Asymmetric Dynamic Price Mechanism for Symmetric Buyers*

Shraman Banerjee† Bo Chen‡

September 17, 2016

Abstract

We consider a dynamic posted price mechanism of a monopolist who must sell a single unit of a good to a number of buyers before a deadline. The monopolist does not have any commitment power in the sense that price in each period is sequentially rational . When the buyers are ex-ante symmetric but non-anonymous to the monopolist, we propose a new asymmetric mechanism that includes a horizontal price discrimination along with the intertemporal price discrimination. We show that this mechanism beats the optimal mechanism (which is symmetric) established in the literature although there is no ex-ante asymmetry to the seller in terms of the buyers' value distributions. Thus even with a payoff-irrelevant observational difference, asymmetric equilibrium can arise in an otherwise ex-ante symmetric case. We change the random tie-breaking allocation rule which is generally used for symmetric mechanisms to generate the asymmetry here. We characterize the equilibrium and the price path of the monopolist over time. We also show that the result holds even for a static version of the model as well which implies that a single static monopoly price is not optimal for the monopolist under non-anonymity.

Keywords: Revenue Management, Intertemporal Price Discrimination, Coase Conjecture, Anonymity

1 Introduction

Revenue management concerns with optimal pricing strategy practised in many industries, such as airline pricing, pricing of movie tickets, pricing of tickets for sports events, rental cars, hotels, and packaged tours etc. There are two distinct features of a revenue management pricing that makes it different from a general dynamic pricing problem. Firstly, there is only a fixed quantity of good

*The authors thank Arunava Sen, Debasis Mishra, Johannes Horner and Rajat Deb for helpful comments. The authors also wish to thank all the seminar participants at Southern Methodist University, Indian Statistical Institute, Delhi, Jawaharlal Nehru University and Indian Institute of Technology, Delhi for helpful discussions and comments.

†O.P. Jindal Global University. Email: sbanerjee@jgu.edu.in

‡Southern Methodist University, Department of Economics, 3300 Dyer Street, Suite 301, Umphrey Lee Center, Dallas TX 75275 − 0496. Email: bochen@smu.edu

that the seller can sell, i.e., the seller has a capacity constraint on the quantity of good that he can sell. Secondly, the good becomes obsolete and not usable after a certain period. When an airline company prices tickets for passenger seats, it has to sell the tickets before the actual date of flight. Similarly a packaged tour company has to price the deal in such a way that it gets sold before the scheduled date of travel. In such situations, the seller cannot bargain indefinitely with the buyers to sell the good.

Most of the standard revenue management literature assumes that the buyers are myopic in that in every period there is a flow of buyers, and the buyers are either served only in that period, or they exit from the market.[1] In contrast to that what we often observe in reality is that some buyers are typically strategic and they face the decision of whether to buy the good in that period or wait to buy the good in another period. Hörner and Samuelson (2011) investigate the dynamics of a revenue management pricing problems with strategic buyers. In their model, the seller (or the monopolist) has to sell an indivisible goods to a group of buys who have private valuations over the good before a deadline. The monopolist adopts a price-posting mechanism where she posts a take-it-or-leave-it price in each period as a dynamic screening device in order to optimally price discriminate the buyers sequentially. The buyers decide whether to buy in the current period at the current price, or wait longer for the price to eventually decrease in future. Waiting is profitable for him if the other buyers have not snatched the good before he does. The buyers' acceptance or rejection of prices in different periods give signals of their possible private types to the seller. The seller updates his belief about the buyer types and sets price accordingly in the next period.

The practice of revenue management pricing, although started initially from the airline industries, has now been applied increasingly by various other industries. These diverse industries have different industry-specific frameworks which should be taken into account while making the pricing decisions optimally. Our paper deals with one such specification in certain revenue management pricing industries and show how the optimal pricing mechanism changes drastically if we include these specifications. While in airlines industries or for packaged tour companies, the buyers are completely anonymous to the seller, there are certain industries like television and radio industries where they sell advertising time slots to different companies in which case the buyers are not completely anonymous. An airline company does not know the information about its large body of potential consumers when it posts its prices in the website. But when a television company sells its advertising time slot in the middle of a popular game or a TV show it may know which companies have actually bid for that slot.

Another example of non-anonymity may be the case when a company is placing classified ads for selling its product to two different newspapers who it knows have disjoint reader groups. The question that we ask in this paper is that in these cases of non-anonymous buyers, should the company ask for same price to all of them or should it price discriminate. It should be noted that we use the term 'non-anonymity' in a strict sense so that we abstract away from any other ex-ante heterogeneity of the buyers apart from the fact that the seller only knows the identities of the buyers with no other additional information.

Standard theories of price discrimination relies on either the ex-ante heterogeneity among the

[1] For a detailed survey of the revenue management literature, see Talluri and van Ryzin (2005).

buyers or some self-selection mechanisms which reveal the buyer types. In contrast our paper claims that even if the buyers are ex-ante homogeneous to the seller with only some payoff-irrelevant observational difference(which in our case is that only the identities of the buyers are known to the seller), it is better for the seller to horizontally price discriminate among the buyers than to charge a single price to both of them.

We consider a monopolist who faces a fixed and known number of strategic buyers. There is a deadline within which the good needs to be sold as the good becomes obsolete after the deadline is over. The buyers are ex-ante homogeneous in the sense that these buyers with private valuations draw their values from an identical and known distribution. In the interim period the seller can set prices in each of the finite instants of time which the buyers can either accept and end the game, or can reject in which case the game moves on to the next period for possible price revisions. The seller cannot ex-ante pre-commit to any fixed price paths, so in our model each price has to be sequentially rational and the equilibrium that we focus on is the perfect Bayesian equilibrium. The allocation rule of the seller is somewhat different from the standard literature. The standard allocation rule in such mechanisms entails a basic norm of equal treatment of equals and unequal treatment of unequals. For example the optimal allocation rule for strategic buyers in Hörner and Samuelson (2011) (since the basic set up of our model is closest to theirs) is to post a single price in each period and allocate the good if any one buyer accepts in that period. If none of the buyers accept in a particular period the game moves to the next period for price revisions. If more than one buyer accept the good in a particular period, the monopolist randomly allocates the good to the accepting buyers. It is in this tie-breaking allocation rule that our mechanism differs. We show that under non-anonymity if the monopolist treats the buyers differently in the sense that instead of randomly allocating the good if he allocates the good deterministically to any one of the accepting buyers, then the monopolist is better off.

This is a kind of a divide and rule policy in the sense that amidst a pool of ex-ante homogeneous buyers with no payoff-relevant heterogeneity among them, this discriminatory treatment stirs up a heterogeneity among them so that even the buyers with same valuations would accept different prices in equilibrium. The buyers start with absolute homogeneity. In this otherwise symmetric environment the asymmetric mechanism from the monopolist generates this asymmetry among the buyers which results in an asymmetric dynamic equilibrium starting from a symmetric environment. So the equilibrium pricing strategy results in a horizontal price discrimination from the monopolist apart from the intertemporal discrimination even in the absence of ex-ante heterogeneity. Also, this divide and rule policy makes the monopolist better off compared to the random tie-breaking allocation rule. The basic intuition is that price discrimination arises here as a way of maximizing seller's return through a diversified portfolio of options. By asking for a high price from one buyer, the seller takes a high risk high return gamble. In order to compensate for that he sets a low price for the other in case the high risk option does not pay off. In fact when there are two buyers, one of the prices is above and another below the price that is set in the single price mechanism.

Another issue that needs to be clarified here is what factor is driving the result in this case. This is to establish the starkness of our result. If the value distributions of the buyers were asymmetric, it would be naturally inferred that the optimal mechanism of the monopolist would have been to treat them differently. This is because the asymmetry of the distributions give the seller some additional information which the seller would want to make use of. In our model once the seller treats the buyers differently in the first period, from the second period onwards the buyers become inherently asymmetric (because the ex-ante conditional distributions of their valuations become different). So in a dynamic model like ours it can be natural to infer that it is because of this asymmetry which is generated right after the first period (or rather the additional information to the seller generated by asymmetry) that makes this asymmetric treatment of the buyers more beneficial to the seller. So we establish our result even in a static environment. We show with an example of a one-period version of the model with two buyers that even in a one period model charging a single price by the monopolist is not optimal. In a posted price scenario like ours, we know that the one period optimal price for the seller is the static monopoly price. But we show that charging the static monopoly price is not optimal when the buyers are non-anonymous. Instead of a single monopoly price it is better for the monopolist to charge two different prices in a static framework. Thus we claim that it is the mechanism itself, not the asymmetry generated out of it, that is driving the result.

This idea of generating asymmetric equilibria in case of a homogeneous population has been treated earlier in evolutionary biology in order to explain the degree of specialization in nature. Nalebuff and Riley('85) has shown that if two completely homogeneous members of a single species engage in a war of attrition kind of contests in continuous times, it can result in an evolutionary stable asymmetric equilibrium (in fact they show that under certain conditions a continuum of equilibria can arise) which results in a division of the species into two different sub-species. In our paper, we have borrowed their basic idea and their proof technique to further illustrate our result in the case of continuous time. In section 3.4 we have shown the existence of an asymmetric equilibrium in our set up even in a continuous time framework. This is to show that our result holds even in the limit when the time difference between the price revisions is sufficiently small.

From the literature on Coase conjecture we know that when a seller cannot pre-commit to any particular mechanism, it is harmful for him. His price goes all the way down to the marginal cost. The seller in our model is also affected adversely from his absence of pre-commitment. He faces a trade-off between setting positive reserve prices and finely price discriminate. We show that if he sets positive reserve prices he has to have non-negligible buyer valuation range whom he charges the same price. Thus he cannot run a Dutch auction-like mechanism with a positive reserve price.

For the full characterization of the equilibrium we take an additional assumption that the valuations of the buyers are drawn from an uniform distribution. This is to ensure that the equilibrium that we get is unique. It can be noted that the buyers' game is a game of strategic complementarity- the more likely the other buyers are to wait, the more incentive a buyer gets from waiting. So there is a possibility of multiple equilibria in the game. So we take uniformly distributed buyer valuations in our model and show that the equilibrium is unique and interior.

The rest of the paper is arranged as follows. In section 2 we give a brief review of the literature. In section 3.1 and 3.2 we set up the model for the anonymous and the non-anonymous buyers respectively. In section 3.3 we illustrate our main result with an example of a two period version of the model. In section 3.4 we derive our main result of defining and characterizing the equilibrium. Section 3.5 shows the existence of the equilibrium in a continuous time setting.

