# A characterization of lexicographic preferences

Mridu Prabal Goswami[*] Manipushpak Mitra[†] Debapriya Sen[‡]

September 15, 2017

**Abstract**

This paper characterizes lexicographic preferences over alternatives that are identified by a finite number of attributes. We say two alternatives are 'totally different' if they are different with respect to every attribute. Our characterization is based on two key concepts: a weaker notion of continuity called 'mild continuity' (strict preference order between any two totally different alternatives is preserved around their small neighborhoods) and an 'unhappy set' (any alternative outside the set is preferred to all alternatives inside).

**Keywords:** lexicographic preference; mild continuity; unhappy set; inclusion of marginally improved alternatives; preference nonreversibility

## 1 Introduction

This paper characterizes lexicographic preferences over alternatives that are identified by a finite number $n$ attributes, so the set of all alternatives is the set of all $n$-tuples of non negative real numbers. The seminal work of Fishburn (1975) axiomatized lexicographic preferences using a technique of proof that is closely connected with Arrow's impossibility theorem. More recently, Petri and Voorneveld (2016) propose an alternative characterization which is based on the robustness of preference ordering between two alternatives for changes in a few rather than a large number of coordinates.

Our approach of characterizing a lexicographic preference uses two key concepts: a weaker version of continuity called mild continuity and the notion of unhappy sets. To define mild continuity, call two alternatives totally different if they are different with respect to *every* attribute. A preference relation is mildly continuous if strict preference order between any two totally different alternatives is preserved around their small neighborhoods. A set of alternatives is called an unhappy set if any alternative outside the set is preferred to all alternatives inside.[1]

[*]Indira Gandhi Institute of Development Research, Mumbai
[†]Indian Statistical Institute, Kolkata
[‡]Ryerson University, Toronto

[1]Unhappy sets were introduced in Mitra and Sen (2017). In a companion paper (Goswami et al., 2017) we use unhappy sets and naive reasoning to give an alternative characterization of lexicographic preferences.

One of our axioms imposes a certain structure on the unhappy sets of the preference relation. Consider two totally different alternatives that belong to an unhappy set. Suppose the second alternative is superior with respect to a subset $S$ of attributes while the first is superior with respect to the complementary subset. We say an unhappy set satisfies inclusion of marginally improved alternatives (IMIA) if it includes a third alternative that does better than the first but worse than the second with respect to the attributes in $S$ but has the same levels as the first alternative for the remaining attributes. First we show that when there are only two attributes, then any rational preference relation is lexicographic if and only if it is strong monotone, mildly continuous and any unhappy set satisfies IMIA (Proposition 3).

Extending to the general case of more than two attributes, the starting point is to compare alternatives for which all but two attributes are missing.[2] Fix any subset of two attributes and consider all alternatives for which all attributes outside that subset have zero level. Interpreting the zero level as the absence of an attribute, one can view these alternatives as having only two attributes. The original preference relation gives an immediate preference order, called an induced preference, between alternatives for which all but two fixed attributes are missing. One of the axioms requires that any closed unhappy set for any such induced preference satisfies IMIA.

Another axiom requires preference nonreversibility. As before consider two alternatives for which all but two attributes are missing. Suppose one of these alternatives is strictly preferred to the other. Now suppose we add one or more attributes to each of these alternatives keeping the levels of additional attributes same across the two. Preference nonreversibility holds if the preference order between such new pairs of alternatives stays the same as before. It can be noted that it is a weaker requirement than independence.

We show that in the the general case of more than two attributes, a rational preference relation is lexicographic if and only if it is (a) strong montone and (b) any induced preference between alternatives for which all but two same attributes are missing satisfies (i) mild continuity, (ii) IMIA for any closed unhappy set and (iii) preference nonreversibility (Theorem 1).

The paper is organized as follows. We present the analytical framework in Section 2. The axioms of lexicographic preference are discussed in Section 3. The main result is presented in Section 4. Section 5 gives examples to illustrate the robustness of the axioms. Section 6 shows how IMIA gives a structure to unhappy sets. Most proofs are presented in the Appendix.

## 2 The analytical framework

Consider an individual who has a preference relation $\succsim$ on a set of alternatives $X$. Each alternative is characterized by $n$ attributes. Let $N = \{1, \ldots, n\}$ be the set of attributes. The domain of any attribute $i \in N$ is $\mathbb{R}_+$. Therefore an alternative is given by a vector

[2] This approach is closely related to choice rules based on "elimination by aspects" proposed by Tversky (1972).

$x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ and the set of all alternatives is $X = \mathbb{R}^n_+$.

For any non empty $S \subseteq N$, denote $X_S = \mathbb{R}^{|S|}_+$. For any $x \in X$ and $S \subseteq N$, we write $x = (x^S, x^{N \setminus S})$ where $x^S \in X_S$ and $x^{N\setminus S} \in X_{N\setminus S}$ (note that $x = x^N$). If $x_i = 0$ for all $i \in S$, we write $x^S = 0^S$.

In the special case when $S$ is the singleton set $\{i\}$, it will be convenient to use the simpler notation $x^S = x_i$, $x^{N\setminus S} = x_{-i}$, $X_S = X_i$ and $X_{N\setminus S} = X_{-i}$. We write $x = (x_i, x_{-i})$ where $x_i \in X_i$ and $x_{-i} \in X_{-i}$.

The distance between $x, y \in X$, denoted by $d(x, y)$, is given by the Euclidean metric. For $x^S, y^S \in X_S$, $d(x^S, y^S)$ is the same metric $d$ restricted to $X_S$. A neighborhood of $x^S$ is a set $B_\varepsilon(x^S)$ consisting of all $y^S \in S$ such that $d(x^S, y^S) < \varepsilon$ for some $\varepsilon > 0$.

The individual's preference on $X$ is defined using the binary relation $\succsim$ . where "$x \succsim y$" stands for "the individual prefers $x$ to $y$". The strict preference is defined as $x \succ y \Leftrightarrow [x \succsim y]$ and not $[y \succsim x]$. The indifference relation is defined as $x \sim y \Leftrightarrow [x \succsim y]$ and $[y \succsim x]$.

