STRATEGY-PROOF AND UNANIMOUS RANDOM RULES ON MIXED SINGLE PEAKED DOMAIN *

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Abstract

We generalize the notion of single-peakedness by allowing existence of preferences that are not single-peaked. We call such domains mixed single-peaked domains. Such domains occur when there is no complete prior ordering, or there are multiple prior orderings over alternatives that agents use to derive their preferences. Examples of such domains include multiple single-peaked domain, partially single-peaked domain etc. The importance of such domains in modeling preferences of agents over political parties, tax policies etc. is well-established in literature. We characterize all strategy-proof and unanimous random social choice rules over these domains. It turns out that, such a rule is partially random dictatorial and partially random min-max rules. Further, we explore the minimal conditions on a domain under which the strategy-proof and unanimous rules will be partially random dictatorial and partially random min-max. We also show that, each strategy-proof and unanimous random rule on mixed single-peaked domain is a probabilistic mixture of the strategy-proof and unanimous deterministic rules on the same domain.

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1 Introduction

We analyze the classical social choice problem of choosing an alternative from a set of feasible alternatives, based on the preferences of the individuals in a society. Such a procedure is called *deterministic social choice function* (DSCF). Desirable properties such as imposed by Arrow, Gibbard, Satterthwaite on such a DSCF are *strategy-proofness* and *unanimity*. A DSCF is strategy-proof if a strategic individual cannot change its outcome in his/her favor by misreporting his/her preferences, and is unanimous if it always choses a unanimously agreed alternative whenever there is one. The classic Gibbard (1973)-Satterthwaite (1975) impossibility theorem shows that, if there are at least three alternatives and the preferences of the individuals are *unrestricted*, then the only DSCFs that are strategy-proof and unanimous are the *dictatorial* ones. That is, there is an individual, called the *dictator*, such that the DSCF always choses the most preferred alternative of that individual.

Although strategy-proofness and unanimity are desirable properties of a DSCF, the unrestricted domain assumption in Gibbard-Satterthwaite Theorem is quite strong. There are many political and economic situations where the preferences of an individual satisfy natural restrictions. Thus, domain restriction turns out to be a natural and useful way in evading the dictatorship result in social choice theory.

One such well-known domain restriction is the *single-peaked* property that occur in an environment where strictly quasi-concave utility functions are maximized over a linear budget set. Other well-known domain restrictions are *single-dipped* and *single-crossing* properties. Single-dipped property is commonly used in public bad location problem. Usefulness of single-crossing property is well-established in literature (see, for example, Romer (1975), p. 181, and Austen-Smith and Banks (2000), pp. 114-115). Single-crossing domains are flexible enough to accommodate the non-convexities that appear in case of majority voting. Such domains arise in models of income taxation and redistribution (Roberts (1977), Meltzer and Richard (1981)), local public goods and stratification (Westhoff (1977), Epple and Platt (1998), Epple et al. (2001)), coalition formation (Demange (1994), KUNG (2006)) and, in models that study the selection of policies in the market for higher education (Epple et al. (2006)) and the choice of constitutional and voting rules (Barbera and Jackson (2004)). Saporiti (2009) has a detailed exposition on various applications, interpretations, and scopes of single-crossing domains.

The study of single-peaked domains at least goes back to Black (1948) where he shows that the majority rules are strategy-proof and unanimous on these domains. Moulin (1980) and Weymark (2011) show that the strategy-proof and unanimous DSCFs on a single-peaked domain are *min-max* rules. A special class of min-max rules, called the *median* rules, satisfy an additional property called *anonymity*. Anonymity implies that every individual in a society is treated equally. The strategy-proof and unanimous DSCFs on single-dipped domains are characterized in Peremans and Storcken (1999) as monotone rules between the left-most and right-most alternatives. Saporiti (2009) characterizes the strategy-proof, unanimous and anonymous DSCFs on single-crossing domains as *peak rules*. Peak rules are median rules where phantom peaks are chosen from the *top-set* of the domain. Top-set of a domain consists of the alternatives that appear as a top in some preference in the domain.

The study of social choice theory is enriched with the notion of *random social choice function* (RSCF). A RSCF, instead of selecting a particular alternative, assigns a probability distribution over the alternatives. Thus, RSCFs are generalization of DSCFs. Importance of RSCFs over DSCFs is well-established in literature (see, for example, Ehlers et al. (2002), Peters et al. (2014)).

The study of RSCFs goes back to Gibbard (1977) where he characterizes all strategy-proof and unanimous RSCFs on the unrestricted domains as the *random dictatorial* rules. Random dictatorial rules are convex combination of dictatorial rules. A domain, where every strategy-proof and unanimous RSCF is a convex combination of DSCFs satisfying those properties, is called a *deterministic extreme point* (DEP) domain. The study of such domains is useful as it entails a connection between the DSCFs and the RSCFs satisfying strategy-proofness and unanimity on those domains. Such a connection is helpful in finding optimum RSCFs for a society, i.e., RSCFs that maximize the total expected utility of a society. Gershkov et al. (2013) characterizes the optimum DSCFs on single-crossing domains. Evidently, if one wants to find the optimum RSCFs on those domains, then such a connection between the DSCFs and RSCFs, if exists, will be useful.

Ehlers et al. (2002) characterizes the strategy-proof and unanimous random rules on single-peaked domains, and Peters et al. (2014) shows that these rules are convex combination of the strategy-proof and unanimous DSCFs on that domain. In a recent work, Peters et al. (2016) characterizes the strategy-proof and unanimous RSCFs on single-dipped domains, and shows that they are convex combination of DSCFs satisfying those properties. However, to the best of our knowledge, the strategy-proof and unanimous RSCFs on single-crossing domains are not characterized yet. We obtain this characterization as an application of our result in this paper.

It is observed that the well-known restricted domains, such as single-peaked, single-dipped, single-crossing, are all based on some *prior ordering* over the alternatives. In tune with this, the strategy-proof and unanimous DSCFs on these domains respect this prior ordering by satisfying a property known as *uncompromisingness*. Uncompromisingness ensures that a DSCF is completely determined by its outcomes at *boundary profiles*. A boundary profile is one where each individual's most preferred alternative is either the maximal or the minimal alternative (w.r.t. the prior ordering).