2 Literature

The first strand of literature to which our paper contributes is that of the revenue maximizing dynamic mechanism. Pai and Vohra (2009) presents a seller with a fixed, finite supply of a homogenous good who faces a population of potential buyers with unit demand who arrive and depart over the course of a finite time horizon. The times at which each agent arrives and departs from the market are his private information, as is his valuation for an object. They use an appropriate formulation of the revelation principle of Myerson (1986) to restrict attention to mechanisms that allocate to buyers, if at all, only in the period of their departure. Gershkov and Moldovanu (2010) examine the allocation of a finite set of heterogeneous durable goods to a dynamic population of randomly arriving buyers. They show that this efficient policy is, in fact, implementable in the presence of incomplete information about the valuations of arriving buyers. Each buyer is charged the expected externality that she imposes on future agents, where expectations are taken with respect to the arrival process and the valuations of future agents. Said (2008, 2009) examines the allocation of a sequence of indivisible goods to a dynamic population of buyers and shows that, when selling a sequence of objects, one in each period, to a stream of buyers who arrive at random times, the sealed-bid second-price auction is no longer efficient. Gallien (2006) shows that when the distribution of buyer inter-arrival times has an increasing failure rate, the optimal mechanism allocates goods to buyers only upon their arrival. If objects are durable instead of perishable Board and Skrzypacz (2010) provides a characterization of the optimal dynamic auction. The revenue-maximizing direct mechanism applies an efficient mechanism to virtual values such that there is a constant cutoff value below which objects are not allocated, and the object is allocated to the buyer with the highest valuation exceeding that cutoff.

All the papers mentioned above deals with varying population of buyers in each period where the buyers are myopic. This is in line with the revenue management literature where the standard practice is to assume myopic buyers. A detailed survey of the revenue management literature can be found in (Talluri and van Ryzin (2005)). In reality, however, the buyers very often decide to wait for longer periods hoping the price to decrease in future while buying, for example, an airline ticket. Hörner and Samuelson(2011) introduced strategic buyers into the revenue management literature. Our paper follows closely to the assumption made in the Hörner-Samuelson paper in assuming the buyers to be strategic. So, in a sense, our paper lies more close to their paper in the basic structure of the model than the earlier papers of the literature.

Secondly, in our setting, since a seller having no commitment power is tempted to lower down the price in subsequent periods in order to tempt the buyers to buy the good, this gives a similarity of the setting with the literature on durable-goods monopoly and Coase conjecture (Ausubel and Deneckere (1989) and Gul, Sonnenschein and Wilson (1986)), but the durable goods literature differs from the revenue management literature in its infinite horizon setting, and also in the fact that in the literature on durable goods there are enough goods compared to the number of buyers. In our model the scarcity of goods induces inherent competition among the buyers for buying the goods. This competition among the buyers incentivizes them to accept the good earlier. Thus the monopolist in our model, even if he is in a Coasian dynamics, will eventually violate the Coase conjecture because the inherent competition among the buyers will not allow the price to go down to marginal cost even if we allow the time interval between price revisions to be close to zero.

3 The Model

3.1 Anonymous Price Posting Mechanisms

Our model is similar to that in Hörner and Samuelson (2011). We consider a general T-period dynamic game where a seller posts take-it-or-leave-it prices to sell an indivisible good to n buyers, where $n \geq 2$. The good has to be consumed at the end of the T periods and after that it becomes valueless. The seller hence has to sell the good within these T periods. We denote time period t as the number of periods remaining in the game, and hence the first period is denoted by T and likewise, $t = T - 1$ denotes the next period while $t = 1$ is the last period.

The timeline for the game is as follows: In each period t, the seller announces a price $p_t \in \mathbb{R}$, and the buyers upon observing the price simultaneously decide whether to accept or to reject the price. If only one of the buyers accepts the price, the game ends and the good is sold to the accepting buyer at price p_t. If more than one buyer accept, then the good is randomly allocated to one of the accepting buyers at the announced price. If no one accepts the good, the game moves to the next period $t - 1$.

Each buyer draws his private valuation v independently and identically from a known distribution $F : [0, 1] \to [0, 1]$ such that F is strictly increasing and continuously differentiable. A buyer with valuation v who obtains the good at price p derives a payoff of $(v - p)$. The seller having no intrinsic valuation over the good has a payoff equal to the price p at which the good is sold.[2]

A non-trivial history $h_t \in H_t$ is the history at period t where the game does not end effectively. A behavior strategy of the seller $\{\sigma_S^t\}_{t=1}^T$ is a sequence of prices p_t which maps from the history to a probability distribution of prices. A behavior strategy of a buyer i, $\{\sigma_i^t\}_{t=1}^T$, is a map from his type, history of prices, and current price to a probability of acceptance, i.e., $\sigma_i^t : [0, 1] \times H_t \times \mathbb{R} \to \{0, 1\}$.

The solution concept we adopt in the paper is perfect Bayesian equilibrium.[3] We assume that the seller does not have any commitment power and each price has to be sequentially rational given the previous history and the belief about the optimal continuation payoff. Although in real world we do find cases where the seller uses different commitment devices, but in the present scenario, a seller without commitment will always do better than a seller without commitment, for at least two buyers. In this section, we shall focus on an anonymous price posting mechanism where the seller posts a single price in each period to all buyers and the buyers use symmetric strategies, $\sigma_i^t = \sigma_j^t$ for $i \neq j$.That is, we assume that the buyers of same type base their strategies on the same conditional distribution. The strategy of a buyer depends only on his valuation but not on his identity. In Subsection 3.2, we shall consider a non-anonymous price posting mechanism where the seller offers different prices to different buyers and accordingly, the buyers adopt different strategies, i.e., $\sigma_i^t \neq \sigma_j^t$ for $i \neq j$.

[2]Without loss of generality, we assume that all parties discount future payoffs using the same discount rate of 0.

[3]Existence of such an equilibrium in our setting is similar to that in Horner and Samuelson (2011), and follows from standard arguments (see Chen (2012)).

In an anonymous price-posting mechanism, the seller posts a single price to all the buyers in each period. Each individual buyer chooses a particular time period (if any) to accept the corresponding prevailing price and ends the game. The buyers face a non-trivial competition problem in each period. In particular, a buyer with higher valuations are more anxious to accept earlier as it is possible that the other buyers may "snatch" the good earlier, leaving him with zero payoff. In particular, the buyers' problem is an optimal stopping problem, where an individual buyer chooses an optimal price in the price path which he can accept, taking his opponent's strategy as given. Consequently, the buyers' game is one with strategic complementarity. The marginal gain from waiting one extra period increases for a buyer, the more likely he believes that his opponents will also wait. In general, in a game of strategic complementarity, there is a possibility of multiple equilibria. (For a particular example of a case where multiple equilibria can arise, see Hörner and Samuelson (2011)). To avoid this issue of multiple equilibria, we shall take a specific case of uniform distribution of buyer valuations while solving the model, in which case we can find a unique equilibrium to the problem.

For the rest of this section, we shall follow Hörner and Samuelson (2011) closely in describing the buyers' and the seller's problems explicitly.

Given our focus on symmetric perfect Bayesian equilibrium, the buyers who accept at time period t are those whose valuations exceed a critical threshold v_t. Our next lemma, taken directly from Hörner and Samuelson (2011) illustrates the seller's posterior beliefs after a history of no sales up to a particular time period.

Lemma 1. (Hörner and Samuelson (2011)) *Let $n \geq 2$. Fix an equilibrium, and suppose period t has been reached without a price having been accepted. Then the seller's posterior belief is that the buyers' valuations are identically and independently drawn from the distribution $F(v)/F(v_{t+1})$, with support $[0, v_{t+1}]$, for some $v_{t+1} \in (0, 1]$.*

In the last period a buyer accepts a price if it is below or equal to his valuation. In the earlier periods each buyer faces a trade-off whether to accept at the posted price, or to wait till the next period . If he waits till the next period, he may get the good at a lower price, but the probability of getting the good decreases. If he accepts, he may get the good at a higher price, compared to waiting till next period, but the probability that he gets the good becomes higher.

Consider an arbitrary time period t and a buyer i with valuation v. Given a critical threshold v_t, buyer i's expected payoff from accepting the price p_t is :

$$\begin{aligned} &F(v_t)^{n-1} \sum_{j=0}^{n-1} \tfrac{1}{j+1} \tbinom{n-1}{j} \left(1 - \tfrac{F(v_t)}{F(v_{t+1})}\right)^j \left(\tfrac{F(v_t)}{F(v_{t+1})}\right)^{n-1-j} (v - p_t) \\ &= F(v_t)^{n-1} \tfrac{1-(F(v_t)/F(v_{t+1}))^n}{1-F(v_t)/F(v_{t+1})} \tfrac{(v-p_t)}{n}. \end{aligned} \tag{1}$$

Notice that here $F(v_t)^{n-1}$ is the probability that no buyer with higher valuations accepts a higher price. The term $1/(j+1)$ is the probability that buyer i receives the good when j other buyers accept the posted price p_t. The binomial expression after $1/(j+1)$ is the probability that exactly j other buyers accept the price p_t : since the valuations of the opponents are drawn identically and independently from the distribution F, the term $\frac{F(v_t)}{F(v_{t+1})}$ is the conditional probability that an opponent's valuation is less than v_t, given that the opponent's valuation is below v_{t+1} (recall that v_t and v_{t+1} are the critical threshold valuations above which a buyer accepts the price p_t). The probability $\left(1 - \frac{F(v_t)}{F(v_{t+1})}\right)$ is hence the corresponding conditional probability that an opponent who has not accepted in period $(t+1)$ accepts the price p_t in period t. The monetary gain of $(v - p_t)$ is buyer i's ex post payoff when i is awarded the good at price p_t.