For any $x \in X$, the lower contour set under $\succsim$ is $L(x) = \{y \in X \mid x \succsim y\}$. The strict lower contour set is $\underline{L}(x) = \{y \in X \mid x \succ y\}$ and the indifference set is $I(x) = \{y \in X \mid x \sim y\}$.

Let $x, y \in X$. If $x_i > y_i$ for all $i \in N$, we write $x > y$. If $x_i \geq y_i$ for all $i \in N$, we write $x \geq y$. Two alternatives $x, y$ are *non-comparable*, denoted by $x \bowtie y$, if there are $i, j \in N$ such that $y_i > x_i$ and $x_j > y_j$.

The preference relation $\succsim$ on $X$ is monotone if for any $x, y \in X$ with $x > y$, we have $x \succ y$. It is strong monotone if for any $x, y \in X$ with $x \geq y$ and $x \neq y$, we have $x \succ y$.

**Definition 1** A preferences $\succsim$ on $X$ is *lexicographic* if there is a linear order $i(1) <_0 i(2) <_0 \ldots <_0 i(n)$ on $N$ such that for any $x, y \in X$, $x \succ y$ if and only if either $[x_{i(1)} > y_{i(1)}]$ or $[x_{i(1)} = y_{i(1)}, x_{i(2)} > y_{i(2)}]$ or ...or $[x_{i(1)} = y_{i(1)}, \ldots, x_{i(n-1)} = y_{i(n-1)}, x_{i(n)} > y_{i(n)}]$.

For such a preference, $i(1)$ will be called the most important attribute, $i(2)$ the next most important attribute and so on.

## 2.1 Some useful concepts

To characterize lexicographic preferences, it will be useful to introduce two concepts: (i) mild continuity of a preference relation, which is a weaker version of continuity and (ii) unhappy sets, which are related to lower contour sets. The notion of induced preferences will be also useful.

### 2.1.1 Mild continuity

**Definition 2** For $x, y \in X$, we say $x$ and $y$ are *totally different*, denoted by $x \mathrel{\hat{\neq}} y$, if $x_i \neq y_i$ for all $i \in N$.

Note that $x \mathrel{\hat{\neq}} y$ if and only if $y \mathrel{\hat{\neq}} x$. For $x^S, y^S \in X_S$, we define $x^S \mathrel{\hat{\neq}} y^S$ similarly. A preference relation is mildly continuous if strict preference order between any two totally different points is preserved around their small neighborhoods.

**Definition 3** A preference relation $\succsim$ is *mildly continuous* on $X$ if for any $x, y \in X$ with $x \mathrel{\hat{\neq}} y$ and $x \succ y$, there exists $\varepsilon > 0$ such that if $\widetilde{x} \in B_\varepsilon(x)$ and $\widetilde{y} \in B_\varepsilon(y)$, then $\widetilde{x} \succ \widetilde{y}$.

We recall that a preference relation $\succsim$ is continuous on $X$ if for any $x, y \in X$ with $x \succ y$, there exists $\varepsilon > 0$ such that if $\widetilde{x} \in B_\varepsilon(x)$ and $\widetilde{y} \in B_\varepsilon(y)$, then $\widetilde{x} \succ \widetilde{y}$. Thus for a continuous preference strict preference order between any two points, totally different or otherwise, is preserved around their small neighborhoods.

### 2.1.2 Unhappy sets

**Definition 4** A set $A \subseteq X$ is an *unhappy set for a preference relation* $\succsim$ on $X$ if for any $y \in X \setminus A$, $y \succ x$ for every $x \in A$.

Observe that lower contour and strict lower contour sets are unhappy sets. So are the sets $X$ and $\varnothing$.

Let $A$ be a subset of a metric space $X$. A point $x \in X$ is a boundary point of $A$ if every neighborhood of $x$ contains at least one point in $A$ and at least one point in $X \setminus A$. The set of all boundary points of $A$ is called the boundary of $A$ and denoted by $\partial A$. To characterize unhappy sets we recall the following result. For the proof, see, e.g., Mendelson (1990: Theorem 4.23, Chapter 3 and Definition 2.1 of Chapter 4).

**Result 1** *If $A$ is a non empty proper subset of a connected space, then* (i) $\partial A \neq \varnothing$ *and* (ii) *the set $A$ cannot be both open and closed.*

Since $\mathbb{R}^n_+$ is a connected set, we can use this result for subsets of $X = \mathbb{R}^n_+$.

**Proposition 1** *Consider a rational preference relation $\succsim$ on $X = \mathbb{R}^n_+$.*

(i) *If $A, B$ are unhappy sets, then either $A \subseteq B$ or $B \subseteq A$.*

(ii) *Let $A$ be a non empty proper subset of $X$. If $A$ is an unhappy set and $\succsim$ is mildly continuous then for any $x \in \partial A$, the following hold for any $y \mathrel{\hat{\neq}} x$.*

  (a) *If $y \in \partial A$, then $x \sim y$.*

  (b) *If $y \in A$, then $x \succsim y$.*

  (c) *If $x \succ y$, then $y \in A$.*

**Proof** See the Appendix. ■

For a continuous preference relation, the last part of the proposition also holds for $y$ that are not totally different from $x$. In that case unhappy sets can be more precisely characterized.

**Corollary 1** *Consider a rational and continuous preference relation $\succsim$ on $X = \mathbb{R}^n_+$. Let $A$ be an unhappy set which is a non empty proper subset of $X$. Then the following hold for any $x \in \partial A$.*

(i) *$x \sim y$ for any $y \in \partial A$.*

(ii) $\underline{L}(x) \subseteq A \subseteq L(x)$.

(iii) *The set $A$ must be either closed or open, but not both. If $A$ is closed, then $A = L(x)$ and if $A$ is open, then $A = \underline{L}(x)$.*

**Proof** See the Appendix. ■

Thus for a continuous preference a set is unhappy set if and only if it is a lower contour or a strict lower contour set. A lexicographic preference is not continuous, although it is mildly continuous. Any lower contour set of a lexicographic preference is neither open nor closed.