This makes it important to understand the extent to which these properties hold. In view of this, we intend to explore what happens when a domain violates single-peakedness in a minimal way. In fact, expecting all the preferences single-peaked with respect to a particular ordering is, we think, a strong requirement. Thus, we allow for situations where some preferences need not be single-peaked with respect to the given prior ordering, or there are multiple prior orderings with respect to which the domain is single-peaked. In this paper, we intend to characterize all strategy-proof and unanimous random social choice rules on mixed single-peaked domains.

In tradition with this literature, stochastic dominance is used to extend preferences over alternatives to preferences over probability distributions. We show that, whenever single-peakedness is violated for a subset of alternatives forming an interval with respect to the prior ordering, every strategy-proof and unanimous random rule behaves like a random dictatorship on that interval and like a random min-max rule outside that interval.

Let $\hat{\mathcal{D}}$ be a single-peaked domain with respect to some prior ordering. Then, a superset $\tilde{\mathcal{D}}$ of $\mathcal{D}$ is called mixed single-peaked if, there is an interval $[a_j, a_{j+1}]$ such that $\tilde{\mathcal{D}}$ restricted to $[a_1, a_j] \cup [a_{j+1}, a_m]$ is single-peaked and $\tilde{\mathcal{D}}$ restricted to $[a_j, a_{j+1}]$ is not single-peaked.

The rest of the paper is organized as follows. We introduce our basic definitions in Section 2. Our main results are presented in Section 3. We conclude the paper in Section 4.

2 Preliminaries

Let $A = \{a_1, a_2, \ldots, a_m\}$ be a finite set of alternatives with a prior ordering $a_1 \prec a_2 \prec \ldots \prec a_m$, and $N = \{1, \ldots, n\}$ be a finite set of agents. Whenever we write minimum or maximum of a subset of A, we mean it w.r.t. the ordering $\prec$ over A. By $a \preceq b$ we mean $a = b$ or $a \prec b$. For $a, b \in A$ define $[a, b] = \{c \mid \text{ either } a \preceq c \preceq b \text{ or } b \preceq c \preceq a\}$. A complete, antisymmetric and transitive binary relation over A (also called a linear order) is called a preference. We denote by

$\mathbb{L}(A)$ the set of all preferences over A. For a preference $P \in \mathbb{L}(A)$, by $r_k(P)$ we mean the kth ranked alternative in P, i.e., $r_k(P) = a$ if and only if $|\{b \in A \mid bPa\}| = k-1$. By P^k we denote a preference such that $r_1(P^k) = a_k$, and by $P^{r,s}$ we denote a preference such that $r_1(P^{r,s}) = a_r$ and $r_2(P^{a,b}) = a_s$. We denote by $\mathcal{D}$ a set of admissible preferences for an(y) agent $i \in N$. As it is clear form the notation, we assume same set of admissible preferences for all the agents. For $a \in A$, let $\mathcal{D}^a = \{P \in \mathcal{D} \mid r_1(P) = a\}$. For any $P \in \mathcal{D}$ and $a \in A$, the *upper contour set* of a at P, denoted by $U(a, P)$, is defined as the set of alternatives that are weakly preferred to a in P, more formally, $U(a, P) = \{b \in A \mid bPa \text{ or } b = a\}$. A preference profile, denoted by $P_N = (P_1, P_2, \ldots, P_n)$, is an element of $\mathcal{D}^n = \mathcal{D} \times \mathcal{D} \times \ldots \times \mathcal{D}$. For $P_N \in \mathcal{D}^n$, let $S(a_k, P_N) = \{i \in N \mid a_k \preceq r_1(P_i)\}$.

A preference $P_i \in \mathbb{L}(A)$ is called single-peaked if P_i has a unique maximal element $\tau(P_i)$, the *peak* of P_i, such that for all $a, b \in A$, $[\tau(P_i) \preceq a \prec b$ or $b \prec a \preceq \tau(P_i)] \Rightarrow aP_ib$. A domain is called single-peaked if each preference in the domain is single-peaked and is called maximal if it contains all single-peaked preferences.

For $P_i \in \mathbb{L}(A)$, and $B \subseteq A$, $P_i|_B \in \mathbb{L}(B)$ is defined as follows: for all $a, b \in B$ $(a, b) \in P_i|_B$ if and only if $(a, b) \in P_i$. For $\mathcal{D} \subseteq \mathbb{L}(A)$, $P_N \in \mathcal{D}^n$, and $B \subseteq A$, define $\mathcal{D}|_B = \{P_i|_B \mid P_i \in \mathcal{D}\}$, and $P_N|_B = (P_1|_B, \ldots, P_n|_B)$.

For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote sets $\{i\}$ by i. By $\triangle A$ we denote the set of probability distributions on A.

A *Random Social Choice Function* (RSCF) is a function $\Phi : \mathcal{D}^n \to \triangle A$. For $S \subseteq A$ and $P_N \in \mathcal{D}^n$, we denote by $\Phi_S(P_N) = \sum_{a \in S} \Phi_a(P_N)$, where $\Phi_a(P_N)$ is the probability of a at $\Phi(P_N)$.

Definition 2.1. A RSCF Φ on $\mathcal{D}^n$ is *unanimous* if for all $P_N \in \mathcal{D}^n$

$$\Phi_{\cap_{i=1}^n r_1(P_i)}(P_N) = 1 \text{ whenever } \cap_{i=1}^n r_1(P_i) \neq \emptyset.$$

Definition 2.2. A RSCF Φ on $\mathcal{D}^n$ is *strategy-proof* if for all $i \in N$, all $P_N \in \mathcal{D}^n$, all $P_i' \in \mathcal{D}$, and all $x \in A$,

$$\sum_{y \in U(x, P_i)} \Phi_y(P_i, P_{-i}) \geq \sum_{y \in U(x, P_i)} \Phi_y(P_i', P_{-i}).$$

REMARK 2.1. A RSCF is called a DSCF if it selects a degenerate probability distribution at every preference profile. More formally, A RSCF Φ on $\mathcal{D}^n$ is called a DSCF if, $\Phi_a(P_N) \in \{0, 1\}$ for all $a \in A$ and all $P_N \in \mathcal{D}^n$. The notions of strategy-proofness and unanimity for DSCFs are special cases of the corresponding definitions for RSCFs.