If the critical threshold v_t is interior, then a buyer with valuation exactly equal to v_t should be indifferent between accepting the current price and waiting for another period to accept. To be explicit, in period t, if buyer i with valuation v_t accepts p_t, his expected payoff can be written as (similar to (1)):

$$\frac{1 - (F(v_t)/F(v_{t+1}))^n}{1 - F(v_t)/F(v_{t+1})} \frac{(v_t - p_t)}{n}. \tag{2}$$

On the other hand, if buyer i with type v_t waits for another period to accept price p_{t-1}, his expected payoff is

$$\begin{aligned} &\left(\frac{F(v_t)}{F(v_{t+1})}\right)^{n-1} \sum_{j=0}^{n-1} \frac{1}{j+1} \binom{n-1}{j} \left(1 - \frac{F(v_{t-1})}{F(v_t)}\right)^j \left(\frac{F(v_{t-1})}{F(v_t)}\right)^{n-1-j} (v_t - p_{t-1}) \\ = &\left(\frac{F(v_t)}{F(v_{t+1})}\right)^{n-1} \frac{1 - (F(v_{t-1})/F(v_t))^n}{1 - F(v_{t-1})/F(v_t)} \frac{(v_t - p_{t-1})}{n}. \end{aligned} \tag{3}$$

In (3), the first term $\left(\frac{F(v_t)}{F(v_{t+1})}\right)^{n-1}$ is the probability that the good is still available for sale in the next period, i.e., the probability that none of his opponents has already bought the good at the start of next period. As before, the term $\frac{F(v_{t-1})}{F(v_t)}$ is the conditional probability that an opponent's valuation is less than v_{t-1}, given that the opponent's valuation is below v_t. All other terms are analogous to the expression in (1). As mentioned previously, if this critical threshold v_t is interior, then a v_t-type buyer is indifferent between accepting at price p_t in this period and waiting for the next period to accept p_{t-1}. In other words, we have

$$\frac{1 - (F(v_t)/F(v_{t+1}))^n}{1 - F(v_t)/F(v_{t+1})} \frac{(v_t - p_t)}{n} = \left(\frac{F(v_t)}{F(v_{t+1})}\right)^{n-1} \frac{1 - (F(v_{t-1})/F(v_t))^n}{1 - F(v_{t-1})/F(v_t)} \frac{(v_t - p_{t-1})}{n}.$$

Thus the above equation recursively defines a set of thresholds v_t such that for a buyer with valuation v if the optimal time period to accept is t, then $v \in [v_t, v_{t+1})$.

The seller's optimization problem is to choose a sequence of prices $\{p_t\}_{t=T}^{1}$ so as to maximize his expected payoff:

$$\max_{\{p_t\}_t} \pi_T(v_T) = \max_{\{p_t\}_t} \sum_{t=1}^{T} [F(v_{t+1})^n - F(v_t)^n] p_t,$$

where $[F(v_{t+1})^n - F(v_t)^n]$ is the probability that the highest valuation of the buyers lies in the interval $[v_t, v_{t+1})$ and the good is sold at price p_t (recall that the seller attaches value 0 to the good).

To solve the problem using a procedure like backward induction, it is convenient to write the seller's expected payoff in t recursively as follows:

$$\pi_t(v_{t+1}) = \left(1 - \left(\frac{F(v_t)}{F(v_{t+1})}\right)^n\right) p_t + \left(\frac{F(v_t)}{F(v_{t+1})}\right)^n \pi_{t-1}(v_t),$$

where as before, $(1 - (F(v_t)/F(v_{t+1}))^n)$ is the probability that a buyer accepts price p_t in period t and $\pi_{t-1}(v_t)$ is the continuation expected payoff.

While conceptually it is straightforward to apply a backward induction process to solve for the seller's optimal sequence of prices, the problem is complicated by the possibility of multiple equilibria in the continuation game, i.e., multiple sequences of critical thresholds $\{v_t\}$ can be consistent with a sequence of equilibrium prices. In addition, which equilibrium prevails in the continuation game can depend arbitrarily on the price p_t offered by the seller in period t and on the entire history of the prices offered by the seller, complicating the issue further.

3.2 Non-Anonymous Price Posting Mechanisms

We now introduce non-anonymous price posting mechanisms in our current framework and we start with formally describing the buyers' game and the seller's maximization problem as we did in Section 3.1. The key difference between an anonymous price posting mechanism and a non-anonymous price posting mechanism is that in a non-anonymous price posting mechanism, the price offered to each buyer in each period can be identity dependent, i.e., different buyers can face different take-it-or-leave-it price offers in each period. Offering such identity dependent posted prices typically requires that the seller can identify different buyers throughout the game. While this is not a particularly strong assumption (i.e., the seller can simply assign each buyer a particular number that will be fixed throughout the T periods), such mechanisms will be typically feasible in settings where the number of the buyers is not too large.

Notice that while the identities of the buyers seemingly provide extra information of the buyers to the seller, such identities can be completely arbitrary and is hence completely payoff-irrelevant from the ex ante point of view. As our main objective is to compare the revenue performance of the anonymous price posting mechanism studied in Hörner and Samuelson (2011) with that of a non-anonymous price posting mechanism, we assume that apart from the identities of the buyers, the buyers are otherwise *ex-ante* symmetric in their valuations. There is no other asymmetry, or payoff relevant information from the buyers. Thus, although the seller can clearly identify the buyers, he does not have any information of the buyer types, just like the previous case.

Notice that in a non-anonymous price posting mechanism, the seller adopts a strategy of unequal treatment of equals even when the buyers types are *ex-ante* identical. From the buyers' perspectives, if different buyers are treated differently, the strategies adopted by the different buyers are necessarily different. As a result, the equilibrium we shall focus on in a non-anonymous price posting mechanism is an asymmetric perfect Bayesian equilibrium where the buyers use asymmetric strategies, $\sigma_i^t \neq \sigma_j^t$, for $i \neq j$. In such an equilibrium, the strategy of a buyer depends not only on the type of the buyer but also on the buyer's identity.

One important issue we have to deal with in a non-anonymous price posting mechanism is the tie-breaking rule when multiple buyers accept the offers from the seller. In the anonymous price posting mechanism, if more than one buyers accept the good at a given period, the good can be randomly allocated to the accepting buyers without affecting the seller's payoff in that given period. In a non-anonymous price posting mechanism, a tie-breaking rule can significantly affect the seller's payoff since different buyers are facing different prices In particular, the seller can modify this random allocation rule to a deterministic allocation rule in order to achieve a higher expected payoff. For example he can specify that among all the buyers accepting the seller's offers, the buyer with the highest price offer is allocated the good with probability 1. It should be noted however that there are many other tie-breaking rules the seller can adopt and the above tie-breaking rule (allocating to the accepting buyer with the highest offered price) is not necessarily the revenue-maximizing rule in the entire dynamic game. We shall however restrict our attention to such an intuitive tie-breaking rule and we shall show that such a rule suffices for the non-anonymous price posting mechanism to generate strictly higher expected payoff for the seller than an anonymous price posting mechanism.

Our above discussion eventually leads to a mechanism where the seller sets different prices to different buyers, and if there is a tie, he allocates the good to the accepting buyer with the highest price in each period. As a result, in a non-anonymous price posting mechanism, we shall have n different price paths, each designed for a particular buyer, for the seller instead of a single one as in the case of anonymous buyers.

To simplify issues, we shall only consider that there are 2 buyers. Qualitatively the analysis will remain the same, but for an n buyers case, the corresponding analysis would become much more cumbersome and difficult to handle. Suppose at each period t the seller sets two different prices p_t (buyer 1) and q_t (buyer 2) to the two different buyers, and without loss of generality, we assume that $p_t > q_t$. As before, the critical valuation thresholds are important, and we denote u_t (buyer 1) and v_t (buyer 2) to be the threshold valuations at time t for the two buyers respectively.

In a non-anonymous price posting mechanism, the buyers are treated differently in equilibrium. As a result, the indifference conditions that pin down the corresponding threshold types will be different for the two buyers. Thus the threshold types in a non-anonymous mechanism will also be different for the two buyers in each period.

In time period t, the incentives for a u_t-type of buyer 1 is given by the indifference condition

$$(u_t - p_t) = \frac{F(v_t)}{F(v_{t+1})}(u_t - p_{t-1}).$$

Notice that in period t, buyer 1 can get the good with certainty if he accepts the offer. On the other hand, if he rejects the offer, the game goes to the next period $(t-1)$ only in the event that buyer 2 has also rejected his own price offer in period t.

Similarly, in time period t, the incentives for a v_t-type of buyer 2 is given by the indifference condition

$$\begin{aligned} \frac{F(u_t)}{F(u_{t+1})}(v_t - q_t) &= \frac{F(u_t)}{F(u_{t+1})}\frac{F(u_{t-1})}{F(u_t)}(v_t - q_{t-1}) \\ &\Rightarrow \quad (v_t - q_t) = \frac{F(u_{t-1})}{F(u_t)}(v_t - q_{t-1}). \end{aligned}$$

Recall that given the tie-breaking rule, buyer 2 can only get the good if buyer 1 rejects the offer. So in period t, he can get the good only with probability $F(u_t)/F(u_{t+1})$. If buyer 2 rejects the offer at period t, and if in the event that the game goes to the next period, again he can win the good with probability $F(u_{t-1})/F(u_t)$, i.e., only if buyer 1 again rejects the offer.

The seller's optimization problem in each time period t is to choose p_t and q_t to maximize $\pi_t(u_t, v_t)$ given the continuation payoff $\pi_{t-1}(v_{t-1})$

$$\begin{aligned} & \max_{p_t,q_t} \pi_t(u_t, v_t) \\ = \quad & \max_{p_t,q_t} \left[\left(1 - \frac{F(u_t)}{F(u_{t+1})}\right) p_t + \frac{F(u_t)}{F(u_{t+1})} \left(1 - \frac{F(v_t)}{F(v_{t+1})}\right) q_t + \frac{F(v_t)}{F(v_{t+1})} \frac{F(u_t)}{F(u_{t+1})} \pi_{t-1}(v_{t-1}) \right] \end{aligned}$$

Suppose the seller charges p_t and q_t to buyers 1 and 2 respectively. The seller gets the lower price q_t if buyer 1 rejects the offer in period t, i.e., only for the event that buyers 1's valuation is lower than his own threshold level, while buyer 2's valuation is above his threshold level in period t. Also, the seller gets the higher price p_t if buyer 1's valuation is higher than his own threshold level no matter what the valuation of buyer 2 is. If both have their valuations below their own threshold levels, the game moves on to the next period.

As mentioned previously, throughout this paper we are concerned with situations where the seller cannot commit to future prices. In the anonymous price posting mechanism, we have assumed that each price chosen by the seller has to be sequentially rational. There is no pre-committed price path that the seller announces beforehand. As we shall see in the next section, if the seller is allowed to treat different buyers differently, the seller might be tempted to do so to increase his expected payoff. So treating them equally can act as a commitment, i.e., an allocation rule of offering the same price to all the buyers and distributing the good with equal probability to any accepting buyer is a commitment on the part of the seller.

