Frequency Based Analysis of Voting Rules

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Abstract

The issue here is on anonymous collective decision making in large populations. Based on the structure on linear orderings induced by the Kemeny distance we study frequency distribution, i.e. (frequency) distributions. In case of a so called unimodal frequency distribution it appears that many if not all reasonable and well-known decision rules yield the same outcome. We also show some robustness conditions for Pairwise rule, Borda rule and Plurality rule.

1 Introduction

Real life voting problems deal with large electorates. There we can not consider individual preferences. Hence putting technical conditions on the set of preferences seems unnatural. More realistic would be to consider the frequency distribution of the preferences and impose conditions on the frequency distributions as a whole. The cultural aspects of an electorate leading to coherency or disharmony among the preferences of its voters are perhaps more easy to describe in terms of frequency distributions and can empirically be tested in this way. We first restrict the frequency distribution in the unimodal class. A set of conditions for (abstract) collective decision rules are found which is sufficient to guarantee that the outcome is the modus at unimodal frequency distributions, where many well-known rules satisfy all the conditions of this set, Then we try to broaden the class of frequency distribution. There we show that Pairwise rule and Borda rule are still selecting one pivotal preference as their outcome. In multimodal frequency distributions the problem of collective decision making is present: different reasonable collective decision rules may lead to different outcomes.

2 Unimodal frequency distributions

Collective decision making is studied within the classical framework where a non-empty and finite set of agents, say $N = \{1, 2, 3, ..., n\}$, collectively orders the alternatives of a non-empty and finite set A from best to worst. Let A consist of m alternatives. Although N is finite its cardinality n is assumed to be large. The collective decision is assumed to be based on the individual preferences, which are formalized here by linear orderings, i.e. complete antisymmetric and transitive relations on A. The set of these linear orderings on A is denoted by $\mathbb{L}$. A profile p (of individual preferences) assigns to every agent i such a linear ordering $p(i)$ in $\mathbb{L}$. It therewith reflects a possible combination of individual preferences at which such a collective decision is taken. Let $\mathbb{L}^N$ denote the set of all these profile that is the set of all the combinations of individual preferences. To exclude discussions on resolving ties in the collective orderings we allow the collective outcomes to be weak orderings, i.e. complete and transitive orderings on A. The set of all these orderings is denoted by $\mathbb{W}$. Linear orderings or more general relations on A will also be denoted by the letter R. Let x and y be alternatives, as usual $(x, y) \in R$ means that at preference (relation) R alternative x is weakly preferred to alternative y. Note that for linear orderings R and different alternatives x and y because of antisymmetry $(x, y) \in R$ implies that $(y, x) \notin R$. In that case we say that x is strictly preferred to y.If R is a complete relation the notion that x is strictly preferred to y can also be expressed by $(y, x) \notin R$. For alternatives x and y let $\mathbb{L}_{xy}$ denote the set of linear orderings R at which x is strictly preferred to y, that is $(y, x) \notin R$. The collective decision is

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formalized by a *preference rule*[1] F a function that assigns to every profile p in $\mathbb{L}^N$ a collective preference $F(p)$ in $\mathbb{W}$.

A frequency distribution is formalized by a type of frequency distribution over the set of preferences. Given a profile p of individual preferences of the agents and a preference R in $\mathbb{L}$, then $f(R,p)$ denotes the number of agents with preference $p(i)$ at profile p, that is

$$f(R,p) = |\{i \in N \mid p(i) = R\}|,$$

where $|S|$ denotes the cardinality of an arbitrary set S. The description of profiles by means of frequency distributions and therewith essentially describing a type of frequency distribution, is based on a metric space on $\mathbb{L}$ induced by the Kemeny distance function δ. This distance function δ is defined for two preferences R^1 and R^2 in $\mathbb{L}$ as follows[2]

$$\delta(R^1, R^2) = \frac{1}{2}\left|(R^1 \triangle\, R^2)\right|,$$

where $\triangle$ denotes the symmetric difference between sets, i.e. $R^1 \triangle\, R^2 = (R^1 - R^2) \cup (R^2 - R^1)$. Kemeny distance function calculates the number of *discordant* pairs or number of *inversions* [3] between two linear orders.

A profile p is called *unimodal* if there exists a preference R^*, the *modus*, such that for every two preferences R^1 and R^2 in $\mathbb{L}$

$$f(R^1,p) > f(R^2,p) \text{ if and only if } \delta(R^*, R^1) < \delta(R^*, R^2).$$

So, a profile p is unimodal if there is a preference R^* in $\mathbb{L}$ with highest frequency such that frequencies for all other preferences in $\mathbb{L}$ strictly decrease in their distance to this modus R^*.

Preference rules will be discussed with respect to the following conditions.

Anonymity means that the preference rule is symmetric in its arguments. Anonymity guarantees that agents' preferences are equal important in determining the outcome. The outcome is based on how many times a preference is announced at a certain profile instead of who announced that preference. Let σ on N be a permutation on N the set of agents. Let p be a profile in $\mathbb{L}^N$. Then profile[4] $q = p \circ \sigma$ is defined for all agents i by

$$q(i) = p(\sigma(i)).$$

Preference rule F is anonymous if for all profiles p and all permutations σ

$$F(p) = F(p \circ \sigma).$$

Neutrality means that the preference rule treats alternative in equal situations equally. Let x and y be alternatives. Let τ_{xy} be the permutation on A such that $\tau_{xy}(x) = y$, $\tau_{xy}(y) = x$ and $\tau_{xy}(z) = z$ for all alternatives z not equal to x and y. So, considering permutations on A as renames, τ_{xy} swaps the names of x and y and leaves all other names unchanged. For a relation R on A let

$$\tau_{xy}R = \{(\tau_{xy}(a), \tau_{xy}(b)) : (a,b) \in R\}.$$

So, in $\tau_{xy}R$ the positions of x and y in R are swapped where all the positions of all the other alternatives are left unchanged. Similarly for a profile p, $\tau_{xy}p$ denotes the profile at which for each agent i the positions of x and y in $p(i)$ are swapped: $\tau_{xy}p = (\tau_{xy}p(1), \tau_{xy}p(2), \ldots, \tau_{xy}p(i), \ldots, \tau_{xy}p(n))$. Preference rule F is neutral[5] if for all alternatives x and y, all profiles p and all permutations τ_{xy} on A

$$F(\tau_{xy}p) = \tau_{xy}F(p).$$

Monotonicity means that if in comparison of profiles p and q the preference "x is preferred to y" among the agents increases when going from p to q, then this preference at the outcomes should not

[1] Preference rules are usually known as Welfare Functions. As the model discussed here also applies to non-welfare issues such as voting we rather use the term preference rule.

[2] Here we actually take half the Kemeny distance because between two linear orderings it is a multiple of two.

[3] Let $x_1, x_2, \ldots, x_m$ be a permutation of $1, 2, \ldots, m$. Then x_i, x_j is called a discordant pair if, $i < j$ but $x_i > x_j$.

[4] Note that we consider a profile to be a function from N to $\mathbb{L}$.

[5] The neutrality notion defined here is equivalent to the usual one, because every permutation can be decomposed into permutations where only two elements are swapped.

decrease. Preference rule F is monotone if for all alternatives x and y , all non-empty subsets S of N and all profiles p and q,

$$\begin{aligned} \text{if } (y,x) &\notin F(p), \\ p(j) &= q(j) \text{ for all } j \in N-S, \\ (y,x) &\in p(i) \text{ for all } i \in S \text{ and} \\ q(i) &= \tau_{xy} p(i) \text{ for all } i \in S, \\ \text{then } (y,x) &\notin F(q). \end{aligned}$$

Discrimination means that at a profile p where for two alternatives x and y every linear ordering R at which x is strictly preferred to y strictly outnumber linear ordering $\tau_{xy}R$ at which y is strictly preferred to x the preference rule cannot be indifferent between x and y. Preference rule F discriminates if for all profiles p and all different alternatives x and y

$$\begin{aligned} \text{if } f(R,p) &> f(\tau_{xy}R,p) \text{ for all } R \in \mathbb{L}_{xy} \ , \\ \text{then } (x,y) &\notin F(p) \text{ or } (y,x) \notin F(p). \end{aligned}$$

It is natural to go one step further and to impose that in those situations x should be strictly preferred to y. This condition is referred to as *positive discrimination.*

3 Decisions in Unimodal frequency distributions

The following examples show that many well-known collective decision rules yield the modus at unimodal profiles. A sufficient condition for this is positive discrimination as will be shown below in Theorem 1. Let $\widehat{p}$ be a unimodal profile with modus $\widehat{R}$.

Lemma 1. *Let x and y be two different preferences and R^1 and R^2 be two linear orderings, such that $(x,y) \in R^1$ and $(y,x) \in R^2$. Then $\delta(R^1, \tau_{xy}R^2) < \delta(R^1, R^2)$.*

Proof. We can calculate number of discordant pairs (the distance) between R^2 and $\tau_{xy}R^2$. Notice that between R^2 and $\tau_{xy}R^2$ changes are due to only changing positions of x and y. Also the alternatives in the set $M = \{a \in A : (x,a),(a,y) \in \tau_{xy}R^2\}$ gets affected by their relative positions with respect to x and y. Suppose $\widehat{M} = \{a \in A : (x,a),(a,y) \in R^1\}$. Then number of discordant pairs between R^2 and $\tau_{xy}R^2$ is

$$1 + 2\left|\widehat{M} \cap M\right|$$

Hence, $\delta(R^1, R^2) = \delta(R^1, \tau_{xy}R^2) + 1 + 2\left|\widehat{M} \cap M\right| > \delta(R^1, \tau_{xy}R^2)$. ∎

Lemma 2. *Let $x,\ y \in \mathbb{A}$, with $(y,x) \notin \widehat{R}$ and let $R \in \mathbb{L}_{xy}$. Then $f(R,\widehat{p}) > f(\tau_{xy}R,\widehat{p})$.*

Proof. By Lemma 1 it follows that $\delta(\widehat{R},R) < \delta(\widehat{R},\tau_{xy}R)$ which easily yields the desired inequality by the unimodality of $\widehat{p}$. ∎

Example 1. Pairwise Preference Rules

Pairwise preference rule depends on the pairwise majority comparisons. In general these comparisons may yield cycles and rules therefore may differ in these situations: different pairwise rules break up cycles in different ways.

If , however, at a certain profile pairwise majority comparison yields a complete strict and transitive ordering from overall winner (the Condorcet winner) to overall looser (the Condorcet looser) then this is the outcome of all these rules at that profile. We argue that at profile $\widehat{p}$ pairwise majority comparisons yield the modus $\widehat{R}$. Herewith we showed that all pairwise preference rules assign the modus at a unimodal profile. Let $(x,y) \in \widehat{R}$, with $x \neq y$. It is sufficient to show that

$$\left|\{i \in N : (x,y) \in \widehat{p}(i)\}\right| > \left|\{i \in N : (y,x) \in \widehat{p}(i)\}\right|.$$

Lemma 2 yields that for all $R \in \mathbb{L}_{xy}$

$$f(R,\widehat{p}) > f(\tau_{xy}R,\widehat{p}).$$

Because

$$\begin{aligned} |\{i \in N : (x,y) \in \widehat{p}(i)\}| &= \sum_{R \in \mathbb{L}_{xy}} f(R, \widehat{p}) \text{ and} \\ |\{i \in N : (y,x) \in \widehat{p}(i)\}| &= \sum_{R \in \mathbb{L}_{xy}} f(\tau_{xy} R, \widehat{p}) \end{aligned}$$

the desired inequality follows.

