac{f(c_1,n-c_1)}{c_1}$ if $\frac{f(c_2,c_1)}{c_2} > \frac{f(c_2',c_2-c_2',c_1)}{c_2'}$ where $c_2' \subset c_2$.

The necessary condition simply formalises the idea that the individual payoff to members of c_2 must be greater than the individual payoff to any defecting subcoalition. In the next sections we consider two different parameters by which agents estimate the winning probabilities and contrast the differences in the results.

5 Estimating Winning Probabilities from Coalition's Power

Consider a setting where agents perform equally at a. By forming a coalition, the performance of the coalition is greater than the aggregate of their abilities. The cooperation enhances their performance by $c^\alpha\delta$, where c is the coalition size and $\alpha, \delta > 0$. The final performance of coalition c is $a_c = ca + \delta c^k$. The true probability of winning the reward K depends on the relative performance of the coalition. Agents estimate the true winning probability of their coalition based on its power: the share of its performance in the economy (Morelli and Park(2015)). Let the coalition structure be $\pi = \{c_1, c_2..., c_m\}$. The winning probability of coalition $c_k \in \pi$ is

$$p_k(\pi) = \frac{\sum\limits_{i\in c_k} a_i}{\sum\limits_{j\in N} a_j} = \frac{c_k a + c_k^\alpha\delta}{(c_1a + c_1^\alpha\delta) + (c_2a + c_2^\alpha\delta) + ...(c_m a + c_m^\alpha\delta)}$$

$p_k(\pi)$ satisfies Assumption 1's property 1 for all values of α: the estimated winning probability, increases with an increases in the coalition size. However, property 2 in Assumption 1: that the more concentrated $\pi \setminus \{c_k\}$ gets, lesser is the winning probability, is satisfied only

