Top inseparability and possibility results outside single peakedness *

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September 15, 2017

Very preliminary draft, do not cite.

Abstract

A domain is dictatorial if the only strategy-proof and unanimous social choice function, defined on that domain, is dictatorial. Aswal et al. (2003) shows that if a domain satisfies unique second condition, then it is not a dictatorial domain. In fact, single peaked domains (Moulin (1980)) satisfy unique second condition and the existence of non-dictatorial, strategy-proof and unanimous rules on these domains are well known. In this paper, we restrict our attention to domains that do not satisfy unique second condition. In particular, we introduce top inseparable domains and show that these domains are not dictatorial. We identify a sub class of top inseparable domains as maximal top inseparable domain. Our main result shows that maximal top inseparable domains are maximal possibility domains.

KEYWORDS. Social Choice Function, Strategy-proofness, Unanimity, Top-k inseparable domains, Dictatorial Domains.

JEL CODES. D71.

*Anup Pramanik acknowledges financial assistance from JSPS KAKENHI - JP15K17021.

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1 Introduction

In strategic social choice, (Gibbard, 1973) and (Satterthwaite, 1975) have shown that under the assumption unrestricted domain, the only strategy-proof and unanimous social choice function is a dictatorial rule. This result critically depends on the assumption of unrestricted domain. Whether by restricting the domain, one can design non dictatorial, strategy-proof rules is a natural follow up question. Previous literature has shown that for some restricted domains one can design non dictatorial strategy-proof rules (Moulin (1980), Barberà et al. (1991)), and there are restricted domains where one cannot (Aswal et al. (2003), Pramanik (2015), Sato (2010)). This paper also pursues a line of enquiry that challenges the assumption of unrestricted domain.

A domain is defined to be dictatorial if the only strategy-proof and unanimous social choice function, defined on that domain, is dictatorial. Aswal et al. (2003) shows that if a domain satisfies unique second condition, then it is not a dictatorial domain. In fact, single peaked domains (Moulin (1980)) and many other domains which are enlargement of single peaked domain satisfy unique second condition (Ching and Serizawa (1998), Barberà et al. (1999), Berga (2002)). The existence of non-dictatorial, strategy-proof and unanimous rules on these domains are well known. In this paper, we restrict our attention to domains that do not satisfy unique second condition. In particular, we introduce top inseparable domains and show that these domains are not dictatorial.

For any subset B of the set of alternatives A, top-$|B|$ inseparable domain is defined as follows. For any ordering included in this domain for which the best ranked alternative is in B, we have all the first $|B|$ best alternatives according to this ordering should also be elements of B. For example, suppose, the set of alternatives contain restaurants of several cuisine. Let there be k Indian restaurants included in the set of alternatives. Then a top-k inseparable domain will consists of orderings as follows. If an ordering has a Indian restaurant as top, then all the k Indian restaurants will be the top k alternatives according to this ordering. On the other hand if the ordering has some restaurant other than any Indian restaurants as top, then there is no restriction on this ordering. We show that for any subset B of the set of alternatives A, top-$|B|$ inseparable domains are not dictatorial. Further we introduce a notion of maximal top-$|B|$ inseparable domains. Our main result shows that any domain containing any maximal top-$|B|$ inseparable domain is dictatorial.

There are many articles which study the maximal possibility domains (see Ching and Serizawa (1998), Barberà et al. (1999), Berga (2002)). The common theme among them is that they enlarge the well-known single-peaked domain to answer the maximal possibility domain. The class of domains illustrated here, do not contain single peaked domain.

This paper is organised as follows. Section 2 introduces the notations and terminology.

Section 3 contains the results. Section 4 concludes.

2 Basic notation and definitions

Let A be the finite set of alternatives/outcomes and $N = \{1, 2, \ldots, n\}$ be the finite set of agents/individuals. We assume that $|A| = m \geq 3$. For each $i \in N$, let P_i denote individual i's strict preference relation on A i.e. P_i is a linear order over A. Let $\mathbb{P}$ be the set of all possible linear order over A. An admissible domain $\mathbb{D}$ is a subset of $\mathbb{P}$ i.e. $\mathbb{D} \subseteq \mathbb{P}$. A preference profile $P = (P_1, \ldots, P_n) \in \mathbb{D}^n$ is an n-tuple of individual preference relations. A preference profile $P = (P_1, P_2, \ldots, P_n)$ is also denoted by (P_i, P_{-i}).