### 2.1.3 Induced preferences

For our analysis it will be useful to consider alternatives for which certain attributes have the minimum level zero. Fix a subset of attributes and consider all points for which attributes in that subset have zero level. Interpreting the zero level as the absence of that attribute, one can view these points as alternatives that have a fewer number of attributes. For example, if built-in camera is considered an attribute for cellphones, a cellphone with no camera has the zero level for that attribute.

The original preference relation gives an immediate preference order between alternatives for which a fixed subset of attributes is missing. This is formalized by an induced preference, which is defined as follows.

**Definition 5** Let $S$ be a non empty subset of $N$. For a preference relation $\succsim$ on $X$, the *induced preference* $\succsim_S$ on $X_S = \mathbb{R}_+^{|S|}$ is defined as follows: for $y^S, z^S \in X_S$, $y^S \succsim_S z^S$ if and only if $(y^S, 0^{N \setminus S}) \succsim (z^S, 0^{N \setminus S})$.

Thus $\succsim_S$ is a preference relation over all alternatives for which the attributes in the set $N \setminus S$ are absent. Note that the induced preference $\succsim_N$ coincides with $\succsim$ . The following observation is immediate.

**Observation 1**

(i) *If $\succsim$ is rational, then $\succsim_S$ is also rational. If $\succsim$ is strong monotone, then $\succsim_S$ is also strong monotone.*

(ii) *If $\succsim$ is lexicographic with linear order $1 <_0 2 <_0 \ldots <_0 n$ and $S = \{i_1, \ldots, i_s\} \subseteq N$ where $i_1 < \ldots < i_s$, then $\succsim_S$ is lexicographic with linear order $i_1 <_0 i_2 <_0 \ldots <_0 i_s$.*

**Remark** Note that the converse of Observation 1 does not hold. We give an example where an induced preference is lexicographic, but the original preference is not.

**Example 1** Let $N = \{1, 2, 3\}$. Consider a preference relation $\succsim$ on $X = \mathbb{R}^3_+$ that is reflexive ($x \sim x$ for all $x \in X$) and for which following hold for any $x_i, y_i \in \mathbb{R}_+$: (a) $(x_1, x_2, x_3) \succ (y_1, y_2, y_3)$ if $x_1 > y_1$, (b) $(0, x_2, x_3) \succ (0, y_2, y_3)$ if $x_2 > y_2$, (c) $(0, x_2, x_3) \succ (0, x_2, y_3)$ if $x_3 > y_3$, (d) $(x_1, x_2, x_3) \succ (x_1, y_2, y_3)$ if $x_1 > 0$ and $x_2 + x_3 > y_2 + y_3$ and (e) $(x_1, x_2, x_3) \sim (x_1, y_2, y_3)$ if $x_1 > 0$ and $x_2 + x_3 = y_2 + y_3$. Note that $\succsim$ is rational and strong monotone. By (a) and (b), for any $S \subseteq N$ with $|S| = 2$, the induced preference $\succsim_S$ is lexicographic, but clearly $\succsim$ is not lexicographic.

#### 2.1.4 Unhappy and lower contour sets for induced preferences

Unhappy sets can be similarly defined for induced preferences. We say a set $A \subseteq X_S$ is an unhappy set for the induced preference $\succsim_S$ if for any $b \in X_S \setminus A$, $b \succ_S a$ for every $a \in A$.

For $x^S \in X_S$, denote by $L^*(x^S)$ the lower contour set of $x^S$ for the induced preference $\succsim_S$, that is,

$$L^*(x^S) = \{y^S \in X_S | x^S \succsim_S y^S\}$$

Observe that $y^S \in L^*(x^S)$ if and only if $(x^S, 0^{N \setminus S}) \succsim (y^S, 0^{N \setminus S})$. Denote by $\overline{L}^*(x^S)$ the closure of the lower contour set $L^*(x^S)$, that is, $\overline{L}^*(x^S) := L^*(x^S) \cup \partial L^*(x^S)$.

**Proposition 2** *Consider a rational and strong monotone preference relation $\succsim$ on $X = \mathbb{R}^n_+$. Let $S \subseteq N$. Suppose for any $T \subseteq S$, the induced preference $\succsim_T$ is mildly continuous on $X_T$. Then for every $x^S \in X_S$, the set $\overline{L}^*(x^S)$ is an unhappy set for $\succsim_S$ .*

**Proof** See the Appendix. ∎

Note that a strong monotone preference relation on $\mathbb{R}_+$ is continuous, and so it is mildly continuous. Suppose $\succsim$ is strong monotone and let $T \subseteq N$ be a singleton set. Then the induced preference $\succsim_T$, which is defined on $\mathbb{R}_+$, is strong monotone and therefore mildly continuous. When $X = \mathbb{R}^2_+$ in Proposition 2, then any non empty proper subset $X$ is a singleton set and mild continuity already holds for the corresponding induced preference. This gives the following result.

**Corollary 2** *For a rational, strong monotone and mildly continuous preference relation on $\mathbb{R}^2_+$, the closure of any lower contour is an unhappy set.*

## 3 Characterization of lexicographic preferences

We begin with a property which together with strong monotonicity and mild continuity can characterize a lexicographic preference in the case where there are only two attributes. This property will also be the basis of an axiom for the characterization of a general lexicographic preference.

### 3.1 Inclusion of marginally improved alternatives

Consider $x, y \in X$ such that $x \hat{\neq} y$ and let $S \subseteq N$. We say (i) $y$ is *$S$-superior* to $x$ if $y^S > x^S$ and (ii) $y$ is *exhaustively $S$-superior* to $x$ if $y^S > x^S$ and $x^{N\setminus S} > y^{N\setminus S}$. If $y > x$, then clearly $y$ is (exhaustively) $N$-superior to $x$.

