Stability in Matching with Groups having Non-Responsive Preferences

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Abstract

The paper studies matching markets where institutions are matched with possibly more than one individuals. The matching market contains some groups who view the pair of jobs as complements. The paper specifies that the groups have a "weak" preference to be matched together. The paper first assumes that that the institutions have the same preference over all the individuals. It then finds out under which weak preferences of groups do stable matching exist and then generalise this idea. It further weakens the assumption of identical preference of institutions and proves existence of stable matching for unrestricted group preferences. Finally, the paper seeks to establish a result on stability by relaxing the condition on identical institution preferences.

Keywords. two-sided matching, couples, stability, weak responsiveness
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1 Introduction

In many different contexts, there is a centralized matching procedure by which individuals on one side of the market are matched with institutions on the other side of the market. These include the market for lawyers in Canada, children in schools in the USA, doctors and senior-level health-care professionals in several countries, etc. There is a huge literature which has been developed on various market designs to find out an "optimal" matching procedure to produce stable matching. A matching is stable if there it ensures that there are no deviations of institution-individual pairs. Thus a stable matching also gives an incentive to agents to correctly reveal their preferences[1] It was shown by *Alvin E. Roth*[2] that it was possible to have mechanisms which induced only one side of the market to correctly reveal their preferences. However, the results on stability have been more encouraging as the received doctrine is that stable matchings do exist under appropriate domain restrictions. But to achieve that, institutions must view individuals as substitutes and individuals also must only care about the institution to which they are matched. It was first pointed out by *Roth*[3] that the presence of couples in the labour market may lead to an impossibility result where no stable matching may exist. This can happen because couples or groups may view pairs of jobs as complements and thus the assumptions which consider the choices individuals to be independent of each other might not apply. *Bettina Klaus and Flip Klijn*[4] identify the maximal domain of preferences of couples under which stable matchings exist. The maximal domain satisfies Responsiveness - so a couple is better off when any member of the couple is matched with a more preferred institution. However, *Fuhito Kojima, Parag A. Pathak, and Alvin E. Roth*[5] point out that Responsiveness is not satisfied in their data

[1] [10] and [11] give illustrating surveys for this literature.
[2] Refer to [9].
[3] Refer to [8].
[4] Refer to [5].
[5] Refer to [6].

sets because couples show strong preference to be matched together in the institutions situated in the same geographical area.

In this paper, we consider a set of doctors who come together to form a group. In particular, if all the group sizes are two, then we get a matching matket with couples. We focus on the issue of existence of stable matchings with groups. Furthermore, we first look at the scenario when all the institutions have identical preferences. The starting point of our analysis is how to model the identical preferences of the institutions and how to model the preference ordering of any group over tuples of positions, given the individual preferences of each member of the group. We look into a setting where the set of institutions is a finite set and thus there is no information about the "distance" between any pair of institutions. But when a group is matched with the same institution, then the distance trivially becomes zero. Thus as assumed by *Bhaskar Dutta and Jordi Masso*[6], we have an option to assume that group prefers to be matched at the same institution rather than being matched with different institutions.

We analyse the situation where groups' preferences violate responsiveness as long as they can be together in any possible institution. We show that under identical preferences of institutions we will have stable matchings if, and only if, the groups' preferences satisfy some joint condition over their allocations.

We then try to restrict the condition of identical institution preferences further and try to check if stable matchings exist with unrestricted joint groups' preferences. We assume that the institutions' preferences are lexicographic and members of the group are adjacent in the identical preference, i.e. there is no other individual in between any two members of a group in the common preference of the institution. We finally show that if groups' preferences are unrestricted, then stable matchings exist if, and only if, the identical preference of institutions follow the above condition.

Finally, we try to find out the consequences of relaxing the condition of

[6] Refer to [3].

identical preferences of the institutions. We restrict the size of each group to two, that is, we consider a matching market with couples. We try to establish a result for the existence of stable matching when institutions' preferences are not identical. We find out, the results proved earlier are not sufficient to prove the existence of stable matching in this scenario. Thus we need to impose further restrictions on couples' preferences in order to get a stable matching.

2 The Framework

We consider many-to-one matching between doctors and hospitals. We denote by H the set of hospitals. We use the notation $\bar{H}$ to denote $H \cup \{\emptyset\}$. If some doctor is matched with with $\{\emptyset\}$, then that doctor is unmatched. Each hospital $h \in H$ has a finite capacity $\kappa_h \geq 1$. We denote by D the set of doctors. We consider a fixed partition of the set of doctors D into subsets, $G^1, \ldots, G^m, S$. Here, for all $j = 1, \ldots, m$, $G^j = \{g_1^j, \ldots, g_{n_j}^j\}$ with $n_j \geq 2$ denotes the set of doctors that are in group j, and S denotes the set of single doctors who do not belong to any of the groups. By $\tilde{G} = \{G^1, \ldots, G^m\}$ we denote the collection of all groups. Throughout this paper, we assume $|H| \geq 2$, $|\tilde{G}| \geq 1$. Furthermore, if $|\tilde{G}| = 1$ then we assume $|S| \geq 2$. We also assume that the total number of vacancies in all hospitals in H is equal to the total number of doctors available, i.e., $\sum_{h \in H} \kappa_h = |D|$.

Consider a group of doctors $G = \{g_1, \ldots, g_n\}$ and let $N = \{1, \ldots, n\}$. Then, an allocation of the group G is an element $\underset{\sim}{h}$ of $\bar{H}^N$ where the hospital $\underset{\sim}{h}_1$ is matched with doctor g_1, hospital $\underset{\sim}{h}_2$ is matched with doctor g_2 and so on. Here, for $i \in N$, by $\underset{\sim}{h}_i$, we mean the i^{th} component of $\underset{\sim}{h}$. Also, for $\underset{\sim}{h}_{-i} \in \bar{H}^{N \setminus i}$ and $h \in \underset{\sim}{H}$, by $(\underset{\sim}{h}_{-i}, (h)_i)$ we denote an allocation of the group G where doctor g_i is matched with h and g_j is matched with $(\underset{\sim}{h}_{-i})_j$ for all $j \neq i$. Furthermore, for hospitals $h, h' \in \bar{H}$, we denote by $((h)_{-i}, (h')_i)$ an allocation of the group G where all doctors in G except g_i are matched with hospital h and g_i is matched with h'. Similarly, for hospitals $h, h', h'' \in \bar{H}$ and

$i, j \in \{1, \ldots, n\}$ with $i \neq j$, we denote by $((h)_{-i-j}, (h')_i, (h'')_j)$ an allocation of the group G where all doctors in G except g_i and g_j are matched with hospital h, g_i is matched with h', and g_j is matched with h'', and so on. Finally, for $h \in \bar{H}$, by $(h, \ldots, h)$ we denote an allocation of the group G where all the doctors in G are matched with h.

For notational convenience, we do not use braces for singleton sets.

2.1 Matching

Definition 1 *A matching is a mapping $\mu : H \cup D \to \bar{H} \cup 2^D$ such that*

(i) for all $h \in H$, $\mu(h) \subseteq D$ with $|\mu(h)| \leq \kappa_h$,

(ii) for all $d \in D$, $\mu(d) \in \bar{H}$, and

(iii) for all $d \in D$ and $h \in H$, $\mu(d) = h$ if and only if $d \in \mu(h)$.

The first condition of the definition says that every hospital $h \in H$ can be matched with at most κ_h many doctors. The second and the third conditions of the definition say that every doctor can be either matched with exactly one hospital in H or be unmatched.

2.2 Preferences

In this section, we introduce the notion of preferences of doctors and hospitals, and present the restrictions on them.

For a set X, we denote by $\mathbb{L}(X)$ the set of linear orders, i.e., complete, transitive, and asymmetric binary relations over X. An element of $\mathbb{L}(X)$ is called a preference over X.

2.2.1 Preferences of Hospitals

A preference of a hospital $h \in H$, denoted by P_h, is a linear order over the feasible sets of doctors $\{D' \subseteq D : |D'| \leq \kappa_h\}$, i.e., $P_h \in \mathbb{L}(\{A \subseteq D : |A| \leq$

$\kappa_h\}$). We assume hospitals' preferences to be responsive which we define below.

Definition **2** *The preference ordering P_h of a hospital $h \in H$ with capacity κ_h satisfies* Responsiveness *if for any $D' \subseteq D$ with $|D'| \leq \kappa_h$ the following hold:*

(i) for any $d' \in D'$ and any $d \in D \setminus D'$, $((D' \cup d) \setminus d')P_h D'$ if and only if $dP_h d'$, and

(ii) for any $D'' \subsetneq D'$, $D'P_h D''$.

Here the first condition says that there are no complementaries across doctors, and the second condition says that all the doctors are acceptable for hospital h. Throughout this paper, we assume that hospitals' preferences satisfy responsiveness.

Definition **3** *The preferences of the hospitals satisfy* Identical Hospital Preferences*(IHP) if for all pairs of hospitals $h, h' \in H$ and for all $d, d' \in D$, $dP_h d'$ if and only if $dP_{h'} d'$.*

Note that, *IHP* implies that all the hospitals have common preferences over individual doctors, however it does not impose any restriction on the preferences of hospitals over larger subsets of doctors.