Example 2. Score Preference Rules

At score preference rules agents assign scores $s_1, s_2, \ldots, s_m$ to the alternative and these are then ordered according to their total scores. It is assumed that $s_m \geq s_{m-1} \geq \ldots \geq s_1$ and that $s_m > s_1$. At a preference $p(i)$ agent i would hand out score s_m to his best alternative, s_{m-1} to his second best and so on. So, his worst alternative receives score s_1. Let $r(x, p(i)) = |\{y \in A : (x,y) \in p(i)\}|$ be the rank of alternative x at profile $p(i)$. Then of course the rank of the best alternative is equal to m the number of alternatives and that of the worst alternative is equal to one. Agent i divides the scores as follows

$$s(x, p(i)) = s_{r(x,p(i))}$$

and total score for alternative x at profile p is now

$$s(x,p) = \sum_{i \in N} s(x, p(i)).$$

Define the score preference rule F_{score} for alternatives x and y and an arbitrary profile p as follows

$$(x,y) \in F_{score}(p) \text{ if and only if } s(x,p) \geq s(y,p)$$

In order to show that $F_{score}(\widehat{p}) = \widehat{R}$ let $(x,y) \in \widehat{R}$, with $x \neq y$. It is sufficient to show that $s(x,\widehat{p}) > s(y,\widehat{p})$. For numbers k and l let $\mathbb{L}_{xy}^{kl} = \{R \in \mathbb{L}_{xy} : s(x,R) = k \text{ and } s(y,R) = l\}$. Then

$$\begin{aligned} s(x,\widehat{p}) &= \sum_{k=2}^{k=m} \sum_{l=1}^{l=k-1} \left(\sum_{R \in \mathbb{L}_{xy}^{kl}} s_k \cdot f(R,\widehat{p}) + \sum_{R \in \mathbb{L}_{yx}^{lk}} s_l \cdot f(R,\widehat{p}) \right) \\ &= \sum_{k=2}^{k=m} \sum_{l=1}^{l=k-1} \left(\sum_{R \in \mathbb{L}_{xy}^{kl}} s_k \cdot f(R,\widehat{p}) + s_l \cdot f(\tau_{xy} R,\widehat{p}) \right) \end{aligned}$$

Similarly

$$s(y,\widehat{p}) = \sum_{k=2}^{k=m} \sum_{l=1}^{l=k-1} \left(\sum_{R \in \mathbb{L}_{xy}^{kl}} s_k \cdot f(\tau_{xy} R,\widehat{p}) + s_l \cdot f(R,\widehat{p}) \right)$$

Therefore it is sufficient to show that for all numbers k and l, with $k > l$, and all $R \in \mathbb{L}_{xy}^{kl}$

$$s_k \cdot f(R,\widehat{p}) + s_l \cdot f(\tau_{xy} R,\widehat{p}) \geq s_k \cdot f(\tau_{xy} R,\widehat{p}) + s_l \cdot f(R,\widehat{p}),$$

and that at least one of these inequalities is strict. The latter equality is equivalent to

$$(s_k - s_l) \cdot (f(R,\widehat{p}) - f(\tau_{xy} R,\widehat{p})) \geq 0$$

Because $k > l$ we have that $s_k \geq s_l$, where by definition $s_m > s_1$. Lemma 2 yields that $f(R,\widehat{p}) - f(\tau_{xy} R,\widehat{p}) > 0$. So, the weak inequality follows readily and the strict in case x is ordered best at R and y is ordered worst.

Example 3. Coombs Preference Rule

At the Coombs preference rule alternatives are successively eliminated and then ordered reverse to their elimination order. The elimination is based on the number of agents which consider the alternative at hand worst. Let x be an alternative and p be a profile. Further, let B be a subset of A. Then $ws(x,B,p) = |\{i \in N : (b,x) \in p(i) \text{ for all } b \in B\}|$ equals the number of agents which order x worst among B at profile p. Let

$$C_0(p) = \{a \in A : ws(a,A,p) \geq ws(x,A,p) \text{ for all } x \in A\}$$

$$\text{and } A_1 = A - C_0(p)$$

Next for all $k > 0$ recursively define

$$\begin{aligned} C_k(p) &= \{a \in A_k : ws(a, A_k, p) \geq ws(x, A_k, p) \text{ for all } x \in A_k\} \\ \text{and } A_{k+1} &= A_k - C_k(p) \end{aligned}$$

Note that there exists a number k_p such that A_k is empty for all k larger than k_p and A_k is non-empty for k smaller than or equal to k_p. The Coombs preference rule F_{Coombs} is now defined for alternatives x and y and profile p as follows

$$(x, y) \in F_{Coombs}(p) \text{ if and only if } y \in A_k \text{ implies } x \in A_k \text{ for all } k.$$

In order to show that $F_{Coombs}(\widehat{p}) = \widehat{R}$ let $(x, y) \in \widehat{R}$, with $x \neq y$. It is sufficient to show that $x \notin C_k(p)$ and $y \in C_k(p)$ for some k. For every subset B of A such that x and y are in B and every preference $R \in \mathbb{L}$ we have

$$\begin{aligned} \text{if } (b, x) &\in R \text{ for all } b \in B, \\ \text{then } R &\in \mathbb{L}_{yx} \text{ and } (b, y) \in \tau_{xy} R \text{ for all } b \in B. \end{aligned}$$

In view of Lemma 2 this yields that

$$ws(x, B, \widehat{p}) < ws(y, B, \widehat{p}).$$

Obviously this yields for all k that

$$\begin{aligned} y &\in A_k \text{ implies } x \in A_k \text{ and} \\ \text{if } y &\in A_k \text{ and } y \notin A_{k+1}, \text{ then } x \in A_{k+1}. \end{aligned}$$

But then $x \notin C_k(p)$ and $y \in C_k(p)$ for some k.

Example 4. Kemeny Like Preference Rule

The Kemeny rule determines its outcome on those linear orderings which are in total closest to all the preferences in the profile. We will show that $\widehat{R}$ is the only solution of the following minimization problem

$$\min_{R \in \mathbb{L}} \sum_{i \in N} \delta(R, \widehat{p}(i)).$$

Then consequently we have that the Kemeny preference rule assigns $\widehat{R}$ to profile $\widehat{p}$. Here we will prove a slightly stronger result that $\widehat{R}$ is the only solution of

$$\min_{R \in \mathbb{L}} \sum_{i \in N} \delta_h(R, \widehat{p}(i)),$$

where $\delta_h(R^1, R^2) = h(\delta(R^1, R^2))$ for R^1, $R^2 \in \mathbb{L}$ and a strictly increasing function h. Let $(x, y) \in \widehat{R}$, with $x \neq y$, and R^1 and R^2 be two preferences such that $R^1 \triangle\, R^2 = \{(x, y), (y, x)\}$, $(x, y) \in R^1$ and $(y, x) \in R^2$. Note that for all $R \in \mathbb{L}_{xy}$ $\delta(R, R^1) < \delta(R, R^2)$ and therewith $\delta_h(\widehat{R}, R^1) < \delta_h(\widehat{R}, R^2)$. Therefore in order to prove that $\widehat{R}$ is the only solution of the above minimization problem it is sufficient to prove that

$$\sum_{i \in N} \delta_h(R^1, \widehat{p}(i)) < \sum_{i \in N} \delta_h(R^2, \widehat{p}(i)).$$

For $k \in \{1, 2\}$

$$\begin{aligned} \sum_{i \in N} \delta_h(R^k, \widehat{p}(i)) &= \sum_{R \in \mathbb{L}_{xy}} \delta_h(R^k, R) \cdot f(R, \widehat{p}) + \sum_{R \in \mathbb{L}_{yx}} \delta_h(R^k, R) \cdot f(R, \widehat{p}) \\ &= \sum_{R \in \mathbb{L}_{xy}} \delta_h(R^k, R) \cdot f(R, \widehat{p}) + \delta_h(R^k, \tau_{xy} R) \cdot f(\tau_{xy} R, \widehat{p}). \end{aligned}$$

So,

$$\sum_{i \in N} \delta_h(R^1, p(i)) - \sum_{i \in N} \delta_h(R^2, p(i)) \text{ equals}$$

$$\sum_{R\in\mathbb{L}_{xy}} (\delta_h(R^1,R)-\delta_h(R^2,R))f(R,\widehat{p})+(\delta_h(R^1,\tau_{xy}R)-\delta_h(R^2,\tau_{xy}R))f(\tau_{xy}R,\widehat{p}).$$

Note that for $R\in\mathbb{L}_{xy}$

$$\begin{aligned}\delta(R^1,R) &= \delta(\tau_{xy}R^1,\tau_{xy}R)=\delta(R^2,\tau_{xy}R)\\ \delta(R^2,R) &= \delta(\tau_{xy}R^2,\tau_{xy}R)=\delta(R^1,\tau_{xy}R).\end{aligned}$$

This implies that $R\in\mathbb{L}_{xy}$

$$\delta_h(R^1,R)-\delta_h(R^2,R)=\delta_h(R^2,\tau_{xy}R)-\delta_h(R^1,\tau_{xy}R).$$

Therefore

$$\begin{aligned}&\sum_{i\in N}\delta_h(R^1,p(i))-\sum_{i\in N}\delta_h(R^2,p(i))\\ =&\sum_{R\in\mathbb{L}_{xy}}(\delta_h(R^1,R)-\delta_h(R^2,R))(f(R,\widehat{p})-f(\tau_{xy}R,\widehat{p})).\end{aligned}$$

which yields by Lemma's 2 and 1 that

$$\sum_{i\in N}\delta_h(R^1,p(i))-\sum_{i\in N}\delta_h(R^2,p(i))<0$$

and

$$\sum_{i\in N}\delta_h(R^1,p(i))<\sum_{i\in N}\delta_h(R^2,p(i)).$$

4 Character of Choosing the Modus

Theorem 1. *Let p be a unimodal profile with modus R^*. Then $F(p)=R^*$ for a preference rule F from $\mathbb{L}^N$ to $\mathbb{W}$ in each of the following two cases:*

1. *F is positively discriminating;*
2. *F is anonymous, neutral, monotone and discriminating.*

Proof. In order to prove the first alternative let F be positively discriminating and let $(x,y)\in R^*$ for some different alternatives x and y. It is sufficient to prove that $(x,y)\in F(p)$ and $(y,x)\notin F(p)$. Let $R\in\mathbb{L}_{xy}$. As Lemma 1 implies that $\delta(R^*,\tau_{xy}R)>\delta(R^*,R)$, because p is unimodal it follows for all $R\in\mathbb{L}_{xy}$ that $f(R,p)>f(\tau_{xy}R,p)$. But then $(x,y)\in F(p)$ and $(y,x)\notin F(p)$ because F is positively discriminating.