REMARK 2.2. Note that, for $L, L' \in \triangle A$ and $P \in \mathbb{L}(A)$ if, $L_{U(x,P)} \geq L'_{U(x,P)}$ and $L'_{U(x,P)} \geq L_{U(x,P)}$ for all $x \in A$, then $L = L'$.

Definition 2.3. Two profiles $P_N, P'_N \in \mathcal{D}^n$ are *tops-equivalent* if $r_1(P_i) = r_1(P'_i)$ for all $i \in N$.

Definition 2.4. A RSCF Φ on $\mathcal{D}^n$ is *tops-only* if $\Phi(P_N) = \Phi(P'_N)$ for all tops-equivalent $P_N, P'_N \in \mathcal{D}^n$.

Definition 2.5. A RSCF Φ on $\mathcal{D}^n$ is *uncompromising* if $\Phi_B(P_N) = \Phi_B(P'_i, P_{-i})$ for all $i \in N$, all $P_N \in \mathcal{D}^n$, all $P'_i \in \mathcal{D}$ and all $B \subseteq A$ such that $B \cap [r_1(P_i), r_1(P'_i)] = \emptyset$.

REMARK 2.3. Note that, an uncompromising RSCF is tops-only by definition.

Definition 2.6. A RSCF Φ is called RANDOM PDGMVS if for all $S \subseteq N$, there exists $\beta_S \in \triangle A$ with the property that

1. $\beta_N = \delta_{a_m}, \beta_\emptyset = \delta_{a_1}$,
2. $\beta_S([a_i, a_m]) \leq \beta_{S \cup T}([a_i, a_m])$ for all $S, T \subseteq N$ and all $a_i \in A$,
3. $\beta_S((a_j, a_{j+l})) = 0$ for all $S \subseteq N$,
4. for all $i \in N$, there exists $\alpha_i \geq 0$ with $\sum_1^n \alpha_i = 1$ such that $\beta_S([a_{j+l}, a_m]) = \sum_i^n \alpha_i$

such that
$\Phi_{a_i}(P_N) = \beta_{S(a_i,P_N)}([a_i, a_m]) - \beta_{S(a_{i+1},P_N)}([a_{i+1}, a_m])$

Definition 2.7. Let $\hat{\mathcal{D}}$ be a left-right single-peaked domain. Then a domain $\tilde{\mathcal{D}} \supseteq \hat{\mathcal{D}}$ is called a mixed single-peaked domain if there exist $a_j, a_{j+l} \in A$ such that

1. $\tilde{\mathcal{D}}|_{[a_1,a_j] \cup [a_{j+l},a_m]}$ is single peaked,
2. there exist $Q, Q' \in \tilde{\mathcal{D}}$ such that $r_1(Q) = a_j$, $r_2(Q) = a_p$ and $r_1(Q') = a_{j+l}$, $r_2(Q) = a_q$ for some a_p, a_q with the property that either $a_{j+1} \prec a_p \prec a_j$ and $a_j \prec a_p \prec a_{j+l-1}$, or $a_p = a_{j+l}$ and $a_q = a_j$.

3 Results

In this section we present all strategy-proof and unanimous RSCFs on mixed single-peaked domains. We begin with a technical lemma that we use in our proof repeatedly.

Lemma 3.1. *Let Φ be a strategy-proof set on a domain $\mathcal{D}^n$. Let $P_1, P_2 \in \mathcal{D}$ and $B \subseteq A$. Suppose $\Phi_a(P_1, P_1, P_{-\{1,2\}}) = \Phi_a(P_2, P_2, P_{-\{1,2\}})$ for all $a \notin B$. Then $\Phi_b(P_1, P_1, P_{-\{1,2\}}) = \Phi_b(P_1, P_2, P_{-\{1,2\}})$ for all $b \notin U(W(B, P_1), P_1) \cap U(W(B, P_2), P_2)$.*

Proof. We show $\Phi_b(P_1, P_1, P_{-\{1,2\}}) = \Phi_b(P_1, P_2, P_{-\{1,2\}})$ for all $b \notin U(W(B, P_1), P_1)$. The proof for the same when $a \notin U(W(B, P_1), P_1)$ follows from the symmetric argument. Take $b \notin U(W(B, P_1), P_1)$. Note that, by strategy-proofness

$$\Phi_{U(b,P_1)}(P_1, P_1, P_{-\{1,2\}}) \geq \Phi_{U(b,P_1)}(P_1, P_2, P_{-\{1,2\}}) \geq \Phi_{U(b,P_1)}(P_2, P_2, P_{-\{1,2\}}). \tag{1}$$

Because $b \notin U(W(B, P_1), P_1)$, $B \subseteq U(b, P_1)$. This means $\Phi_c(P_1, P_1, P_{-\{1,2\}}) = \Phi_c(P_2, P_2, P_{-\{1,2\}})$ for all $c \notin U(b, P_1)$. Hence, $\Phi_{U(b,P_1)}(P_1, P_1, P_{-\{1,2\}}) = 1 - \sum_{c \notin U(b,P_1)} \Phi_c(P_1, P_1, P_{-\{1,2\}}) = 1 - \sum_{c \notin U(b,P_1)} \Phi_c(P_2, P_2, P_{-\{1,2\}}) = \Phi_{U(b.P_1)}(P_2, P_2, 1 - \sum_{c \notin U(b,P_1)} \Phi_c(P_1, P_1, P_{-\{1,2\}}))$. By (1), this means

$$\Phi_{U(b,P_1)}(P_1, P_1, P_{-\{1,2\}}) = \Phi_{U(b,P_1)}(P_2, P_2, P_{-\{1,2\}}). \tag{2}$$

Let c be the alternative that appears just above b in P_1. Then, using the fact that $B \subseteq U(c, P_1)$, it follows from similar argument that

$$\Phi_{U(b,P_1)}(P_1, P_1, P_{-\{1,2\}}) = \Phi_{U(b,P_1)}(P_2, P_2, P_{-\{1,2\}}). \tag{3}$$

Subtracting (3) from (2), we have $\Phi_b(P_1, P_1, P_{-\{1,2\}}) = \Phi_b(P_1, P_2, P_{-\{1,2\}})$. ■

The following theorem is the main theorem of this paper. It says that, a RSCF is strategy-proof and unanimous on mixed single-peaked domain if and only if it is a Random PDGMVS.