Unless mentioned otherwise, we assume that hospitals' preferences satisfy IHP. Under IHP, the common restriction of $(P_h)_{h \in H}$ over individual doctors is defined as $P^0 \in \mathbb{L}(D)$ such that for all $d, d' \in D$, $dP^0 d'$ if $dP_h d'$ for all $h \in H$. Throughout this paper P^0 denotes an IHP. Whenever we consider an IHP P^0 we assume for ease of presentation that the indexation of the doctors in groups is such that $g_i^j P^0 g_{i+1}^j$ for all $i < n_j$ and $j \leq m$, and $g_{n_j}^j P^0 g_{n_k}^k$ for all $j < k \in \{1, \ldots, m\}$. This is without of loss of generality as we consider only one IHP at every given context.

2.2.2 Preferences of Doctors

A preference of a doctor $d \in D$, denoted by P_d, is a linear order over $\bar{H}$, i.e., $P_d \in \mathbb{L}(\bar{H})$. We assume $hP_d\emptyset$ for all $h \in H$ and all $d \in D$. For a doctor $d \in D$, by h_d, we denote the top ranked hospital of the doctor d according to preference P_d. Having defined the preferences of the doctors (as singles), now we proceed to define the preferences of the groups of doctors.

Preferences of Groups of Doctors

In this paper we intend to deviate from responsiveness in a 'minimal' way and check what happens to stability. We assume that a groups' preferences are almost responsive except in the situations where all the members of the group get to stay together in the same hospital. The usual definition of responsive group preference means that for two group allocations, that differ in the allocation for only one group member, then the group prefers the allocation where that member is assigned to his more preferred hospital. However, here we allow for the group to violate responsiveness only if all the members of the group can be matched with the same hospital. We call this as preference for togetherness.

We think some preference for togetherness should be adopted in a matching model with groups, otherwise the fact that a few doctors act as group will not have any impact on the model. Moreover, this is a very natural situation that can occur in reality. However, we also do not wish to deviate far from the assumption of responsiveness.

In the following, we define responsiveness for a preference of a group of doctors. The notion of responsiveness is in principle same as that for a preference of a hospital. However, for the sake of clarity, we present the formal definition below.

Definition 4 *Let $G = \{g_1, \ldots, g_n\}$ be any group of doctors and $N = \{1, \ldots, n\}$. Let for all $i \in N$, P_{g_i} be a preference of g_i. Then, a preference $P_G \in \mathbb{L}(\bar{H}^N)$ of the group G is called responsive if, for all $i \in N$, for all $\underset{\sim}{h}_{-i} \in \bar{H}^{N \setminus i}$, and all $h, h' \in \bar{H}$, we have $(\underset{\sim}{h}_{-i}, (h)_i)P_G(\underset{\sim}{h}_{-i}, (h')_i)$ if and only if*

$hP_{g_i}h'$. For a group G, by $\mathcal{D}^R_G$ we denote the set of responsive preferences of G.

DEFINITION **5** *Let $G = \{g_1, \ldots, g_n\}$ be any group of doctors and $N = \{1, \ldots, n\}$. Then, a preference ordering $\bar{P}_G \in \mathbb{L}(\bar{H}^N)$ of the group G satisfies Responsiveness Violated for Togetherness (RVT) if and only if there is a responsive preference $P_G \in \mathcal{D}^R_G$ of the group G such that*

(i) for all $h \in H$ and all $\underset{\sim}{h} \in \bar{H}^N$, $(h, \ldots, h)P_G\underset{\sim}{h}$ implies $(h, \ldots, h)\bar{P}_G\underset{\sim}{h}$, and

(ii) for all $\underset{\sim}{h}, \underset{\sim}{h}' \in \bar{H}^N$ such that $\underset{\sim}{h}_i \neq \underset{\sim}{h}_j$ and $\underset{\sim}{h}'_k \neq \underset{\sim}{h}'_l$ for some $i, j, k, l \in N$, $\underset{\sim}{h}P_G\underset{\sim}{h}'$ if and only if $\underset{\sim}{h}\bar{P}_G\underset{\sim}{h}'$.

For a group G, by $\mathcal{D}^{RVT}_G$ we denote the set of RVT preferences of G.

Note that, RVT implies that groups' preferences can violate responsiveness only in order for all of them to be together in some hospital. Further note that, by taking $\underset{\sim}{h}$ such that $\underset{\sim}{h}_1 = \ldots = \underset{\sim}{h}_n$ in Condition (i) of Definition 5, it follows that the relative ordering amongst the allocations where all the doctors of a group G are in the same hospital does not change from P_G to $\bar{P}_G$.

REMARK **1** *In the rest of the paper, we assume that there exists a hospital $h \in H$ and a group $G \in \tilde{G}$ such that $\kappa_h \geq |G|$. We assume this because if $\kappa_h < |G|$ for all $h \in H$ and all $G \in \tilde{G}$, then the violation of responsiveness would be on an infeasible set which will not play any role in the matching mechanism.*

2.2.3 Preference Profiles

In this section, we define the notion of a preference profile.

DEFINITION **6** *A preference profile $\underset{\sim}{P}$ for hospitals in H and doctors in D with groups $\tilde{G}$ is defined as a collection of preferences $(\{\underset{\sim}{P}_d\}_{d \in D}, \{\underset{\sim}{P}_G\}_{G \in \tilde{G}}, \{\underset{\sim}{P}_h\}_{h \in H})$*

where $\underset{\sim}{P}_d$ *is a preference of doctor* d*,* $\underset{\sim}{P}_G$ *is a preference of group* G*, and* $\underset{\sim}{P}_h$ *is a preference of hospital* h *for all* $d \in D$*, all* $G \in \tilde{G}$*, and all* $h \in H$ *respectively.*

By a matching problem, we mean the set of hospitals with corresponding capacities, the set of doctors with its partition into groups, and a preference profile.

2.3 Stability

Our model is formally equivalent to a many-to-many matching market as any group with n doctors looks for n positions and hospitals may have more than one positions. Thus, one can have different notions of stability based on different types of permissible blocking coalitions. [7]

Blocking pairs can be a hospital and a single doctor, or a set of hospitals and a group of doctors.

Let $s \in S$ be a single doctor and $h \in H$ be a hospital. Then, for two matchings μ, μ', we write $\mu \to_{\{h,s\}} \mu'$ if,

(i) $\mu'(h) = (\mu(h) \setminus D') \cup s$ for some (possibly empty) $D' \subseteq \mu(h)$, and

(ii) $\mu'(h') = \mu(h') \setminus s$ for all $h' \neq h$.

Let $G = \{g_1, \ldots, g_n\} \in \tilde{G}$ be a group of doctors and $N = \{1, \ldots, n\}$. Let $\underset{\sim}{h} = (h_1, \ldots, h_n) \in H^N$. For $h \in \{h_1, \ldots, h_n\}$, let $G^h = \{g_i : \tilde{h}_i = h\}$ Then, for two matchings μ, μ', we write $\mu \to_{\{\underset{\sim}{h},G\}} \mu'$ if,

(i) $\mu'(\underset{\sim}{h}) = (\mu(\underset{\sim}{h}) \setminus D^h) \cup G^h$ for all $i \in N$, for all $h \in \{h_1, \ldots, h_n\}$ and some (possibly empty) $D^h \subseteq \mu(\underset{\sim}{h})$, and

(ii) $\mu'(h) = \mu(h) \setminus \{g_1, \ldots, g_n\}$ for all $h \in H \setminus \{h_1, \ldots, h_n\}$.

[7] Refer to [8, 9, 7, 4] for some alternative notions of stability in many-to-many matchings.

The statement $\mu \rightarrow_{\{h,s\}} \mu'$ captures the idea that the hospital h and the single doctor s can change their allocation under μ to that under μ' because h can release some doctors and hire s. Moreover, hospitals other than h continue to retain their allocations unless they were matched with s. Similarly, the statement $\mu \rightarrow_{\{\underset{\sim}{h},G\}} \mu'$ captures the idea that the set of hospitals $\underset{\sim}{h} = (h_1, \ldots, h_n)$ and the group of doctors G can change their allocation under μ to that under μ' because each of the hospital can release some doctors and hire a doctor from the group. Moreover, hospitals other than the hospitals in $\{h_1, \ldots, h_n\}$ continue to retain their allocations unless they were matched with any member of the group G.

Definition 7 *Let $h \in H$ be a hospital and $s \in S$ be a single doctor. Then, (h, s) blocks μ through μ' if $\mu \rightarrow_{\{h,s\}} \mu'$, $\mu'(h)P_h\mu(h)$, and $hP_s\mu(s)$.*

Definition 8 *Let $G = \{g_1, \ldots, g_n\}$ be a group of doctors and $N = \{1, \ldots, n\}$. Let $\underset{\sim}{h} = (h_1, \ldots, h_n) \in H^N$. Then, $(\underset{\sim}{h}, G)$ blocks μ through μ' if $\mu \rightarrow_{\{\underset{\sim}{h},G\}} \mu'$, $\underset{\sim}{h}P_G(\mu(g_1), \ldots, \mu(g_n))$, and $\mu'(h)P_h\mu(h)$ for all $h \in \{h_1, \ldots, h_n\}$ with $|\mu(h) \cap G| \leq 1$.*

Thus for a hospital h and doctor s, (h, s) blocks μ, if there exists a matching μ' with $\mu \rightarrow_{\{h,s\}} \mu'$ such that h and s are better off in μ' as compared to μ. Similarly, for hospitals $\underset{\sim}{h} = (h_1, \ldots, h_n)$ and group G, $(\underset{\sim}{h}, G)$ blocks μ, ig there exists a matching μ' with $\mu \rightarrow_{\{\underset{\sim}{h},G\}} \mu'$ such that G is better off in μ' than in μ. Further, every hospital in $\{h_1, \ldots, h_n\}$ which has at most one doctor from group G in μ is better off in μ' than in μ.