In order to prove the second alternative let F be anonymous, neutral, monotone and discriminating. Let $(x,y)\in R^*$ for some different alternatives x and y. It is sufficient to prove that $(x,y)\in F(p)$ and $(y,x)\notin F(p)$. Unimdularity implies for all $R\in\mathbb{L}_{xy}$ that $f(R,p)>f(\tau_{xy}R,p)$. By discrimination of F we have that $(x,y)\notin F(p)$ or $(y,x)\notin F(p)$. Suppose that $(x,y)\notin F(p)$. It is sufficient to prove that this assumption yields a contradiction. So, as $F(p)$ is a weak ordering, $(y,x)\in F(p)$ and $(x,y)\notin F(p)$. Furthermore, neutrality implies $(y,x)\notin F(\tau_{xy}p)$ and $(x,y)\in F(\tau_{xy}p)$. Now for all $R\in\mathbb{L}_{xy}$

$$f(\tau_{xy}R,\tau_{xy}p)=f(R,p)>f(\tau_{xy}R,p)=f(R,\tau_{xy}p).$$

This means that at profile $\tau_{xy}p$ every $R\in\mathbb{L}_{xy}$ is outnumbered by $\tau_{xy}R$ which is in $\mathbb{L}_{yx}$. Therefore we can take a non-empty subset S of N such that for all $R\in\mathbb{L}_{xy}$ $f(R,(\tau_{xy}p)|_{N-S})=f(\tau_{xy}R,(\tau_{xy}p)|_{N-S})$ and for all $i\in S$ we have $(x,y)\notin\tau_{xy}p(i)$. Consider profile q such that $q(j)=\tau_{xy}p(j)$ for all $j\in N-S$ and $q(i)=\tau_{xy}(\tau_{xy}p(i))=p(i)$ for all $i\in S$. Monotonicity implies that $(y,x)\notin F(q)$. Hence $(x,y)\in F(q)$. Now for all $R\in\mathbb{L}_{xy}$

$$\begin{aligned}f(R,q) &= f(R,q|_S)+f(R,q|_{N-S})\\ &= f(R,p|_S)+f(R,(\tau_{xy}p)|_{N-S})\\ &= f(R,p|_S)+f(\tau_{xy}R,(\tau_{xy}p)|_{N-S})\end{aligned}$$

$$\begin{aligned} &= f(R, p|_S) + f(R, p|_{N-S}) \\ &= f(R, p). \end{aligned}$$

As every $R \in \mathbb{L}_{xy}$ outnumbers $\tau_{xy}R$ in p it follows for all $R \in \mathbb{L}$ that $f(R, q) = f(R, p)$. But then by anonymity we have $F(q) = F(p)$, which cannot be because $(x, y) \in F(q)$ and $(x, y) \notin F(p)$. ■

The following example shows that the necessary conditions spelled out in Theorem 1 case 2 are logically independent.

Example 5. Let the dictatorial preference rule $F_{dict,i}$ with dictator i be defined for a profile p as follows

$$F_{dict,i}(p) = p(i).$$

So, this preference rule assigns the preference of agent i independent of the preferences of all other agents. Note that $F_{dict,i}$ is neutral, monotone and discriminating but of course not anonymous.

For a weak ordering R on A let the constant preference rule $F_{const,R}$ be defined for a profile p as follows

$$F_{const,R}(p) = R.$$

So, this preference rule assigns relation R independent of the preferences of the agents. Note that if $R \in \mathbb{L}$, then $F_{const,R}$ is anonymous, monotone and discriminating, where if $R = A \times A$, then $F_{const,A\times A}$ is anonymous neutral and monotone.

Let the reverse transitive closure of pairwise majority preference rule F_{odd} be defined for a profile p and a pair of alternatives x and y as follows $(x, y) \in F_{odd}(p)$ if and only if there are $y = z_0, z_1, \ldots, z_k = x$ such that for all $0 < j \leqslant k$

$$|\{i \in N : (z_j, z_{j-1}) \in p(i)\}| \geq |\{i \in N : (z_{j-1}, z_j) \in p(i)\}| .$$

It is straightforward to prove that F_{odd} is neutral, anonymous and discriminating. It is clearly not monotone.

5 Multimodal frequency distributions

It is clear that in large electorates relatively small distortions of the frequency requirements for unimodal frequency distribution will have almost no effect on the outcomes as derived above. This will especially hold if these perturbations in these frequencies involve preferences remote from the modus. Considerable distortions, as we will see, defect these finding. Such distortions are modelled here with a special type of multimodal frequency distribution: the superposition of different unimodal frequency distributions.

Let N_1 and N_2 be two disjoint sets of agents. Let $p^1 \in \mathbb{L}^{N_1}$ and $p^2 \in \mathbb{L}^{N_2}$ such that p^k is a unimodal profile on N_k with modus R^k for $k \in \{1, 2\}$. Let $N = N_1 \cup N_2$ and $p \in \mathbb{L}^N$ defined by $p(i) = p^k(i)$ if $i \in N_k$ for $k \in \{1, 2\}$. In that case p is said to be a *superposition of unimodal* profiles p^1 and p^2. It is straight forward to see that on such superposed profiles different rules may yield different outcomes. On the other hand the following theorem shows that positively discriminating preference rules agree on the intersection of these two modi.

Theorem 2. *Let N_1 and N_2 be two disjoint non-empty sets of agents, such that $N = N_1 \cup N_2$. For $k \in \{1, 2\}$ and let p^k be a unimodal profile on N_k with modus R^k. Let p be the superposition of these two unimodal profiles. Let F be a positively discriminating preference rule from $\mathbb{L}^N$ to $\mathbb{W}$.*

Then $(x, y) \in F(p)$ and $(y, x) \notin F(p)$ for all $(x, y) \in R^1 \cap R^2$, such that $x \neq y$.

Proof. Let $(x, y) \in R^1 \cap R^2$, with $x \neq y$. It is sufficient to prove that $(x, y) \in F(p)$ and $(y, x) \notin F(p)$. Because $R^1$and R^2 are linear orderings it follows that $(y, x) \notin R^1$ and $(y, x) \notin R^2$. Lemma 2 implies for all $R \in \mathbb{L}_{xy}$ that

$$f(R, p^1) > f(\tau_{xy}R, p^1) \text{ and } f(R, p^2) > f(\tau_{xy}R, p^2),$$

Therefore

$$\begin{aligned} f(R, p) &= f(R, p^1) + f(R, p^2) \\ &> f(\tau_{xy}R, p^1) + f(\tau_{xy}R, p^2) \\ &= f(\tau_{xy}R, p). \end{aligned}$$

Positive discrimination implies that $(x, y) \in F(p)$ and $(y, x) \notin F(p)$. ■

Note that this result generalizes to any arbitrary number of superposed unimodal profiles.

Although it is not difficult to find superposed unimodal profiles at which the standard problems of Social Choice appear, such as for instance Condorcet cycles, investigating these further might be fruitful. The following Theorem shows that at a superposition of two unimodal profiles intransitivity of pairwise majority can only occur on pairs which are not in the intersection of the modi. An alternative x is said to *weakly beat* an alternative y *at* a profile p if

$$|\{i \in N : (x, y) \in p(i)\}| \geq |\{i \in N : (y, x) \in p(i)\}| .$$

Theorem 3. *Let N_1 and N_2 be two disjoint non-empty sets of agents, such that $N = N_1 \cup N_2$. For $k \in \{1, 2\}$ and let p^k be a unimodal profile at N_k with modus R^k. Let p be the superposition of these two unimodal profiles. Let x, y be two different alternatives with $(x, y) \in R^1 \cap R^2$. Then there is no alternative z such that z weakly beats x and y weakly beats z at p.*

Proof. To the contrary assume that some alternative z weakly beats x and y weakly beats z. We will deduce a contraction and are done. For different alternatives a, b and c let $\mathbb{L}_{abc} = \mathbb{L}_{ab} \cap \mathbb{L}_{bc} (\cap \mathbb{L}_{ac})$. Hence, $\mathbb{L}_{abc}$ consists of all linear orderings which order a strictly above b and b strictly above c. Consequently in those orderings a is strictly ordered above c . Furthermore, for $k \in \{1, 2\}$ let

$$n^k_{abc} = \sum_{R \in \mathbb{L}_{abc}} f(R, p^k).$$

Note that for $k \in \{1, 2\}$

$$\left|\{i \in N : (x, y) \in p^k(i)\}\right| = n^k_{xyz} + n^k_{xzy} + n^k_{zxy}.$$

Therefore the assumption z weakly beats x yields

$$n^1_{zxy} + n^1_{zyx} + n^1_{yzx} + n^2_{zxy} + n^2_{zyx} + n^2_{yzx} \geq n^1_{xzy} + n^1_{xyz} + n^1_{yxz} + n^2_{xzy} + n^2_{xyz} + n^2_{yxz} \text{ and}$$

the assumption y weakly beats z yields

$$n^1_{yzx} + n^1_{yxz} + n^1_{xyz} + n^2_{yzx} + n^2_{yxz} + n^2_{xyz} \geq n^1_{zyx} + n^1_{zxy} + n^1_{xzy} + n^2_{zyx} + n^2_{zxy} + n^2_{xzy} \ .$$

Adding and simplifying these latter two inequalities yields

$$n^1_{yzx} + n^2_{yzx} \geq n^1_{xzy} + n^2_{xzy}.$$

But as $(x, y) \in R^1 \cap R^2$ Lemma 2 implies for $k \in \{1, 2\}$

$$n^k_{xzy} = \sum_{R \in \mathbb{L}_{xzy}} f(R, p^k) > \sum_{R \in \mathbb{L}_{xzy}} f(\tau_{xy} R, p^k) = \sum_{R \in \mathbb{L}_{yzx}} f(R, p^k) = n^k_{yzx}.$$

This however yields the contradiction

$$n^1_{yzx} + n^2_{yzx} < n^1_{xzy} + n^2_{xzy},$$

which ends the proof. ■

6 Relaxation of the Unimodal Assumption

In this section we assume that the frequency distribution has equal frequency at equal distances from a particular linear order, say $\widehat{R} \in \mathbb{L}$. We also assume that linear orders that are close to $\widehat{R}$ have higher frequencies than the frequencies of the linear orders that are close to $-\widehat{R}$, where $-R$ denotes the inverse linear order of R . Essentially we assume that, $f(k) > f(\binom{m}{2} - k)$ for all $k < \frac{1}{2}\binom{m}{2}$. So the frequency of a linear order depends only on the distance from $\widehat{R}$. We show that under this assumption, Pairwise majority rule and Borda rule choose $\widehat{R}$ as the outcome.

We assume that $\widehat{R} = 1, 2, 3, \ldots, m$ and $-\widehat{R} = m, m - 1, \ldots, 1$.

- $\mathbb{L}(m)$ is the set of all permutations of $\widehat{R}$.
- $\mathbb{L}_k(m) = \{R \in \mathbb{L}(m) | \delta(R, \widehat{R}) = k\}$.

- $\mathbb{L}^{i,j}(m) = \{R \in \mathbb{L}(m) | i < j \in R\}$.
- $\mathbb{L}_k^{i,j}(m) = \{R \in \mathbb{L}(m) | i < j \in R \text{ and } \delta(R, \widehat{R}) = k\}$.
- $d(m,k) = |\mathbb{L}_k(m)|$.
- $\mathbb{L}'_k(m) = \{R \in \mathbb{L}(m) | \delta(R, -\widehat{R}) = k\}$
- $d'(m,k) = |\mathbb{L}'_k(m)|$.
- $\mathbb{L}_k^{i,j}(m) = \{R \in \mathbb{L}_k(m) | (i,j) \in R\}$, i.e. i is on the left of j in R.
- $d^{i,j}(m,k) = |\mathbb{L}_k^{i,j}(m)|$

Before analysing different rules under relaxed assumptions we first present some required combinatorial results.