**Definition 6** For a preference relation $\succsim$ on $X$, an unhappy set $A$ satisfies *inclusion of marginally improved alternatives* (IMIA) if the following hold for any $x \hat{\neq} y$ and any $S \subseteq N$: if $x, y \in A$ with $y^S > x^S$ and $x^{N\setminus S} > y^{N\setminus S}$, then $\exists\ \widetilde{x}^S \in X_S$ with $y^S > \widetilde{x}^S > x^S$ such that $(\widetilde{x}^S, x^{N\setminus S}) \in A$.

That is, for an unhappy set $A$ satisfying IMIA, if $x, y \in A$ and $y$ is $S$-superior to $x$, then any new alternative that does better than $x$ but worse than $y$ with respect to the attributes in $S$ but is the same as $x$ for the remaining attributes does not necessarily take us outside of the unhappy set $A$.

**Remark 1** When $S = N$ in Definition 6, we have $y > x$. So for a monotone preference, the requirement of IMIA always holds when $S = N$. IMIA imposes additional structure on a monotone preferences only when $S \subset N$.

**Remark 2** If $x$ is an interior point of $A$ then there is a neighborhood $B_\varepsilon(x) \subset A$ and we can find a point $\widetilde{x} = (\widetilde{x}^S, x^{N\setminus S}) \in B_\varepsilon(x)$ such that $y^S > \widetilde{x}^S > x^S$. Thus IMIA always holds for open unhappy sets such as strict lower contour sets. IMIA imposes additional structure on the preference only for unhappy sets that are not open. For example, for continuous preferences such as Cobb-Douglas, perfect substitutes or perfect complements IMIA does not hold for lower contour sets. For a lexicographic preference IMIA does not hold for lower contour sets either (which are neither open nor closed). However, any closed unhappy set of a lexicographic preference satisfies this property.

**Lemma 1** *Any closed unhappy set for a lexicographic preference satisfies IMIA.*

**Proof** See the Appendix. ■

When a closed unhappy set satisfies IMIA, it can be bounded with respect to at most one attribute. This gives a structure to such sets, which is stated in the next lemma. See Section 6 for more details of this structure. For a preference relation $\succsim$, let $\mathcal{A}_{\succsim}$ be a family of subsets of $X$ defined as follows

$$\mathcal{A}_{\succsim} = \{A | A \subset X; \exists y \in A \text{ with } y > 0^N; A \text{ is a closed unhappy set that satisfies IMIA}\} \quad (1)$$

**Lemma 2** *Consider a rational, strong monotone and mildly continuous preference relation $\succsim$ on $X$. Let $A \in \mathcal{A}_{\succsim}$. Then $\exists$ a positive number $\alpha^A$ and an attribute $i^* \in N$ such that $A = \{x \in X | 0 \leq x_{i^*} \leq \alpha^A\}$. Moreover the attribute $i^*$ is the same for all sets in $\mathcal{A}_{\succsim}$.*

**Proof** See the Appendix. ■

### 3.2 Lexicographic preference with two attributes

Applying Lemma 2 for the case $X = \mathbb{R}^2_+$ and using Corollary 2, we can characterize a lexicographic preference with two attributes. This result is of independent interest. Moreover, it will be also useful for characterizing lexicographic preferences with more than two attributes.

**Proposition 3** *A rational preference relation $\succsim$ on $\mathbb{R}^2_+$ is lexicographic if and only if it is strong monotone, mildly continuous and any unhappy set of $\succsim$ satisfies IMIA.*

**Proof** See the Appendix. ■

### 3.3 Lexicographic preference with more than two attributes

#### 3.3.1 Strong monotonicity, IMIA

**Axiom 1** The preference relation $\succsim$ is strong monotone.

**Axiom 2** Consider any $S \subseteq N$ with $|S| = 2$. Any closed subset of $X_S = \mathbb{R}^2_+$ that is an unhappy set for the preference $\succsim_S$, satisfies IMIA.

**Remark** Let $A$ be an unhappy set for a monotone preference relation on $\mathbb{R}^2_+$. Then $A$ satisfies IMIA if for any $x, y \in A$ with $y_1 > x_1$ and $x_2 > y_2$, there is $\widetilde{x}_1$ with $y_1 > \widetilde{x}_1 > x_1$ such that $(\widetilde{x}_1, x_2) \in A$.

#### 3.3.2 Mild continuity of induced preference

Consider two points both of which have zero levels for all but two attributes, say 1 and 2. Also suppose these two points are different with respect to both attributes $1, 2$ and one of them is strictly preferred to other. The next axiom requires that the initial preference order is preserved when we move to new pairs of points that are found by small changes in attributes 1 or 2, still keeping all other attributes at zero level. This is simply the requirement that the induced preference $\succsim_S$ is mildly continuous for any $S$ that contains only two attributes.

Note that when two points are same with respect to all but one attribute, say 1, by strong monotonicity preference orders are preserved for small changes with respect to attribute 1.

**Axiom 3** For any $S \subseteq N$ with $|S| = 2$, the induced preference $\succsim_S$ is mildly continuous on $X_S = \mathbb{R}^2_+$.

#### 3.3.3 Nonreversibility under additional attributes

As before consider two points for which all but two attributes are missing. Suppose one of these points is strictly preferred to the other. Now suppose we add one or more attributes to each of these points keeping the levels of additional attributes same across the two. Preference nonreversibility holds if the preference order between such new pairs of points stays the same as before.

**Definition 7** A preference relation $\succsim$ on $X$ satisfies *nonreversibility under additional attributes* (NRAA) if for any $S \subseteq N$ with $|S| = 2$, the following hold: if $(x^S, 0^{N/S}) \succ (y^S, 0^{N/S})$, then $(x^S, z^{N/S}) \succ (y^S, z^{N/S})$ for any $z^{N/S} \in X_{N\setminus S}$.

**Remark** Independence would require for any $z^{N\setminus S}, \tilde{z}^{N\setminus S} \in X_{N\setminus S}$: $(x^S, z^{N/S}) \succ (y^S, z^{N/S})$ if and only if $(x^S, \tilde{z}^{N/S}) \succ (y^S, \tilde{z}^{N/S})$. Thus NRAA is a weaker condition than independence. Also note that for a preference relation that has an additively separable utility function, NRAA holds.