Definition 9 *A matching μ is stable if it is not blocked by any pair (h, s) where $h \in H$ and $s \in S$, or by any pair $(\underset{\sim}{h}, G)$ where $\underset{\sim}{h} \in H^{|G|}$ and $G \in \tilde{G}$.*

Note that, members of a group move according to their group preference, in particular, a member of a group does not block according his/her individual preference.

Remark **2** *By our assumptions on the preferences of hospitals and doctors, hospitals find all doctors acceptable and doctors find all hospitals acceptable. Therefore,* all *matchings are individually rational.*

2.4 Algorithm

In this section we present a well-known doctor proposing deferred acceptance algorithm (DPDA) that we will use throughout the paper to match hospitals with doctors. Our existence proof uses a modification of the Gale-Shapley deferred acceptance algorithm with the doctors making the proposals (DPDA). We give a very short description of DPDA.

DPDA: In stage 1 of the algorithm, all doctors simultaneously propose to their most preferred hospitals. Each hospital h provisionally accepts up to κ_h most preferred doctors. If a hospital has received more than κ_h proposals, then it rejects all the doctors after its κ_h most preferred doctors. In any step k, the unmatched doctors propose to their most preferred hospital from the remaining set of hospitals who have not rejected them in any of the earlier steps. In any stage of DPDA, since each hospital accepts κ_h most preferred doctors, it may reject some doctors that it had provisionally accepted earlier. Hospitals whose provisional list of accepted doctors is less than their maximum capacity can still add to their accepted list if they have received fresh proposals. Thus the algorithm finally terminates when each doctor is matched or has been rejected by all hospitals.

Now we present another well-known algorithm called Serial Dictatorship Algorithm (SDA). In the SDA, the highest-ranked doctor according to the identical hospital preference chooses his/her most-preferred hospital, and in general the k-highest ranked doctor chooses his/her most preferred hospital among the hospitals with available vacancy after all the higher ranked doctors have made their choices.

Remark **3** *DPDA and SDA produce the same matching under IHP.*

2.5 Conditions for Stability under RVT

In this section, we provide conditions on groups' preferences satisfying RVT that guarantee the existence of stable matchings.

Let $P^0_{\tilde{G}} = (\{P^0_g\}_{g\in D\setminus S}, \{P^0_G\}_{G\in\tilde{G}})$ be a given collection of preferences of the doctors that are in some group, and of the groups in $\tilde{G}$. Let P^0 be an IHP. Recall that, we assume the indexation of the doctors in groups to be such that $g^j_i P^0 g^j_{i+1}$ for all $i < n_j$ and all $j \leq m$. Then, by $\underset{\sim}{\mathcal{D}}(P^0_{\tilde{G}}, P^0)$ we denote the set of preference profiles where doctors $g \in D \setminus S$ and groups $G \in \tilde{G}$ have preferences as in $P^0_{\tilde{G}}$ and the IHP is P^0, i.e., $\underset{\sim}{\mathcal{D}}(P^0_{\tilde{G}}, P^0) = \{\underset{\sim}{P} : \underset{\sim}{P}_g = P^0_g$ for all $g \in D \setminus S$, $\underset{\sim}{P}_G = P^0_G$ for all $G \in \tilde{G},$ and P^0 is the IHP of $\underset{\sim}{P}\}$.

Condition **1** *Suppose $P^0_{\tilde{G}}$ is such that $P^0_G \in \mathcal{D}^{RVT}_G$ for all $G \in \tilde{G}$. Then, the collection of preferences $(P^0_{\tilde{G}}, P^0)$ satisfies Condition 1 if: for each group $G = \{g_1, \ldots, g_n\} \in \tilde{G}$ and each $g_i \in \{g_1, \ldots, g_{n-1}\}$, $(h, \ldots, h)P^0_G((h)_{-i}, (h')_i)$ and $h' P^0_{g_i} h$ imply that there is be some $j < i$ such that $((h)_{-j}, (h')_j)P^0_G(h, \ldots, h)$.*

Condition 1 says the following. Consider a group G and let g_i be a doctor in G who is not the least preferred doctor according to the IHP. Suppose g_i prefers some hospital h to another hospital h'. Suppose further that a RVT preference of G prefers an allocation where all the doctors of G are in h to another allocation where all the doctors of G except g_i are in h and g_i is in h'. Then, there must exist a doctor g_j who is preferred to g_i according to the IHP such that the RVT preference of G prefers the allocation where all the doctors of G except g_j are in h and g_j is in h' to the allocation where all the doctors of G are in h. Note that, Condition 1 does not impose any restriction on the preference of the least preferred doctor in G according to the IHP. Further note that, Condition 1 implies that responsiveness is not violated for togetherness with a compromise from the most preferred doctor of G according to IHP.

Theorem **1** *Suppose $P^0_{\tilde{G}}$ is such that $P^0_G \in \mathcal{D}^{RVT}_G$ for all $G \in \tilde{G}$. Then, a stable matching exists at every preference profile in $\underset{\sim}{\mathcal{D}}(P^0_{\tilde{G}}, P^0)$ if and only if $(P^0_{\tilde{G}}, P^0)$ satisfies Condition 1.*

Proof: [Necessity] Suppose $P^0_{\tilde{G}}$ is such that $P^0_G \in \mathcal{D}^{RVT}_G$ for all $G \in \tilde{G}$. Suppose further that $(P^0_{\tilde{G}}, P^0)$ does not satisfy Condition 1. We show that there is a preference profile in $\underset{\sim}{D}(P^0_{\tilde{G}}, P^0)$ with no stable matching. Since $(P^0_{\tilde{G}}, P^0)$ does not satisfy Condition 1, there exists a group $G = \{g_1, \dots g_n\}$, two hospitals $h_1, h_2 \in H$, and $i \in \{1, \dots, n-1\}$ such that $(h_1, \dots, h_1)P^0_G((h_1)_{-i}, (h_2)_i)$, $h_2 P^0_{g_i} h_1$, and $(h_1, \dots, h_1)P^0_G((h_1)_{-j}, (h_2)_j)$for all $j < i$. Without loss of generality, we assume that g_i is the highest ranked doctor of the group according to IHP for which Condition 1 is violated. That is, for all $k < i$, $(h_1, \dots, h_1)P^0_G((h_1)_{-k}, (h_2)_k)$ and $h_2 P^0_{g_k} h_1$ imply $((h_1)_{-j}, (h_2)_j)P^0_G(h_1, \dots, h_1)$ for some $j < k$. Note that, this and the fact that $(h_1, \dots, h_1)P^0_G((h_1)_{-j}, (h_2)_j)$ for all $j < i$ imply that $h_1 P^0_{g_j} h_2$ for all $j < i$.

Since $h_2 P^0_{g_i} h_1$, it follows from the definition of RVT that for all $i' \neq i$, we have $((h_1)_{-i-i'}, (h_2)_i, (h_2)_{i'})P^0_G((h_1)_{-i'}, (h_2)_{i'})$. Also, as $h_1 P^0_{g_j} h_2$ for all $j < i$, RVT implies $((h_1)_{-i}, (h_2)_i)P^0_G((h_1)_{-j-i}, (h_2)_j, (h_2)_i)$. Take doctors d_1 and d_2 such that $d_1, d_2 \notin G$. Consider a preference profile $\underset{\sim}{P}$ in $\underset{\sim}{\mathcal{D}}(P^0_{\tilde{G}}, P^0)$ such that the IHP P^0 satisfies $g_i P^0 d_1 P^0 d_2 P^0 g_{i+1}$, and $h_{d_1} = h_1$, $h_{d_2} = h_2$. Suppose $|\{d : dP^0 g_1 \text{ and } h_d = h_2\}| = \kappa_{h_2} - 2$, $|\{d : dP^0 g_1 \text{ and } h_d = h_1\}| = \kappa_{h_1} - n$, and $|\{d : dP^0 g_1 \text{ and } h_d = h\}| = \kappa_h$ for all $h \neq h_1, h_2$. Suppose further that the preferences of all groups other than G satisfy responsiveness. We show that there is no stable matching at $\underset{\sim}{P}$. Suppose μ is a stable matching at $\underset{\sim}{P}$. Since μ is stable, it must be that $\mu(d) = h_d$ for all $dP^0 g_1$. Moreover, since $|\{d : dP^0 g_1 \text{ and } h_d = h_2\}| = \kappa_{h_2} - 2$ and $|\{d : dP^0 g_1 \text{ and } h_d = h_1\}| = \kappa_{h_1} - n$, there are exactly 2 positions left in h_2 and exactly n positions left in h_1 after matching all the doctors d such that $dP^0 g_1$. We distinguish the following cases.