Theorem 3.1. *Let $\tilde{\mathcal{D}}$ be a mixed single-peaked domain. Then, Φ is a strategy-proof and unanimous RSCF on $\mathcal{D}^n$ if and only if Φ is a Random PDGMVS.*

Proof. Let Φ be a strategy-proof and unanimous RSCF on $\tilde{\mathcal{D}}^n$. If $n = 1$ then by unanimity $\Phi_a(P_N) = 1$ where $r_1(P_1) = a$ for all $P_N \in \tilde{\mathcal{D}}^n$ and hence, Φ is Random PDGMVS. We prove the theorem by induction on the number of agents. Assume that the theorem holds for all sets with $k < n$ agents.

Let $|N| = n$ and $N^* = N \setminus \{1\}$. Define the RSCF $g : \tilde{\mathcal{D}}^{n-1} \to \triangle A$ for the set of voters N^* as follows: for all $P_{N^*} = (P_2, P_3, \ldots, P_n) \in \tilde{\mathcal{D}}^{n-1}$,

$$g(P_2, P_3, \ldots, P_n) = \Phi(P_2, P_2, P_3, P_4, \ldots, P_n).$$

Evidently, g is a well defined RSCF, satisfying strategy-proofness and unanimity (See Lemma 3 in Sen (2011) for a detailed argument). Hence, by induction hypothesis g is a Random PDG-MVS.

Lemma 3.2. *Let $P_N, P'_N \tilde{\mathcal{D}}^n$ be tops-equivalent with $r_1(P_1) = r_1(P_2)$. Then $\Phi(P_N) = \Phi_(P'_N)$.*

Proof. First we show that

$$\Phi(P_1, P_1, P_{-\{1,2\}}) = \Phi(P_1, P_2, P_{-\{1,2\}}). \tag{4}$$

By the definition of g, $\Phi(P_1, P_1, P_{-\{1,2\}}) = g(P_1, P_{-\{1,2\}})$. Since $r_1(P_1) = r_1(P_2)$, $g(P_1, P_{-\{1,2\}}) = g(P_2, P_{-\{1,2\}}) = \Phi(P_2, P_2, P_{-\{1,2\}})$. Using lemma 3.1 with $B = \emptyset$, $\Phi(P_1, P_1, P_{-\{1,2\}}) = \Phi(P_1, P_2, P_{-\{1,2\}})$. This shows (4). USing similar argument, we have

$$\Phi(P'_1, P'_1, P'_{-\{1,2\}}) = \Phi(P'_1, P'_2, P'_{-\{1,2\}}). \tag{5}$$

Moreover, by our IH $\Phi(P_1, P_1, P_{-\{1,2\}}) = g(P_1, P_{-\{1,2\}}) = g(P'_1, P'_{-\{1,2\}}) = \Phi(P'_1, P'_1, P'_{-\{1,2\}})$. Hence, it comes from (4) and (5) that $\Phi(P_N) = \Phi(P'_N)$. This completes the proof of the lemma. ■

Lemma 3.3. *Let $P_1, P_2 \in \hat{\mathcal{D}}$ and $P_i \in \tilde{\mathcal{D}}$ for all $i \geq 3$. Then $\Phi_a(P_1, P_1, P_{-\{1,2\}}) = \Phi_a(P_1, P_2, P_{-\{1,2\}})$ for all $a \notin [r_1(P_1), r_1(P_2)]$*

Proof. The proof of this lemma follows from the application of Lemma 3.1 with $B = [r_1(P_1), r_1(P_2)]$ using the fact that P_1 and P_2 are single-peaked. ■

Lemma 3.4. *Let $P_N, P'_N \in \tilde{\mathcal{D}}^n$ be such that $P_1, P_2, P'_1, P'_2 \in \hat{\mathcal{D}}$, $r_1(P_i), r_1(P'_i) \in [a_j, a_{j+l}]$ for $i \in \{1,2\}$, $r_1(P_i), r_1(P'_i) \notin (a_j, a_{j+l})$ for $i \geq 3$, and $r_1(P_i) \leq a_j$ if and only if $r_1(P'_i) \leq a_j$ for all $i \geq 3$. Then*

$\Phi_a(P_N) = 0 = \Phi_b(P'_N)$ for all $a \in (a_j, a_{j+l}) \setminus \{r_1(P_1), r_1(P_2)\}$ and all $b \in (a_j, a_{j+l}) \setminus \{r_1(P'_1), r_1(P'_2)\}$, and $\Phi_{r_1(P_i)}(P_N) = \Phi_{r_1(P'_i)}(P'_N)$ for all $i \in \{1,2\}$ if $r_1(P_i), r_1(P'_i) \in (a_j, a_{j+l})$.

Proof. Note that by the IH, there exists $\alpha \geq 0$ such that $\Phi_{r_1(P_1)}(P_1, P_1, P_{-\{1,2\}}) = \alpha$ for all $P_N \in \tilde{\mathcal{D}}^n$ with $r_1(P_1) \in (a_j, a_{j+l})$ and $r_1(P_i) \notin (a_j, a_{j+l})$ for all $i \geq 3$. For $k = j, \ldots, j+l-1$ let

$$\alpha_1(k) = \Phi_{a_k}(P^k, P^{k+1}, P_{-\{1,2\}}) - \Phi_{a_k}(P^{k+1}, P^{k+1}, P_{-\{1,2\}}),$$

and

$$\alpha_2(k) = \Phi_{a_k}(P^{k+1}, P^k, P_{-\{1,2\}}) - \Phi_{a_k}(P^{k+1}, P^{k+1}, P_{-\{1,2\}})$$

Claim. $\alpha_i(k) \leq \alpha_i(k+1)$ for all $i \in 1,2$ and all $k = j, \ldots, j+l-2$.