For any $P_i \in \mathbb{D}$, let $r_k(P_i)$ denote the k^{th} ranked alternative in P_i, where $k \in \{1, 2, \ldots, m\}$. For any $P_i \in \mathbb{D}$ and any $B \subseteq A$, let $P_i|_B = (B \times B) \cap P_i$ denote the preference P_i restricted to B. As usual, let $r_l(P_i|_B)$ denote the l^{th} ranked alternative according to P_i restricted to B, where $l \in \{1, 2, \ldots, |B|\}$.

Definition **1.** *A social choice function (SCF) f is a mapping from $\mathbb{D}^n$ to A i.e. $f : \mathbb{D}^n \longrightarrow A$.*

In the following, we introduce two well-known properties of a SCF.

Definition **2.** *A SCF $f : \mathbb{D}^n \longrightarrow A$ satisfies unanimity if for any $a \in A$ and any $P \in \mathbb{D}^n$, whenever $r_1(P_i) = a$ for all $i \in N$, we have $f(P) = a$*

Whenever individuals agree on their top ranked alternative, a unanimous SCF must select that alternative.

Definition **3.** *A SCF $f : \mathbb{D}^n \longrightarrow A$ is strategy-proof if for any $i \in N$, for any $P \in \mathbb{D}^n$ and for any $P_i' \in \mathbb{D}$, we have $f(P) P_i f(P_i', P_{-i})$ or $f(P) = f(P_i', P_{-i})$.*

A SCF is strategy-proof if no individual can obtain a preferred alternative by misrepresenting her preferences for any announcement of the preferences of the other individuals. Strategy-proofness ensures that that for every agent truth-telling is a weakly dominant strategy.

It is well-known that a dictatorial SCF satisfies both unanimity and strategy-proofness.

Definition **4.** *A SCF $f : \mathbb{D}^n \longrightarrow A$ satisfies dictatorship if there exists an individual $i \in N$ such that for any $P \in \mathbb{D}^n$, we have $f(P) = r_1(P_i)$.*

A SCF f satisfies non-dictatorship if it does not satisfy dictatorship. The celebrated Gibbard-Satterthwaite theorem tells us that we cannot design a SCF (defined on $\mathbb{D} = \mathbb{P}$), which satisfies unanimity, strategy-proofness and non-dictatorship. Whether a domain restriction would allow us to design a unanimous, strategy-proof and non-dictatorial SCF, has been a central question in this area of research.

Throughout this paper, we consider admissible domains that satisfy following richness conditions.

DEFINITION **5.** *An admissible domain $\mathbb{D} \subseteq \mathbb{P}$ is minimally rich if for any $a \in A$, there exists a linear order $P_i \in \mathbb{D}$ such that $r_1(P_i) = a$.*

DEFINITION **6.** *An admissible domain $\mathbb{D} \subseteq \mathbb{P}$ is rich if for any $a \in A$, there exist two distinct linear order $P_i, P_i' \in \mathbb{D}$ such that $r_1(P_i) = a$ and $r_2(P_i) \neq r_2(P_i')$.*

Note that a domain is rich if and only if it is minimally rich and does not satisfy unique second property (in Aswal et al. (2003)). Since dictatorial domains do not satisfy unique second condition, therefore we can conclude that among the class of minimally rich domains, all dictatorial domains are rich domain. However, richness does not imply dictatorship (see examples in Pramanik (2015)). The formal definition of dictatorial domains is introduced below.

DEFINITION **7.** *Let $f : \mathbb{D}^n \longrightarrow A$. If f satisfies unanimity and strategy-proofness implies f satisfies dictatorship, then $\mathbb{D}$ is a dictatorial domain.*

A domain is non dictatorial if it is not dictatorial. In this paper, we restrict our attention to rich domains and our objective is to see where we can design non dictatorial strategy-proof and unanimous rules among these domains and where we cannot. We conclude this section by making the following remark.