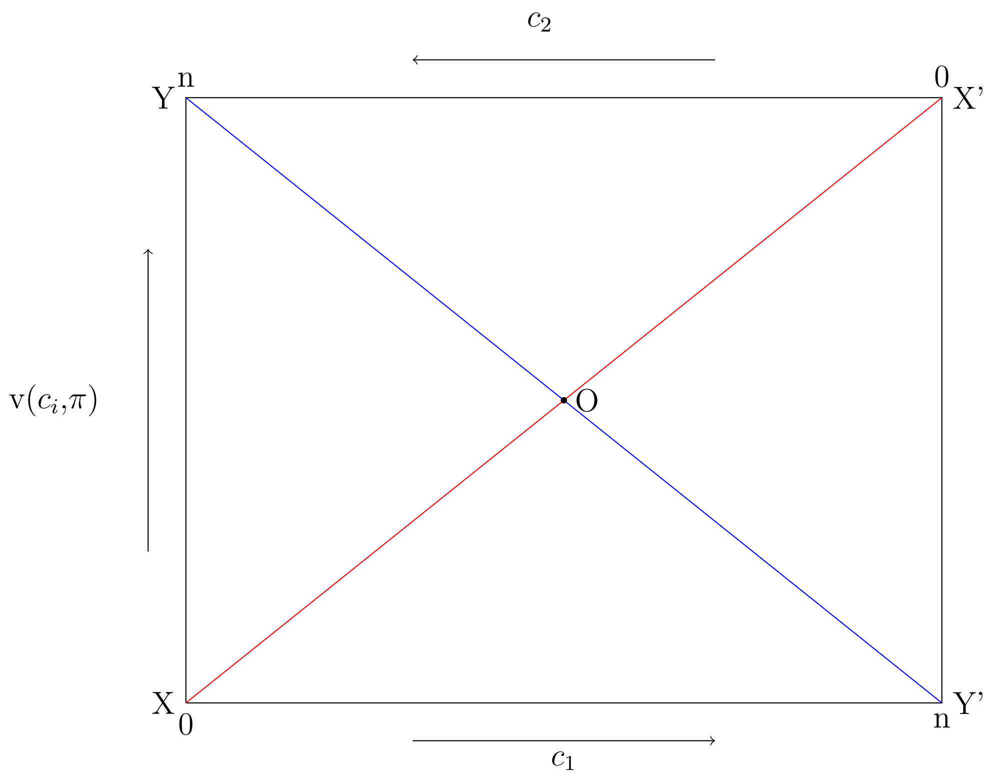

Figure 2: Linear Coalition worth beyond $c_1 = \frac{n}{2}$

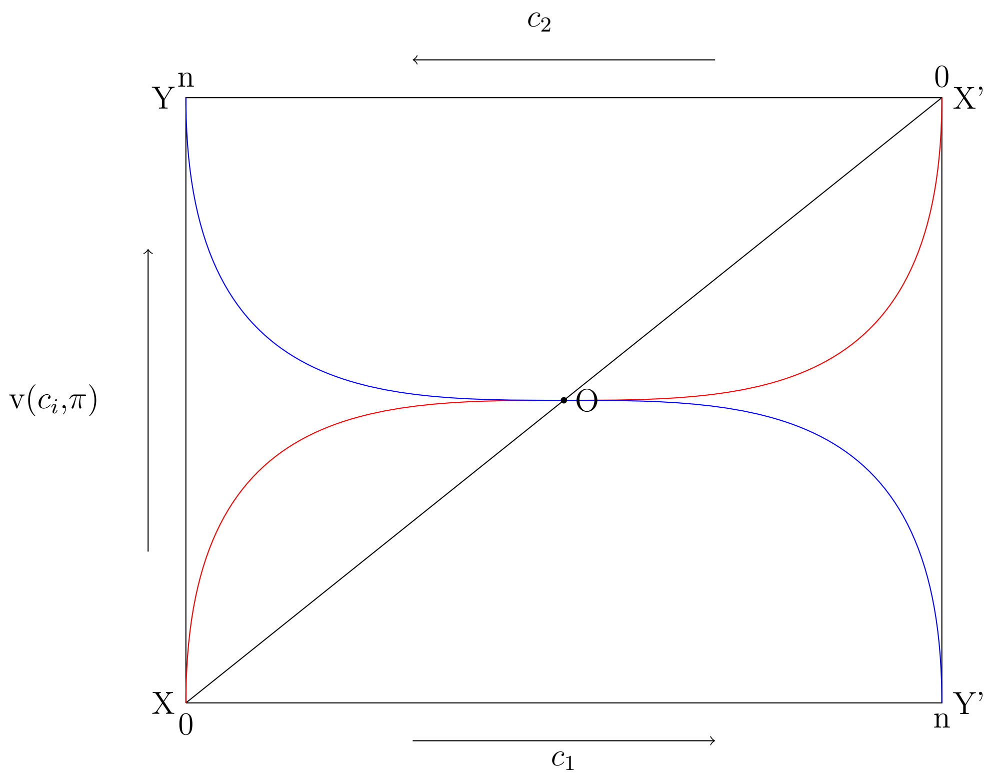

Figure 3: Convex Coalition worth beyond $c_1 = \frac{n}{2}$

for $\alpha > 1$. Hence, we restrict this example only to the case $\alpha > 1$. The expected reward or coalition's worth is $v(c_k, \pi) = p_k(\pi)K$ and individual payoff is $v_i(c_k, \pi) = \frac{p_k(\pi)K}{c_k}$. In the result below, we predict the coalition structure and the size of the coalitions formed.

Proposition 1: Normalising endowed ability to zero, the coalition structure $\pi = \{c*, n - c*\}$ is the SPE where $c*$ is the solution to $(\frac{n}{c} - 1)^{\alpha-1}\big(\frac{n}{c}(\alpha - 1) + 1\big) = 1$. The maximum value of $c*$ tends to $0.8n$ when $\alpha \to 1$ and $c* \to 0.5n$ as $\alpha \to \infty$

Proof. Assume that coalition c_1 forms where every individual maximises his individual payoff $v_i(c_1, \pi) = \frac{p_1(\pi)K}{c_1}$. $v_i(c_1, \pi)$ will be minimum when $n - c_1$ agents all form a coalition c_2. Thus, the minimum winning probability is

$$p_k(\pi) = \frac{c_1 a + c_1^{\alpha}\delta}{(c_1 a + c_1^{\alpha}\delta) + \Big((n - c_1)a + (n - c_1)^{\alpha}\delta\Big)}$$

And the individual payoff at best response is

$$v_i(c_1, \pi) = \frac{(c_1 a + c_1^{\alpha}\delta)K}{na + c_1^{\alpha}\delta + (n - c_1)^{\alpha}\delta}$$

Maximizing $v_i(c_1, \pi)$ with respect to c_1 and normalising $a = 0$ we have $c_1 = c*$ where $c*$ is the solution to

$$(\frac{n}{c} - 1)^{\alpha-1}\big(\frac{n}{c}(\alpha - 1) + 1\big) = 1$$

Next, to show that the remaining $n - c_1$ agents do not benefit from forming two or more coalitions it must be the case that

$$v(n - c*, \pi) > v(n - c* - d, \pi') \text{ for all interger values of } d \in (0, n - c*)$$

Solving the equation

$$\frac{(n - c*)^{\alpha}}{(c*)^{\alpha} - (n - c*)^{\alpha}} - \frac{(n - c* - d)^{\alpha}}{(c*)^{\alpha} + (n - c* - d)^{\alpha} + d^{\alpha}}$$

we have

$$(c*)^\alpha + d^\alpha - (c*)^\alpha\Big(1 - \frac{d}{n - c*}\Big)^\alpha$$

It can be easily shown that

$$(c*)^\alpha + d^\alpha > (c*)^\alpha\Big(1 - \frac{d}{n - c*}\Big)^\alpha \ \forall\ \alpha > 1$$

Thus, the remaining $n - c*$ agents form a coalition.