We prove this claim for $i = 1$, the proof for $i = 2$ follows from symmetric argument. First we show $\alpha_1(j) \leq \alpha_1(j+1)$. By Lemma 3.3 $\Phi_{a_{j+2}}(P^j, P^{j+1}, P_{-\{1,2\}}) = \Phi_{a_{j+2}}(P^j, P^j, P_{-\{1,2\}})$ and $\Phi_{\{a_{j+1}, a_{j+2}\}}(P^j, P^{j+1}, P_{-\{1,2\}}) = \Phi_{\{a_{j+1}, a_{j+2}\}}(P^j, P^{j+2}, P_{-\{1,2\}})$. This means $\Phi_{a_{j+1}}(P^j, P^{j+1}, P_{-\{1,2\}}) \geq \Phi_{a_{j+2}}(P^j, P^{j+2}, P_{-\{1,2\}})$. Now using strategy-proofness we get, $\alpha + \Phi_{a_j}(P^{j+1}, P^{j+1}, P_{-\{1,2\}}) = \Phi_{\{a_j, a_{j+1}\}}(P^j, P^{j+1}, P_{-\{1,2\}}) = \Phi_{a_j}(P^j, P^{j+1}, P_{-\{1,2\}}) + \Phi_{a_{j+1}}(P^j, P^{j+1}, P_{-\{1,2\}})$. Therefore, $\alpha_1(j) = \alpha - \Phi_{a_{j+1}}(P^j, P^{j+1}, P_{-\{1,2\}})$. Similarly, we can show $\alpha_1(j+1) = \alpha - \Phi_{a_{j+2}}(P^{j+1}, P^{j+2}, P_{-\{1,2\}}) = \alpha - \Phi_{a_{j+2}}(P^j, P^{j+2}, P_{-\{1,2\}})$. Since $\Phi_{a_{j+1}}(P^j, P^{j+1}, P_{-\{1,2\}}) \geq \Phi_{a_{j+2}}(P^j, P^{j+2}, P_{-\{1,2\}})$, $\alpha_1(j) \geq \alpha_1(j+1)$. Now we show $\alpha_1(k) \geq \alpha_1(k+1)$ for all $k = j+1, \ldots, j+l-2$. Take $k \in \{j+1, \ldots, j+l-2\}$. Since $\Phi_{a_k}(P^{k+1}, P^{k+1}, P_{-\{1,2\}}) = \Phi_{a_{k+1}}(P^{k+2}, P^{k+2}, P_{-\{1,2\}}) = 0$, $\alpha_1(k) = \Phi_{a_k}(P^k, P^{k+1}, P_{-\{1,2\}})$ and $\alpha_1(k+1) = \Phi_{a_{k+1}}(P^{k+1}, P^{k+2}, P_{-\{1,2\}})$. By Lemma 3.3, $\Phi_{a_k}(P^{k+1}, P^{k+2}, P_{-\{1,2\}}) = 0$ and $\Phi_{\{a_k, a_{k+1}\}}(P^{k+1}, P^{k+2}, P_{-\{1,2\}}) = \Phi_{\{a_k, a_{k+1}\}}(P^k, P^{k+2}, P_{-\{1,2\}})$. Hence it follows that, $\alpha_1(k+1) = \Phi_{\{a_k, a_{k+1}\}}(P^{k+1}, P^{k+2}, P_{-\{1,2\}}) = \Phi_{\{a_k, a_{k+1}\}}(P^k, P^{k+2}, P_{-\{1,2\}}) = \alpha_1(k) + \Phi_{a_{k+1}}(P^k, P^{k+2}, P_{-\{1,2\}})$. This means $\alpha_1(k+1) \geq \alpha_1(k)$. This completes the proof of the claim.

Now, we complete the proof of the lemma. Strategy-proof implies $\Phi_{a_j}(P^j, P^{j+1,j}, P_{-\{1,2\}}) = \Phi_{a_j}(Q, P^{j+1,j}, P_{-\{1,2\}})$. By Lemma 3.3, $\Phi_{a_j}(P^j, P^j, P_{-\{1,2\}}) = \Phi_{a_j}(P^j, P^{j+1,j}, P_{-\{1,2\}})$ for all $a \notin \{a_j, a_{j+1}\}$. Moreover, by Lemma 3.1 $\Phi_a(Q, Q, P_{-\{1,2\}}) = \Phi_a(Q, P^{j+1,j}, P_{-\{1,2\}})$ for all $a \notin \{a_j, a_{j+1}\}$. By the IH, $\Phi(P^j, P^j, P_{-\{1,2\}}) = \Phi(Q, Q, P_{-\{1,2\}})$. Hence, $\Phi_a(P^j, P^{j+1,j}, P_{-\{1,2\}}) = \Phi_a(Q, P^{j+1,j}, P_{-\{1,2\}})$ for all $a \notin \{a_j, a_{j+1}\}$. Since $\Phi_{a_j}(P^j, P^{j+1,j}, P_{-\{1,2\}}) = \Phi_{a_j}(Q, P^{j+1,j}, P_{-\{1,2\}})$, $\Phi_{a_{j+1}}(P^j, P^{j+1,j}, P_{-\{1,2\}}) = \Phi_{a_{j+1}}(Q, P^{j+1,j}, P_{-\{1,2\}})$. Thus we have

$$\Phi(P^j, P^{j+1,j}, P_{-\{1,2\}}) = \Phi(Q, P^{j+1,j}, P_{-\{1,2\}}). \tag{6}$$

By strategy-proofness, $\Phi_{\{a_j,a_p\}}(Q, P^{j+1,j}, P_{-\{1,2\}}) \geq \Phi_{\{a_j,a_p\}}(P^p, P^{j+1,j}, P_{-\{1,2\}})$. Using (6), we have

$$\Phi_{\{a_j,a_p\}}(P^j, P^{j+1,j}, P_{-\{1,2\}}) \geq \Phi_{\{a_j,a_p\}}(P^p, P^{j+1,j}, P_{-\{1,2\}}). \tag{7}$$