6.1 Useful combinatorial results

In this subsection we prove all the required lemma.

Lemma 3.

$$d(m,k) = \begin{cases} d(m-1,k) + d(m,k-1) & \textit{if } k \leq m-1 \\ d(m-1,k) + d(m-1,k-1) + \ldots + d(m-1,k-m+1) & \textit{if } k \geq m \end{cases}$$

Proof. Let $(x_1 x_2 \ldots x_m) \in \mathbb{L}_k(m)$ be such a permutation. Now possible choices of x_1 are $1, 2, \ldots, k+1$.(when $k \leq m-1$) For $x_1 = 1$, $(x_2, \ldots, x_m)$ has k discordant pairs with $(2, \ldots, m)$. For $x_1 = 2$, $(x_2, \ldots, x_m)$ has $k-1$ discordant pairs with $(1, 3, 4, \ldots, m)$. Finally for $x_1 = k+1$, $(x_2, \ldots, x_m)$ has 0 discordant pairs with $(1, 2, \ldots, k, k+2, \ldots, m)$. Hence,

$$d(m,k) = d(m-1,k) + d(m-1,k-1) + \ldots + d(m-1,1) + d(m-1,0) = d(m-1,k) + d(m,k-1).$$

Now for $k \geq m$, possible choices for x_1 are $1, 2, \ldots, m$. Hence the recursive relation becomes,

$$d(m,k) = d(m-1,k) + d(m-1,k-1) + \ldots + d(m-1,k-m+2) + d(m-1,k-m+1).$$

Thus we get the following formula

$$d(m,k) = \begin{cases} d(m-1,k) + d(m,k-1) & \text{if } k \leq m-1 \\ d(m-1,k) + d(m-1,k-1) + \ldots + d(m-1,k-m+1) & \text{if } k \geq m \end{cases}$$

■

Corollary 1. *$d(m,k-1) \leq d(m,k)$ for all $k \leq \frac{1}{2}\binom{m}{2}$.*

Lemma 4.

$$d'(m,k) = \begin{cases} d'(m-1,k) + d'(m,k-1) & \textit{if } k \leq m-1 \\ d'(m-1,k) + d'(m-1,k-1) + \ldots + d'(m-1,k-m+1) & \textit{if } k \geq m \end{cases}$$

Proof. For this proof we start with a permutation $(x_1 x_2 \ldots x_m) \in \mathbb{L}'_k(m)$. A similar reasoning as the previous proof leads to the result. ■

Hence we get the same recursive relations. Now the initial values are also the same, viz.

$d(1,0) = 1 = d'(1,0)$ for all m.

$d(1,1) = 0 = d'(1,1)$ for all $k > 0$.

$d(m,k) = 0 = d'(m,k)$ for any $k < 0$ or $k > \binom{m}{2}$.

So, $d(m,k) = d'(m,k)$ for all m and for all k.

Corollary 2. *Having k concordant pairs is the same as having $\binom{m}{2} - k$ discordant pairs. Hence, $d(m,k) = d(m, \binom{m}{2} - k)$.*

Remark 1. Note that from Lemma 3 and Lemma 4 it is clear that $d(m,k) \geq d(m,k-1)$ for all $k < \frac{1}{2}\binom{m}{2}$ and $d'(m,k) \geq d'(m,k+1)$ for all $k > \frac{1}{2}\binom{m}{2}$. We also have that $d(m, \frac{1}{2}\binom{m}{2}) \geq d(m,k)$ for all $k \in \{1, 2, \ldots, \binom{m}{2}\}$.

Lemma 5. $d^{i,j}(m,k) = d^{j,i}(m, \binom{m}{2} - k)$

Proof. For every preference R at distance k from $\widehat{R}$ there is a preference $-R$ at distance k from $-\widehat{R}$. Hence $-R$ is at distance $\binom{m}{2} - k$ from $\widehat{R}$. So, $d^{i,j}(m,k) = d'^{j,i}(m,k)$. As $d'^{j,i}(m,k) = d'^{j,i}(m, \binom{m}{2} - k)$, the result follows. ■

Lemma 6. $D^{i,j}(m,k) = d^{i,j}(m,k) - d^{j,i}(m,k) > 0$ *for all* $i < j$, $k < \frac{1}{2}\binom{m}{2}$.

Proof. We prove this by induction on m.

Step 1:(Base case)

For $m = 2$. Clearly $D^{i,j}(m,k) = 1$ for $k = 0$.

Step 2: (Inductive step)

Now we assume that

$$D^{i,j}(n,k) > 0 \text{ for some } n \in \mathbb{N}. \tag{1}$$

We have to show that $D^{i,j}(n+1,k) > 0$. We break the proof in different cases.

Case 1. $j < n+1$.

Consider a permutation $x_1, x_2, \ldots, x_{n+1} \in \mathbb{L}_k^{i,j}(n+1)$. Suppose $x_{n+1-l} = n+1$. In this permutation $n+1$ contributes in l discordant pairs. So if we remove $n+1$ from the permutation we are left with n numbers with $k-l$ discordant pairs. l can take values $1, 2, \ldots, n+1$. Thus we get the following recursive relation:

$$d^{i,j}(n+1,k) = d^{i,j}(n,k) + d^{i,j}(n,k-1) + \ldots + d^{i,j}(n,k-n).$$

With the same line of argument we have

$$d^{j,i}(n+1,k) = d^{j,i}(n,k) + d^{j,i}(n,k-1) + \ldots + d^{j,i}(n,k-n).$$

Hence,

$$D^{i,j}(n+1,k) = D^{i,j}(n,k) + D^{i,j}(n,k-1) + \ldots + D^{i,j}(n,k-n). \tag{2}$$

In case $k < \frac{1}{2}\binom{n}{2}$, applying equation 1 we directly have $D^{i,j}(n+1,k) > 0$.

Now consider, $\frac{1}{2}\binom{n}{2} < k < \frac{1}{2}\binom{n+1}{2}$. Since, $\frac{1}{2}\binom{n+1}{2} - \frac{1}{2}\binom{n}{2} = \frac{n}{2}$. In RHS of equation 2 out of $n+1$ terms, we can have at most $\frac{n}{2}$ terms with a k-argument more than $\frac{1}{2}\binom{n}{2}$. But those terms get cancelled out because of $D^{i,j}(n, \binom{n}{2} - k) = -D^{i,j}(n,k)$ by Lemma 5. We are still left with $D^{i,j}(n,k')$ for at least one $k' < \frac{1}{2}\binom{n}{2}$ and thus RHS > 0.

Case 2. $j = n+1$ and $i > 1$.

This case can be proved by similar arguments if we consider removing 1 from the permutations instead of $n+1$.

Case 3. $i = 1, j = n+1$.

Now we need to show that $D^{1,n+1}(n+1,k) > 0$. Let $y_1, y_2, \ldots, y_{n+1} \in \mathbb{L}_k^{i,j}(n+1)$ be such a permutation. Suppose $y_{l_1} = 1$ and $y_{l_{n+1}} = n+1$. So, 1 contributes in $l_1 - 1$ discordant pairs and $n+1$ contributes in $n+1-l_{n+1}$. So, if we remove 1 and $n+1$ from $y_1, y_2, \ldots, y_{n+1}$ we are left with $k - (l_1 - 1) - (n+1-l_{n+1}) = k - n + l_{n+1} - l_1$. Hence as long as $l_{n+1} - l_1$ remains the same they will have the same effect after removing. Now possible choices for l_1 and l_{n+1} are anything but maintaining $l_1 < l_{n+1}$. Clearly $1 \leq l_{n+1} - l_1 \leq n$. For $l_{n+1} - l_1 = t$, possible values for l_1 are $1, \ldots, n+1-t$. So there are $n+1-t$ cases with $l_{n+1} - l_1 = t$. Thus we have the following recursion relation

$$d^{i,j}(n+1,k) = d(n-1,k) + 2d(n-1,k-1) + \ldots + nd(n-1,k-n+1) \tag{3}$$

Similar reasoning as above yields the following recursion relation

$$d^{j,i}(n+1,k) = d(n-1,k+1-2n) + 2d(n-1,k+2-2n) + \ldots + nd(n-1,k-n) \tag{4}$$

Subtracting Equation 4 from Equation 3 we have

$$\begin{aligned} & D^{i,j}(n+1,k) \\ = \; & [d(n-1,k) - d(n-1,k+1-2n)] + 2[d(n-1,k-1) - d(n-1,k+2-2n)] \\ & + \ldots + n[d(n-1,k-n+1) - d(n-1,k-n)] \end{aligned}$$

Now by Remark 1 we have $D^{i,j}(n+1,k) > 0$ as long as $k < \frac{1}{2}\binom{n-1}{2}$. Now if $k \geq \frac{1}{2}\binom{m}{2}$ we consider the extreme case with $k = \lceil \frac{1}{2}\binom{n+1}{2} \rceil - 1$. Even then we have to check only the first $\frac{1}{2}\binom{n+1}{2} - \frac{1}{2}\binom{n-1}{2} = n - \frac{1}{2}$ terms in the RHS of the above equation because rest of them are positive directly by Remark 1. Now consider the extreme case when $k = \frac{1}{2}\binom{n+1}{2}$. Notice that $d(n-1,k) = d(n-1, \frac{1}{2}\binom{n+1}{2}) = d(n-1, \frac{1}{2}\binom{n}{2} + n - \frac{1}{2})$ and $d(n-1, k-(2n-1)) = d(n-1, \frac{1}{2}\binom{n+1}{2} - (2n-1)) = d(n-1, \frac{1}{2}\binom{n-1}{2} + n - \frac{1}{2} - (2n-1)) = d(n-1, \frac{1}{2}\binom{n-1}{2} - (n - \frac{1}{2}))$. From Remark 1 it can be easily seen that $d(n-1,k)$ and $d(n-1, k-(2n-1))$ get cancelled out. Similarly all the other terms with $k > \frac{1}{2}\binom{n-1}{2}$ get cancelled out. Again by remark 1 we are left with some positive terms. This completes the proof.

∎

Corollary 3. *$D^{i,j}(m,k) = d^{i,j}(m,k) - d^{j,i}(m,k) < 0$ for all $i < j$, $k > \frac{1}{2}\binom{m}{2}$.*

Lemma 7. *Let i, j, i', j' be such that $i - j = i' - j'$. Then $d^{i,j}(m,k) = d^{i',j'}(m,k)$ for all m and for all $k \in \{0, 1, 2, \ldots, \binom{m}{2}\}$.*

Proof. We prove this by induction on m.

Step 1:(Base case)

For $m = 3$. We have to show that $d^{1,2}(3,k) = d^{2,3}(3,k)$ for all $k = 0, 1, 2, 3$.

k=0 $d^{1,2}(3,k) = 1 = d^{2,3}(3,k)$

k=1 $d^{1,2}(3,k) = 1 = d^{2,3}(3,k)$

k=2 $d^{1,2}(3,k) = 1 = d^{2,3}(3,k)$

k=3 $d^{1,2}(3,k) = 0 = d^{2,3}(3,k)$

Step 2: (Inductive step) Here we assume that

$$d^{i,j}(n,k) = d^{i',j'}(n,k) \tag{5}$$

for some n and we have to show that $d^{i,j}(n+1,k) = d^{i',j'}(n+1,k)$. We break the proof in different cases.