Case 1: Suppose $i = 1$.

- If $\mu(G) = ((h_1)_{-1-j}, (h_2)_1, (h_2)_j)$ for any $j \in \{2, \dots, n\}$, then (h_2, d_2) blocks μ as $h_{d_2} = h_2$ and $d_2 P^0 g_j$ for all $j \in \{2, \dots, n\}$.
- If $\mu(G) = ((h_1)_{-j}, (h_2)_j)$, then $(((h_1)_{-1-j}, (h_2)_1, (h_2)_j), G)$ blocks μ as $g_1 P^0 d_1 P^0 d_2$ and by RVT, $((h_1)_{-1-j}, (h_2)_1, (h_2)_j)P^0_G((h_1)_{-j}, (h_2)_j)$.

- If $\mu(G) = (h_1, \ldots, h_1)$, then (h_1, d_1) blocks μ as $h_{d_1} = h_1$ and $d_1 P^0 g_j$ for all $j \in \{2, \ldots, n\}$.

- If $\mu(G) = ((h_1)_{-1}, (h_2)_1)$, then $((h_1, \ldots, h_1), G)$ blocks μ as $g_1 P^0 d_1 P^0 d_2$ and by RVT, $(h_1, \ldots, h_1) P^0_G ((h_1)_{-1}, (h_2)_1)$.

Case 2: Suppose $i \in \{2, \ldots, n-1\}$.

- Suppose $\mu(g_i) = h_2$. If $\mu(g_k) = h_2$ for any $k > i$, then μ is blocked by (h_2, d_2) as $d_2 P^0 g_k$ and $h_{d_2} = h_2$. Let $\mu(g_j) = h_2$ for some $j < i$. Note that, by RVT $((h_1)_{-i}, (h_2)_i) P^0_G ((h_1)_{-j-i}, (h_2)_j, (h_2)_i)$. This, together with the fact that $g_j P^0 g_i P^0 d_1 P^0 d_2$, means μ is blocked by $(((h_1)_{-i}, (h_2)_i), G)$. Finally, let $\mu(G) = ((h_1)_{-i}, (h_2)_i)$. By RVT, we have $(h_1, \ldots, h_1) P^0_G ((h_1)_{-i}, (h_2)_i)$. This, together with the fact that $g_i P^0 d_1 P^0 d_2$, means μ is blocked by $((h_1, \ldots, h_1), G)$.

- Suppose $\mu(g_i) = h_1$. Let $\mu(G) = ((h_1)_{-k}, (h_2)_k)$ for some $k \neq i$. By RVT, we have $((h_1)_{-i-k}, (h_2)_i, (h_2)_k) P^0_G ((h_1)_{-k}, (h_2)_k)$. This, together with the fact that $g_i P^0 d_1 P^0 d_2$, means μ is blocked by $(((h_1)_{-i-k}, (h_2)_i, (h_2)_k), G)$. Finally, let $\mu(G) = (h_1, \ldots, h_1)$. Then, μ is blocked by (h_1, d_1) as $d_1 P^0 g_{i+1}$ and $h_{d_1} = h_1$.

This completes the proof of the necessity part.

[Sufficiency] The proof of this part is constructive. Suppose $P^0_{\tilde{G}}$ is such that $P^0_G \in \mathcal{D}^{RVT}_G$ for all $G \in \tilde{G}$. Suppose further that $(P^0_{\tilde{G}}, P^0)$ satisfies Condition 1. Take $\underset{\sim}{P} \in \underset{\sim}{\mathcal{D}}(P^0_{\tilde{G}}, P^0)$. We construct an algorithm that produces a stable matching in $\underset{\sim}{P}$. For each group $G = \{g_1, \ldots, g_n\}$ and each hospital h, define the conditional preference of g_n given h, $P^0_{g_n|h} \in \mathbb{L}(H)$, in the following way: $h' P^0_{g_n|h} h''$ if and only if $((h)_{-n}, (h')_n) P^0_G ((h)_{-n}, (h'')_n)$. Recall that, by our initial assumption on IHP, $g^j_{n_j} P^0 g^k_{n_k}$ for all $j < k \in \{1, \ldots, m\}$. In the following, we present our algorithm.

Algorithm: The algorithm involves $m + 1$ steps. We present the 1^{st} step, and a general step of the algorithm.

Step 1: Use SDA to match all doctors ranked above $g^1_{n_1}$ according to P^0. If $g^1_1, \ldots, g^1_{n_1-1}$ are all matched with same hospital, say h, then match $g^1_{n_1}$ using SDA, where $g^1_{n_1}$ bids according to $P^0_{g^1_{n_1}|h}$. Else match $g^1_{n_1}$ using SDA where $g^1_{n_1}$ bids according to $P^0_{g^1_{n_1}}$.

$$\vdots$$

Step j: Use SDA to match all doctors ranked below $g^{j-1}_{n_{j-1}}$ and above $g^j_{n_j}$ according to P^0. If $g^j_1, \ldots, g^j_{n_j-1}$ are all matched with same hospital, say h, then match $g^j_{n_j}$ by SDA where $g^j_{n_j}$ bids according to $P^0_{g^j_{n_j}|h}$. Else match $g^j_{n_j}$ using SDA where $g^j_{n_j}$ bids according to $P^0_{g^j_{n_j}}$.

$$\vdots$$

Continue this process till Step m and then match the remaining doctors by SDA at step $m+1$.

We show that the above algorithm produces a stable matching at $\underset{\sim}{P}$. Let μ be the outcome of the above mentioned algorithm. We distinguish the following cases.

Case 1: Suppose (h, s) blocks μ through μ'. Note that, by the nature of our algorithm, all doctors that propose before s are more preferred to s according to the IHP. Since $s \notin \mu(h)$, by the nature of our algorithm, this means either dP^0s for all $d \in \mu(h)$ and $|\mu(h)| = \kappa_h$, or $\mu(s)\underset{\sim}{P}_s h$. If dP^0s for all $d \in \mu(h)$ and $|\mu(h)| = \kappa_h$, then by responsiveness of hospitals' preferences, we have $\mu(h)\underset{\sim}{P}_h\mu'(h)$. Thus hospital h does not block with s. On the other hand, if $\mu(s)\underset{\sim}{P}_s h$, then clearly s does not block with hospital h. This contradicts that (h, s) blocks μ.

Case 2: Suppose $(\underset{\sim}{h}, G)$ blocks μ where $G = \{g_1, \ldots, g_n\}$ and $\underset{\sim}{h} = (h_1, \ldots, h_n)$.

We show $((\mu(g_1), \ldots, \mu(g_{n-1}), h_n), G)$ blocks μ. We show this by induction. Take $i \in \{1, \ldots, n-1\}$. Suppose $((\mu(g_1), \ldots, \mu(g_{i-1}), h_i, h_{i+1}, \ldots, h_n), G)$ blocks μ through μ'. Here, for $i = 1$, we mean $((h_1, \ldots, h_n), G)$ blocks μ. We show that $((\mu(g_1), \ldots, \mu(g_i), h_{i+1}, \ldots, h_n), G)$ blocks μ.

Note that, if $\mu(g_i) = h_i$, then there is nothing to show. We claim $\mu(g_i)P^0_{g_i}h_i$. Assume for contradiction that $h_iP^0_{g_i}\mu(g_i)$. Note that, by the nature of our algorithm, all the doctors that propose before g_i are preferred to g_i according to the IHP P^0. Since $g_i \notin \mu(h_i)$, it must be that dP^0g_i for all $d \in \mu(h_i)$ and $|\mu(h_i)| = \kappa_{h_i}$. By responsiveness of hospitals' preferences, this means $\mu(h_i)\underset{\sim}{P}_{h_i}\mu'(h_i)$. This is a contradiction to the fact that $((\mu(g_1), \ldots, \mu(g_{i-1}), h_i, h_{i+1}, \ldots, h_n), G)$ blocks μ through μ'. Therefore, $\mu(g_i)P^0_{g_i}h_i$.

Now, we show that $(\mu(g_1), \ldots, \mu(g_i), h_{i+1}, \ldots h_n)P^0_G(\mu(g_1), \ldots, \mu(g_{i-1}), h_i, \ldots h_n)$. Suppose $h' \neq h''$ for some $h', h'' \in (\mu(g_1), \ldots, \mu(g_{i-1}), h_i, \ldots h_n)$. Then by RVT, it follows that $(\mu(g_1), \ldots, \mu(g_i), h_{i+1}, \ldots h_n)P^0_G(\mu(g_1), \ldots, \mu(g_{i-1}), h_i, \ldots h_n)$. Now suppose $\mu(g_1) = \ldots = \mu(g_{i-1}) = h_i = \ldots = h_n = h$. Since $\mu(g_i)P^0_{g_i}h_i$, by Condition 1 we have either $((h)_{-i}, (\mu(g_i))_i)P^0_G(h, \ldots, h)$, or $((h)_{-j}, (\mu(g_i))_j)P^0_G(h, \ldots, h)$ for some $j < i$. Note that, if $i = 1$, then $((h)_{-i}, (\mu(g_i))_i)P^0_G(h, \ldots, h)$. Suppose $i \neq 1$ and $((h)_{-j}, (\mu(g_i))_j)P^0_G(h, \ldots, h)$ for some $j < i$. Since $j < i$, we have $g_jP^0g_i$. Then, according to SDA, g_j proposes before g_i. Since g_i is matched to $\mu(g_i)$, this means either $\mu(g_j) = \mu(g_i)$ or $\mu(g_j)P^0_{g_j}\mu(g_i)$. Since $\mu(g_j) = h$ and $\mu(g_i)P^0_{g_i}h$, it must be that $hP^0_{g_j}\mu(g_i)$. This contradicts $((h)_{-j}, (\mu(g_i))_j)P^0_G(h, \ldots, h)$. Therefore, $(\mu(g_1), \ldots, \mu(g_i), h_{i+1}, \ldots h_n)P^0_G(\mu(g_1), \ldots, \mu(g_{i-1}), h_i,$.