**Axiom 4** The preference relation $\succsim$ satisfies nonreversibility under additional attributes.

# 4 The main result

**Theorem 1** *Consider any rational preference relation $\succsim$ on $X$. The following statements are equivalent.*

(LP1) *The preference relation $\succsim$ satisfy Axiom* 1, *Axiom* 2, *Axiom* 3 *and Axiom* 4.

(LP2) *The preference relation $\succsim$ is lexicographic.*

## 4.1 Proof of Theorem 1

**Proof of (LP2)⇒(LP1)**

Consider a lexicographic preference relation $\succsim$. It is strong monotone, so Axiom 1 holds. Since for $\succsim$, the induced preference $\succsim_S$ is mildly continuous for any $S \subseteq N$, Axiom 3 holds. Since for any $S \subseteq N$, whenever $(x^S, 0^{N\setminus S}) \succ (y^S, 0^{N\setminus S})$, we have $(x^S, z^{N\setminus S}) \succ (y^S, z^{N\setminus S})$ for any $z^{N\setminus S} \in X_{N\setminus S}$, Axiom 4 also holds.

Finally recall from Observation 1 that for a lexicographic preference $\succsim$, the induced preference $\succsim_S$ is also lexicographic for any $S \subseteq N$ (and in particular when $|S| = 2$). Applying Lemma 1 for $\succsim_S$ it follows that any closed unhappy set for $\succsim_S$ satisfies IMIA. Hence Axiom 2 also holds. This shows that (LP2)⇒(LP1). ■

**Proof of (LP1)⇒(LP2)**

We begin with a lemma, which will be useful for the proof.

**Notation** Let $i, j \in N$ and $S = \{i, j\}$. We write "$i \succ^* j$" to mean "$\succsim_S$ is a lexicographic preference with linear order $i <_0 j$".

**Lemma 3** *Consider a rational preference relation $\succsim$ that satisfy Axiom* 1, *Axiom* 2, *Axiom* 3 *and Axiom* 4.

(i) *Consider any $S \subseteq N$ with $|S| = 2$. Then $\exists\, i, j \in S$ such that*

(a) *$i \succ^* j$ so that $(x^S, 0^{N\setminus S}) \succ (y^S, 0^{N\setminus S})$ if and only if either $x_i > y_i$ or ($x_i = y_i$ and $x_j > y_j$).*

(b) *For any $z^{N\setminus S} \in X_{N\setminus S}$ : $(x^S, z^{N\setminus S}) \succ (y^S, z^{N\setminus S})$ if and only if either $x_i > y_i$ or ($x_i = y_i$ and $x_j > y_j$).*

(ii) *Let $i, j, k \in N$. If $i \succ^* j$ and $j \succ^* k$, then $i \succ^* k$.*

(iii) *Based on $\succ^*$, the attributes of $N$ can be ordered. That is, we can write $N = \{i_1, \ldots, i_n\}$ such that $i_1 \succ^* \ldots \succ^* i_n$.*

**Proof** See the Appendix. ■

**Proof of (LP1)⇒(LP2)** Consider a rational preference relation $\succsim$ that satisfies Axiom 1, Axiom 2, Axiom 3, and Axiom 4. By Lemma 3 we can write $N = \{i_1, \ldots, i_n\}$ such that $i_1 \succ^* \ldots \succ^* i_n$. Without loss of generality let $i_1 = 1, \ldots, i_n = n$. Fix any $m \in \{1, \ldots, n-1\}$. Let $S = \{i \in N | i < m\}$ and $T = \{i \in N | i > m\}$. Let $x, y \in X$ such that $x^S = y^S$ and $x_m > y_m$. We can find numbers $x_m^0, \ldots, x_m^{n-m}$ such that $x_m = x_m^0 > x_m^1 > \ldots > x_m^{n-m} = y_m$. Construct $z(k) \in X$ recursively as follows: $z(0) = x$ and for $k = 1, \ldots, n-m$, $z(k)$ is such that

$$z(k)_m = x_m^k, z(k)_{m+k} = y_{m+k}, z(k)^S = z(0)^S = x^S \text{ and } z(k)^{T\setminus\{m+k\}} = z(k-1)^{T\setminus\{m+k\}}$$

Observe that $z(k)$ and $z(k-1)$ have same levels for all but two attributes ($m$ and $m+k$). Since $m \succ^* m+k$ and $z(k-1)_m = x_{m-1}^k > z(k)_m = x_m^k$, by Lemma 3(i)(b) we have $z(k-1) \succ z(k)$. Noting that $z(n-m) = y$, we conclude that $x = z(0) \succ \ldots \succ z(n-m) = y$. Hence $x \succ y$.

Thus for any $x, y$ with $x_i = y_i$ for all $i < m$ and $x_m > y_m$, we have $x \succ y$. Applying this result for $m = 1, \ldots, n-1$ it follows that $\succsim$ is a lexicographic preference with linear order $1 <_0 \ldots <_0 n$. ■

## 5 Robustness of the axioms

We give examples of rational preferences to show that if one of the four axioms does not hold, then we do not get a lexicographic preference.

**Example 2** Let $N = \{1, 2, 3\}$. Consider a rational and continuous preference relation $\succsim$ on $X = \mathbb{R}_+^3$ that has utility function $u(x_1, x_2, x_3) = x_1$. Note that this is not a strong monotone preference, so Axiom 1 is violated. We show the remaining three axioms hold.