Case 4. $j < n+1$.

Consider a permutation $x_1, x_2, \ldots, x_{n+1} \in \mathbb{L}_k^{i,j}(n+1)$. Suppose $x_{n+1-l} = n+1$. In this permutation $n+1$ contributes in l discordant pairs. So if we remove $n+1$ from the permutation we are left with n numbers with $k - l$ discordant pairs. l can take values $1, 2, \ldots, n+1$. Thus we get the following recursive relation:

$$d^{i,j}(n+1,k) = d^{i,j}(n,k) + d^{i,j}(n,k-1) + \ldots + d^{i,j}(n,k-n).$$

Now for all $j' < n+1$ from Equation 5 we have

$$d^{i,j}(n+1,k) = d^{i',j'}(n,k) + d^{i',j'}(n,k-1) + \ldots + d^{i',j'}(n,k-n).$$

So for $j' < n+1$ we have $d^{i,j}(n+1,k) = d^{i',j'}(n+1,k)$.

Now for $j' = n+1$, $i' > 1$(since, $j < n+1$). By similar arguments as above we can derive the formula for $d^{i',n+1}$ as follows, (if we consider removing 1 from the permutations instead of $n+1$.)

$$d^{i',n+1}(n+1,k) = d^{i',n+1}(n,k) + d^{i',n+1}(n,k-1) + \ldots + d^{i',n+1}(n,k-n).$$

Thus in this case also we have $d^{i,j}(n+1,k) = d^{i',n+1}(n+1,k)$, where i' is such that $i-j = i'-(n+1)$.

Case 5. $i > 1$.

This case is similar to the previous case the only difference being we start with removing 1 instead of $n+1$.

Case 6. $i = 1, j = n+1$.

In this case we do not have any $i' \neq i$ and $j' \neq j$ satisfying $i - j = i' - j'$.

Hence in all the above cases $d^{i,j}(n+1,k) = d^{i',j'}(n+1,k)$.

This completes the proof. ■

Lemma 8. *Let i, j, i', j' be such that $j - i < j' - i'$. Then $d^{i,j}(m,k) \leq d^{i',j'}(m,k)$ for all m and for all $k < \frac{1}{2}\binom{m}{2}$.*

Proof. We prove this by induction on m.

Step 1:(Base case)

For $m = 3$. We have to show that $d^{1,2}(3,k) \leq d^{1,3}(3,k)$ and $d^{2,3}(3,k) \leq d^{1,3}(3,k)$, for all $k = 0, 1$.

k=0 $d^{1,2}(3,k) = 1 = d^{2,3}(3,k)$ and $d^{1,3}(3,k) = 1$.

k=1 $d^{1,2}(3,k) = 1 = d^{2,3}(3,k)$ and $d^{1,3}(3,k) = 2$.

Step 2: (Inductive step) Here we assume that

$$d^{i,j}(n,k) \leq d^{i',j'}(n,k) \tag{6}$$

for some n and we have to show that $d^{i,j}(n+1,k) = d^{i',j'}(n+1,k)$. We break the proof in different cases.

Case 7. $j < n+1$.

Consider a permutation $x_1, x_2, \ldots, x_{n+1} \in \mathbb{L}_k^{i,j}(n+1)$. Suppose $x_{n+1-l} = n+1$. In this permutation $n+1$ contributes in l discordant pairs. So if we remove $n+1$ from the permutation we are left with n numbers with $k - l$ discordant pairs. l can take values $1, 2, \ldots, n+1$. Thus we get the following recursive relation:

$$d^{i,j}(n+1,k) = d^{i,j}(n,k) + d^{i,j}(n,k-1) + \ldots + d^{i,j}(n,k-n).$$

Now for all $j' < n+1$ from Equation 6 we have

$$d^{i,j}(n+1,k) \leq d^{i',j'}(n,k) + d^{i',j'}(n,k-1) + \ldots + d^{i',j'}(n,k-n).$$

So for $j' < n+1$ we have $d^{i,j}(n+1,k) \leq d^{i',j'}(n+1,k)$.

Now for $j' = n+1$, $i' > 1$(since, $j < n+1$). By similar arguments as above we can derive the formula for $d^{i',n+1}$ as follows, (if we consider removing 1 from the permutations instead of $n+1$.)

$$d^{i',n+1}(n+1,k) = d^{i',n+1}(n,k) + d^{i',n+1}(n,k-1) + \ldots + d^{i',n+1}(n,k-n).$$

Thus in this case also we have $d^{i,j}(n+1,k) \leq d^{i',n+1}(n+1,k)$, where i' is such that $i-j = i'-(n+1)$.

Case 8. $i > 1$.

This case is similar to the previous case the only difference being we start with removing 1 instead of $n+1$.

Case 9. $i = 1, j = n + 1$.

In this case we do not have any $i' \neq i$ and $j' \neq j$ satisfying $i - j < i' - j'$.

Hence in all the above cases $d^{i,j}(n+1,k) = d^{i',j'}(n+1,k)$.

This completes the proof. ■

Corollary 4.

1. Let i, j, i', j' be such that $j - i < j' - i'$. Then $d^{j,i}(m,k) \geq d^{j',i'}(m,k)$ for all m and for all $k < \frac{1}{2}\binom{m}{2}$.
2. Let i, j, i', j' be such that $j - i < j' - i'$. Then $d^{i,j}(m,k) \geq d^{i',j'}(m,k)$ for all m and for all $k > \frac{1}{2}\binom{m}{2}$.
3. Let i, j, i', j' be such that $j - i < j' - i'$. Then $d^{j,i}(m,k) \leq d^{j',i'}(m,k)$ for all m and for all $k > \frac{1}{2}\binom{m}{2}$.

Let $d^i_j(m,k)$ denote the number of permutations with exactly k discordant pairs and with i at the j^{th} position.

Lemma 9.

$$\sum_{r=1}^{m} r(d^j_r(m,k) - d^i_r(m,k)) > 0$$

for all $i < j$ and for all $k < \frac{1}{2}\binom{m}{2}$.

Proof. It is easy to see that $\sum_{r=1}^{m}(m-r)d^i_r(m,k) = \sum_{l \neq i} d^{i,l}(m,k)$. Now

$$\begin{aligned}
& \sum_{r=1}^{m}(m-r)d^i_r(m,k) - \sum_{r=1}^{m}(m-r)d^j_r(m,k) \\
= \; & \sum_{l \neq i} d^{i,l}(m,k) - \sum_{l \neq j} d^{j,l}(m,k) \\
= \; & \sum_{l<i}(d^{i,l}(m,k) - d^{j,l}(m,k)) + \sum_{i<l<j}(d^{i,l}(m,k) - d^{j,l}(m,k)) + \sum_{l>j}(d^{i,l}(m,k) - d^{j,l}(m,k))
\end{aligned}$$

Now we show that, Lemma 6,7,8 and Corollaries 3,4 implies above three sums are all negative. By Corollary 4 we have $d^{i,l}(m,k) - d^{j,l}(m,k) > 0$ for all $l < i$. By Lemma 8 we have $d^{i,l}(m,k) - d^{j,l}(m,k) > 0$ for all $l > j$.

Suppose all the numbers between i and j are like following $i < l_1 < l_2 < \ldots < l_t < j$. By Lemma 7 $d^{i,l_1}(m,k) = d^{l_t,j}(m,k)$. By Lemma 6 $d^{l_t,j}(m,k) > d^{j,l_t}(m,k)$. Hence $d^{i,l_1}(m,k) > d^{j,l_t}(m,k)$. Similarly $d^{i,l_u}(m,k) > d^{j,l_{t+1-u}}(m,k)$. It is clear to see $\sum_{i<l<j}(d^{i,l}(m,k) - d^{j,l}(m,k)) > 0$. This completes the proof. ■

Corollary 5. *In case $\binom{m}{2}$ is even, from Lemma 5 we have for $k = \frac{1}{2}$,*

$$\begin{aligned}
& \sum_{l<i}(d^{i,l}(m,k) - d^{j,l}(m,k)) + \sum_{i<l<j}(d^{i,l}(m,k) - d^{j,l}(m,k)) + \sum_{l>j}(d^{i,l}(m,k) - d^{j,l}(m,k)) \\
= \; & \sum_{l<i}(d^{l,i}(m,k) - d^{l,j}(m,k)) + \sum_{i<l<j}(d^{i,l}(m,k) - d^{l,j}(m,k)) + \sum_{l>j}(d^{i,l}(m,k) - d^{j,l}(m,k)) \\
= \; & \sum_{l<i} d^{l,i}(m,k) + \sum_{i<l<j} d^{i,l}(m,k) + \sum_{l>j} d^{i,l}(m,k) - \sum_{l<i} d^{l,j}(m,k) - \sum_{i<l<j} d^{l,j}(m,k) - \sum_{l>j} d^{j,l}(m,k) \\
= \; & d(m,k) - d(m,k)
\end{aligned}$$

Hence, $\sum_{r=1}^{m} r(d^j_r(m,k) - d^i_r(m,k)) = 0$.

Lemma 10.

$$d^i_j(m,k) = \begin{cases} d(m-1, k-j+1) & \text{if } i = 1 \\ d(m-1, k-i+1) & \text{if } j = 1 \\ \sum_{l=1}^{i-1} d^{i-1}_{j-1}(m-1, k-l+1) + & \\ \sum_{l=i+1}^{\min(k+1,m)} d^i_{j-1}(m-1, k-l+1) & \text{if } i \neq 1 \text{ and } j \neq 1 \end{cases}$$

Proof. Let $(x_1x_2\dots x_m)$ be such a permutation with k discordant pairs and with $x_j = i$. Now possible choices of x_1 are $1, 2, \dots, i-1, i+1, \dots, k+1$.(when $k \le m-1$) For $x_1 = 1$, $(x_2, \dots, x_m)$ has k discordant pairs with $(2, \dots, m)$. But for this $(x_2, \dots, x_m)$ permutation x_j is playing the role of x_{j-1} if we consider $(x_1, \dots, x_{m-1})$, and also i plays the role of $i-1$ if we consider $(1, \dots, m-1)$ instead of $(2, \dots, m)$. For $x_1 = 2$, $(x_2, \dots, x_m)$ has $k-1$ discordant pairs with $(1, 3, 4, \dots, m)$. Similarly x_j plays the role of x_{j-1} in $(x_1, \dots, x_{m-1})$ and also i plays the role of $i-1$ in $(1, 3, 4, \dots, m)$. The same thing goes on till $x_1 = i-1$. After that when $x_1 = i+1$, we again have x_j playing the role of x_{j-1} in $(x_2, \dots, x_m)$. But now i plays the role of i itself in $(1, 2, \dots, i, i+2, \dots, m)$. This goes on till $x_1 = k+1$. Finally for $x_1 = k+1$, $(x_2, \dots, x_m)$ has 0 discordant pairs with $(1, 2, \dots, k, k+2, \dots, m)$, where x_j plays the role of x_{j-1} and i plays the role of i. Hence,

$$\begin{aligned} & d^i_j(m,k) \\ = \; & d^{i-1}_{j-1}(m-1,k) + d^{i-1}_{j-1}(m-1,k-1) + \ldots + d^{i-1}_{j-1}(m-1,k-(i-1)+1) \\ & + d^i_{j-1}(m-1,k-(i+1)+1) + \ldots + d^i_{j-1}(m-1,0) \end{aligned}$$