Remark 1. If $\mathbb{D}$ is rich then it is minimally rich. However, the converse is not true. For example, single peaked domains (Moulin (1980)), single peaked domains on a tree (Demange (1982)), single crossing domains (Saporiti (2009)) and many other domains are minimally rich, but not rich and the existence of non-dictatorial, strategy-proof and unanimous rules on these domains are well known.

3 Results

In this section, among the class of rich domains, we introduce the top-k inseparable domains. To that end, we provide the following definitions.

DEFINITION **8.** *A rich domain $\mathbb{D}$ is top inseparable to the set of alternatives B, where $\emptyset \neq B \subseteq A$, if for any $P_i \in \mathbb{D}$ such that $r_1(P_i) \in B$, we have $\{r_1(P_i), r_2(P_i), \dots, r_{|B|}(P_i)\} = B$.*

Definition 9. *A rich domain $\mathbb{D}$ is top-k inseparable if there exists a set of alternatives B, where $\emptyset \neq B \subseteq A$ such that*

(i). $3 \leq |B| = k < m$ *and*

(ii). $\mathbb{D}$ *is top inseparable to B,*

Remark 2. Since we restrict our attention to rich domains, so these domains are minimally rich by definition. Further, the justification behind the bounds on k is as follows. For cases, where $k \in \{1, m\}$, every domain satisfies top-k inseparability. For the case where $k = 2$, no rich domain satisfies top-k inseparability. Our objective here is to partition the class of rich domains into dictatorial and non-dictatorial domains. So we impose the bounds on k.

Theorem 1. *If a domain $\mathbb{D}$ is top-k inseparable, then it is not a dictatorial domain.*

Proof. Let $\mathbb{D}$ be a top-k inseparable domain. So there exists a set of alternatives B, where $\emptyset \neq B \subseteq A$ such that $3 \leq |B| = k < m$ and $\mathbb{D}$ is top inseparable to B. Fix two agents $i, j \in N$. We define a SCF $f^B_{i,j} : \mathbb{D}^n \longrightarrow A$ as follows.

$$f^B_{i,j}(P) = \begin{cases} r_1(P_i) & \text{if } r_1(P_i) \notin B \\ r_1(P_j|_B) & \text{if } r_1(P_i) \in B \end{cases}$$

Note that $f^B_{i,j}$ is unanimous. To see this consider any alternative $a \in A$ and any profile $P \in \mathbb{D}$ such that $r_1(P_m) = a$ for all $m \in N$. If $a \notin B$, then it follows that $f^B_{i,j}(P) = r_1(P_i) = a$. Otherwise, if $a \in B$, then $f^B_{i,j}(P) = r_1(P_j|_B) = r_1(P_j) = a$. This shows that $f^B_{i,j}$ satisfies unanimity. Next, we show that $f^B_{i,j}$ is strategy-proof.

Proof of strategy-proofness of $f^B_{i,j}$. Note that any agent other than i and j cannot affect $f^B_{i,j}$. So they cannot gain by unilaterally misreporting their preferences. Also agent j cannot affect $f^B_{i,j}$ if $r_1(P_i) \notin B$. But when $r_1(P_i) \in B$, $f^B_{i,j}(P)$ is the top ranked alternative of agent j in B. Note that agent j cannot change B. So it follows that agent j cannot gain by unilaterally misreporting his preference. Now consider agent i. If $r_1(P_i) \notin B$, then $f^B_{i,j}(P) P_i f^B_{i,j}(P'_i, P_{-i})$ for any $P'_i \in \mathbb{D}$, as in this case $f^B_{i,j}(P) = r_1(P_i)$. Otherwise, suppose $r_1(P_i) \in B$. Note that in this case $f^B_{i,j}(P) \in B$. Now consider any $P'_i \in \mathbb{D}$. If $r_1(P'_i) \in B$, then $f^B_{i,j}(P) = f^B_{i,j}(P'_i, P_{-i})$. So suppose that $r_1(P'_i) \notin B$. This implies that $f^B_{i,j}(P'_i, P_{-i}) \in A \setminus B$. As $\mathbb{D}$ is top inseparable to B and $r_1(P_i) \in B$, it follows that $f^B_{i,j}(P) P_i f^B_{i,j}(P'_i, P_{-i})$.