Note that for any $S \subseteq N$ with $|S| = 2$, the induced preference $\succsim_S$ is continuous, so Axiom 3 holds. As $\succsim_S$ is continuous, any closed unhappy set for $\succsim_S$ is a lower contour set. To see Axiom 2 and Axiom 4 hold, first let $S = \{1, 2\}$. Since $x^S = (x_1, x_2) \succ_S y^S = (y_1, y_2)$ if and only if $x_1 > y_1$, we have $L^*(x^S) = \{y^S \in \mathbb{R}_+^2 | 0 \leq y_1 \leq x_1\}$. If $y, z \in L^*(x^S)$ such that $z_1 < y_1$, then we can find $z_1 < \widetilde{z}_1 < y_1 \leq x_1$, so $(\widetilde{z}_1, z_2) \in L^*(x^S)$. If $y, z \in L^*(x^S)$ such that $z_2 < y_2$, then for any $z_2 < \widetilde{z}_2 < y_2$, $(z_1, \widetilde{z}_2) \in L^*(x^S)$. This shows IMIA holds for any closed unhappy set for $\succsim_S$ . Similar reasoning applies when $S = \{1, 3\}$.

Note that if $(x_1, x_2, 0) \succ (y_1, y_2, 0)$, then $x_1 > y_1$. So for any $x_3$ we have $u(x_1, x_2, x_3) = x_1 > u(y_1, y_2, x_3) = y_1$. This shows the requirement for NRAA is satisfied when $S = \{1, 2\}$. Similar reasoning applies when $S = \{1, 3\}$.

Finally let $S = \{2, 3\}$. Then $u(0, x_2, x_3) = 0$. Hence for any $x^S, y^S \in X_S$ we have $x^S \sim_S y^S$ so that $L^*(x^S) = \mathbb{R}^2_+$. This shows IMIA trivially holds for any closed unhappy set for $\succsim_S$. Since for any $x^S = (x_2, x_3), y^S = (y_2, y_3)$, we have $(0, x_2, x_3) \sim_S (0, y_2, y_3)$, the requirement for NRAA is satisfied vacuously when $S = \{2, 3\}$. This shows both Axiom 2 and Axiom 4 hold.

**Example 3** Let $N = \{1, 2, 3\}$. Consider a rational, continuous preference relation $\succsim$ on $X = \mathbb{R}^3_+$ with utility function $u(x) = x_1 + x_2 + x_3$ (perfect substitutes). This preference is strong monotone, so Axiom 1 holds. Since it is continuous, Axiom 3 also holds. Since the utility function is additively separable, NRAA (Axiom 4) holds.

However, Axiom 2 does not hold. For instance, let $S = \{1, 2\}$. Consider the induced preference $\succsim_S$, which has utility function $u_S(x^S) = x_1 + x_2$. It is continuous, so by Corollary 2 any closed unhappy set of $\succsim_S$ is a lower contour set. Consider the lower contour set $L^*(x^S)$ for $x^S = (1, 3)$. Let $y^S = (3, 1)$. Then $x^S, y^S \in L^*(x^S)$. Also note that $x^S \hat{\ngeq} y^S$ with $y_1 > x_1$ and $x_2 > y_2$. But there is no $\widetilde{x}_1$ with $y_1 > \widetilde{x}_1 > x_1$ such that $(\widetilde{x}_1, x_2) \in L^*(x^S)$. This is because for any $\widetilde{x}_1 > x_1$, we have $\widetilde{x}_1 + x_2 > x_1 + x_2 = u_S(x^S)$, so such a point will be outside $L^*(x^S)$.

**Example 4** Let $N = \{1, 2, 3\}$. Consider the lexi-max preference $\succsim$ introduced by Bossert, Pattanaik and Xu (1994). For $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+$, denote by $x^*_i$ the $i$-th highest order statistics of $x$ so that $x^*_1 \geq x^*_2 \geq x^*_3$. For the lexi-max preference, for any $x_i, y_i \in \mathbb{R}_+$ we have: (a) $x \succ y$ if either $[x^*_1 > y^*_1]$, or $[x^*_1 = y^*_1,\ x^*_2 > y^*_2]$, or $[x^*_1 = y^*_1,\ x^*_2 = y^*_2,\ x^*_3 > y^*_3]$ and (b) $x \sim y$ if $[x^*_1 = y^*_1,\ x^*_2 = y^*_2,\ x^*_3 = y^*_3]$.

Note that $\succsim$ is rational. It does not satisfy Axiom 3, but satisfies the remaining three axioms. To see Axiom 3 is violated, let $S = \{1, 2\}$ and consider the induced preference $\succsim_S$ . Let $x = (3, 2, 0)$, $y = (1, 3, 0)$, so that $x^S = (3, 2)$, $y^S = (1, 3)$. Note that $x^S \hat{\ngeq} y^S$. Since $x \succ y$, we have $x^S \succ_S y^S$. Any neighborhood of $y^S$ contains a point $z^S = (1, z_2)$ such that $z_2 > 3$, so that $z^*_1 = z_2 > 3 = x^*_1$ and we have $z^S \succ_S x^S$. This shows $\succsim_S$ is not mildly continuous, so Axiom 3 does not hold.

To see that the remaining three axioms hold, note that $\succsim$ is strong monotone, so Axiom 1 holds. To see Axiom 2 and Axiom 4 hold, consider any $S \subseteq N$ with $|S| = 2$. Without loss of generality let $S = \{1, 2\}$. For $x^S = (x_1, x_2)$, $y^S = (y_1, y_2)$, recall that the induced preference $\succsim_S$ is defined as

$$x^S \succsim_S y^S \text{ if and only if } (x_1, x_2, 0) \succsim (y_1, y_2, 0)$$

So the induced preference $\succsim_S$ is a lexi-max preference on $X_S = \mathbb{R}^2_+$. That is, (a) $x^S \succ_S y^S$ if either $[x_1^* > y_1^*]$, or $[x_1^* = y_1^*,\ x_2^* > y_2^*]$ and (b) $x^S \sim_S y^S$ if $[x_1^* = y_1^*,\ x_2^* = y_2^*]$. Let $A$ be any closed unhappy set for $\succsim_S$ . In what follows we show $A$ satisfies IMIA.