In the next subsection we shall use a simple motivating example to illustrate how the seller can increase his expected payoff by treating different buyers differently. To ease exposition, we shall consider a model with two buyers and two periods. We will explicitly solve the two-period model to derive the price paths for anonymous and non-anonymous price posting mechanisms respectively. To do so, we shall assume that the valuations of the buyers are drawn from uniform distribution over $[0, 1]$. This is done not only to avoid computational complexity but also to abstract away from the issues of multiple equilibria in the present buyers game of strategic complementarity. We will show that when the seller has the option of treating different buyers differently, his expected payoff can be actually strictly better if he opts for a non-anonymous price posting mechanism.

3.3 A Two-Period Example

Consider a two-period model where a seller sells an indivisible good to two buyers. The valuation of each buyer is drawn independently from a uniform distribution over [0,1]. The seller posts take-it-or-leave-it prices to the buyers in both periods, and the seller's prices are such that they are sequentially rational.

We first analyze the optimal sequence of prices when the seller chooses an anonymous price posting mechanism, i.e., he sets a single price in each period to both buyers. We solve the model using backward induction, starting from the last period, i.e., $t = 1$.

In $t = 1$, the seller maximizes his expected payoff:

$$\begin{aligned} \max_{v_1} \pi_1(v_1) &= \left(1 - \left(\frac{v_1}{v_2}\right)^2\right) p_1 \\ \text{s.t.} &: \quad p_1 \leq v_1. \end{aligned}$$

In the maximization problem, v_1 and v_2 are the equilibrium critical valuation thresholds in the two periods. The constraint implies that a v_1-type buyer accepts the price in the last period only if his valuation is at least as high as the price. In the objective function, $\left[1 - (\frac{v_1}{v_2})^2\right]$ is the probability that at least one of the buyers have a valuation greater than v_1, conditional on that they both had valuations less than v_2, which comes from the fact that the good remained unsold after the first period. So, this is the probability that the good is sold in the last period. Since this is the last period, the constraint is binding. The seller finds no reason to charge a price less than v_1 in the last period. He then chooses the optimal v_1-type buyer whom he wants to target so that the objective function is maximized. The above discussion implies that the seller faces the following problem:

$$\max_{v_1} \pi_1(v_1) = \left(1 - \left(\frac{v_1}{v_2}\right)^2\right) v_1.$$

The corresponding first-order condition is

$$\frac{\partial \pi_1(v_1)}{\partial v_1} = 0 \Rightarrow v_1^* = p_1^* = \frac{v_2}{\sqrt{3}},$$

implying that the optimal continuation payoff is

$$\pi_1(v_1^*) = \frac{2v_2}{3\sqrt{3}}. \tag{4}$$

In the first period, i.e., $t = 2$, denote the seller's price to be p_2. First consider the buyers' problem. The incentive constraint for the buyers is (i.e., the indifference condition for a type-v_2 buyer):

$$\frac{1 - v_2^2}{1 - v_2}(v_2 - p_2) = v_2 \frac{1 - (v_1/v_2)^2}{1 - v_1/v_2}(v_2 - v_1). \tag{5}$$

Given our discussion in Section 3.1, we know that the probability that a buyer accepts the price and obtains the good in the first period can be obtained via a binomial expression $\sum_{j=0}^{1} \frac{1}{j+1}(1 - v_2)^j (v_2)^{1-j} = \frac{1}{2}\frac{1-v_2^2}{1-v_2}$. So, the left hand side of the constraint (5) is the expected payoff of a v_2-type buyer in the first period when he accepts the offered price. The right hand side of the constraint (5) is the expected payoff to the buyer if he waits till the last period to buy the good. Notice that given our uniform assumption, v_2 is the probability that the opponent's valuation is less than v_2, i.e., the good has remained unsold after the first period. The expression $\frac{1-v_2^2}{1-v_2}$, as described, is the probability that the buyer gets the good in the last period, if he accepts the last period price offer, given that none of the buyers have accepted the first-period price. The v_1 and v_2 in the probabilities are the types of the opponent buyer who is indifferent between accepting and rejecting the price offers in the last and first period respectively. Since both the buyers are using symmetric strategies, so the left hand and right hand sides of (5) should be equal to each other.

We now consider the seller's problem in the last period. The seller maximizes:

$$\begin{aligned}
\max_{v_2} \pi_2(v_2) &= [(1-v_2^2)p_2 + v_2^2\pi_1(v_1)] \\
\text{s.t. } \frac{1-v_2^2}{1-v_2}(v_2-p_2) &= v_2\frac{1-(v_1/v_2)^2}{1-v_1/v_2}(v_2-v_1).
\end{aligned}$$

The seller chooses the optimal v_2 threshold to maximize his expected payoff. If any of the buyers accept the price (this happens with probability $(1-v_2^2)$), he gets p_2, otherwise the game proceeds to the last period, in which case he gets $\pi_1(v_1)$.

Using our results in (4), we can solve for the optimal prices explicitly:

$$p_2^* = 0.58 \text{ and } p_1^* = 0.479.$$

In particular, notice that the optimal prices are decreasing over time. The optimal prices together with (4) imply that the seller's optimal expected revenue is $\pi_2(v_2^*) = 0.4$.

Observe that the optimal price in the last period $p_1^* = v_1^* = v_2^*/\sqrt{3} > 0$, i.e., the optimal price in the last period is above the marginal cost. While such a result is similar to the standard result of a durable-goods monopolist's pricing strategy in a similar two-period model, the price path obtained in our setting is intrinsically different. Similar to the Coase Conjecture, which states that when a monopolist does not have any commitment power in setting prices in a dynamic framework, the prices chosen by the seller should gradually decrease over time (and towards the marginal cost, which is zero here, in an infinite horizon model),. the optimal price path $\{p_1^*, p_2^*\}$ is also driven by the fact that the good is limited relative to the demand, i.e., there is excess demand in the market and the buyers compete with each other to acquire the good: Intuitively, a buyer may wait for one extra period for the price to fall, but at the same time he fears that the good might be snatched by his opponent in the current period, in which case he gets nothing. This provides him an incentive to buy the good earlier. This inherent competition among the buyers drives the optimal price path to be different from that in a Coasian framework.

We now consider the case where the seller adopts a non-anonymous price posting mechanism. We denote the prices offered by the seller as p_t (buyer 1) and q_t (buyer 2) with $p_t > q_t$, and the critical valuation thresholds as u_t (buyer 1) and v_t (buyer 2) in period $t = 1, 2$. As described in Section 3.2, the incentive constraint for buyer 1 is:

$$(u_2 - p_2) = v_2(u_2 - u_1),$$

while the incentive constraint for buyer 2 is:

$$u_2(v_2 - q_2) = u_1(v_2 - v_1).$$

Next, we consider the seller's maximization problem in the last period. The objective function of the seller is:

$$\pi_2(v_2) = [(1-u_2)v_2p_2 + (1-u_2)(1-v_2)p_2 + u_2(1-v_2)q_2 + u_2v_2\pi_1(v_1)]$$

Thus the seller gets the lower price q_2 only in the event that buyer 1 rejects the offer while buyer 2 accepts his offer. Similarly, the seller gets the higher price p_2 when buyer 1 accepts the offer regardless of the decisions of buyer 2, and when nobody accepts in the first period, the price offering game moves on to the last period.

Hence the seller's optimization problem can be written as:

$$\begin{aligned} \max_{u_2,v_2} \pi_2(v_2) &= [(1-u_2)v_2p_2 + (1-u_2)(1-v_2)p_2 + u_2(1-v_2)q_2 + u_2v_2\pi_1(v_1)] \\ \text{s.t.} &: (u_2 - p_2) = v_2(u_2 - u_1) \text{ and } u_2(v_2 - q_2) = u_1(v_2 - v_1). \end{aligned}$$

Using a similar approach as that of the anonymous price posting mechanism, we find that the optimal prices chosen by the seller in the two periods are

$$p_2^* = 0.62, q_2^* = 0.56, \text{ and } p_1^* = 0.55, q_1^* = 0.402.$$

The optimal price paths lead to an optimal expected revenue of $\pi_2(v_2^*) = 0.404$. There are some interesting observations to be noted here. It is easy to see that the expected revenues are such that $\pi_2^A = 0.404 > \pi_2^N = 0.4$. In other words, the possibility of unequal treatment of equals strictly increases the payoff of the seller. The following table compares the performance of the optimal anonymous mechanism with the optimal non-anonymous mechanism of the two-period model more explicitly.

	Anonymous Mechanism	Non-Anonymous Mechanism
Price in period 2	0.58 (p_2)	0.62 (p_2^1), 0.56 (p_2^2)
Price in period 1	0.48 (p_1)	0.55 (p_1^1), 0.40 (p_1^2)
Expected payoff	0.4	0.404

Denote p_t as the optimal price in period t under the anonymous mechanism, $t \in \{1,2\}$ and denote p_t^i as the optimal price in period t for buyer $i \in \{1,2\}$, $t \in \{1,2\}$ under the non-anonymous mechanism. A first observation from the table is that the two prices in each period under the non-anonymous mechanism are a "spread" from the corresponding price under the anonymous mechanism, i.e., $p_t^1 > p_t > p_t^2$ for each t. Hence buyer 1 is charged with a price higher than the anonymous mechanism price while buyer 2 is charged a price lower than the anonymous mechanism price. One possible explanation is that we can view this as risk sharing motive for the seller: The seller charges

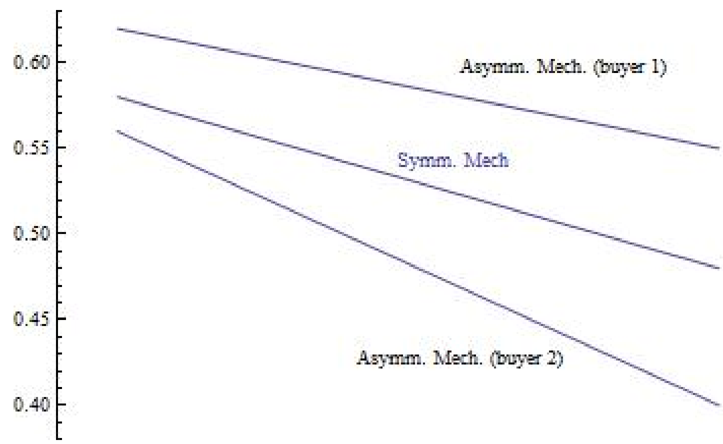

Figure 1: Single Price Path for Symmetric Mechanism and Two Price Paths for Asymmetric Mechanism for buyers 1 and 2 respectively

a higher price to buyer 1 to take a high risk, high return gamble, while at the same time, the seller charges a lower price to buyer 2 as a fallback option in case the high price gamble does not work out.