This shows that $f^B_{i,j}$ is strategy-proof and concludes the proof of Theorem 1 as evidently $f^B_{i,j}$ is not a dictatorial SCF. □

Corollary 1. *If a domain $\mathbb{D}$ is dictatorial, then it is not a top-k inseparable domain.*

Proof. Let $\mathbb{D}$ be a dictatorial domain. If $\mathbb{D}$ is a top-k inseparable domain, then we immediately contradict theorem 1. Therefore, $\mathbb{D}$ is not a top-k inseparable domain. $\square$

DEFINITION **10.** *A domain $\mathbb{D}$ is maximal top-k inseparable if*

(i). $\mathbb{D}$ is top-k inseparable and

(ii). for any $P_i \in \mathbb{P} \setminus \mathbb{D}$, $\mathbb{D} \cup P_i$ is not a top-k inseparable domain.

In what follows, we provide some implications of a domain $\mathbb{D}$, which is maximal top-k inseparable.

LEMMA **1.** *Let $\mathbb{D}$ be a maximal top-k inseparable domain. Also suppose that $\emptyset \neq B \subsetneq A$ be such that $\mathbb{D}$ is top inseparable to B. Then for any $P_i \in \mathbb{P}$ such that $r_1(P_i) \in A \setminus B$, we have $P_i \in \mathbb{D}$.*

Proof. Suppose for contradiction that $P_i \notin \mathbb{D}$ for any $P_i \in \mathcal{P}$ such that $r_1(P_i) = a \in A \setminus B$. Note that as $r_1(P_i) \in A \setminus B$, we have $\mathbb{D} \cup \{P_i\}$ is top inseparable to B. This contradicts the fact that $\mathbb{D}$ is a maximal top-k inseparable domain. This concludes the proof of Lemma 1. $\square$

LEMMA **2.** *Let $\mathbb{D}$ be a maximal top-k inseparable domain. Then there exists a unique $\emptyset \neq B \subsetneq A$ such that $\mathbb{D}$ is top inseparable to B.*

Proof. As $\mathbb{D}$ is a maximal top-k inseparable domain, so there exists a $\emptyset \neq B \subsetneq A$ such that $\mathbb{D}$ is top inseparable to B. Suppose for contradiction, that this B is not unique. So suppose that there exists two non-trivial subsets B_1, B_2 of A such that $\mathbb{D}$ is top inseparable to both B_1 and B_2 and $B_1 \neq B_2$. Without loss of generality, assume that there exists two distinct alternatives $b_1, b_2 \in A$ such that $b_1 \in B_1 \setminus B_2$ and $b_2 \notin B_1$. Now consider an ordering $P_i \in \mathbb{P}$ such that $r_1(P_i) = b_1$ and $r_2(P_i) = b_2$. As $r_1(P_i) \notin B_2$ and $\mathbb{D}$ is a maximal top-k inseparable domain and $\mathbb{D}$ is top inseparable to B_2, Lemma 1 implies that $P_i \in \mathbb{D}$. Then the fact that $r_1(P_i) \in B_1$ and $r_2(P_i) \notin B_1$ and $\mathbb{D}$ is top inseparable to B_1 implies that $P_i \notin \mathbb{D}$, which contradicts the fact that $P_i \in \mathbb{D}$ and concludes the proof of Lemma 2. $\square$

DEFINITION **11.** *A domain $\mathbb{D}$ is a possibility domain of order* 1 *if*

(i). $\mathbb{D}$ is not a dictatorial domain and

(ii). for any $P_i \in \mathbb{P} \setminus \mathbb{D}$, $\mathbb{D} \cup P_i$ is a dictatorial domain.

DEFINITION **12.** *A domain $\mathbb{D}$ is a maximal possibility domain if*

(i). $\mathbb{D}$ is not a dictatorial domain and

(ii). for any $\mathbb{D}' \subseteq \mathbb{P}$ such that $\mathbb{D} \subsetneq \mathbb{D}'$, we have $\mathbb{D}'$ is a dictatorial domain.

First we show a relation between a maximal possibility domain and a possibility domain of order 1. in the following corollary.