Since $((\mu(g_1), \ldots, \mu(g_{i-1}), h_i, \ldots h_n), G)$ blocks μ, $(\mu(g_1), \ldots, \mu(g_i), h_{i+1}, \ldots h_n)P^0_G(\mu(g_1), \ldots,$ implies $((\mu(g_1), \ldots, \mu(g_i), h_{i+1}, \ldots h_n), G)$ blocks μ. Hence, by the induction argument we have $((\mu(g_1), \ldots, \mu(g_{n-1}), h_n), G)$ blocks μ.

Now we proceed to show that $((\mu(g_1), \ldots, \mu(g_{n-1}), h_n), G)$ cannot block μ.

Case 2.1: Suppose $\mu(g_i) \neq \mu(g_j)$ for some $i \neq j \in \{1, \ldots, n-1\}$. Then, by our algorithm, $h_nP^0_{g_n}\mu(g_n)$. Note that, by the nature of the algorithm, all doctors that propose before g_n are preferred to g_n according to the IHP P^0. Since $g_n \notin \mu(h_n)$, it must be that dP^0g_n for all $d \in \mu(h_n)$ and $|\mu(h_n)| = \kappa_{h_n}$. By responsiveness of hospitals' preferences $\mu(h_n)\underset{\sim}{P}_{h_n}\mu'(h_n)$. This contradicts that $((\mu(g_1), \ldots, \mu(g_{n-1}), h_n), G)$ is a block.

Case 2.2: Suppose $\mu(g_1) = \ldots = \mu(g_{n-1}) = h$. Then, by our algorithm and

the definition of $P^0_{g_n|h}$, we have $h_n P^0_{g_n|h} \mu(g_n)$. By the nature of the algorithm, all doctors that propose before g_n are preferred to g_n according to the IHP P^0. Since $g_n \notin \mu(h_n)$, it must be that $dP^0 g_n$ for all $d \in \mu(h_n)$ and $|\mu(h_n)| = \kappa_{h_n}$. By responsiveness of hospitals' preferences $\mu(h_n) \underset{\sim}{P}_{h_n} \mu'(h_n)$. This contradicts that $((\mu(g_1), \ldots, \mu(g_{n-1}), h_n), G)$ is a block.
This completes the proof of the theorem. ■

2.6 Existence of Stable Matching with Adjacent IHP

In this section, we consider restrictions on the Identical Hospital Preference and investigate the existence of stable matching under those restrictions. We relax the RVT condition on the preferences of the groups of doctors. By assuming that a group of doctors can have any preference over the sets of hospitals irrespective of the preferences of the individual doctors in that group over individual hospitals. Such preferences of the groups of doctors are called unrestricted preferences. More formally, the set of unrestricted preferences of a group $G \in \tilde{G}$ is $\mathbb{L}(\bar{H}^{|G|})$.

In the following, we define lexicographic preferences of the hospitals. Let P_h be a preference of a hospital h and $D' \subseteq D$. Then, define $r_k(P_h, D') = d$ if and only if $|\{d' \in D' : d' P_h d\}| = k - 1$.

Definition **10** *A preference P_h of a hospital h with capacity κ_h is called lexicographic if for all $D', D'' \subseteq D$ with $|D''| = |D'| \leq \kappa_h$, $D' P_h D''$ if and only if there exists $k \in \{1, \ldots, |D'|\}$ such that $r_k(P_h, D') P_h r_k(P_h, D'')$ and $r_l(P_h, D') = r_l(P_h, D'')$ for all $l < k$. The set of lexicographic preferences of a hospital h is denoted by $\mathcal{D}^L_h$.*

Definition **11** *A preference profile $\underset{\sim}{P}$ with lexicographic hospitals' preferences and RVT groups' preferences is defined as $(\{\underset{\sim}{P}_d\}_{d \in D}, \{\underset{\sim}{P}_G\}_{G \in \tilde{G}}, \{\underset{\sim}{P}_h\}_{h \in H})$ where $\underset{\sim}{P}_d \in \mathbb{L}(\bar{H})$ for all $d \in D$, $\underset{\sim}{P}_G \in \mathcal{D}^{RVT}_G$ for all $G \in \tilde{G}$, $\underset{\sim}{P}_h \in \mathcal{D}^L_h$ for all $h \in H$, and hospitals' preferences satisfy IHP. The set of preference profiles with lexicographic hospitals' preferences and RVT groups' preferences is denoted by $\underset{\sim}{\mathcal{D}}^{LR}$.*

Definition **12** *A preference profile $\underset{\sim}{P}$ with lexicographic hospitals' preferences and unrestricted groups' preferences is defined as $(\{\underset{\sim}{P}_d\}_{d\in D}, \{\underset{\sim}{P}_G\}_{G\in\tilde{G}}, \{\underset{\sim}{P}_h\}_{h\in H})$ where $\underset{\sim}{P}_d \in \mathbb{L}(\bar{H})$ for all $d \in D$, $\underset{\sim}{P}_G \in \mathbb{L}(\bar{H}^{|G|})$ for all $G \in \tilde{G}$, $\underset{\sim}{P}_h \in \mathcal{D}^L_h$ for all $h \in H$, and hospitals' preferences satisfy IHP. The set of all preference profiles with lexicographic hospitals' preferences and unrestricted groups' preferences is denoted by $\underset{\sim}{\mathcal{D}}^{LU}$.*

In the following, we introduce the notion of Adjacent IHP (AIHP). AIHP implies that for any two doctors in any group, there cannot be a doctor outside that group that lies in-between those two doctors in the identical hospital preference. Recall that, whenever we consider an IHP P^0, we assume the indexation of the doctors in groups to be such that $g^j_i P^0 g^j_{i+1}$ for all $i < n_j$ and all $j \leq m$. Below, we provide a formal definition of AIHP.

Definition **13** *Let P^0 be an IHP. Then, P^0 satisfies Adjacent IHP (AIHP) if for any group $G = \{g_1, \ldots, g_n\} \in \tilde{G}$ and any $d \in D$, $g_1 P^0 d P^0 g_n$ implies $d \in G$.*

Definition **14** *Let P^0 be an IHP. Then, the collection of preference profiles where*

- *hospitals in H have lexicographic preferences and groups' preferences satisfy RVT, denoted by $\underset{\sim}{\mathcal{D}}^{LR}(P^0)$, is defined as $\underset{\sim}{\mathcal{D}}^{LR}(P^0) = \{\underset{\sim}{P} \in \underset{\sim}{\mathcal{D}}^{LR} : P^0$ is the IHP of $\underset{\sim}{P}\}$,*
- *hospitals in H have lexicographic preferences and groups' preferences are unrestricted, denoted by $\underset{\sim}{\mathcal{D}}^{LU}(P^0)$, is defined as $\underset{\sim}{\mathcal{D}}^{LU}(P^0) = \{\underset{\sim}{P} \in \underset{\sim}{\mathcal{D}}^{LU} : P^0$ is the IHP of $\underset{\sim}{P}\}$.*

Note that, for any IHP $P^0 \in \mathbb{L}(D)$, $\underset{\sim}{\mathcal{D}}^{LR}(P^0) \subseteq \underset{\sim}{\mathcal{D}}^{LU}(P^0)$.

Our next theorem says that AIHP is a necessary condition for the existence of stable matching at every preference profile where hospitals have lexicographic preferences and groups' preferences satisfy RVT.