A second useful observation is that $|p_t^i - p_t|$ is decreasing in t for each $t = 1, 2$ and $i = 1, 2$. In other words, in the earlier period the spread of the prices is less than that in the final period. In the second period which is the final period to sell the good, the seller tends to diversify even more (i.e., reducing the "risk") so that it is more likely that at least one of the buyers accepts the good in the final period. To be more explicit, let's consider the price variations for the buyers in the two mechanisms in detail. It can be shown that the line of the price path for buyer 2 in the non-anonymous mechanism is steeper than that of buyer 1 in the non-anonymous mechanism, while the slope of the line of the price path for the anonymous mechanism lies in the mid-way. In addition, we can see that the price difference (between the two mechanisms) for buyer 1 is relatively higher in the first period than that of buyer 2, i.e., $|p_2^1 - p_2| > |p_2^2 - p_2|$, while in the final period the price difference for buyer 2 is higher, i.e. $|p_1^1 - p_1| < |p_1^2 - p_1|$.

A final important issue that still needs to be clarified is exactly what factor is driving the difference in the performance of the two mechanisms. If the distributions of valuations of the buyers were different, it would be intuitive that the seller should adopt a non-anonymous price posting mechanism if such a "horizontal price discrimination" is also allowed on top of the intertemporal price discrimination. This is because the asymmetry of the distributions would give the seller additional payoff-relevant information on the buyers which the seller would want to make use of. So it would had been natural for him to treat different buyers differently. It is less clear why treating the buyers differently can increase the seller's expected revenue when the buyers are ex-ante symmetric. In our dynamic framework, when the seller treats the buyers differently in the first period and the game moves on to the next period, the buyers will be ex ante different in the next period due to their different treatments in the first period although they started with symmetry. Hence the asymmetry between the buyers in the last period comes purely from their different treatments in the first period. The additional information in the second period generated by the asymmetric treatments can to some degree drive the difference between the two mechanisms.

We argue, however, that the ultimate driving force of the performance difference of the two mechanisms comes from the fact that different treatments of the buyers intensify the competition of the buyers. This can be more clearly seen from comparing the two mechanisms in the static version of the model. The following table shows the comparison results of the two competing mechanisms for the one-period model. The detailed result of the one-period model is omitted here as it is straightforward and the basic structure follows from the two-period model above.

	Anonymous Mechanism	Non-Anonymous Mechanism
Price	0.58	$0.5, 0.63$
Expected payoff	0.385	0.39

Hence, even without any dynamic considerations, a non-anonymous price posting mechanism performs better than an anonymous price posting mechanism. As discussed, posting different prices to different buyers intensifies the competition between the buyers, which in turn drives up the seller's expected revenue.

Thus the implication that we get from these exercises is that asymmetric equilibrium exists even in an *ex-ante* symmetric setting, and moreover the asymmetric mechanism is the optimal one under posted price domain. Another interesting implication that our static version of the example gives is that under the assumption of non-anonymity, in an otherwise *ex-ante* symmetric framework, the setting of the single static monopoly price is not the optimal price mechanism for the seller. It is quite common in the standard monopoly pricing literature that the monopolist price discriminates to extract the maximum producer surplus. The horizontal discrimination happens when the buyers come from different segments of population which have different demand structure and the monopolist has some information over the respective demands or the valuations, *i.e.* when there is an asymmetry in the distribution of valuations. Our example shows that even in a symmetric setting horizontal discrimination is the optimal one should the monopolist know about the identities of the buyers.

3.4 T period Characterization of Equilibrium

3.4.1 Non-anonymity

This subsection characterizes the equilibrium price path of the seller in a general T period case in the case for non-anonymous buyers, and then in the next subsection we compare the equilibrium with that in the case of anonymous buyers. We assume that the valuations of the buyers are drawn independently from an identical distribution $F[0, 1]$.

The characterization of the equilibrium comes from its recursive formulation. Suppose period t is reached without the good being sold in the earlier periods. We can recall that the seller's t^{th}

period continuation payoff is

$$\pi_t(u_{t+1}, v_{t+1}) = (1 - \frac{F(u_t)}{F(u_{t+1})})p_t + \frac{F(u_t)}{F(u_{t+1})}(1 - \frac{F(v_t)}{F(v_{t+1})})q_t + \frac{F(v_t)}{F(v_{t+1})}\frac{F(u_t)}{F(u_{t+1})}\pi_{t-1}(u_{t-1}, v_{t-1}).$$

Suppose that for any time period t, and for every set of valuations (u_t, v_t), there is an unique and interior equilibrium for the continuation game with $t-1$ periods remaining and the buyers' valuations being drawn from $[0, u_t]$ and $[0, v_t]$. The threshold valuation buyers in each period of the continuation game (u_{t-k}, v_{t-k}) for any $k \in [1, t-1]$, are indifferent between accepting the current prices and waiting for the next period, rendering the interior solution of the game. In each period t, the seller maximizes $\pi_t(u_{t+1}, v_{t+1})$ given his continuation payoff. The buyers' incentive constraints fix (u_t, v_t) in period t, $\pi_{t-1}(u_{t-1}, v_{t-1})$, (u_{t-1}, v_{t-1}), and (p_{t-1}, q_{t-1}) are fixed by the continuation payoff, and the seller then maximizes his current payoff by choosing (p_t, q_t). The entire model can then be solved recursively by backward induction.

In the final period, *i.e.* with 1 period to go, the problem is a static problem and the optimal prices for the monopolist are the two static monopoly prices instead of a single monopoly price as he discriminates among the non-anonymous buyers taking u_2 and v_2 as given. Then, given the payoff in the last period, we can backwardly solve for the prices in all the previous periods, and thus the entire price paths of the monopolist can be traced. There will be two prices in each period, one higher than the other, thus generating two price paths over the period.

Another interesting feature of the problem that needs to be discussed is how it differs from a standard optimal auction design in the case of posted prices. It is well-known that under the case of posted prices, the optimal mechanism for the seller is a Dutch auction with a positive reserve price when the virtual valuations are increasing. (For the detailed discussion, see Myerson '81). But this would not be an optimal mechanism under the present scenario. A Dutch auction results in a fine discrimination among the buyers' valuation types while the positive reserve price excludes the lower valuation buyers from being allocated. In a discriminatory mechanism, it would extend to two parallel Dutch auctions along with two optimal reserve prices. But in our case, we show that this will not be the case as the positive terminal prices would not allow a fine discrimination of the buyer types as there will always be two non-negligible buyer valuation ranges (for two buyers respectively) for which the same prices would be charged in each period.

To illustrate the idea, we consider the last period where the monopolist sets the discriminatory static prices to determine the final period threshold valuations. These threshold valuations are non-zero if they are lower than their respective upper bound of their posterior distributions, which are nothing but the threshold valuations of the previous period. Thus, for each buyer, there is a non-negligible gap between the two threshold valuations. The range of valuations within this gap was charged the same price in the previous period. With an induction logic we can claim that in every period under positive terminal prices, there would be two respective ranges of buyers' types who would be charged the same prices. This is stated formally below.

Proposition 1: *Suppose that the distribution function F has no atoms. If* $\lim_{\Delta \to 0} u_{\Delta 1} > 0$ *and* $\lim_{\Delta \to 0} v_{\Delta 1} > 0$, *then for all* k,

$$\lim_{\Delta \to 0} u_{\Delta k+1} > \lim_{\Delta \to 0} u_{\Delta k}$$

$$\lim_{\Delta\to 0} v_{\Delta k+1} > \lim_{\Delta\to 0} v_{\Delta k}$$

where $u_{\Delta k}$ and $v_{\Delta k}$ are the threshold buyers' valuation types who are indifferent between accepting and rejecting the period-k price.

Proof: Suppose that $u_{\Delta 1} > 0$ and $v_{\Delta 1} > 0$. Thus given $u_{\Delta 2}$ and $v_{\Delta 2}$,

$$(u_{\Delta 1}, v_{\Delta 1}) = \arg\max (1 - F_2(u))u + F_2(u)(1 - F_2(v))v,$$

where $F_2(u)$ and $F_2(v)$ are the posterior distributions such that $(u_{\Delta 1}, v_{\Delta 1})$ is contained in $(u_{\Delta 2}, v_{\Delta 2})$. Now if $F_2(u)$ and $F_2(v)$ have strictly positive density, then

$$u_{\Delta 2} > u_{\Delta 1}$$

$$v_{\Delta 2} > v_{\Delta 1}.$$

Since $\lim_{\Delta\to 0} u_{\Delta 1} > 0$ and $\lim_{\Delta\to 0} v_{\Delta 1} > 0$, thus we can get $\lim_{\Delta\to 0} u_{\Delta 2} > \lim_{\Delta\to 0} u_{\Delta 1}$ and $\lim_{\Delta\to 0} v_{\Delta 2} > \lim_{\Delta\to 0} v_{\Delta 1}$. Again, by an argument of induction we can establish this inequality for any earlier period k.

Thus a fine discrimination of buyer types by running a Dutch auction as well as setting positive terminal prices is not possible for the seller.