Now for $k \ge m$, possible choices for x_1 are $1, 2, \dots, i-1, i+1, \dots, m$. Hence the recursive relation becomes,

$$\begin{aligned} & d^i_j(m,k) \\ = \; & d^{i-1}_{j-1}(m-1,k) + d^{i-1}_{j-1}(m-1,k-1) + \ldots + d^{i-1}_{j-1}(m-1,k-(i-1)+1) \\ & + d^i_{j-1}(m-1,k-(i+1)+1) + \ldots + d^i_{j-1}(m-1,k-m+1) \end{aligned}$$

For $i = 1$, number 1 being at position j contributes to $j-1$ discordant pairs. So, $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m)$ has $k-j+1$ discordant pairs with $(2, \dots, m)$. Hence,

$$d^1_j(m,k) = d(m-1,k-j+1).$$

For $j = 1$, number i contributes to $i-1$ discordant pairs. So,$(x_2, \dots, x_m)$ has $k-i+1$ discordant pairs with $(1, \dots, i-1, i+1, \dots, m)$. Hence,

$$d^i_1(m,k) = d(m-1,k-i+1).$$

Thus we get the following formula

$$d^i_j(m,k) = \begin{cases} d(m-1,k-j+1) & \text{if } i = 1 \\ d(m-1,k-i+1) & \text{if } j = 1 \\ \sum_{l=1}^{i-1} d^{i-1}_{j-1}(m-1,k-l+1) + & \\ \sum_{l=i+1}^{\min(k+1,m)} d^i_{j-1}(m-1,k-l+1) & \text{if } i \neq 1 \text{ and } j \neq 1 \end{cases}$$

■

Lemma 11. *$d(m-1,k-i+1) - d(m-1,k-j+1) \ge 0$ for all $k \le k^{max}_{i,j}$ for all $1 \le i < j \le m$, where $k^{max}_{i,j} = \lceil \frac{\binom{m-1}{2}+(i+j)-1}{2} \rceil - 1$.*

Proof. For all k, i such that $k-i+1 \le \frac{\binom{m-1}{2}}{2}$ it trivially holds because of Remark 1. Also

$$k^{max}_{i,j} - i + 1 = \binom{m-1}{2} - (k^{max}_{i,j} - j + 1).$$

$i < j$ implies $k^{max}_{i,j} - i + 1 > k^{max}_{i,j} - j + 1$, therefore $k^{max}_{i,j} - i + 1 > \frac{\binom{m-1}{2}}{2} > k^{max}_{i,j} - j + 1$. Hence for all $\frac{\binom{m}{2}}{2} \le k \le k^{max}_{i,j}$, $d(k-i+1) > d(k-j+1)$. This completes the proof. ■

Since these combinatorial results are based on permutations of a set with m numbers, m acts like a parameter in these notations. Sometimes we drop the parameter m without creating confusion. Here $i < j \in R$ meaning that i is more preferred than j in preference R is the same as $(i,j) \in R$, which essentially means i comes before j in permutation R.

Now we analyse behaviour of different rules under relaxed assumptions.

6.2 Condorcet-like Rules

Pairwise preference rule depends on the pairwise majority comparisons. In general these comparisons may yield cycles and rules therefore may differ in these situations: different pairwise rules break up cycles in different ways. If , however, at a certain profile pairwise majority comparison yields a complete strict and transitive ordering from overall winner (the Condorcet winner) to overall looser (the Condorcet looser) then this is the outcome of all these rules at that profile.

For the next result we do not assume that frequencies at a particular distance from $\widehat{R}$ are the same. Rather we want to see what happens to the previous result if we assume that $f(R) > f(R')$ where $\delta(\widehat{R}, R) = k$ and $\delta(\widehat{R}, R') = \binom{m}{2} - k$. Then,

Theorem 4. *Suppose there is a preference $\widehat{R}$ such that for all R and R' with $\delta(\widehat{R}, R) = k$ and $\delta(\widehat{R}, R') = \binom{m}{2} - k$ we have $f(R) > f(R')$ for all $k < \frac{1}{2}\binom{m}{2}$. And the distribution satisfies the following assumption*

$$\min_{R:R\in L^{i,j}(m,k)} f(R) + \min_{R:R\in L^{i,j}(m,\binom{m}{2}-k)} f(R) \geq \max_{R:R\in L^{j,i}(m,\binom{m}{2}-k)} f(R) + \max_{R:R\in L^{j,i}(m,k)} f(R)$$

for all $x, y \in A$ and for all $k < \frac{1}{2}\binom{m}{2}$. Then Pairwise majority rule selects $\widehat{R}$ as the outcome.

Proof. Let $i < j \in \widehat{R}$. The proof is similar to the last proof. So, we present the case when $\binom{m}{2}$ is odd. [6]

$$\begin{aligned}
& \sum_{R\in\mathbb{L}^{i,j}} f(R) - f(\tau_{i,j}R) \\
= & \sum_{k=0}^{\binom{m}{2}} \{ \sum_{R:R\in\mathbb{L}^{i,j}(k)} f(R) - \sum_{R:R\in\mathbb{L}^{j,i}(k)} f(R)\} \\
= & \sum_{k=0}^{\lceil\frac{1}{2}\binom{m}{2}\rceil-1} \{ \sum_{R:R\in\mathbb{L}^{i,j}(k)} f(R) - \sum_{R:R\in\mathbb{L}^{j,i}(k)} f(R)\} + \sum_{k=\lceil\frac{1}{2}\binom{m}{2}\rceil}^{\binom{m}{2}} \{ \sum_{R:R\in\mathbb{L}^{i,j}(k)} f(R) - \sum_{R:R\in\mathbb{L}^{j,i}(k)} f(R)\} \\
= & \sum_{k=0}^{\lceil\frac{1}{2}\binom{m}{2}\rceil-1} \sum_{R:R\in\mathbb{L}^{i,j}(k)} f(R) - \sum_{k=\lceil\frac{1}{2}\binom{m}{2}\rceil}^{\binom{m}{2}} \sum_{R:R\in\mathbb{L}^{j,i}(k)} f(R)\} \\
& + \sum_{k=\lceil\frac{1}{2}\binom{m}{2}\rceil}^{\binom{m}{2}} \sum_{R:R\in\mathbb{L}^{i,j}(k)} f(R) - \sum_{k=0}^{\lceil\frac{1}{2}\binom{m}{2}\rceil-1} \sum_{R:R\in\mathbb{L}^{j,i}(k)} f(R) \\
= & \sum_{k=0}^{\lceil\frac{1}{2}\binom{m}{2}\rceil-1} \{ \sum_{R:R\in\mathbb{L}^{i,j}(k)} f(R) - \sum_{R:R\in\mathbb{L}^{i,j}(\binom{m}{2}-k)} f(R)\} + \sum_{k=0}^{\lceil\frac{1}{2}\binom{m}{2}\rceil-1} \{ \sum_{R:R\in\mathbb{L}^{i,j}(\binom{m}{2}-k)} f(R) - \sum_{R:R\in\mathbb{L}^{j,i}(k)} f(R)\} \\
\geq & \sum_{k=0}^{\lceil\frac{1}{2}\binom{m}{2}\rceil-1} \{ \sum_{R:R\in\mathbb{L}^{i,j}(k)} \min f(R) - \sum_{R:R\in\mathbb{L}^{j,i}(\binom{m}{2}-k)} \max f(R)\} \\
& + \sum_{k=0}^{\lceil\frac{1}{2}\binom{m}{2}\rceil-1} \{ \sum_{R:R\in\mathbb{L}^{i,j}(\binom{m}{2}-k)} \min f(R) - \sum_{R:R\in\mathbb{L}^{j,i}(k)} \max f(R)\} \\
= & \sum_{k=0}^{\lceil\frac{1}{2}\binom{m}{2}\rceil-1} \{d^{i,j}(m,k) \min_{R:R\in\mathbb{L}^{i,j}(k)} f(R) - d^{j,i}(m, \binom{m}{2} - k) \max_{R:R\in\mathbb{L}^{j,i}(\binom{m}{2}-k)} f(R)\} \\
& + \sum_{k=0}^{\lceil\frac{1}{2}\binom{m}{2}\rceil-1} \{d^{i,j}(m, \binom{m}{2} - k) \min_{R:R\in\mathbb{L}^{i,j}(\binom{m}{2}-k)} f(R) - d^{j,i}(m,k) \max_{R:R\in\mathbb{L}^{j,i}(k)} f(R)\}
\end{aligned}$$

[6] In case $\binom{m}{2}$ is even for $k = \frac{1}{2}\binom{m}{2}$, $\min_{R:R\in L^{i,j}(m,\frac{1}{2}\binom{m}{2})} f(R) + \min_{R:R\in L^{i,j}(m,\frac{1}{2}\binom{m}{2})} f(R) \geq \max_{R:R\in L^{j,i}(m,\frac{1}{2}\binom{m}{2})} f(R) + \max_{R:R\in L^{j,i}(m,\frac{1}{2}\binom{m}{2})} f(R)$. This implies $f(R)$ is constant if $\delta(\hat{R}, R) = \frac{1}{2}\binom{m}{2}$.

$$
\begin{aligned}
&= \sum_{k=0}^{\lceil \frac{1}{2}\binom{m}{2} \rceil - 1} \{d^{i,j}(m,k) \min_{R:R\in\mathbb{L}^{i,j}(k)} f(R) - d^{i,j}(m,k) \max_{R:R\in\mathbb{L}^{j,i}(\binom{m}{2}-k)} f(R)\} \\
&\quad + \sum_{k=0}^{\lceil \frac{1}{2}\binom{m}{2} \rceil - 1} \{d^{j,i}(m,k) \min_{R:R\in\mathbb{L}^{i,j}(\binom{m}{2}-k)} f(R) - d^{yx}_{\widehat{R}}(k) \max_{R:R\in\mathbb{L}^{j,i}(k)} f(R)\} \\
&= \sum_{k=0}^{\lceil \frac{1}{2}\binom{m}{2} \rceil - 1} \{d^{i,j}(m,k)(\min_{R:R\in\mathbb{L}^{i,j}(k)} f(R) - \max_{R:R\in\mathbb{L}^{j,i}(\binom{m}{2}-k)} f(R))\} \\
&\quad + \sum_{k=0}^{\lceil \frac{1}{2}\binom{m}{2} \rceil - 1} \{d^{j,i}(m,k)(\min_{R:R\in\mathbb{L}^{i,j}(\binom{m}{2}-k)} f(R) - \max_{R:R\in\mathbb{L}^{j,i}(k)} f(R))\}
\end{aligned}
$$

We know that $d^{i,j}(m,k) > d^{j,i}(m,k)$ for all $k < \frac{1}{2}\binom{m}{2}$ and $d^{i,j}(m,k) = d^{j,i}(m, \binom{m}{2}-k)$. By assumption we have $f(R) > f(R')$ whenever $\delta(\widehat{R}, R) = k$ and $\delta(\widehat{R}, R') = \binom{m}{2} - k$. Hence the above expression is positive and thus $(x,y) \in F$. Since the result holds for any $i < j \in \widehat{R}$, we can conclude that $F(p) = \widehat{R}$. ■

The condition in the above theorem means that the minimum support in favor of $i < j$ at distance k plus distance $\binom{m}{2} - k$ must be more than the maximum support in favor of $j < i$ at distance k plus distance $\binom{m}{2} - k$.