Theorem 2 *Let P^0 be an IHP. Suppose stable matching exists at every preference profile in $\underset{\sim}{\mathcal{D}}^{LR}(P^0)$. Then, P^0 satisfies AIHP.*

Proof: Consider an IHP P^0. Suppose P^0 does not satisfy AIHP. We show that there exists a preference profile in $\underset{\sim}{\mathcal{D}}^{LR}(P^0)$ with no stable matching. Since P^0 does not satisfy AIHP, there exists a group $G = \{g_1, \ldots, g_n\}$ and doctors $d_1 \notin G$ such that $g_i P^0 d_1 P^0 g_{i+1}$ for some $i \in \{1, \ldots, n-1\}$ and $g_n P^0 d_2$. Take two hospitals $h_1, h_2 \in H$. Consider a preference profile $\underset{\sim}{P}$ in $\underset{\sim}{\mathcal{D}}^{LR}(P^0)$ such that $h_{d_1} = h_1$ and $h_2 \underset{\sim}{P}_{d_1} h$ for all $h \in H \setminus \{h_1\}$. Further, $h_{d_2} = h_2$ and $h_1 \underset{\sim}{P}_{d_2} h$ for all $h \in H \setminus \{h_2\}$. Let the preference of group G be such that $h_d = h_1$ and $h_2 \underset{\sim}{P}_d h$ for all $h \in H \setminus \{h_1\}$ for all $d \in \{g_2, \ldots, g_n\}$. Also, $h_{g_1} = h_2$ and $h_1 \underset{\sim}{P}_{g_1} h$ for all $h \in H \setminus \{h_2\}$ but $(h_1, \ldots, h_1) \underset{\sim}{P}_G ((h_1)_{-1}, (h_2)_1)$. An allocation of the group G, where at least one doctor in G is matched to a hospital $h \notin \{h_1, h_2\}$ is assumed to responsive and is ranked below all the allocations, where all the members of the group are either matched to h_1 or h_2. Let $|\{d : h_d = h_2\}| = \kappa_{h_2} - 2$, $|\{d : h_d = h_1\}| = \kappa_{h_1} - n$, and $|\{d : h_d = h\}| = \kappa_h$ for all $h \neq h_1, h_2$. Finally, we assume that the preferences of all groups other than G satisfy responsiveness. We show that there is no stable matching at this preference profile. Let μ be a stable matching at this preference profile. Since μ is stable and $\sum_{h \in H} \kappa_h = |D|$, it must be that $\mu(d) = h_d$ for all $d \notin \{g_1, \ldots, g_n, d_1, d_2\}$. Now we distinguish the following cases for the allocation of the group G.

- Suppose $\mu(G) = (h_1, \ldots, h_1)$. Then, (h_1, d_1) blocks μ as $h_1 \underset{\sim}{P}_{d_1} h_2$ and $d_1 P^0 g_{i+1}$.
- Suppose $\mu(G) = ((h_1)_{-1}, (h_2)_1)$. Then, $((h_1, \ldots, h_1), G)$ blocks μ as $g_1 P^0 d_1 P^0 d_2$ and $(h_1, \ldots, h_1) \underset{\sim}{P}_G ((h_1)_{-1}, (h_2)_1)$.
- Suppose $\mu(G) = ((h_1)_{-j}, (h_2)_j)$ for some $j \neq 1$. Note that since $h_2 \underset{\sim}{P}_{g_1} h_1$, by RVT $((h_1)_{-1-j}, (h_2)_1, (h_2)_j) \underset{\sim}{P}_G ((h_1)_{-j}, (h_2)_j)$. This, together with the fact that $g_1 P^0 d_1 P^0 d_2$, means μ is blocked by $(((h_1)_{-1-j}, (h_2)_1, (h_2)_j), G)$.

- Suppose $\mu(G) = ((h_1)_{-j-k}, (h_2)_j, (h_2)_k)$ for some $j < i+1$ and for some $k \neq j$. Since $(h_1, \ldots, h_1) \underset{\sim}{P}_G ((h_1)_{-j-k}, (h_2)_j, (h_2)_k)$, $g_j P^0 d_1 P^0 d_2$ and hospitals' preferences are lexicographic, μ is blocked by $(((h_1)_{-k}, (h_2)_k), G)$.

- Suppose $\mu(G) = ((h_1)_{-j-k}, (h_2)_j, (h_2)_k)$ for some $j > k > i+1$. Note that, since $h_1 \underset{\sim}{P}_{g_i} h_2$, by RVT we have $((h_1)_{-k}, (h_2)_k) \underset{\sim}{P}_G ((h_1)_{-j-k}, (h_2)_j, (h_2)_k)$. This together with the fact that $g_k P^0 d_2$, means μ is blocked by $(((h_1)_{-k}, (h_2)_k), G)$.

This completes the proof of Theorem 2. ■

Now we prove the converse of Theorem 2 which states that if the hospitals' preferences satisfy AIHP, then stable matching exists at every preference profile where hospitals' preferences are lexicographic and groups' preferences satisfy RVT. However, we prove a stronger version of this, where we show that if the hospitals' preferences satisfy AIHP, then stable matching exists at every preference profile where hospitals' preferences are lexicographic and groups' preferences are unrestricted.

THEOREM **3** *Let P^0 be an IHP. Suppose P^0 satisfies AIHP. Then, stable matching exists at every preference profile in $\underset{\sim}{\mathcal{D}}^{LU}(P^0)$.*

Proof: The proof of Theorem 3 is constructive. Suppose P^0 satisfies AIHP. We construct an algorithm that produces a stable matching at every preference profile in $\underset{\sim}{\mathcal{D}}^{LU}(P^0)$. Take $\underset{\sim}{P} \in \underset{\sim}{\mathcal{D}}^{LU}(P^0)$. Recall that, by our initial assumption on IHP, $g^j_{n_j} P^0 g^k_{n_k}$ for all $j < k \in \{1, \ldots, m\}$. Since P^0 is AIHP, this means $g^j_1 P^0 g^k_1$ for all $j < k \in \{1, \ldots, m\}$. Now we present our algorithm.
Algorithm: The involves $m+1$ steps. We present the 1^{st} step, and a general step of the algorithm. At every step, a doctor from a group proposes to a set of hospitals, one for each member of the group. Whenever a hospital receives a set of proposals at some step, it accepts all proposals if it has adequate vacancies, otherwise it rejects all the proposals.
Step 1: Use SDA to match all the doctors ranked above g^1_1 according to P^0. Let g^1_1 propose on behalf of G^1 to the top n_1-tuple of hospitals according to

$\underset{\sim}{P}_{G^1}$. If at least one doctor of the group is rejected, then let g_1^1 propose to the next ranked n_1-tuple of hospitals according to $\underset{\sim}{P}_{G^1}$. Continue this process till the whole group G^1 is accepted by a set of hospitals.

$$\vdots$$

Step j: Use SDA to match all the doctors that ranked below $g_{n_{j-1}}^{j-1}$ and above g_1^j according to P^0. Let g_1^j propose on behalf of G^j to the top n_j-tuple of hospitals according to $\underset{\sim}{P}_{G^j}$. If at least one doctor of the group is rejected, then let g_1^j propose to the next ranked n_j-tuple of hospitals according to $\underset{\sim}{P}_{G^j}$. Continue this process till the whole group G^j is accepted by a set of hospitals.

$$\vdots$$

Continue this process till Step m and then match the remaining doctors by SDA at the $m+1^{th}$ step.

We show that the above algorithm produces a stable matching at $\underset{\sim}{P}$. Let μ be the outcome of the above mentioned algorithm. We distinguish the following cases.

Case 1: Suppose (h,s) blocks μ through μ'. Note that, by the nature of our algorithm, all the doctors that propose before s are more preferred to s according to the AIHP P^0. Moreover, for any $G=(g_1,\ldots,g_n)$, if g_1P^0s, then AIHP implies g_nP^0s. Since $s\notin\mu(h)$, by the nature of the algorithm, we have either dP^0s for all $d\in\mu(h)$ and $|\mu(h)|=\kappa_h$, or $\mu(s)\underset{\sim}{P}_sh$. If dP^0s for all $d\in\mu(h)$ and $|\mu(h)|=\kappa_h$, then by responsiveness of hospitals' preferences, we have $\mu(h)\underset{\sim}{P}_h\mu'(h)$. This means hospital h does not block with s. On the other hand, if $\mu(s)\underset{\sim}{P}_sh$, then s does not block with hospital h. This contradicts that (h,s) blocks μ.

Case 2: Suppose $(\underset{\sim}{h},G)$ blocks μ through μ' where $G=\{g_1,\ldots,g_n\}$ and $\underset{\sim}{h}=(h_1,\ldots,h_n)$. Then, it must be that $\underset{\sim}{h}\underset{\sim}{P}_G(\mu(g_1),\ldots,\mu(g_n))$. By the nature of the algorithm, this means g_1 proposes to $(h_1,\ldots,h_n)$ before proposing to $(\mu(g_1),\ldots,\mu(g_n))$, and some hospital, say h_i, rejects at least one member of the group G. Let $\{g_{i_1}, \ldots, g_{i_l}\}$ be the set of doctors of group G that apply to h_i and get rejected. Because h_i rejects those doctors, it must be that h_i has less than l vacancies when g_1 proposes to $(h_1, \ldots, h_n)$ on behalf of G. Let D' be the set of doctors that are present in h_i at that time. By AIHP assumption and the nature of the algorithm, this means each doctor in D' is preferred to all the doctors of the group G. Again, by the nature of the algorithm, it follows that $D' \subseteq \mu(h_i)$. This means h_i must release some doctors from D' for the block $(\mu', \underset{\sim}{h}, G)$ to μ. Since $d \underset{\sim}{P}_{h_i} g_{i_k}$ for all $d \in D'$ and $k \in \{1, \ldots, l\}$, and $\underset{\sim}{P}_{h_i}$ is lexicographic, we have $\mu(h_i) \underset{\sim}{P}_{h_i} \mu'(h_i)$. This contradicts that $(\underset{\sim}{h}, G)$ blocks μ. This completes the proof of Theorem 3. ■

In what follows, we show by means of an example that the lexicographic assumption on the hospitals' preferences is necessary for Theorem 3. In other words, we show that if hospitals' preferences are not lexicographic, then existence of stable matching is not guaranteed even if hospitals' preferences follow AIHP. In fact, we show a stronger version where the existence of stable matching is not guaranteed under the additional assumption that groups' preferences satisfy RVT.