By Lemma 3.3 and IH, $\Phi_{a_p}(P^j, P^{j+1,j}, P_{-\{1,2\}}) = 0$. Recall that, $\alpha_1(j) = \Phi_{a_j}(P^j, P^{j+1}, P_{-\{1,2\}}) - \Phi_{a_j}(P^j, P^j, P_{-\{1,2\}})$. Therefore, $\Phi_{\{a_j,a_p\}}(P^j, P^{j+1,j}, P_{-\{1,2\}}) = \Phi_{a_j}(P^j, P^{j+1,j}, P_{-\{1,2\}}) = \alpha_1(j) + \Phi_{a_j}(P^{j+1}, P^{j+1}, P_{-\{1,2\}})$. Moreover, by Lemma 3.3 it follows that, $\Phi_{\{a_j,a_p\}}(P^p, P^{j+1,j}, P_{-\{1,2\}}) = \Phi_{a_j}(P^p, P^{j+1,j}, P_{-\{1,2\}}) + \Phi_{a_p}(P^p, P^{j+1,j}, P_{-\{1,2\}}) = \Phi_{a_j}(P^{j+1}, P^{j+1,j}, P_{-\{1,2\}}) + \Phi_{a_p}(P^p, P^{p-1}, P_{-\{1,2\}})$ and $\alpha = \Phi_{a_p}(P^p, P^{p-1}, P_{-\{1,2\}}) + \Phi_{a_{p-1}}(P^p, P^{p-1}, P_{-\{1,2\}})$. By the IH, $\Phi_{a_{p-1}}(P^p, P^p, P_{-\{1,2\}}) = 0$. Hence, $\alpha_2(p-1) = \Phi_{a_{p-1}}(P^p, P^{p-1}, P_{-\{1,2\}})$. Thus, $\Phi_{a_p}(P^p, P^{p-1}, P_{-\{1,2\}}) = \alpha - \Phi_{a_{p-1}}(P^p, P^{p-1}, P_{-\{1,2\}})$ $\alpha - \alpha_2(p-1)$. Plugging these values in (7), we have

$$\alpha_1(j) + \Phi_{a_j}(P^{j+1}, P^{j+1}, P_{-\{1,2\}}) \geq \Phi_{a_j}(P^{j+1}, P^{j+1,j}, P_{-\{1,2\}}) + \alpha - \alpha_2(p-1).$$

This means

$$\alpha_1(j) + \alpha_2(p-1) \geq \alpha. \tag{8}$$

By changing the roles of agent 1 and 2

$$\alpha_1(p-1) + \alpha_2(j) \geq \alpha. \tag{9}$$

Using similar logic as in the derivation of (6),

$$\Phi(P^{j+l}, P^{j+l-1,j+l}, P_{-\{1,2\}}) = \Phi(Q, P^{j+l-1,j+l}, P_{-\{1,2\}}). \tag{10}$$

By strategy-proofness, $\Phi_{\{a_{j+l},a_q\}}(Q', P^{j+l-1,j+1}, P_{-\{1,2\}}) \geq \Phi_{\{a_{j+l},a_q\}}(P^q, P^{j+l-1,j+l}, P_{-\{1,2\}})$. Using (10) we have

$$\Phi_{\{a_{j+l},a_q\}}(P^{j+l}, P^{j+l-1,j+1}, P_{-\{1,2\}}) \geq \Phi_{\{a_{j+l},a_q\}}(P^q, P^{j+l-1,j+l}, P_{-\{1,2\}}). \tag{11}$$

Now using argument similar to the derivation of (8), we have

$$\alpha_1(q) + \alpha_2(j+l-1) \leq \alpha. \tag{12}$$

Again, by changing the role of agent 1 and 2, we have

$$\alpha_1(j+l-1) + \alpha_2(q) \leq \alpha. \tag{13}$$

Now we prove the lemma by considering different cases with respect to the values of p and q.

CASE 1. Suppose $p \leq q$.

Since $j \leq j+l-1$ and $p-1 \leq q$, $\alpha_1(j+l-1) \geq \alpha_1(j)$ and $\alpha_2(q) \geq \alpha_2(p-1)$. By (8) and (13), this means $\alpha_1(j) = \alpha_1(j+l-1)$ and $\alpha_2(p-1) = \alpha_(q)$. Using similar logic, (9) and (12) imply $\alpha_1(p-1) = \alpha_1(q)$ and $\alpha_2(j) = \alpha_(j+l-1)$. Since $\alpha_1(j) = \alpha_1(j+l-1)$ and $\alpha_2(j) = \alpha_2(j+l-1)$, $\alpha_i(k) = \alpha_i(k+1)$ for all $i \in \{1,2\}$ and all $k = j, \ldots, j+l-1$. Let $\alpha_1(j) = \alpha_1$ and $\alpha_2(j) = \alpha_2$. Then (8) implies $\alpha_1 + \alpha_2 = \alpha$.

CASE 2. Suppose $p > q$ and $p \neq j+l$, $q \neq j$.

Since $\alpha_1(j) \leq \alpha_1(q)$ and $\alpha_2(p-1) \leq \alpha_2(j+l-1)$, (8) and (12) together imply

$$\alpha_1(j) = \alpha_1(j+1) = \ldots = \alpha_1(q)$$

and

$$\alpha_2(p-1) = \alpha_2(p) = \ldots = \alpha_1(j+l-1).$$

Moreover, since $\alpha_1(p-1) \leq \alpha_1(j+l-1)$ and $\alpha_2(j) \leq \alpha_2(q)$, (9) and (13) together imply

$$\alpha_1(p-1) = \alpha_1(p) = \ldots = \alpha_1(j+l-1)$$

and

$$\alpha_2(j) = \alpha_2(j+1) = \ldots = \alpha_2(q).$$

Recall that $\alpha_1(j) = \Phi_{a_j}(P^j, P^{j+1}, P_{-\{1,2\}}) - \Phi_{a_j}(P^{j+1}, P^{j+1}, P_{-\{1,2\}})$. By Lemma 3.3 it follows that $\Phi_{a_j}(P^j, P^{j+1}, P_{-\{1,2\}}) = \Phi_{a_j}(P^j, P^{j+l}, P_{-\{1,2\}})$. Hence,

$$\Phi_{a_j}(P^j, P^{j+1}, P_{-\{1,2\}}) = \alpha_1(j) + \Phi_{a_j}(P^{j+1}, P^{j+1}, P_{-\{1,2\}}). \tag{14}$$