COROLLARY **2.** *Restricted to minimally rich domains, any possibility domain of order* 1 *is a maximal possibility domain.*

Proof. Let $\mathbb{D}$ be a possibility domain of order 1. So for any $P_i \in \mathbb{P} \setminus \mathbb{D}^\star$, $\mathbb{D}^\star \cup \{P_i\}$ is dictatorial. Under our assumption, $\mathbb{D}$ is minimally rich. So, using Sanver (2007), it follows that for any $\mathbb{D}' \subseteq \mathbb{P}$ such that $\mathbb{D} \subsetneq \mathbb{D}'$, we have $\mathbb{D}'$ is a dictatorial. This concludes the proof of Corollary 2. □

This brings us to our main result.

THEOREM **2.** *A maximal top-k inseparable domain is a maximal possibility domain.*

In what follows, we will denote a maximal top-k inseparable domain by $\mathbb{D}^\star$. Note that $\mathbb{D}^\star$ is rich (minimally rich). To prove Theorem 2, we first show that for any $P_i \in \mathbb{P} \setminus \mathbb{D}^\star$, we have $\mathbb{D}^\star \cup \{P_i\}$ is a dictatorial domain. From Theorem 1, it follows that $\mathbb{D}^\star$ is not a dictatorial domain. Then, using Corollary 2, we can conclude that $\mathbb{D}^\star$ is a maximal possibility domain. First, we introduce the following well known result.

COROLLARY **3.** *Let $\Omega \subset \mathbb{P}$ be a minimally rich domain. Then, the following two statements are equivalent:*

a. $f : \Omega^2 \longrightarrow A$ is strategy-proof and satisfies unanimity $\Longrightarrow$ f is dictatorial.

b. $f : \Omega^| N| \longrightarrow A$ is strategy-proof and satisfies unanimity $\Longrightarrow$ f is dictatorial, $|N| \geq 2$.

Proof. See Aswal et al. (2003). □

Note that $\mathbb{D}^\star$ is minimally rich. Then in view of Corollary 3 it is sufficient to show that for any $P_i \in \mathbb{P} \setminus \mathbb{D}^\star$, we have $\mathbb{D}^\star \cup \{P_i\}$ is a dictatorial domain assuming $|N| = 2$.

Proof of Theorem 2. As $\mathbb{D}^\star$ is a maximal top-k inseparable domain, then Lemma 2 implies that there exists a unique non-trivial subset B of A, such that $\mathbb{D}^\star$ is top inseparable to B. Lemma 1 implies that $\mathbb{D}^\star$ is a minimally rich domain. As we restrict our attention to only rich domains, it follows that $k \geq 3$. Also it follows that for any $P_i \notin \mathbb{D}^\star$, we have the following.

i. $r_1(P_i) := \overline{a} \in B$. Follows from Lemma 1.

ii. $r_1(P_i|_{A\setminus B}) := x^\star P_i a^\star := r_k(P_i|_B)$. Follows from the fact that $\mathbb{D}^\star$ is a maximal top-k inseparable domain.

We are going to show that for any $P_i \notin \mathbb{D}^\star$, $\mathbb{D}^\star \cup \{P_i\}$ is a dictatorial domain for $|N| = 2$. In other words, let $f : \mathbb{D}^\star \cup \{P_i\}^2 \longrightarrow A$ be a strategy-proof and unanimous SCF. Then we are going to show that f is a dictatorial rule.To this end, we introduce the following notions. We define an agent $i \in N$ to be decisive over an alternative $a \in A$ at any SCF $\varphi : \mathbb{D}^2 \longrightarrow A$, if for all profile P such that $r_1(P_i) = a$, we have $\varphi(P) = a$. It follows that if an agent i is decisive over all alternatives in A at any SCF φ, then φ is a dictatorial rule and agent i is the dictator. In what follows, we are going to show that there exists an agent $i \in N$, who is decisive over all alternatives in A at the SCF f. First consider a sub-domain $\mathbb{D}'$ of $\mathbb{D}^\star$ as follows.