Corollary 6. *Suppose we have a preference profile $\widehat{p}$, such that there is a preference $\widehat{R}$ such that $f(k) > f(\binom{m}{2} - k)$ for all distances $k < \frac{1}{2}\binom{m}{2}$ from $\widehat{R}$. Then pairwise majority rule chooses $\widehat{R}$ as the output.*

Corollary 7. *In general any distribution with $\min_{k<\frac{1}{2}\binom{m}{2}} f(k) > \max_{k>\frac{1}{2}\binom{m}{2}} f(k)$(where k is the distance from $\widehat{R}$) will select $\widehat{R}$ as the output of pairwise majority rule. As a special case, any distribution which has unimodal frequencies till $\frac{1}{2}\binom{m}{2}$ distance from the mode and after that all the frequencies have maximum value at most the minimum frequency from the unimodal part of the distribution, pairwise rule will choose the mode.*

Corollary 8. *Pairwise rule applied on a preference profile gives a particular output. So if there is a preference $\widehat{R}$ such that $f(k) > f(\binom{m}{2} - k)$ for all distances $k < \frac{1}{2}\binom{m}{2}$ from $\widehat{R}$, then that $\widehat{R}$ is unique. Thus the distribution can be arranged in the above mentioned way in only one way.*

6.3 Borda Rule

Now we are back at the case with constant frequencies at fixed distances. Let $d^i_r(m,k)$ be the number of preferences at distance k from $\widehat{R}$ and with alternative i at rank r.

Lemma 12.

$$d^i_r(m,k) = d^i_{m+1-r}(m, \binom{m}{2} - k).$$

Proof. Any preference at distance k from $\widehat{R}$ has the same but opposite ordered preference counterpart at distance k from $-\widehat{R}$. So a preference at distance k from $\widehat{R}$ which has alternative i at rank r, has a counter preference in just the opposite order at distance k from $-\widehat{R}$. So, in that preference alternative i has rank $m+1-r$. Hence, $d^i_r(m,k) = {d'}^i_{m+1-r}(m,k)$. Now we use the fact that ${d'}^i_{m+1-r}(m,k) = d^i_{m+1-r}(m, \binom{m}{2} - k)$, which concludes the proof. ■

Remark 2. From Lemma 9 we have

$$\sum_{r=1}^{m} r(d^j_r(m,k) - d^i_r(m,k)) > 0$$

for all $i < j$ and for all $k < \frac{1}{2}\binom{m}{2}$. As, $\sum_{r=1}^m d^i_r(m,k) = d(m,k) = \sum_{r=1}^m d^j_r(m,k)$, we have

$$\sum_{r=1}^{m}(m-r)d^i_r(m,k) > \sum_{r=1}^{m}(m-r)d^j_r(m,k))$$

for all $i < j$ and for all $k < \frac{1}{2}\binom{m}{2}$. Thus Borda rule on $\mathbb{L}_k$ selects $\widehat{R}$ as the outcome for all $k < \frac{1}{2}\binom{m}{2}$.

Theorem 5. *Suppose there is a preference $\widehat{R}$ such that $f(k) > f(\binom{m}{2} - k)$ for all distances $k < \frac{1}{2}\binom{m}{2}$ from $\widehat{R}$. Then Borda rule picks $\widehat{R}$ as the outcome.*

Proof. The proof is similar to the proof for Pairwise rule. So, we present the case when $\binom{m}{2}$ is odd. Let us denote the Borda score of an alternative i by $BS(i)$. Then by definition

$$\begin{aligned} BS(i) = & \sum_{k=0}^{\binom{m}{2}} (\sum_{r=1}^{m} (m-r) d_r^i(m,k)) f(k) \\ = & \quad m \sum_{k=0}^{\binom{m}{2}} (\sum_{r=1}^{m} d_r^i(m,k)) f(k) - \sum_{k=0}^{\binom{m}{2}} (\sum_{r=1}^{m} r d_r^i(m,k)) f(k) \\ = & \quad m \sum_{k=0}^{\binom{m}{2}} d(m,k) f(k) - \sum_{k=0}^{\binom{m}{2}} (\sum_{r=1}^{m} r d_r^i(m,k)) f(k) \end{aligned}$$

Similarly we have

$$BS(j) = m \sum_{k=0}^{\binom{m}{2}} d(m,k) f(k) - \sum_{k=0}^{\binom{m}{2}} (\sum_{r=1}^{m} r d_r^j(m,k)) f(k)$$

Hence,

$$\begin{aligned} & BS(i) - BS(j) \\ = & \sum_{k=0}^{\binom{m}{2}} \left\{ \sum_{r=1}^{m} r(d_r^j(m,k) - d_r^i(m,k)) \right\} f(k) \end{aligned}$$

From Corollary 5 we have, $\sum_{r=1}^{m} r(d_r^j(m,k) - d_r^i(m,k)) = 0$ for $k = \frac{1}{2}\binom{m}{2}$ if $\binom{m}{2}$ is even [7], we have

$$\begin{aligned} & BS(i) - BS(j) \\ = & \sum_{k=0}^{\frac{1}{2}\binom{m}{2}-1} \left\{ \sum_{r=1}^{m} r(d_r^j(m,k) - d_r^i(m,k)) \right\} f(k) + \sum_{k=\frac{1}{2}\binom{m}{2}+1}^{\binom{m}{2}} \left\{ \sum_{r=1}^{m} r(d_r^j(m,k) - d_r^i(m,k)) \right\} f(k) \end{aligned}$$

From Lemma 12 we have that,

$$d_r^i(m,k) = d_{m+1-r}^i(m, \binom{m}{2} - k).$$

Thus, [8]

$$\begin{aligned} & BS(i) - BS(j) \\ = & \sum_{k=0}^{\frac{1}{2}\binom{m}{2}-1} \left\{ \sum_{r=1}^{m} r(d_r^j(m,k) - d_r^i(m,k)) \right\} f(k) \\ & + \sum_{k=\frac{1}{2}\binom{m}{2}+1}^{\binom{m}{2}} \left\{ \sum_{r=1}^{m} r(d_{m+1-r}^j(m, \binom{m}{2} - k) - d_{m+1-r}^i(m, \binom{m}{2} - k)) \right\} f(k) \\ = & \sum_{k=0}^{\frac{1}{2}\binom{m}{2}-1} \left\{ \sum_{r=1}^{m} r(d_r^j(m,k) - d_r^i(m,k)) \right\} f(k) + \end{aligned}$$

[7] In case $\binom{m}{2}$ is an odd number we can split the sums as $0 \le k \le \lceil \frac{1}{2}\binom{m}{2} \rceil - 1$ and $\lceil \frac{1}{2}\binom{m}{2} \rceil \le k \le \binom{m}{2}$.

[8] Changing indexes in the summation term: $k' = \binom{m}{2} - k$ and $r' = m + 1 - r$. But we replace the notations by k and r.

$$\sum_{k=0}^{\frac{1}{2}\binom{m}{2}} \left\{\sum_{r=1}^{m}(m+1-r)(d_r^j(m,k)-d_r^i(m,k))\right\} f(\binom{m}{2}-k)$$

$$= \sum_{k=0}^{\frac{1}{2}\binom{m}{2}-1} \left\{\sum_{r=1}^{m} r(d_r^j(m,k)-d_r^i(m,k))\right\} \{f(k)-f(\binom{m}{2}-k)\}$$

$$+(m+1)\sum_{k=0}^{\frac{1}{2}\binom{m}{2}} \left\{\sum_{r=1}^{m}(d_r^j(m,k)-d_r^i(m,k))\right\} f(\binom{m}{2}-k)$$

$$= \sum_{k=0}^{\frac{1}{2}\binom{m}{2}-1} \left\{\sum_{r=1}^{m} r(d_r^j(m,k)-d_r^i(m,k))\right\} \{f(k)-f(\binom{m}{2}-k)\}$$

$$+(m+1)\sum_{k=0}^{\frac{1}{2}\binom{m}{2}} (d(m,k)-d(m,k))f(\binom{m}{2}-k)$$

$$= \sum_{k=0}^{\frac{1}{2}\binom{m}{2}-1} \left\{\sum_{r=1}^{m} r(d_r^j(m,k)-d_r^i(m,k))\right\} \{f(k)-f(\binom{m}{2}-k)\}$$

applying Lemma 9, we have $BS(i)-BS(j)>0$. Since the result holds for any $i<j\in\widehat{R}$, we can conclude that $F(p)=\widehat{R}$. ■

Now we will check whether Borda rule provides a result similar to pairwise rule with non-constant frequencies at fixed distance.

The proof is similar to the proof for Pairwise rule. So, we present the case when $\binom{m}{2}$ is odd. Let us denote the Borda score of an alternative i by $BS(i)$. Then by definition

$$\begin{aligned}
BS(i) = &\sum_{k=0}^{\binom{m}{2}}\sum_{R\in} f(k)\\
= \quad & m\sum_{k=0}^{\binom{m}{2}}(\sum_{r=1}^{m} d_r^i(m,k))f(k)-\sum_{k=0}^{\binom{m}{2}}(\sum_{r=1}^{m} r d_r^i(m,k))f(k)\\
= \quad & m\sum_{k=0}^{\binom{m}{2}} d(m,k)f(k)-\sum_{k=0}^{\binom{m}{2}}(\sum_{r=1}^{m} r d_r^i(m,k))f(k)
\end{aligned}$$

Similarly we have

$$BS(j)=m\sum_{k=0}^{\binom{m}{2}} d(m,k)f(k)-\sum_{k=0}^{\binom{m}{2}}(\sum_{r=1}^{m} r d_r^j(m,k))f(k)$$

Hence,

$$\begin{aligned}
& BS(i)-BS(j)\\
= \quad & \sum_{k=0}^{\binom{m}{2}}\left\{\sum_{r=1}^{m} r(d_r^j(m,k)-d_r^i(m,k))\right\} f(k)
\end{aligned}$$

From Corollary 5 we have, $\sum_{r=1}^{m} r(d_r^j(m,k)-d_r^i(m,k))=0$ for $k=\frac{1}{2}\binom{m}{2}$ if $\binom{m}{2}$ is even [9], we have

$$BS(i)-BS(j)$$

[9]In case $\binom{m}{2}$ is an odd number we can split the sums as $0\le k\le\lceil\frac{1}{2}\binom{m}{2}\rceil-1$ and $\lceil\frac{1}{2}\binom{m}{2}\rceil\le k\le\binom{m}{2}$.

$$= \sum_{k=0}^{\frac{1}{2}\binom{m}{2}-1} \left\{ \sum_{r=1}^{m} r(d_r^j(m,k) - d_r^i(m,k)) \right\} f(k) + \sum_{k=\frac{1}{2}\binom{m}{2}+1}^{\binom{m}{2}} \left\{ \sum_{r=1}^{m} r(d_r^j(m,k) - d_r^i(m,k)) \right\} f(k)$$