Recall that $\alpha_1(j+l-1) = \Phi_{a_{j+l-1}}(P^{j+l-1}, P^{j+l}, P_{-\{1,2\}})$. Since $\alpha = \Phi_{a_{j+l-1}}(P^{j+l-1}, P^{j+l-1}, P_{-\{1,2\}})$, $\alpha - \alpha_1(j+l-1) = \Phi_{a_{j+l-1}}(P^{j+l-1}, P^{j+l-1}, P_{-\{1,2\}}) - \Phi_{a_{j+l-1}}(P^{j+l-1}, P^{j+l}, P_{-\{1,2\}})$. By Lemma 3.3, $\Phi_{\{a_{j+l-1}, a_{j+l}\}}(P^{j+l-1}, P^{j+l}, P_{-\{1,2\}}) = \Phi_{\{a_{j+l-1}, a_{j+l}\}}(P^{j+l-1}, P^{j+l-1}, P_{-\{1,2\}})$. Therefore it follows that, $\Phi_{a_{j+l}}(P^{j+l-1}, P^{j+l}, P_{-\{1,2\}}) = \Phi_{a_{j+l}}(P^{j+l-1}, P^{j+l-1}, P_{-\{1,2\}}) + \Phi_{a_{j+l-1}}(P^{j+l-1}, P^{j+l-1}, P_{-\{1,2\}}) - \Phi_{a_{j+l}}(P^{j+l-1}, P^{j+l}, P_{-\{1,2\}})$. This implies $\Phi_{a_{j+l}}(P^{j+l-1}, P^{j+l}, P_{-\{1,2\}}) = \Phi_{a_{j+l}}(P^{j+l-1}, P^{j+l-1}, P_{-\{1,2\}}) + \alpha - \alpha_1(j+l-1)$. By Lemma 3.3, $\Phi_{a_{j+l}}(P^{j+l-1}, P^{j+l}, P_{-\{1,2\}}) = \Phi_{a_{j+l}}(P^j, P^{j+l}, P_{-\{1,2\}})$. Hence,

$$\Phi_{a_{j+l}}(P^j, P^{j+l}, P_{-\{1,2\}}) = \Phi_{a_{j+l}}(P^{j+l-1}, P^{j+l-1}, P_{-\{1,2\}}) + \alpha - \alpha_1(j+l-1). \tag{15}$$

Now take $k \in \{j+1, \ldots, j+l-1\}$. By Lemma 3.3, $\Phi_{a_k}(P^j, P^{j+l}, P_{-\{1,2\}}) = \Phi_{a_k}(P^{k-1}, P^{k+1}, P_{-\{1,2\}})$ and $\Phi_{a_{k-1}}(P^{k-1}, P^{k+l}, P_{-\{1,2\}}) = \Phi_{a_{k-1}}(P^{k-1}, P^k, P_{-\{1,2\}})$. Moreover, by IH and Lemma 3.3, it follows that $\Phi_{a_{k-1}}(P^k, P^{k+1}, P_{-\{1,2\}}) = 0$. Hence, $\Phi_{a_k}(P^k, P^{k+l}, P_{-\{1,2\}}) = \Phi_{a_k}(P^{k-1}, P^{k+1}, P_{-\{1,2\}}) + \Phi_{a_{k-1}}(P^{k-1}, P^{k+l}, P_{-\{1,2\}})$. Thus it follows that $\Phi_{a_k}(P^k, P^{k+l}, P_{-\{1,2\}}) - \Phi_{a_{k-1}}(P^{k-1}, P^k, P_{-\{1,2\}}) = \Phi_{a_k}(P^{k-1}, P^{k+1}, P_{-\{1,2\}})$. By the definition of $\alpha_1(k)$ and $\alpha_1(k-1)$, this means

$$\alpha_1(k) - \alpha_1(k-1) = \Phi_{a_k}(P^{k-1}, P^{k+1}, P_{-\{1,2\}}) = \Phi_{a_k}(P^j, P^{j+1}, P_{-\{1,2\}}). \tag{16}$$

Since $\alpha_1(j) = \ldots = \alpha_1(q)$ and $\alpha_1(p-1) = \ldots = \alpha_1(j+l-1)$, we have

$$\Phi_{a_k}(P^j, P^{j+1}, P_{-\{1,2\}}) = 0 \forall k \in \{j+1, \ldots, q\} \cup \{p, \ldots, j+l-1\}. \tag{17}$$

Since $r_1(P^j) = r_1(Q) = a_j$, by strategy-proofness

$$\Phi_{a_j}(P^j, P^{j+1}, P_{-\{1,2\}}) = \Phi_{a_j}(Q, P^{j+1}, P_{-\{1,2\}}). \tag{18}$$

By (14) this means

$$\Phi_{a_j}(P^j, P^{j+1}, P_{-\{1,2\}}) = \alpha_1(j) + \Phi_{a_j}(P^{j+1}, P^{j+1}, P_{-\{1,2\}}). \tag{19}$$

By Lemma 3.3,

$$\Phi_{a_j}(P^p, P^{j+l}, P_{-\{1,2\}}) = \Phi_{a_j}(P^{j+1}, P^{j+1}, P_{-\{1,2\}}) \tag{20}$$

and

$$\Phi_{a_p}(P^p, P^{j+l}, P_{-\{1,2\}}) = \Phi_{a_p}(P^p, P^{p+1}, P_{-\{1,2\}}) = \alpha_1(p). \tag{21}$$

Since $r_1(Q) = a_j$ and $r_2(Q) = a_p$, by using strategy-proofness we get $\Phi_{\{a_j,a_p\}}(Q, P^{j+l}, P_{-\{1,2\}}) \geq \Phi_{\{a_j,a_p\}}(P^p, P^{j+l}, P_{-\{1,2\}})$. By means of (18), (19) and (20) and the fact that $\alpha_1(p) = \alpha_1(p-1)$, this means

$$\Phi_{a_p}(Q, P^{j+l}, P_{-\{1,2\}}) \geq \alpha_1(p-1) - \alpha_1(q). \tag{22}$$

By Lemma 3.3, $\Phi_a(P^j, P^{j+l}, P_{-\{1,2\}}) = \Phi_a(P^j, P^j, P_{-\{1,2\}})$ for all $a \notin [a_j, a_{j+l}]$. By the IH $\Phi_a(Q, Q, P_{-\{1,2\}})$ $\Phi_a(P^{j+l}, P^{j+l}, P_{-\{1,2\}})$ for all $a \notin [a_j, a_{j+l}]$. Using Lemma 3.1 and the fact that $U(a_j, P^{j+l}) \cap U(a_{j+l}, Q) = [a_j, a_{j+l}]$, it follows that $\Phi_a(Q, P^{j+l}, P_{-\{1,2\}}) = \Phi_a(Q, Q, P_{-\{1,2\}})$ for all $a \notin [a_j, a_{j+l}]$. By the IH and Lemma 3.2, $\Phi(Q, Q, P_{-\{1,2\}}) = \Phi_a(P^j, P^j, P_{-\{1,2\}})$. Combining all these,

$$\Phi_a(P^j, P^{j+l}, P_{-\{1,2\}}) = \Phi_a(Q, P^{j+l}, P_{-\{1,2\}}) \forall a \notin [a_j, a_{j+l}]. \tag{23}$$