$$\mathbb{D}' = \{P_i \in \mathbb{D}^\star : r_1(P_i) \in B\}.$$

Let $g : \mathbb{D}'^2 \longrightarrow A$ be an SCF defined as $g(P) = f(P)$. As f is strategy-proof and unanimous and $\mathbb{D}' \subset \mathbb{D}^\star$, it follows that g is strategy-proof and unanimous. Next, we show that $R(g) = \{a \in A :$ there exists $P \in \mathbb{D}'^2$ with $g(P) = a\} = B$. Suppose for contradiction that, for some $P \in \mathbb{D}'^2$, we have $g(P) \notin B$. Now consider another profile $(P_i', P_{-i}) \in \mathbb{D}'^2$ such that $r_1(P_i') = r_1(P_{-i})$. As g satisfies unanimity, so $g(P_i', P_{-i}) = r_1(P_{-i}) \in B$. The last inclusion follows from the fact that $P \in \mathbb{D}'^2$. Now $P_i \in \mathbb{D}'$ and $g(P) \notin B$ implies that $g(P_i', P_{-i}) P_i g(P)$, which is a violation of strategy-proofness of g. So it follows that $R(g) = B$, i.e., $|R(g)| = k \geq 3$. As g is strategy-proof and unanimous, Gibbard (1973) and Satterthwaite (1975) implies that g is a dictatorial rule. Let agent $1 \in N$ be the dictator. Also as $|N| = 2$, let the other agent be denoted by 2. In what follows, we show that agent 1 is decisive for all alternatives $a \in A$ at f. To this end, we consider the following cases.

Case 1 $a \in B$: Here we consider the following two subcases.

Sub case 1 $a = a^\star$: In this case, we define the following orderings.

	P_i^1	P_j^1	P_i'	P_j'
$r_1()$	$a^\star$	a_1	$a^\star$	$x^\star$
$r_2()$	$\overline{a}$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r_k()$	$\vdots$	$a^\star$	$\vdots$	$\vdots$
$r_{k+1}()$	$\vdots$	$\vdots$	$x^\star$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r_m()$	$\vdots$	$\vdots$	$\vdots$	$a^\star$

Here a_1 is any alternative in B. Note that as $\mathbb{D}^\star$ maximal top-k inseparable domain, so it follows that $P_i^1, P_j^1, P_i', P_j' \in \mathbb{D}^\star$. In particular, $P_i^1, P_j^1, P_i' \in \mathbb{D}'$. Now consider the profile $P \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $P_1 = P_i^1$ and $P_2 = P_j^1$. As $P_i^1, P_j^1 \in \mathbb{D}'$, $g(P) = a^\star$. This follows from the fact that agent 1 is the dictator in g. So, it follows that $f(P) = a^\star$. Now consider another profile $P' \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $P_1' = P_1 = P_i^1$ and $P_2' = P_i$. Note that $r_1(P_i) = \overline{a}$. We are going to show that $f(P') \in \{a^\star, \overline{a}\}$. Suppose for contradiction, that $f(P') \in A \setminus \{a^\star, \overline{a}\}$. Note that $\overline{a} P_i^1 f(P')$ for any $f(P') \in A \setminus \{a^\star, \overline{a}\}$. Then we have a violation of strategy-proofness, as $\overline{a} = f(P_i, P_i) P_i^1 f(P')$. Here $f(P_i, P_i) = \overline{a}$ follows from unanimity of f. So it follows that $f(P') \in \{a^\star, \overline{a}\}$. Note that $P_j^1 \in \mathbb{D}'$ and $r_k(P_j^1) = a^\star$. So it follows that $\overline{a} P_j^1 a^\star$. Then strategy-proofness for the deviation from P to P' implies that $f(P') \neq \overline{a}$. Hence $f(P') = a^\star$. Now, consider another profile $\overline{P} \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $\overline{P}_2 = P_2' = P_i$ and $\overline{P}_1 = P_i'$. As $f(P') = a^\star$, strategy-proofness for the deviation from P' to $\overline{P}$ implies that $f(\overline{P}) = a^\star$. Finally consider the profile $P^\star \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $P_1^\star = \overline{P}_1 = P_i'$ and $P_2^\star = P_j'$. Note that $a^\star = r_k(P_i|_B)$. So it follows that $b P_i a^\star$ for all $b \in B \setminus \{a^\star\}$. So strategy-proofness for the deviation from $\overline{P}$ to $P^\star$ and $f(\overline{P}) = a^\star$ implies that $f(P^\star) \notin B \setminus \{a^\star\}$. Next we show that $f(P^\star) \notin (A \setminus B) \setminus \{x^\star\}$. So, suppose for contradiction that $f(P^\star) \in (A \setminus B) \setminus \{x^\star\}$. Note that $x^\star P_i' b$ for any $b \in (A \setminus B) \setminus \{x^\star\}$. This contradicts strategy-proofness as $x^\star = f(P_j', P_j') P_i' f(P^\star)$, where $f(P_j', P_j') = x^\star$ follows from unanimity of f. So it follows that $f(P^\star) \notin (A \setminus B) \setminus \{x^\star\}$. Combining, we have $f(P^\star) \in \{a^\star, x^\star\}$. As $x^\star P_i a^\star$ and $f(\overline{P}) = a^\star$, then strategy-proofness for the deviation from $\overline{P}$ to $P^\star$ implies that $f(P^\star) = a^\star$. Then strategy-proofness implies that $f(P) = a^\star$ for any $P \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $r_1(P_1) = a^\star$. So we can conclude that agent 1 is decisive for $a^\star$ at f.