Example **1** Consider a matching problem where $H = \{h_1, h_2\}$ with $\kappa_{h_1} = \kappa_{h_2} = 2$, $D = \{g_1, g_2, s_1, s_2\}$ and there is exactly one group $G = \{g_1, g_2\}$ in $\tilde{G}$. The preferences of individual doctors, preference of the group and AIHP of hospitals on the set of individual doctors is given in Table 1. The group's preferences over pairs where one member is matched with a hospital $h \in H$ and the other one is unmatched is not shown in the table, but assumed to be responsive and ranked below the shown pairs. Finally, we assume that for hospital h_1, $\{g_1, g_2\} P_{h_1} \{s_1, s_2\}$. Note that, the preference of the hospital h_1 is not lexicographic. Further note that, the preference of the group G satisfies RVT.

Clearly, for a stable matching μ, each hospital should get exactly 2 doctors. We consider all 4 possible cases of group matching.

P^0	P_{s_1}	P_{s_2}	P_{g_1}	P_{g_2}	P_G
s_1	h_1	h_2	h_1	h_2	(h_1, h_1)
g_1	h_2	h_1	h_2	h_1	(h_2, h_2)
g_2					(h_1, h_2)
s_2					(h_2, h_1)

Table 1: Preferences

- Suppose $\mu(G) = (h_1, h_1)$. Since $s_1 P^0 g_2$ and $h_1 P_{s_1} h_2$, μ is blocked by (h_1, s_1).

- Suppose $\mu(G) = (h_2, h_2)$. Since $\{g_1, g_2\} P_{h_1} \{s_1, s_2\}$ and $(h_1, h_1) P_G (h_2, h_2)$, it follows that μ is blocked by $((h_1, h_1), G)$.

- Suppose $\mu(G) = (h_1, h_2)$ or (h_2, h_1). We show μ is blocked when $\mu(G) = (h_1, h_2)$, the proof of the same for $\mu(G) = (h_2, h_1)$ can be obtained by changing the roles of g_1 and g_2. Since $h_1 P_{s_1} h_2$ and $s_1 P^0 s_2$, if $\mu(h_1) = s_2$, then μ is blocked by (h_1, s_1). Now suppose $\mu(s_1) = h_1$. Since $\mu(g_1) = h_1$, this means $\mu(s_2) = h_2$. Then, $(h_2, h_2) P_G (h_1, h_2)$ and $g_1 P^0 s_2$ imply μ is blocked by $((h_2, h_2), G)$.

This shows that there is no stable matching at the above mentioned preference profile.

2.7 Matching Market with Non-Identical Hospital Preference and Couples

In this section, we investigate what happens if the IHP condition is slightly relaxed. We further assume that the groups are couples. Thus $D = S \cup M \cup F$ with $|M| = |F|$, where $M = \{m_1, \ldots, m_k\}$ is the set of males and $F = \{f_1, \ldots, f_k\}$ is the set of females. Further, we denote the set of couples by $C = \{\{m_1, f_1\}, \ldots, \{m_k, f_k\}\}$. We assume that for all couples $c = \{m, f\} \in C$ and for all hospitals $h \in H$, $m P_h f$.

Recall that, for a given set of hospitals H and set of doctors D with couples C, a preference profile $\underset{\sim}{P}$ is defined by a collection of preferences $(\{\underset{\sim}{P}_d\}_{d\in D}, \{\underset{\sim}{P}_c\}_{c\in C}, \{\underset{\sim}{P}_h\}_{h\in H})$ where $\underset{\sim}{P}_d$ is a preference of doctor d, $\underset{\sim}{P}_c$ is a preference of couple c and $\underset{\sim}{P}_h$ is a preference of hospital h for all $d \in D$, all $c \in C$, and all $h \in H$.

In what follows, we show by the means of an example that IHP assumption on hospitals' preferences is necessary for Theorem 1. In other words, we show that if hospitals' preferences do not satisfy IHP, then stable matching is not guaranteed even if couples' preferences satisfy Condition 1.

EXAMPLE **2** Consider a matching problem where $H = \{h_1, h_2, h_3\}$ with $\kappa_{h_1} = \kappa_{h_2} = 1$, $\kappa_{h_3} = 2$, $D = \{m, f, s_1, s_2\}$ and there is exactly one couple $c = \{m, f\}$ in C. The preferences of individual doctors, preference of the group and preferences of hospitals on the set of individual doctors is given in Table 2. The couple's preference over pairs where one member is matched with a hospital and the other one is unmatched is not shown in the table, but assumed to be responsive and ranked below the shown pairs. Note that, the preference of the couple c satisfies Condition 1.

P_{h_1}	P_{h_2}	P_{h_3}	P_{s_1}	P_{s_2}	P_m	P_f	P_c
s_1	m	m	h_2	h_3	h_1	h_2	(h_1, h_2)
m	f	f	h_1	h_1	h_3	h_1	(h_1, h_3)
f	s_1	s_1	h_3	h_2	h_2	h_3	(h_3, h_3)
s_2	s_2	s_2					(h_3, h_2)
							(h_3, h_1)
							(h_2, h_1)
							(h_2, h_3)

Table 2: Preferences

Let μ be a stable matching at the preference profile given in Table 2. Since $s_1 P_h s_2$ for all $h \in H$, it must be that either $\mu(s_1) = \mu(s_2)$ or $\mu(s_1) P_{s_1} \mu(s_2)$.

- Suppose $\mu(c) = (h_1, h_2)$. Since $h_1 P_{s_1} h_3$ and $s_1 P_{h_1} m$, μ is blocked by (h_1, s_1).

- Suppose $\mu(c) = (h_1, h_3)$. Since $(h_1, h_2)P_c(h_1, h_3)$ and $fP_{h_2}s_1$, μ is blocked by (h_2, f).

- Suppose $\mu(c) = (h_3, h_3)$. Since $(h_1, h_2)P_c(h_3, h_3)$, $fP_{h_2}s_1$ and $mP_{h_1}s_2$, μ is blocked by $((h_1, h_2), c)$.

- Suppose $\mu(c) = (h_3, h_2)$. Since $(h_3, h_3)P_c(h_3, h_2)$ and $fP_{h_3}s_2$, μ is blocked by (h_3, f).

- Suppose $\mu(c) = (h_3, h_1)$. Since $(h_3, h_3)P_c(h_3, h_1)$ and $fP_{h_3}s_2$, μ is blocked by (h_3, f).

- Suppose $\mu(c) = (h_2, h_1)$. Since $h_1P_{s_1}h_3$ and $s_1P_{h_1}f$, μ is blocked by (h_1, s_1).

- Suppose $\mu(c) = (h_2, h_3)$. Since $(h_3, h_3)P_c(h_2, h_3)$ and $mP_{h_3}s_2$, μ is blocked by (h_3, m).

Thus, there is no stable matching at this preference profile.

In view of the above example, we look for condition on couples' preferences that is sufficient to ensure existence of stable matching when hospitals' preferences are non-identical over individual doctors. Recall that, we have a mild condition on hospitals' preferences that m is preferred to f for all the hospitals h.

Let $P_C^0 = (\{P_d^0\}_{d \in D \setminus S}, \{P_c^0\}_{c \in C})$ be a given collection of preferences of the doctors that are in some couple, and of the couples in C. Then, by $\underset{\sim}{\mathcal{D}}(P_C^0)$ we denote the set of preference profiles where doctors $d \in D \setminus S$ and couples in $c \in C$ have preferences as in P_C^0, i.e., $\underset{\sim}{\mathcal{D}}(P_C^0) = \{\underset{\sim}{P} : \underset{\sim}{P}_d = P_d^0$ for all $d \in D \setminus S$ and $\underset{\sim}{P}_c = P_c^0$ for all $c \in C\}$.

Condition **2** *Suppose P_C^0 is such that $P_c^0 \in \mathcal{D}_c^{RVT}$ for all $c \in C$. Then, P_C^0 satisfies Condition 2 if there exists responsive preference $P_c \in \mathcal{D}_c^R$ for all $c \in C$ such that for all $c = \{m, f\} \in C$ and all $(h_1, h_2), (h_3, h_4) \in (\bar{H} \times \bar{H}) \setminus (h_m, h_m)$, we have $(h_1, h_2)P_c(h_3, h_4)$ if and only if $(h_1, h_2)P_c^0(h_3, h_4)$.*

Note that, Condition 2 implies that for a couple $c = \{m, f\}$, P_c^0 satisfies responsiveness over all pairs of hospitals except (h_m, h_m). Furthermore, P_c^0 violates responsiveness for togetherness only when both members of the couple get a position at h_m. In the following theorem, we show that existence of a stable matching is guaranteed at a preference profile if the couples' preferences satisfy Condition 2.