By strategy-proofness $\Phi_{U(a_{j+l-1}, P^j)}(P^j, P^{j+l}, P_{-\{1,2\}}) \geq \Phi_{U(a_{j+l-1}, P^j)}(Q, P^{j+l}, P_{-\{1,2\}})$. By (23) this means

$$\begin{aligned} &\Phi_{a_j}(P^j, P^{j+l}, P_{-\{1,2\}}) + \ldots + \Phi_{a_{j+l-1}}(P^j, P^{j+l}, P_{-\{1,2\}}) \\ \geq\ &\Phi_{a_j}(Q, P^{j+l}, P_{-\{1,2\}}) + \ldots + \Phi_{a_{j+l-1}}(Q, P^{j+l}, P_{-\{1,2\}}). \end{aligned}$$

By (17) and (18) this gives

$$\begin{aligned} &\Phi_{a_q}(P^j, P^{j+l}, P_{-\{1,2\}}) + \ldots + \Phi_{a_{p-1}}(P^j, P^{j+l}, P_{-\{1,2\}}) \\ \geq\ &\Phi_{a_{j+1}}(Q, P^{j+l}, P_{-\{1,2\}}) + \ldots + \Phi_{a_{j+l-1}}(Q, P^{j+l}, P_{-\{1,2\}}). \end{aligned}$$

By means of (17), this implies

$$\alpha_1(p-1) - \alpha_1(q) \geq \Phi_{a_{j+1}}(Q, P^{j+l}, P_{-\{1,2\}}) + \ldots + \Phi_{a_{j+l-1}}(Q, P^{j+l}, P_{-\{1,2\}}). \tag{24}$$

Thus it follows from (22) that,

$$\Phi_{a_p}(Q, P^{j+l}, P_{-\{1,2\}}) \geq \alpha_1(p-1) - \alpha_1(q) \text{ and } \Phi_{a_k}(Q, P^{j+l}, P_{-\{1,2\}}) = 0 \forall a_k \in [a_{j+1}, a_{p-1}] \cup [a_{p+1}, a_{j+l-1}] \tag{25}$$

Since $\alpha_1(j) = \ldots = \alpha_1(q)$ and $\alpha_1(p-1) = \ldots = \alpha_1(j+l-1)$, by (14), (16), (17)

$$\sum_{j}^{j+l-1} \Phi_{a_k}(P^j, P^{j+l}, P_{-\{1,2\}}) = \Phi_{a_j}(P^{j+1}, P^{j+l}, P_{-\{1,2\}}) + \alpha_1(p-1). \tag{26}$$

and by (19) and(25)

$$\sum_{j}^{j+l-1} \Phi_{a_k}(Q, P^{j+l}, P_{-\{1,2\}}) = \Phi_{a_j}(P^{j+1}, P^{j+l}, P_{-\{1,2\}}) + \alpha_1(p-1). \tag{27}$$

This means

$$\sum_{j}^{j+l-1} \Phi_{a_k}(P^j, P^{j+l}, P_{-\{1,2\}}) = \sum_{j}^{j+l-1} \Phi_{a_k}(Q, P^{j+l}, P_{-\{1,2\}}). \tag{28}$$

By (23)

$$\sum_{j}^{j+l} \Phi_{a_k}(P^j, P^{j+l}, P_{-\{1,2\}}) = \sum_{j}^{j+l} \Phi_{a_k}(Q, P^{j+l}, P_{-\{1,2\}}). \tag{29}$$

Subtracting (22) from (29) we have

$$\Phi_{a_{j+l}}(P^j, P^{j+l}, P_{-\{1,2\}}) = \Phi_{a_{j+l}}(Q, P^{j+l}, P_{-\{1,2\}}). \tag{30}$$

By (15), $\Phi_{a_{j+l}}(P^{j+l-1}, P^{j+l-1}, P_{-\{1,2\}}) + \alpha - \alpha_1(j+l-1) = \Phi_{a_{j+l}}(Q, P^{j+l}, P_{-\{1,2\}})$. Thus we have by (18), (23), (25) and (30)

$$\begin{aligned} \Phi_{a_k}(P^j, P^{j+l}, P_{-\{1,2\}}) = \Phi_{a_k}(Q, P^{j+l}, P_{-\{1,2\}}) \forall a_k \notin (a_j, a_{j+l}), \\ \Phi_{a_p}(Q, P^{j+l}, P_{-\{1,2\}}) = \alpha_1(p-1) - \alpha_1(q), \\ \text{and } \Phi_{a_p}(Q, P^{j+l}, P_{-\{1,2\}}) = 0 \;\; \forall a_k \in (a_j, a_{j+l}) \setminus p. \end{aligned} \tag{31}$$

Using similar logic as in the derivation of (23), we have $\Phi(P^j, Q', P_{-\{1,2\}}) = \Phi(Q, Q', P_{-\{1,2\}}) = \Phi(P^j, P^{j+l}, P_{-\{1,2\}})$ for all $a \notin [a_j, a_{j+l}]$. Again using logic similar to the derivation of (31),

$$\begin{aligned} \Phi_{a_k}(P^j, P^{j+l}, P_{-\{1,2\}}) = \Phi_{a_k}(P^j, Q', P_{-\{1,2\}}) \forall a_k \notin (a_j, a_{j+l}), \\ \Phi_{a_q}(P^j, Q', P_{-\{1,2\}}) = \alpha_1(p-1) - \alpha_1(q), \\ \text{and } \Phi_{a_q}(P^j, Q', P_{-\{1,2\}}) = 0 \;\; \forall a_k \in (a_j, a_{j+l}) \setminus q. \end{aligned} \tag{32}$$

This completes the proof of the lemma. ■

Now the proof of the theorem follows from Lemmas 3.1-3.4. ■

4 Conclusion

We have shown in this paper that, every strategy-proof and unanimous RSCF on a partially single-peaked domain is a Random PDGMVS. We further show that, such a RSCF can be written as a convex combination min-max DSCFs with some restriction on that domain. Many domains satisfy the condition of mixed single-peaked domain, including well-known domains like multiple single-peaked domain, multidimensional single peaked domain. Thus, our results provide a characterization of strategy-proof and unanimous RSCFs on those domains.

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