Sub case 2 $a \in B \setminus \{a^\star\}$: In this case, we define the following orderings.

	P_i^1	P_j^1	P_j^2
$r_1()$	a	a_2	$\vdots$
$r_2()$	$a^\star$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r_{m-1}()$	$\vdots$	a	$a^\star$
$r_m()$	$\vdots$	$a^\star$	a

Here a_2 is any alternative in $A \setminus B$. Note that as $\mathbb{D}^\star$ maximal top-k inseparable domain, so it follows that $P_i^1, P_j^1, P_j^2 \in \mathbb{D}^\star$. Now consider the profile $P \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $P_1 = P_i^1$ and $P_2 = P_j^1$. As agent 1 is decisive for $a^\star$ at f, strategy-proofness implies that $f(P) \in \{a, a^\star\}$. Suppose $f(P) = a^\star$. Then there is a violation of strategy-proofness as $a = f(P_j^1, P_j^1) P_i^1 f(P)$. Here $f(P_j^1, P_j^1) = a$ follows from unanimity of f. So it follows that $f(P) = a$. Next, consider the profile $P' \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $P'_1 = P_1 = P_i^1$ and $P'_2 = P_j^2$. As $f(P) = a$, strategy-proofness for the deviation from P to P' implies that $f(P') \in \{a, a^\star\}$. Suppose $f(P') = a^\star$. This contradicts strategy-proofness as $f(P') = a^\star P_j^3 f(P_i^1, P_j^3) = a$, where $P_j^3 \in \mathbb{D}^\star$ is such that $r_1(P_j^3) = a^\star$. This follows form the fact that $(P_i^1, P_j^3)\mathbb{D}'^2$ and we have shown that $f(P_i^1, P_j^3) = g(P_i^1, P_j^3) = r_1(P_i^1) = a \neq a^\star$. So, we have $f(P') = a$. Then strategy-proofness implies that $f(P) = a$ for any $P \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $r_1(P_1) = a$. So we can conclude that agent 1 is decisive for a at f.

From these subcases, it follows that agent 1 is decisive for all alternatives in B at f.

Case 2 $a \in A \setminus B$: Here we consider the following two sub cases.

Sub case 1 $a = x^\star$: In this case, we define the following orderings.

	P_i^1	P_i^2	P_j^1
$r_1()$	$x^\star$	$x^\star$	$\overline{a}$
$r_2()$	$a^\star$	$\overline{a}$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r_m()$	$\vdots$	$\vdots$	$x^\star$