THEOREM **4** *Suppose P_C^0 is such that $P_c^0 \in \mathcal{D}_c^{RVT}$ for all $c \in C$. Then, stable matching exists at every preference profile in $\underset{\sim}{\mathcal{D}}(P_C^0)$ if P_C^0 satisfies Condition 2.*

Proof: The proof of Theorem 4 is constructive. We construct an algorithm that produces a stable matching at every preference profile where couples' preferences satisfy Condition 2. Suppose P_C^0 is such that $P_c^0 \in \mathcal{D}_c^{RVT}$ for all $c \in C$. Suppose further that P_C^0 satisfies Condition 2. Then, for each $c = \{m, f\} \in C$ there exists some $P_c \in \mathcal{D}_c^R$ such that for all $(h_1, h_2), (h_3, h_4) \in (\bar{H} \times \bar{H}) \setminus (h_m, h_m)$, we have $(h_1, h_2) P_c (h_3, h_4)$ if and only if $(h_1, h_2) P_c^0 (h_3, h_4)$. For each couple $c = \{m, f\}$, define a conditional preference $P_{f|h}^0 \in \mathbb{L}(H)$ of f in the following way: $h' P_{f|h}^0 h''$ if and only if $(h, h') P_c^0 (h, h'')$. In the following lemma we establish a connection between P_f^0 and $P_{f|h_m}^0$.

LEMMA **1** *Suppose $c = \{m, f\}$ is a couple and h_1, h_2, h_m are all distinct hospitals. Then, $h_1 P_f^0 h_2$ implies $h_1 P_{f|h_m}^0 h_2$.*

Proof: Assume for contradiction that $h_1 P_f^0 h_2$ and $h_2 P_{f|h_m}^0 h_1$. Since $h_2 P_{f|h_m}^0 h_1$, we have $(h_m, h_2) P_c^0 (h_m, h_1)$. Because h_1, h_2, h_m are all distinct, by Condition 2 we have $(h_m, h_2) P_c (h_m, h_1)$. Because $P_c \in \mathcal{D}_c^R$, this means $h_2 P_f^0 h_1$, which is a contradiction. This completes the proof. ■

Take $\underset{\sim}{P} \in \underset{\sim}{\mathcal{D}}(P_C^0)$. In the following, we present our algorithm that produces a stable matching at $\underset{\sim}{P}$.

Algorithm: Use DPDA where every doctor bids as a single doctor. For all $c = \{m, f\}$, m proposes according to P_m^0 and f proposes according to $P_{f|h_m}^0$

where h_m is the P_m^0 maximal hospital. For all $s \in S$, s proposes according to $\underset{\sim}{P}_s$.

The following lemma establishes an important property of DPDA. The proof of the lemma is elementary, however we present the proof for the sake of completeness. Let μ be the outcome of the above mentioned algorithm.

LEMMA **2** *Suppose a doctor d is rejected by hospital h at some stage of the algorithm. Then μ cannot be blocked such that d moves to the hospital h from $\mu(d)$.*

Proof: Since h has rejected d during some stage of the algorithm, hospital h had κ_h many better proposals from doctors that are better than d according to P_h at the time when h rejected d. Therefore, by the nature of DPDA all the doctors that are matched with h at the end of the algorithm must be better than d according to P_h. So h will not block with d. This completes the proof. ■

Now, we show that the above mentioned algorithm produces a stable matching at $\underset{\sim}{P}$. We distinguish the following cases.

Case 1: Suppose (h, s) blocks μ through μ'. Since s blocks with h, we have $s\underset{\sim}{P}_s\mu(s)$. Therefore, it must be that h rejected s earlier in the algorithm. Hence by Lemma 2, (h, s) cannot block μ.

Case 2: Suppose $(\underset{\sim}{h}, c)$ blocks μ through μ' where $c = \{m, f\}$ and $\underset{\sim}{h} = (h_1, h_2)$. Note that, h_1 and h_2 are not necessarily different. Since c blocks with (h_1, h_2), we have $(h_1, h_2)P_c^0(\mu(m), \mu(f))$.

Case 2.1: Suppose $\mu(m) = h_m$. By the definition of P_c^0, this means $(h_m, h_2)P_c^0(h_1, h_2)$. Since $(h_1, h_2)P_c^0(h_m, \mu(f))$, this implies $(h_m, h_2)P_c^0(h_m, \mu(f))$. Because $(\mu', \underset{\sim}{h}, c)$ blocks μ where $\mu(m) = h_m$, it follows that $(\mu', (h_m, h_2), c)$ also blocks μ. Note that, $(h_m, h_2)P_c^0(h_m, \mu(f))$ implies $h_2P_{f|h_m}^0\mu(f)$. Therefore, by the definition of the algorithm, it must be that f proposed to h_2 and got rejected at an earlier stage of the algorithm. Hence, by Lemma 2, $(\mu', (h_m, h_2), c)$ cannot block μ.

Case 2.2: Suppose $\mu(m) \neq h_m$. Since m bids according to P^0_m, using similar logic as before, it follows that either $h_1 = \mu(m)$ or $\mu(m)P^0_m h_1$. This, together with the facts that $(h_1, h_2)P^0_c(\mu(m), \mu(f))$ and $\mu(m) \neq h_m$, implies that $h_2 P^0_f \mu(f)$. Because $\mu(m) \neq h_m$, it must be that $|\mu(h_m)| = \kappa_{h_m}$ and $d \underset{\sim}{P}_{h_m} m$ for all $d \in \mu(h_m)$. Since $m \underset{\sim}{P}_h f$ for all $h \in H$, this means $d \underset{\sim}{P}_{h_m} f$ for all $d \in \mu(h_m)$. Therefore $\mu(f) \neq h_m$. Moreover, since $(\mu', \underset{\sim}{h}, c)$ blocks μ, it follows that $h_2 \neq h_m$. Because $h_2 P^0_f \mu(f)$, it must be that $h_2 \neq \mu(f)$. As $\mu(f), h_2, h_m$ are all distinct and $h_2 P^0_f \mu(f)$, by Lemma 1 we have $h_2 P^0_{f|h_m} \mu(f)$. Therefore, by the definition of the algorithm, it must be that f proposed to h_2 and got rejected at an earlier stage of the algorithm. Hence, by Lemma 2, $(\underset{\sim}{h}, c)$ cannot block μ.

This completes the proof. ■

3 Conclusion

We consider many-to-one matching problem between doctors and hospitals where doctors are divided into groups. We assume that the preferences of the groups do not follow responsiveness and hospitals have identical preferences over the individual doctors. We show by means of an example that existence of stable matching is not guaranteed in such scenarios. In view of this, we find restriction on the preferences of groups that is necessary and sufficient for the existence of stable matching. We further consider an additional restriction on the identical hospital preference that we call adjacent identical hospital preference, and show that stable matching always exists under this restriction. Afterwards, we relax the condition of identical hospital preferences. However, in order to make the problem tractable, we restrict the groups to be couples. Finally, we provide a sufficient condition for the existence of stable matching for such matching problems.

References

[1] B. Douglas Bernheim, Bezalel Peleg, and Michael D. Whinston, *Coalition-proof Nash equilibria I. concepts*, Journal of Economic Theory 42 (1987), no. 1, 1–12.

[2] Francis Bloch and Bhaskar Dutta, *Formation of networks and coalitions*, Handbook of Social Economics, North Holland: Amsterdam, 2011, Edited by J. Benhabib, A. Bisin, and M. O. Jackson.

[3] Bhaskar Dutta and Jordi Massó, *Stability of matchings when individuals have preferences over colleagues*, Journal of Economic Theory 75 (1997), no. 2, 464–475.

[4] Federico Echenique and Jorge Oviedo, *A theory of stability in many-to-many matching markets*, Theoretical Economics 1 (2006), no. 2, 233âĂŞ-273.

[5] Bettina Klaus and Flip Klijn, *Stable matchings and preferences of couples*, Journal of Economic Theory 121 (2005), no. 1, 75–106.

[6] Fuhito Kojima, Parag A. Pathak, and Alvin E. Roth, *Matching with couples: Stability and incentives in large markets*, Working Paper, Stanford University, 2010.

[7] Hideo Konishi and M. Utku Ünver, *Credible group stability in many-to-many matching problems*, Journal of Economic Theory 129 (2006), no. 1, 57–80.

[8] Alvin E. Roth, *The evolution of the labor market for medical interns and residents: A case study in game theory*, The Journal of Political Economy (1984), 991–1016.

[9] Alvin E. Roth, *Misrepresentation and stability in the marriage problem*, Journal of Economic Theory **34** (1984), no. 2, 383–387.

[10] Alvin E. Roth and Marilda A. Oliveira Sotomayor, *Two-sided matching: A study in game-theoretic modelling and analysis*, vol. 18, Cambridge University Press, 1990.

[11] Tayfun Sonmez and M. Utku Ünver, *Matching, allocation, and exchange of discrete resources*, Handbook of Social Economics, vol. 1, North-Holland, 2011, Edited by Jess Benhabib, Alberto Bisin, and Matthew O. Jackson, pp. 781–852.

[12] Marilda Sotomayor, *Three remarks on the many-to-many stable matching problem, Mathematical social sciences* 38 (1999), no. 1, 55–70.