From Lemma 12 we have that,

$$d_r^i(m,k) = d_{m+1-r}^i(m, \binom{m}{2} - k).$$

Thus, [10]

$$\begin{aligned}
& BS(i) - BS(j) \\
= & \sum_{k=0}^{\frac{1}{2}\binom{m}{2}-1} \left\{ \sum_{r=1}^{m} r(d_r^j(m,k) - d_r^i(m,k)) \right\} f(k) \\
& + \sum_{k=\frac{1}{2}\binom{m}{2}+1}^{\binom{m}{2}} \left\{ \sum_{r=1}^{m} r(d_{m+1-r}^j(m, \binom{m}{2} - k) - d_{m+1-r}^i(m, \binom{m}{2} - k)) \right\} f(k) \\
= & \sum_{k=0}^{\frac{1}{2}\binom{m}{2}-1} \left\{ \sum_{r=1}^{m} r(d_r^j(m,k) - d_r^i(m,k)) \right\} f(k) + \\
& \sum_{k=0}^{\frac{1}{2}\binom{m}{2}} \left\{ \sum_{r=1}^{m} (m+1-r)(d_r^j(m,k) - d_r^i(m,k)) \right\} f(\binom{m}{2} - k) \\
= & \sum_{k=0}^{\frac{1}{2}\binom{m}{2}-1} \left\{ \sum_{r=1}^{m} r(d_r^j(m,k) - d_r^i(m,k)) \right\} \{f(k) - f(\binom{m}{2} - k)\} \\
& +(m+1) \sum_{k=0}^{\frac{1}{2}\binom{m}{2}} \left\{ \sum_{r=1}^{m} (d_r^j(m,k) - d_r^i(m,k)) \right\} f(\binom{m}{2} - k) \\
= & \sum_{k=0}^{\frac{1}{2}\binom{m}{2}-1} \left\{ \sum_{r=1}^{m} r(d_r^j(m,k) - d_r^i(m,k)) \right\} \{f(k) - f(\binom{m}{2} - k)\} \\
& +(m+1) \sum_{k=0}^{\frac{1}{2}\binom{m}{2}} (d(m,k) - d(m,k)) f(\binom{m}{2} - k) \\
= & \sum_{k=0}^{\frac{1}{2}\binom{m}{2}-1} \left\{ \sum_{r=1}^{m} r(d_r^j(m,k) - d_r^i(m,k)) \right\} \{f(k) - f(\binom{m}{2} - k)\}
\end{aligned}$$

applying Lemma 9 we have $BS(i) - BS(j) > 0$. Since the result holds for any $i < j \in \widehat{R}$, we can conclude that $F(p) = \widehat{R}$.

6.4 Plurality Rule

Plurality rule counts(plurality score) the number of times an alternative comes at the top of agent's preferences and ranks the alternatives in the decreasing order of their plurality scores. Let $pl(i)$ denote the plurality score for alternative i. So,

$$pl(i) = \sum_{k=0}^{\binom{m}{2}} d_1^i(m,k) f(k).$$

[10]Changing indexes in the summation term: $k' = \binom{m}{2} - k$ and $r' = m + 1 - r$. But we replace the notations by k and r.

Theorem 6. *Suppose there is a preference $\widehat{R}$ such that $f(k) > f(k+1)$ for all $k < k^{max}$ and $f(k) < f(k^{max})$ for all $k > k^{max}$, where k is the distance from $\widehat{R}$ from $\widehat{R}$ and $k^{max} = \lceil \frac{1}{2}\binom{m-1}{2} \rceil + m - 2$. Then Plurality rule picks $\widehat{R}$ as the outcome.*

Proof. Let $i < j$. We need to show that $pl(i) > pl(j)$. Now,

$$pl(i) - pl(j) = \sum_{k=0}^{\binom{m}{2}} \left\{ d_1^i(m,k) - d_1^j(m,k) \right\} f(k).$$

By lemma 10 we have

$$pl(i) - pl(j) = \sum_{k=0}^{\binom{m}{2}} \{ d(m-1, k-i+1) - d(m-1, k-j+1) \} f(k).$$

Let $0 \le k_{i,j} \le k'_{i,j}$ be two integers such that the following holds

$$k_{i,j} - i + 1 = \binom{m-1}{2} - (k'_{i,j} - j + 1)$$

For any such $(k_{i,j}, k'_{i,j})$ pair we have

$$k_{i,j} + k'_{i,j} = \binom{m-1}{2} + (i+j) - 2 \tag{7}$$

Let $k_{i,j}^{max}$ be the highest integer value of $k_{i,j}$ satisfying equation (7). Then $k_{i,j}^{max} = \lceil \frac{\binom{m-1}{2} + (i+j) - 2 - 1}{2} \rceil = \lceil \frac{\binom{m-1}{2} + (i+j) - 1}{2} \rceil - 1$. It is easy to see that $k_{i,j}^{max} = k_{j,i}^{max}$.

Since $3 \le i + j \le 2m - 1$, the minimum value of $k_{i,j}^{max}$, i.e. $\min_{i,j} k_{i,j}^{max} = k^{min} = \lceil \frac{\binom{m-1}{2} + 2}{2} \rceil - 1 = \lceil \frac{1}{2}\binom{m-1}{2} \rceil$. Similarly, $\max_{i,j} k_{i,j}^{max} = k^{max} = \lceil \frac{\binom{m-1}{2} + 2m - 2}{2} \rceil - 1 = \lceil \frac{1}{2}\binom{m-1}{2} \rceil + m - 2$.

Suppose $f(k) > f(k+1)$ for all $k < k^{max}$ and $f(k) < f(k^{max})$ for all $k > k^{max}$. Then for all $i < j$ we can break the sum as follows

$$\begin{aligned}
& pl(i) - pl(j) \\
= & \sum_{k=0}^{\binom{m}{2}} \{ d(m-1, k-i+1) - d(m-1, k-j+1) \} f(k) \\
= & \sum_{k=0}^{k_{i,j}^{max}} \{ d(m-1, k-i+1) - d(m-1, k-j+1) \} f(k) + \\
& \sum_{k=k_{i,j}^{max}+1}^{\binom{m}{2}} \{ d(m-1, k-i+1) - d(m-1, k-j+1) \} f(k) \\
= & \sum_{k=0}^{k_{i,j}^{max}} \{ d(m-1, k-i+1) - d(m-1, k-j+1) \} f(k) + \\
& \sum_{k=k_{i,j}^{max}+1}^{\binom{m}{2}} \left\{ d(m-1, \binom{m-1}{2} - (k-i+1)) - d(m-1, \binom{m-1}{2} - (k-j+1)) \right\} f(k)
\end{aligned}$$

Case 1. If $\binom{m-1}{2} + (i+j) - 2$ is even

$$= \sum_{k=0}^{k_{i,j}^{max}} \{ d(m-1, k-i+1) - d(m-1, k-j+1) \} f(k) +$$

$$
\begin{aligned}
& \sum_{k=k_{i,j}^{max}+1}^{\binom{m}{2}} \left\{d(m-1, 2k_{i,j}^{max}-k-j+1) - d(m-1, 2k_{i,j}^{max}-k-i+1)\right\} f(k) \\
= & \sum_{k=0}^{k_{i,j}^{max}} \{d(m-1, k-i+1) - d(m-1, k-j+1)\} f(k) + \\
& \sum_{k=0}^{k_{i,j}^{max}} \{d(m-1, k-j+1) - d(m-1, k-i+1)\} f(2k_{i,j}^{max}-k) \\
= & \sum_{k=0}^{k_{i,j}^{max}} \{d(m-1, k-i+1) - d(m-1, k-j+1)\} \left\{f(k) - f(2k_{i,j}^{max}-k)\right\} \\
> & \; 0
\end{aligned}
$$

Because by assumptions $f(k) > f(k+1)$ for all $k < k^{max}$ and $f(k) < f(k^{max})$ for all $k > k^{max}$ and from Lemma 11 we have $d(m-1, k-i+1) - d(m-1, k-j+1) \geq 0$ for all $k \leq k_{i,j}^{max}$.

Case 2. If $\binom{m-1}{2} + (i+j) - 2$ is odd

$$
\begin{aligned}
= & \sum_{k=0}^{k_{i,j}^{max}} \{d(m-1, k-i+1) - d(m-1, k-j+1)\} f(k) + \\
& \sum_{k=k_{i,j}^{max}+1}^{\binom{m}{2}} \left\{d(m-1, 2k_{i,j}^{max}-k-j+1) - d(m-1, 2k_{i,j}^{max}-k-i+1)\right\} f(k) \\
= & \sum_{k=0}^{k_{i,j}^{max}} \{d(m-1, k-i+1) - d(m-1, k-j+1)\} f(k) + \\
& \sum_{k=0}^{k_{i,j}^{max}-1} \{d(m-1, k-j+1) - d(m-1, k-i+1)\} f(2k_{i,j}^{max}-k) \\
= & \sum_{k=0}^{k_{i,j}^{max}-1} \{d(m-1, k-i+1) - d(m-1, k-j+1)\} \left\{f(k) - f(2k_{i,j}^{max}-k)\right\} + \\
& \left\{d(m-1, k_{i,j}^{max}-i+1) - d(m-1, k_{i,j}^{max}-j+1)\right\} \left\{f(k_{i,j}^{max}) - f(2k_{i,j}^{max}-k_{i,j}^{max})\right\} \\
> & \; 0
\end{aligned}
$$

Because by assumptions $f(k) > f(k+1)$ for all $k < k^{max}$ and $f(k) < f(k^{max})$ for all $k > k^{max}$ and from Lemma 11 we have that $d(m-1, k-i+1) - d(m-1, k-j+1) \geq 0$ for all $k \leq k_{i,j}^{max}$.

Hence we have shown that $pl(i) > pl(j)$. ■

6.5 Examples

In this subsection we show that conditions in Theorems 4,5,6 are not necessary. We consider a voting scenario with three alternatives a, b and c. The preference distribution is given in the table below:

Preferences	abc	acb	cab	cba	bca	bac
Frequencies	$f(0)$	$f(1)$	$f(2)$	$f(3)$	$f(2)$	$f(1)$

Table 1: Frequency distribution with three alternatives

Thus we have the following table showing Pairwise comparisons.

Alternative \ Alternative	a	b	c
a	$-$	$f(0)+f(1)+f(2)$	$f(0)+2f(1)$
b	$f(1)+f(2)+f(3)$	$-$	$f(0)+f(1)+f(2)$
c	$2f(2)+f(3)$	$f(1)+f(2)+f(3)$	$-$

Table 2: Pairwise comparison

Example 6. Suppose that the following conditions are satisfied

1. $f(0) > f(3)$,
2. $f(2) > f(1)$,
3. $f(0) - f(3) > 2(f(2) - f(1))$.

Condition 3 does not comply with the conditions in Theorems 4,5. But, from Table 2 it is clear that Condorcet-like rules and Borda rule ranks the alternatives as $a \succ b \succ c$ as it was in the pivotal preference abc.

Example 7. Suppose that the frequencies satisfy the following conditions:

1. $f(0) > f(2)$,
2. $f(1) > f(3)$,
3. $f(1) < f(2)$.

Condition 3 does not comply with the conditions in Theorems 6. But plurality rule ranks the alternatives as $a \succ b \succ c$ as it was in the pivotal preference abc.

7 Conclusion

8 References

- The Strategy of Social Choice-By H. Moulin.