Let $x^S, y^S \in A$ be such that $y_1 > x_1$ and $x_2 > y_2$. If $x_1 \geq x_2$, then we have $y_1^* = y_1 > x_1^* = x_1$, so that $y^S \succ_S x^S$. Consider any $\widetilde{x}_1$ with $y_1 > \widetilde{x}_1 > x_1$ and let $\widetilde{x}^S = (\widetilde{x}_1, x_2)$. Then $y^S \succ_S \widetilde{x}^S$, so we have $\widetilde{x}^S \in A$.

Finally let $x^S, y^S \in A$ be such that $y_1 > x_1$, $x_2 > y_2$ and $x_2 > x_1$. Consider any $\widetilde{x}_1$ with $\min\{x_2, y_1\} > \widetilde{x}_1 > x_1$ and let $\widetilde{x}^S = (\widetilde{x}_1, x_2)$. Then any neighborhood of $\widetilde{x}^S$ contains a point $z^S = (z_1, z_2)$ such that $z_1 = \widetilde{x}_1 < z_2 < x_2$. Then $z_1^* = z_2 < x_1^* = x_2$ and we have $x^S \succ_S z^S$, so $z^S \in A$. This shows $\widetilde{x}^S$ is a limit point of $A$. Since $A$ is closed, we must have $\widetilde{x}^S \in A$. This shows $A$ satisfies IMIA, so Axiom 2 holds.

To verify Axiom 4 (NRAA) holds, let $(x_1, x_2, 0) \succ (y_1, y_2, 0)$. Then either $[x_1^* > y_1^*]$ or $[x_1^* = y_1^*,\ x_2^* > y_2^*]$. Take any $x_3 \in \mathbb{R}_+$ and let $a, b \in \mathbb{R}^3_+$ be such that

$$a_1 = x_1, a_2 = x_2, a_3 = x_3 \text{ and } b_1 = y_1, b_2 = y_2, b_3 = x_3$$

We have to show $a \succ b$. First let $x_3 \geq x_1^* \geq y_1^*$. Then $a_1^* = b_1^* = x_3$, $a_2^* = x_1^*$, $b_2^* = y_1^*$, $a_3^* = x_2^*$ and $b_3^* = y_2^*$. Since either $[a_2^* > b_2^*]$ or $[a_2^* = b_2^*,\ a_3^* > b_3^*]$, we conclude that $a \succ b$.

Next suppose $x_3 < x_1^*$. If $x_1^* > y_1^*$, then $a_1^* = x_1^* > b_1^* = \max\{y_1^*, x_3\}$, so we have $a \succ b$.

Finally suppose $x_3 < x_1^*$ and let $x_1^* = y_1^*$, $x_2^* > y_2^*$. Then $a_1^* = b_1^* = x_1^*$. If $x_2^* \leq x_3$, then $a_2^* = b_2^* = x_3$ and $a_3^* = x_2^* > b_3^* = y_2^*$, so we have $a \succ b$. If $x_2^* > x_3$, then $a_2^* = x_2^*$ and $b_2^* = \max\{y_2^*, x_3\} < a_2^*$, so we have $a \succ b$. This shows that Axiom 4 (NRAA) also holds.

**The preference of Example 1** This preference is rational. It is strong monotone, so Axiom 1 holds. For any $S \subseteq N$ with $|S| = 2$, the induced preference $\succsim_S$ is lexicographic, so it is mildly continuous and Axiom 3 holds. By Lemma 1, any closed unhappy set of $\succsim_S$ satisfies IMIA, so Axiom 2 also holds. Note from (c) and (d) that $(0, 6, 4) \succ (0, 3, 8)$ but $(1, 3, 8) \succ (1, 6, 4)$ so it violates Axiom 4 (NRAA).

The last two examples show without rationality we do not get lexicographic preference even with all four axioms. For these two examples, it will be useful to recall $x, y \in X$, we say $x, y$ are non-comparable, denoted by $x \bowtie y$, if $\exists\ i, j$ such that $y_i > x_i$ and $x_j > y_j$.

**Example 5** Let $N = \{1, 2, 3\}$. Consider a preference relation $\succsim$ on $X = \mathbb{R}^3_+$ that is reflexive, strong monotone and $x \sim y$ whenever $x \bowtie y$. Note that $\succsim$ is complete but not transitive. For example, let $x = (3, 2, 0)$, $y = (2, 2, 0)$ and $z = (4, 1, 0)$. By strong monotonicity, $x \succ y$. Since $y \bowtie z$, we have $y \sim z$. For transitivity to hold we must have $x \succ z$. However, since $x \bowtie z$, we have $x \sim z$, so transitivity does not hold.

The preference relation is strong monotone, so Axiom 1 holds. To see the remaining axioms also hold, consider any $S \subseteq N$ with $|S| = 2$. Without loss of generality let $S = \{1, 2\}$. Let $A$ be an unhappy set for $\succsim_S$ and let $x^S, y^S \in A$ such that $y_1 > x_1$, $x_2 > y_2$. Then $(x_1, x_2, 0) \bowtie (y_1, y_2, 0)$, so we have $x^S \sim_S y^S$. For any $\widetilde{x}_1$ with $y_1 > \widetilde{x}_1 > x_1$, let $\widetilde{x}^S = (\widetilde{x}_1, x_2)$.

Since $(\widetilde{x}_1, x_2, 0) \bowtie (y_1, y_2, 0)$, we have $\widetilde{x}^S \sim_S y^S$. As $A$ is an unhappy set, we must have $\widetilde{x}^S \in A$. This shows any unhappy set for $\succsim_S$ satisfies IMIA, so Axiom 2 holds.

To see Axiom 3 holds, let $x^S \hat{\not\geq} y^S$ and $x^S \succ_S y^S$. Then we must have $x^S > y^S$. Since there are neighborhoods $B_\varepsilon(x^S), B_\varepsilon(y^S)$ such that $\widetilde{x}^S > \widetilde{y}^S$ for all $\widetilde{x}^S \in B_\varepsilon(x^S)$ and $\widetilde{y}^S \in B_\varepsilon(y^S)$, by strong monotonicity of $\succsim_S$ it follows that $\succsim_S$ is mildly continuous, so Axiom 3 holds.