Uniform Distribution : We now assume the distribution of buyers' valuation to be uniformly distributed on $[0, 1]$. The specification of uniform distribution helps to find the unique solution to the problem and would allow us to find an explicit characterization of the equilibrium. We can pin down the unique equilibrium from the buyers' indifference conditions. Buyer 2's indifference condition gives

$$\begin{aligned}
v_t - q_t &= \frac{u_{t-1}}{u_t}(v_t - q_{t-1}) \\
&= \beta_{t-1}(v_t - q_{t-1}) \\
&= \beta_{t-1}(v_t - v_{t-1}) + \beta_{t-1}(v_{t-1} - q_{t-1}) \\
&= \beta_{t-1}(1 - \gamma_t)v_t + \beta_t(v_{t-1} - q_{t-1}), \text{ where } \beta_t = \frac{u_t}{u_{t+1}} \text{ and } \gamma_t = \frac{v_t}{v_{t+1}}.
\end{aligned}$$

Proceeding recursively, we can write the indifference condition as

$$v_t - q_t = \sum_{\tau=1}^{t-1} (1 - \gamma_\tau)(\Pi_{l=\tau}^{t-1} \beta_l) v_{\tau+1}.$$

Again, writing $v_t - q_t$ as $v_t(1 - \frac{q_t}{v_{t+1}} \frac{1}{\gamma_t})$, we can rewrite the above equation as

$$(1 - \frac{q_t}{v_{t+1}} \frac{1}{\gamma_t}) = \beta_{t-1} + \sum_{\tau=1}^{t} \beta_{t-\tau} \Pi_{\tau=1}^{t} \gamma_{t-\tau}^2 - \sum_{\tau=2}^{t} \beta_{t-\tau}\gamma_{t-\tau} \Pi_{\tau=2}^{t} \gamma_{t-\tau+1}^2 - \beta_{t-1}\gamma_{t-1}.$$

The left hand side of the equation is monotonic in γ_t while the right hand side is independent of

γ_t. Thus the equation can pin down γ_t as a function of $\frac{q_t}{v_{t+1}}$. Thus in the continuation game with t periods to go, given the price offered by the monopolist, there can be only one threshold type of Buyer 2 who is indifferent between accepting the price and waiting for the next period.

Similarly we can write down buyer 1's indifference condition and substitute recursively as

$$\begin{aligned} u_t - p_t &= \frac{v_t}{v_{t+1}}(u_t - p_{t-1}) \\ &= \gamma_t \sum_{\tau=1}^{t-1}(1-\beta_\tau)(\Pi_{l=\tau}^{t-1}\gamma_l)u_{\tau+1}. \end{aligned}$$

Similarly writing $u_t - p_t$ as $u_t(1 - \frac{p_t}{u_{t+1}}\frac{1}{\beta_t})$, the above equation can be rewritten as

$$\frac{1}{\gamma_t}(1 - \frac{p_t}{u_{t+1}}\frac{1}{\beta_t}) = \sum_{\tau=1}^{t}\Pi_{\tau=1}^{t}\gamma_{t-\tau}\beta_{t-\tau} - \sum_{\tau=1}^{t}\Pi_{\tau=1}^{t}\beta_{t-\tau}\gamma_{t-\tau+1}\gamma_{t-\tau} - \beta_{t-1}.$$

The left side of the equation is a function of β_t and γ_t, and since γ_t is pinned down from buyer 2's indifferent condition, thus the left side becomes monotonic in only β_t, while the right hand side is independent of it, thus pinning down β_t.Thus we can claim that the equilibrium of the monopolist's problem is unique.

The monopolist's problem is then to maximize his expected payoff

$$\pi_t(u_{t+1}, v_{t+1}) = (1 - \frac{u_t}{u_{t+1}})p_t + \frac{u_t}{u_{t+1}}(1 - \frac{v_t}{v_{t+1}})q_t + \frac{u_t}{u_{t+1}}\frac{v_t}{v_{t+1}}\pi_{t-1}(u_t, v_t).$$

This along with the indifference conditions of the buyers gives $\pi_t(u_{t+1}, v_{t+1})$ as a linear function of u_{t+1} and v_{t+1}. This again suggests that the solution is unique. This can be stated formally in the following lemma:

Lemma 2: *In the continuation game with t periods remaining, the prices for the two buyers p_t and q_t, and the monopolist's payoff function are linear functions of u_{t+1} and v_{t+1} for every t.*

Thus we can see that in this equilibrium the prices that the monopolist sets at any period t and the t^{th} period revenue of the monopolist are linear functions of u_{t+1} and v_{t+1}. From the buyers' problem we can ensure that the solution to this problem is unique in the sense that in each period we get two unique threshold valuations for the two buyers respectively, and thus the two prices that the monopolist sets for the two buyers respectively in each period are unique. A detailed proof of it is shown in the Appendix. The idea is to start from the last period. In the last period it is straightforward to show that the solution is unique. Then we apply the logic of induction on the number of periods and show that this is the case for any general t^{th} period. In any period the solution is unique given the continuation game.

The seller's problem on the other hand shows that the solution is indeed interior. The first order conditions from the seller's maximization problem characterize the price path of the monopolist in any general t^{th} period, and the second order condition shows that the solution is interior. The interior solution implies that in each period there exist some buyer valuations that do accept the prices in that period. The following set of first order conditions define the price paths of the monopolist and show the very existence of asymmetric equilibria in our otherwise symmetric setting. The second order condition along with Proposition 1 would show that the solution is interior, while the buyers' problem pins down the solution to be unique. The corresponding t^{th} period first order conditions that define the price paths are

$$\beta_t : -2(1-\beta_t)\beta_t[\Sigma_{\tau=1}^{t-1}(1-\beta_\tau)\Pi_{l=\tau+1}^{t-1}\beta_l^2]u_{t+1} - \gamma_t(1-\gamma_t)\Sigma_{\tau=1}^{t-1}(1-\gamma_\tau)\Pi_{l=\tau+1}^{t-1}\gamma_l^2\gamma_\tau v_{t+1}$$

$$+ [\Sigma_{\tau=1}^{t-1}(1-\beta_\tau)\Pi_{l=\tau+1}^{t-1}\beta_l^2]u_{t+1} + (1-2\beta_t)u_{t+1} + (1-\gamma_\tau)\gamma_t v_{t+1} + \gamma_t\pi_{t-1} = 0$$

and

$$\gamma_t : -\beta_t(1-2\gamma_t)[\Sigma_{\tau=1}^{t-1}(1-\gamma_\tau)\Pi_{l=\tau+1}^{t-1}\gamma_l^2\gamma_\tau v_{t+1}] + \beta_t(1-2\gamma_t)v_{t+1} + \beta_t\pi_{t-1} = 0$$

The monopolist thus sets prices in each period according to the threshold cut-off rules such that the corresponding cut-off types are indifferent between accepting the price and waiting for the next period. The buyers on the other hand follow the strategy in any period to accept the price if their valuations(or types) are strictly greater than the respective cutoff valuations in that period, otherwise they wait for the next period. This gives the unique perfect Bayesian equilibrium of the continuation game, which is stated in the following proposition.

Proposition 2: *When the buyers are non-anonymous, at any period t, if the monopolist's posterior beliefs are $[0, u_{t+1}]$ and $[0, v_{t+1}]$, then in the unique perfect Bayesian equilibrium, the t^{th} period prices are given by*

$$p_t = u_t - \gamma_t \sum_{\tau=1}^{t-1}(1-\beta_\tau)(\Pi_{l=\tau}^{t-1}\gamma_l)u_{\tau+1}$$

and

$$q_t = v_t - \sum_{\tau=1}^{t-1}(1-\gamma_\tau)(\Pi_{l=\tau}^{t-1}\beta_l)v_{\tau+1},$$

and given prices $\widetilde{p_t}$ and $\widetilde{q_t}$, buyers 1 and 2 with their respective valuations $u > u_t(\widetilde{p_t}, u_{t+1})$ and $v > v_t(\widetilde{p_t}, v_{t+1})$, the threshold types at time period t, accept their prices, and buyers 1 and 2 with their respective valuations $u < u_t(\widetilde{p_t}, u_{t+1})$ and $v < v_t(\widetilde{p_t}, v_{t+1})$ reject the prices, where $u_t(\widetilde{p_t}, u_{t+1})$ and $v_t(\widetilde{p_t}, v_{t+1})$ are given by

$$(1 - \frac{\widetilde{q_t}}{v}) = \beta_{t-1} + \sum_{\tau=1}^{t}\beta_{t-\tau}\Pi_{\tau=1}^{t}\gamma_{t-\tau}^2 - \sum_{\tau=2}^{t}\beta_{t-\tau}\gamma_{t-\tau}\Pi_{\tau=2}^{t}\gamma_{t-\tau+1}^2 - \beta_{t-1}\gamma_{t-1}.$$

and

$$\frac{1}{\gamma_t}(1 - \frac{\widetilde{p_t}}{u}) = \sum_{\tau=1}^{t}\Pi_{\tau=1}^{t}\gamma_{t-\tau}\beta_{t-\tau} - \sum_{\tau=1}^{t}\Pi_{\tau=1}^{t}\beta_{t-\tau}\gamma_{t-\tau+1}\gamma_{t-\tau} - \beta_{t-1}.$$

Proof: See the Appendix.

3.4.2 Anonymity

This subsection deals with the benchmark case of anonymity of the buyers to the monopolist. The monopolist cannot distinguish among the buyers so he treats the buyers symmetrically. In each period he posts a single price. If one of the buyers accept the price he gives the good to that buyer. If none of them accepts, the game moves on to the next period. In the event that ore than one buyer accept the good in a given period he randomly allocates the good to all the accepting buyers. If there are 2 buyers and the buyers' valuations are drawn from the distribution $F(.)$, the monopolist's t^{th} period maximization problem is:

$$Max_{p_t}\pi_t(v_{t+1}) = Max_{p_t}[(1-(\frac{F(v_t)}{F(v_{t+1})})^2)p_t + (\frac{F(v_t)}{F(v_{t+1})})^2\pi_{t-1}(v_{t-1})],$$

where v_t is the threshold valuation of the buyers in period t and p_t is the price in period t. Similar to the previous case in each period t the seller maximizes $\pi_t(v_{t+1})$ given his continuation payoff. The buyers' incentive constraints fix v_t in period t, $\pi_{t-1}(v_{t-1})$, v_{t-1}, and p_{t-1}, are fixed by the continuation payoff, and the seller then maximizes his current payoff by choosing p_t. The entire model can again be solved recursively by backward induction where the last period price is the static monopoly price. Thus the entire price path can be traced.