Note that as $\mathbb{D}^\star$ maximal top-k inseparable domain, so it follows that $P_i^1, P_i^2, P_j^1 \in \mathbb{D}^\star$. Now consider the profile $P \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $P_1 = P_i^1$ and $P_2 = P_i$. As agent 1 is decisive for $a^\star$ at f, so strategy-proofness implies that $f(P) \in \{x^\star, a^\star\}$. Suppose $f(P) = a^\star$. Then we have a violation of strategy-proofness as $x^\star = f(P_i^1, P_i^1) P_i f(P) = a^\star$. Here $f(P_i^1, P_i^1) = x^\star$ follows from unanimity of f. So it follows that $f(P) = x^\star$. Now consider the profile $P' \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $P_1' = P_i^2$ and $P_2' = P_2 = P_i$. As $x^\star = r_1(P_i^1) = r_1(P_i^2)$, strategy-proofness and $f(P) = x^\star$ implies that $f(P') = x^\star$. Finally consider the profile $P^\star \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $P_1^\star = P_1' = P_i^2$ and $P_2^\star = P_j^1$. As $f(P_j^1, P_j^1) = \overline{a}$ by unanimity of f, it follows that $f(P^\star) \in \{\overline{a}, x^\star\}$ by strategy-proofness. Suppose that $f(P^\star) = \overline{a}$. This contradicts strategy-proofness as $\overline{a} = f(P^\star) P_i f(P') = x^\star$. So we have $f(P^\star) = x^\star$. Then strategy-proofness implies that $f(P) = x^\star$ for any $P \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $r_1(P_1) = x^\star$. So we can conclude that agent 1 is decisive for $x^\star$ at f.

Sub case 2 $a \in (A \setminus B) \setminus \{x^\star\}$: In this case, we define the following orderings.

	P_i^1	P_j^1	P_i^2	P_j^2
$r_1()$	a	a_3	a	a_3
$r_2()$	$x^\star$	$\vdots$	a_3	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r_{k-1}()$	$\vdots$	a	$\vdots$	$x^\star$
$r_m()$	$\vdots$	$x^\star$	$\vdots$	a

Here a_3 is any alternative in A. Note that as $\mathbb{D}^\star$ maximal top-k inseparable domain, so it follows that $P_i^1, P_i^2, P_j^1, P_j^2 \in \mathbb{D}^\star$. Now consider the profile $P \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $P_1 = P_i^1$ and $P_2 = P_j^1$. As agent 1 is decisive for $x^\star$ at f, so strategy-proofness implies that $f(P) \in \{a, x^\star\}$. Suppose $f(P) = x^\star$. This violets strategy-proofness as $a = f(P_i^1, P_i^1) P_j^2 f(P) = x^\star$. Here $f(P_i^1, P_i^1) = a$ follows from unanimity of f. So it follows that $f(P) = a$. Now consider the profile $P' \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $P_1' = P_i^2$ and $P_2' = P_2 = P_j^1$. As $a = r_1(P_i^1) = r_1(P_i^2)$, strategy-proofness and $f(P) = a$ implies that $f(P') = a$. Finally consider the profile $P^\star \in \mathbb{D}^\star \cup \{P_i\}^2$ such that $P_1^\star = P_1' = P_i^2$ and $P_2^\star = P_j^2$. As $f(P') = a$, strategy-proofness for the deviation from P' to $P^\star$ implies that $f(P^\star) \in \{a, x^\star\}$. Suppose $f(P^\star) = x^\star$. This violets strategy-proofness as $a_3 = f(P_j^2, P_j^2) P_i^2 f(P^\star) = x^\star$. Here $f(P_j^2, P_j^2) = a_3$ follows from unanimity of f. So we have $f(P^\star) = a$. Then strategy-proofness implies that $f(P) = a$ for any $P \in \mathbb{D}^\star \cup \{P_i\}^2$ such that

$r_1(P_1) = a$. So we can conclude that agent 1 is decisive for a at f.

From these subcases, it follows that agent 1 is decisive for all alternatives in $A \setminus B$ at f.

Combining this cases, it follows that agent 1 is decisive for all alternatives in A at f. This shows that if agent 1 is the dictator in g, then agent 1 is the dictator in f. Similarly we can show that if agent 2 is the dictator in g, then agent 2 is the dictator in f. Then we can conclude, using Corollary 3, that $\mathbb{D}^{\star} \cup \{P_i\}$ is a dictatorial domain. This concludes the proof of Theorem 2. □

4 Conclusion

In this paper, we introduce the notion of a top-$|B|$ inseparable domain for any $B \subset A$. We show that such domains are not dictatorial. Further, we introduce the notion of maximal top-$|B|$ inseparable domain for any $B \subset A$. We show that any domain, which is a super set of the maximal top-$|B|$ inseparable domain for any $B \subset A$, is dictatorial.

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