To prove the convergence of the coalition $c*$, rewriting equation (3)

$$\frac{n}{c*} - 1 = \Big((\alpha - 1)\frac{n}{c*} + 1\Big)^{\frac{1}{1-\alpha}}$$

Thus, as $\alpha \to \infty$, $c* \to \frac{n}{2}$ □

The result in *Proposition 1* is a derivative of Corollary 1 and we see that the largest coalition is always $\frac{n}{2} < c* < n$ which is in accordance with *Theorem 1*. Note that when the benefit of forming a coalition, $c^\alpha\delta$, increases the size of the majority coalition, $c*$, decreases. However, this phenomenon depends on the method by which agents estimate the winning probability. As we see in the next section, estimating winning probability from relative rank produces a completely different effect on the increase of coalition benefit.

6 Estimating Winning Probabilities from Relative Ranks

Using the same setting as in the previous section, now assume agents to estimate their winning probabilities based on their relative rank: the sum of the difference between an agent's ability and all other agents. The relative rank of agent i given by

$$R_i = \sum_{j=1}^{n}(a_i - a_j)$$

. As before, $a_i = a + c^\alpha\delta$ where c is the size of the coalition agent i belongs to. Unlike the previous example, agents here form individual probabilities based on their relative rank. Let

$p(.)$ be a probability distribution over all agents such that the function $p(.)$ strictly increases in rank and $\sum_{i=1}^{n} p(R_i) = 1$. The expected individual payoff is $p(R_i)K$. The coalition worth is the sum of the expected rewards of each member: $v(c, pi) = \sum_{i \in C} p_{R_i} K$. As Thus, this model to conforms with the general theory presented earlier. The two properties in *Assumption 1* is satisfied for all positive values of α for this example. Agents now form coalition c to maximise their individual payoff.

Proposition 2: The coalition structure $\pi = \{c*, n - c*\}$ is the SPE where $c*$ is the solution to $(\frac{n}{c} - 1)^\alpha = 1 - \frac{\alpha n}{(\alpha+1)c}$. The miminum value of $c*$ is $0.75n$ when $\alpha = 1$. Also, $c* \to n$ as $\alpha \to \infty$ and $c* \to 0.8n$ as $\alpha \to 0$

Proof. We begin by maximising $p(R_i)K$ with respect to c. As $p(.)$ strictly increases with relative rank, let agents form a coalition c to maximise their relative rank, $\frac{dR_i}{dc} = 0$ to get the equation

$$(\frac{n}{c} - 1)^\alpha = 1 - \frac{\alpha n}{(\alpha + 1)c} \tag{1}$$

Next, to show that the remaining $n - c_1$ agents do not benefit from forming two or more coalitions it must be the case that

$$v(n - c*, \pi) > v(n - c * - d, \pi') \text{ for all integer values of } d \in (0, n - c*)$$

Solving the equation

$$c\Big((n - c*)^\alpha - (c*)^\alpha\Big) - c * \Big((n - c * - d)^\alpha - (c*)^\alpha\Big) + d\Big((n - c * - d)^\alpha - d^\alpha\Big)$$

we have

$$c * - d \frac{\Big[(1 - \frac{d}{n-c})^\alpha - (\frac{d}{n-c})^\alpha\Big]}{1 - \Big[1 - \frac{d}{n-c})^\alpha\Big]}$$

It can easily be shown that the
numerator $= \Big[(1 - \frac{d}{n-c})^\alpha - (\frac{d}{n-c})^\alpha\Big] <$ denominator $= 1 - \Big[1 - \frac{d}{n-c})^\alpha\Big]$

Also, $d < c*$ as $c* . \frac{n}{2}$. Therefore,

$$c * -d \frac{\left[(1-\frac{d}{n-c})^{\alpha} - (\frac{d}{n-c})^{\alpha}\right]}{1-\left[1-\frac{d}{n-c})^{\alpha}\right]} > 0$$

and

$$v(n-c*, \pi) > v(n-c*-d, \pi') \text{ for all integer values of } d \in (0, n-c*)$$

To prove the convergence of the coalition $c*$, rewriting equation (2).

$$\frac{n}{c} - 1 = 1 - \left(\frac{n \setminus c}{1+\frac{1}{\alpha}}\right)^{\frac{1}{\alpha}}$$

Thus as $\alpha \to \infty$, $c \to n$. □

Note that in this proof, the endowed skill is homogeneous, but not normalised to zero. Further, it can be shown that this result holds true for the case of heterogeneous agents too. *Proposition 2* contrasts with *Proposition 1*: as the benefit of forming a coalition increases, the coalition with the majority of the members tend to forming a grand coalition. However, in the previous example, the majority of the members tend to form a coalition of size $\frac{n}{2}$ with an increase the benefit. This shows the method used by the agents to estimate the winning probability greatly affects the way they form coalitions.

7 Future Research

The directions I plan to proceed is the following. There is a cost associated with forming a coalition such that benefit of a coalition increases with size till an optimal point and then declines. This is modelled by the benefit of a coalition being $h(c)\delta$, where $h(c)$ is a quadratic concave function. The question here is that when agents maximise individual payoff do they form coalitions above this optimal point. This set up can also be extended to a social network where agents estimate winning probabilities based on their neighbours' performance.

However, the estimate here is an independent probability, it is not a distribution, as agents are aware that their estimate based on localised information. Agents form bilateral cooperative links to maximise their winning probability. The problem here is what network structures are formed at equilibrium.

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