We can directly switch to the assumption of uniform distribution of the buyers' valuations and uniquely pin down the equilibrium solution to the benchmark problem. The buyers' indifference condition is given by the following equation

$$\frac{1-\gamma_t^2}{1-\gamma_t}(v_t - p_t) = \gamma_t\frac{1-\gamma_{t-1}^2}{1-\gamma_{t-1}}(v_t - p_{t-1})$$

By recursive substitution, the equation can be rewritten as

$$\frac{1-\gamma_t^2}{1-\gamma_t}(1-\frac{p_t}{v_t}) = \gamma_t(1-\Pi_{\tau=1}^{t-1}\gamma_\tau^2)$$

Again, writing $\frac{p_t}{v_t}$ as $\frac{p_t}{v_{t+1}}\frac{1}{\gamma_t}$, we can rewrite the above equation as

$$\frac{1-\gamma_t^2}{1-\gamma_t}(1-\frac{p_t}{v_{t+1}}\frac{1}{\gamma_t}) = \gamma_t(1-\Pi_{\tau=1}^{t-1}\gamma_\tau^2)$$

Dividing both sides by γ_t the left hand side is monotonic in γ_t while the right hand side is independent of γ_t.Thus the equation pins down γ_t as a function of $\frac{p_t}{v_{t+1}}$ and the solution is unique. The monopolist's problem is then to maximize his expected payoff subject to the buyers' indifference condition.

$$Max_{p_t}\pi_t(v_{t+1}) = Max_{p_t}[(1-(\frac{v_t}{v_{t+1}})^2)p_t + (\frac{v_t}{v_{t+1}})^2\pi_{t-1}(v_{t-1})].$$

This along with the indifference condition on the buyers again gives $\pi_t(v_{t+1})$ as a linear function of v_{t+1} which suggests that the solution is unique. This is stated formally in the following lemma which is the corresponding lemma to Lemma 2.

Lemma 3: *In the continuation game with t periods remaining, the price for the two buyers p_t*

and the monopolist's payoff function are linear functions of v_{t+1} *for every* t.

The seller's problem shows that the solution is interior, i.e. there exists some buyer valuations in each period who accept the good.The following proposition formally defines the perfect Bayesian equilibrium in the anonymous case. The difference with the non-anonymous buyers case is that there is only one price in each period. Not only are the value distributions of the two buyers same but also they are not observationally different and the seller treats the buyers symmetrically. As we have noted earlier this single price mechanism is not optimal for the seller when the buyers are non-anonymous. The mechanism of treating differently beats the random allocation mechanism. We define the equilibrium formally in the Proposition below.

Proposition 3: *When the buyers are anonymous, at any period t, if the monopolist's posterior belief is* $[0, v_{t+1}]$*, then in the unique perfect Bayesian equilibrium, the* t^{th} *period price is given by*

$$p_t = 1 - \frac{1-\gamma_t}{1-{\gamma_t}^2}\gamma_t(1 - \sum_{\tau=1}^{t} \Pi_{\tau=1}^{t}\gamma_{t-\tau}^2)$$

and given price $\widetilde{p_t}$*, buyers 1 and 2 with their valuations* $v > v_t(\widetilde{p_t}, v_{t+1})$*, the threshold type at time period t, accept their prices, and buyers 1 and 2 with their valuation* $v < v_t(\widetilde{p_t}, v_{t+1})$ *reject the price, where* $v_t(\widetilde{p_t}, v_{t+1})$ *is given by*

$$1 - \frac{\widetilde{p_t}}{v} = \frac{1-\gamma_t}{1-{\gamma_t}^2}\gamma_t(1 - \sum_{\tau=1}^{t-1} \Pi_{l=\tau}^{t-1}\gamma_l^n)$$

3.5 Continuous Time Approach

To start with we let the seller to post prices at infinitesimally small time intervals, Δt. Then we shall examine the equilibria in the continuous case when $\Delta t \to 0$. The length of the finite horizon that the seller and the buyers face is $T = \frac{1}{\Delta t}$. Thus, in a similar way as we have assumed previously, $t = 0, ...t - \Delta t, t, t + \Delta t, ...T$ denote the number of periods remaining to complete the game. At each period the seller can post two prices p_t and q_t with $p_t > q_t$, to the two buyers at each period t. We let the buyers to draw their valuations from a strictly increasing and continuously differentiable distribution F from $[0, 1] \to [0, 1]$.

We start with buyer 2 first. For buyer 2 with valuation v, if he accepts his price in at time period t, the payoff he receives is

$$Y_2(t) = \frac{F(u(t))}{F(u(t+\Delta t))}(v - q(t)).$$

Buyer 2 gets a payoff $v - q(t)$ in the event that his opponent buyer 1's valuation is less than his threshold valuation $u(t)$ at period t , given that his valuation was less than the previous threshold $u(t + \Delta t)$, Δt period before. Again, if he rejects the current offer and waits for Δt period more, his payoff is

$$Y_2(t - \Delta t) = \frac{F(u(t-\Delta t))}{F(u(t+\Delta t))}(v - q(t - \Delta t)).$$

Thus the change in expected utility for buyer 2 for not accepting the offer at time t and waiting till $t - \Delta t$ is

$$\Delta Y_2(t) = \frac{F(u(t - \Delta t))}{F(u(t + \Delta t))}(v - q(t - \Delta t)) - .\frac{F(u(t))}{F(u(t + \Delta t))}(v - q(t))$$

We assume differentiability of the objective function. Thus we can write the incentive to wait of buyer 2 at each instant t when the time period is continuous.

$$\begin{aligned} Y_2'(t) &= Lim_{\Delta t \to 0} \frac{\Delta Y_2(t)}{\Delta t} \\ &= \frac{q'(t)F(u(t)) + q(t)F'(u(t))u'(t) - vF'(u(t))u'(t)}{F(u(t))} \end{aligned}$$

If we assume that the game starts at $t = T$ and ends at $t = 0$, and if the optimal time for buyer 2 to stop the game and accept the price offered at that instant is t_0, then t_0 will satisfy the first order condition $Y_2'(t_0) = 0$, provided $t_0 \in (0, T)$ and $Y_2''(t_0) < 0$. Thus the problem for buyer 2 is an optimal stopping problem, where his strategy in the continuous time boils down to choosing an optimal time to stop the game and accept the price offered at that instant.

Similarly for buyer 1 with valuation u, again assuming differentiability of the objective function, we can find the incentive to wait at instant t.

$$Y_1'(t) = \frac{p'(t)F(v(t)) + p(t)F'(v(t))v'(t) - uF'(v(t))v'(t)}{F(v(t))}.$$

Thus in the given optimal stopping problems for the buyers, their strategies are choosing their respective optimal t's where their payoff become maximum. Since we can claim that **Lemma 1** also applies in the continuous time case, (*i.e.* the regular dynamic single-crossing condition holds), the threshold valuations $u(t)$ and $v(t)$ are increasing functions of t. Thus, although the choice variables for the buyers are the optimal instants of time, we can work directly with the threshold valuations $u(t)$ and $v(t)$ instead of working with their inverse functions. As they are both increasing functions of t, optimal $u(t)$ and $v(t)$ can constitute the equilibria of the game.

The following proposition shows that a stream of $u(t)$ and $v(t)$ satisfying a system of ordinary differential equations and associated boundary conditions can constitute the equilibria of the game. Thus we can see the existence asymmetric equilibria although this asymmetry does not arise from any difference in *ex-ante* probability distributions.

Proposition 4: *If* $(u(t), v(t))$ *is a solution to the following system of differential equations*

$$\begin{cases} (a)\ uF'(v(t))v'(t) = p'(t)F(v(t)) + p(t)F'(v(t))v'(t) \\ (b)\ vF'(u(t))u'(t) = q'(t)F(u(t)) + q(t)F'(u(t))u'(t) \end{cases}$$

such that

$$\min\{u(0), v(0)\} = 0$$

then $(u(t), v(t))$ *is a pair of equilibrium threshold valuations.*

Proof : Suppose $(u(t), v(t))$ satisfies all the hypotheses of the Proposition. From the differential equations it follows that

$$u'(t) = \frac{q'(t)F(u(t)) + q(t)F'(u(t))u'(t)}{vF'(u(t))} \quad > 0, \text{ for all } t > 0,$$

$i..e$, for $t > 0$ $u(t)$ is strictly increasing and differentiable. The same is true for $v'(t)$, $i.e.$ $v(t)$ is also strictly increasing and differentiable. Thus the distribution functions of the threshold valuations of buyer 1 and buyer 2 can be written as $F(u(t))$ and $F(v(t))$ respectively.

If buyer 1 accepts the good at period t, his expected payoff is

$$Y_1(t) = (v - q(t))F(u(t)) \quad > 0, \text{ for all } t > 0,$$

Differentiating, we get buyer 1's expected gain from waiting infinitesimally more time

$$\frac{\partial Y_1(t)}{\partial t} = vF'(u(t))u'(t) - q'(t)F(u(t)) - q(t)F'(u(t))u'(t)$$

Substituting $u'(t)$ from Proposition 2, we get

$$\frac{\partial Y_1(t)}{\partial t} = \frac{v - v(t)}{v(t)}(q'(t)F(u(t)) + q(t)F'(u(t))u'(t) \ )$$

As $u(t)$ is strictly increasing, the expression $q'(t)F(u(t)) + q(t)F'(u(t))u'(t) > 0$ for ant $t > 0$. Thus

$$(\frac{v - v(t)}{v(t)})\frac{\partial Y_1(t)}{\partial t} \geq 0$$

The inequality is strict for all the cases when $v(t) \neq v$. Thus buyer 1's optimal decision involves choosing t such that $v(t) = v$. A similar argument can show that $u(t) = u$ defines the optimal behavior for buyer 2.

For complete characterization, it remains to consider the end points of the threshold valuation functions. Since both $u(t)$ and $v(t)$ are strictly increasing functions in t, either of the following two conditions should hold :

1) Either $u(t^*) = 1$ or $v(t^{**}) = 1$ or both for any $t^*, t^{**} \in (0, \infty]$,

or

2) Both $u(t^*)$ and $v(t^{**})$ are bounded away from 1 for all t.

We claim that condition 2 cannot hold. To show this we consider any interval $[a, b]$ from $(0, \infty)$. Since $u(t)$ is increasing, it follows from the differential equations that

$$\frac{u(b)F'(v(t))v'(t)}{p'(t)F(v(t)) + p(t)F'(v(t))v'(t)} > 1 > \frac{u(a)F'(v(t))v'(t)}{p'(t)F(v(t)) + p(t)F'(v(t))v'(t)}.$$

Let us denote $\int_a^b \frac{F'(v(t))v'(t)}{p'(t)F(v(t))+p(t)F'(v(t))v'(t)}dt = A(a,b)$ with the property that $\frac{\partial}{\partial b}A(a,b) = \frac{F'(v(b))v'(b)}{p'(b)F(v(b))+p(b)F'(v(b))v'(b)} > 0$. Then integrating the above inequality we can obtain

$$u(b)A(a,b) > b - a > u(a)A(a,b).$$

If Condition 2 holds, then the left hand side if the expression is bounded from above. But this is not possible since b can be made large enough to contradict the first inequality. We can thus claim that $u(t^*) = 1$.

If we consider the second inequality, as $b \to t^*$, the right hand side of the expression increases without bound. It follows from the second inequality that $t^* = \infty$.

4 Conclusion

The main contribution of our paper is the introduction of non-anonymity of buyer valuations in the revenue management literature and the generation of an asymmetric equilibrium in an otherwise ex-ante symmetric environment. Although there has been an earlier work in the evolutionary biology literature where asymmetric equilibria arise from a symmetric environment in a war of attrition type of game (Nalebuff and Riley (1985)), but there is no general result as such in the mechanism design and revenue management literature at present. In this respect this paper is the first one to give a general characterization of the pricing behavior of a monopolist, and introduces the concept of non-anonymity for the first time into the revenue management literature.

This paper shows that in an ex-ante symmetric environment if the seller uses the asymmetric price mechanism the expected revenue is higher than that with the symmetric mechanism in Hörner and Samuelson (2011). The future extension of this work would be a t-period general characterization of the price path of the monopolist from the two period model in the present paper.

Reference

Alderighi, Marco, A. Cento and C. Piga (2011), "A case study of pricing strategies in European airline markets: The London – Amsterdam route." *Journal of Air Transport Management*, 17(6), 369-373.

Ausubel, Lawrence M. and Raymond J. Deneckere (1989), "Reputation in bargaining and durable goods monopoly." *Econometrica*, 57(3), 511-531,

Board, S. and A. Skrypacz (2014), "Revenue Management for Forward Looking Buyers". *Journal of Political Economy.*

Gallien, J. (2006), "Dynamic Mechanism Design for Online Commerce". *Operations Research*, 54(2), 291-310.

Gershkov, A. and B. Moldvanu (2010), "Efficient Seuential Assignment with Incomplete Information". *Games and Economic Behavior*, 68(1), 144-154.

Gul, Faruk , Hugo Sonnenschein, and Robert Wilson. "Foundations of dynamic monopoly and the Coase conjecture." *Journal of Economic Theory*, 39(1):155-190, 1986

Horner, Johannes, and Larry Samuelson (2011): "Managing Strategic Buyers," *Journal of Political Economy*, 119, 379-425.

Jain, Sanjay (2012) "Marketing of Vice Goods: A Strategic Analysis of the Package Size Decision." *Marketing Science* 31, (1), , 36–51

Myerson, R. (1981). "Optimal Auction Design", *Mathematics of Operations Research*, 6(1), 58-73.

Nalebuff, B. and J. Riley (1985). "Asymmetric Equilibria in the War of Attrition". *Journal of Theoretical Biology*, 113, 517-527

O'Donaghue, T. and M. Rabin (1999). "Doing it Now or Later", *American Economic Review*, 89 (1), 103-125.

O'Donaghue, T. and M. Rabin (1999). "Incentives for Procrastinators", *Quarterly Journal of Economics*, 114 (3), 769-816.

O'Donaghue, T. and M. Rabin (2001). "Choice and Procrastination", *Quarterly Journal of Economics*, 116 (1), 121-160.

Pai, M. amd R. Vohra (2013). "Optimal Dynamic Auctions and Simple Index Rules", *Mathematics of Operations Research*, 38(4), 682-697

Talluri, Kalyan T. and Garrett J. van Ryzin. "The Theory and Practice of Revenue Management." *Springer*, New York, 2005.

Thaler, Richard H. (1981), "Some Empirical Evidence on Dynamic Inconsistency," *Economic Letters* 8, 201–207.

Appendix

Proof of Proposition 2:

We have assumed without loss of generality that the two buyers face t^{th} period prices p_t and q_t respectively with $p_t \geq q_t$. We start from the last period, *i.e.* $t = 1$. In the last period, the buyers accept the price if and only if their valuations are at least the prices they face in that period *i.e.* $u_1 \geq p_1$ and $v_1 \geq q_1$ respectively for buyers 1 and 2. The seller updates his posterior belief that the buyers' valuations are drawn from uniform distributions in the range $[0, u_2]$ and $[0, v_2]$ respectively.

The seller sets $u_1 = p_1$ and $v_1 = q_1$. The objective function of the seller is:

$$\begin{aligned} & (1 - \frac{u_1}{u_2})u_1 + \frac{u_1}{u_2}(1 - \frac{v_1}{v_2})v_1 \\ = \; & (1 - \beta_1)\beta_1 u_2 + \beta_1(1 - \gamma_1)\gamma_1 v_2 \end{aligned}$$

where $\beta_1 = \frac{u_1}{u_2}$ and $\gamma_1 = \frac{v_1}{v_2}$.

From the first order conditions we get:

$$\begin{aligned} \beta_1 &: \; (1 - 2\beta_1)u_2 + ((1 - \gamma_1)\gamma_1)v_2 = 0 \\ \gamma_1 &: \; (1 - 2\gamma_1)\beta_1 v_2 = 0 \end{aligned}$$

Solving the first order conditions,

$$\begin{aligned} u_1 &= \frac{4u_2 + v_2}{8} \\ v_1 &= \frac{v_2}{2} \end{aligned}$$

As we can see, in the last period, u_1 and v_1 can be expressed as linear functions of u_2 and v_2.

The value of the problem is

$$\pi_1 = \mu_1 u_2 + \upsilon_1 v_2$$

where $\mu_1 = (1 - \beta_1)\beta_1$ and $\upsilon_1 = \frac{\beta_1}{4}$. Thus in the last period the solution is linear in u_2 and v_2.

Now we use the logic of induction on the number of time periods to show that the solution is unique for any general t^{th} period problem. Let us first fix t and assume that for all periods upto $t - 1$, the solution is unique and is characterized by μ_{t-1}, β_{t-1} and γ_{t-1}. Now let us consider the t^{th} period problem where the posterior beliefs are that the valuations of the two buyers are drawn from uniform distributions in $[0, u_{t+1}]$ and $[0, v_{t+1}]$ respectively.

The indifference conditions of the two buyers in the t^{th} period are:

$$u_t - p_t = \frac{v_t}{v_{t+1}}(u_t - p_{t-1})$$

and

$$v_t - q_t = \frac{u_{t-1}}{u_t}(v_t - q_{t-1}).$$

Writing $\frac{v_t}{v_{t+1}} = \gamma_t$ and $\frac{u_t}{u_{t+1}} = \beta_t$, we can write for buyer 1,

$$\begin{aligned} u_t - p_t &= \gamma_t(u_t - p_{t-1}) \\ &= \gamma_t(u_t - u_{t-1}) + \gamma_t(u_{t-1} - p_{t-1}) \\ &= \gamma_t(1-\beta_{t-1})u_t + \gamma_t(\gamma_{t-1}(1-\beta_{t-2})u_{t-1} + \gamma_{t-1}(u_{t-2} - p_{t-2})) \\ &= \sum_{\tau=1}^{t-1}(1-\beta_\tau)(\Pi_{l=\tau+1}^{t}\gamma_l)v_{\tau+1}. \end{aligned}$$

Similarly, for buyer 2,

$$\begin{aligned} v_t - q_t &= \gamma_{t-1}(v_t - q_{t-1}) \\ &= \gamma_{t-1}(v_t - v_{t-1}) + \gamma_{t-1}(v_{t-1} - q_{t-1}) \\ &= \gamma_{t-1}(1-\gamma_{t-1})v_t + \gamma_{t-1}(\gamma_{t-2}(1-\gamma_{t-2})v_{t-1} + \gamma_{t-2}(v_{t-2} - q_{t-2})) \\ &= \sum_{\tau=1}^{t-1}(1-\gamma_\tau)(\Pi_{l=\tau}^{t-1}\gamma_l)v_{\tau+1}. \end{aligned}$$

Now again let us consider buyer 1. Buyer 1's indifference condition can also be written as:

$$(1 - \frac{p_t}{u_{t+1}}\frac{1}{\beta_t}) = (\sum_{\tau=1}^{t-1}\Pi_{l=t-1}^{t-\tau+1}\gamma_l^2(1-\beta_{t-\tau})$$

We can write a similar expression for buyer 2 as well. Thus we have characterized the buyers' behavior completely and uniquely. Given the sequences $\{\beta_t\}_{t=t-1}^{T}$ and $\{\gamma_t\}_{t=t-1}^{T}$ in each period t, we can pin down β_t and γ_t uniqely as functions of $\frac{p_t}{u_{t+1}}$ and $\frac{p_t}{v_{t+1}}$.

In the above equation, the left hand side is monotonic in β_t while the right hand side is independent of it. Thus β_t can be pinned down uniquely given u_{t+1} and the values in the continuation game.

Next we come to the seller's problem. The seller's expected payoff is:

$$\pi_t(u_{t+1}, v_{t+1}) = (1 - \frac{u_t}{u_{t+1}})p_t + \frac{u_t}{u_{t+1}}(1 - \frac{v_t}{v_{t+1}})q_t + \frac{u_t}{u_{t+1}}\frac{v_t}{v_{t+1}}\pi_{t-1}(u_t, v_t).$$

The seller maximizes the objective function subject to the indifference conditions of the buyers.

The first order conditions from the seller's maximization problem characterize the price path of the monopolist in the t^{th} period.

$$\beta_t : -2(1-\beta_t)\beta_t[\Sigma_{\tau=1}^{t-1}(1-\beta_\tau)\Pi_{l=\tau+1}^{t-1}\beta_l^2]u_{t+1} - \gamma_t(1-\gamma_t)\Sigma_{\tau=1}^{t-1}(1-\gamma_\tau)\Pi_{l=\tau+1}^{t-1}\gamma_l^2\gamma_\tau v_{t+1}$$

$$+ [\Sigma_{\tau=1}^{t-1}(1-\beta_\tau)\Pi_{l=\tau+1}^{t-1}\beta_l^2]u_{t+1} + (1-2\beta_t)u_{t+1} + (1-\gamma_\tau)\gamma_t v_{t+1} + \gamma_t\pi_{t-1} = 0$$

and

$$\gamma_t : -\beta_t(1-2\gamma_t)[\Sigma_{\tau=1}^{t-1}(1-\gamma_\tau)\Pi_{l=\tau+1}^{t-1}\gamma_l^2\gamma_\tau v_{t+1}] + \beta_t(1-2\gamma_t)v_{t+1} + \beta_t\pi_{t-1} = 0$$

From the second order condition it can be shown that the solution is also interior. Thus the solution to the t^{th} period problem is unique and interior given the continuation game.