Finally to see Axiom 4 (NRAA) holds, let $(x_1, x_2, 0) \succ (y_1, y_2, 0)$. Then we must have $x^S = (x_1, x_2) \geq y^S = (y_1, y_2)$ and $x^S \neq y^S$. So for any $x_3$, $(x_1, x_2, x_3) \geq (y_1, y_2, x_3)$ and $(x_1, x_2, x_3) \neq (y_1, y_2, x_3)$. This shows Axiom 4 also holds.

**Example 6** Let $N = \{1, 2, 3\}$. Consider a preference relation $\succsim$ on $X = \mathbb{R}^3_+$ that is reflexive, strong monotone and whenever $x \bowtie y$, we have neither $x \succsim y$ nor $y \succsim x$. This preference relation is transitive but not complete.

The preference relation is strong monotone, so Axiom 1 holds. By the same reasoning as in Example 5, Axiom 3 and Axiom 4 (NRAA) also hold. To see Axiom 2 holds, consider any $S \subseteq N$ with $|S| = 2$. Without loss of generality let $S = \{1, 2\}$. Let $A$ be an unhappy set for $\succsim_S$ and let $x^S, y^S \in A$ such that $y_1 > x_1$, $x_2 > y_2$. For any $\widetilde{x}_1$ with $y_1 > \widetilde{x}_1 > x_1$, let $\widetilde{x}^S = (\widetilde{x}_1, x_2)$. Since $(\widetilde{x}_1, x_2, 0) \bowtie (y_1, y_2, 0)$, we have neither $\widetilde{x}^S \succsim_S y^S$ nor $y^S \succsim_S \widetilde{x}^S$. So in particular we do not have $\widetilde{x}^S \succ_S y^S$. Since $A$ is an unhappy set, we must have $\widetilde{x}^S \in A$. This shows Axiom 2 holds.

## 6 Boundedness of unhappy sets under IMIA

When an unhappy set is inclusive of marginally improved alternatives, there are some useful implications with respect to boundedness of that set. The following observation will be useful to understand the properties of unhappy sets under IMIA.

**Observation 2** *Let $A$ be an unhappy set for a monotone preference relation $\succsim$ on $X$. If $A$ is inclusive of marginally improved alternatives, then the following hold for $x, y \in A$:*

(i) *If $y^S > x^S$ and $x^{N\setminus S} > y^{N\setminus S}$, then for any $T \subseteq S$, $\exists\ \widetilde{x}^T \in X_T$ with $y^T > \widetilde{x}^T > x^T$ such that $(\widetilde{x}^T, x^{N\setminus T}) \in A$.*

(ii) *If $y_i > x_i$ and $x_{-i} \hat{\not\geq} y_{-i}$, then $\exists\ \widetilde{x}_i$ with $y_i > \widetilde{x}_i > x_i$ such that $(\widetilde{x}_i, x_{-i}) \in A$.*

**Proof** (i) By IMIA, $\exists\ \widetilde{x}^S \in X_S$ with $y^S > \widetilde{x}^S > x^S$ such that $(\widetilde{x}^S, x^{N\setminus S}) \in A$. As the preference is monotone, for any $T \subseteq S$, we have $(\widetilde{x}^S, x^{N\setminus S}) \succsim (\widetilde{x}^T, x^{N\setminus T})$. As $A$ is an unhappy set, the result follows.

(ii) Since $y_i > x_i$ and $x_{-i} \hat{\not\geq} y_{-i}$, there is a non empty set $S$ (in particular, $i \in S$) such that $y^S > x^S$ and $x^{N\setminus S} > y^{N\setminus S}$ (the set $N \setminus S$ can be possibly empty). Then the result follows by taking $T = \{i\}$ in part (i). ■

Let $A \subseteq X$ and $x = (x_i, x_{-i}) \in A$. Denote $A(x_{-i}) := \{y_i \in X_i | (y_i, x_{-i}) \in A\}$. We say $A(x_{-i})$ is bounded above if there is a number $k$ such that $y_i \leq k$ for all $y_i \in A(x_{-i})$.

**Proposition 4** *Consider a rational and strong monotone preference relation $\succsim$ on $X$. Let $A$ be a proper subset of $X$. Suppose $A$ is a closed unhappy set which satisfies IMIA. Let $x \in A$ be such that $x > 0^N$.*

(i) *Let $i \in N$. If $A(x_{-i})$ is bounded above, then*

 (a) $\exists$ $\alpha(x_{-i})$ *(the least upper bound of $A(x_{-i})$) such that $A(x_{-i}) = [0, \alpha(x_{-i})]$.*

 (b) *For any $z \in A$, $A(z_{-i})$ is also bounded above with the same least upper bound. That is, $\alpha(x_{-i}) = \alpha(z_{-i}) = \alpha_i > 0$ and $A(x_{-i}) = A(z_{-i}) = [0, \alpha_i]$. Moreover the points $(\alpha_i, x_{-i}), (\alpha_i, z_{-i})$, are both boundary points of $A$.*

(ii) *Let $i \in N$. If $A(x_{-i})$ is not bounded above, then for any $z \in A$, $A(z_{-i})$ is also not bounded above and $A(x_{-i}) = A(z_{-i}) = \mathbb{R}_+$.*

(iii) *Suppose $\succsim$ is also mildly continuous. Then there can be at most one $i \in N$ where the following hold: $\exists$ $z \in A$ with $z > 0^N$ such that $A(z_{-i})$ is bounded above.*

**Proof** (i)(a) Since $x = (x_i, x_{-i}) \in A$, we have $x_i \in A(x_{-i})$, so $A(x_{-i})$ is a non empty subset of $\mathbb{R}$. As $A(x_{-i})$ is bounded above, by the least-upper-bound property of $\mathbb{R}$ (see, e.g., Theorem 1.19, Rudin, 1976) $A(x_{-i})$ has a least upper bound $\alpha(x_{-i}) \geq x_i > 0$. As $A$ is an unhappy set, strong monotonicity of the preference implies whenever $y_i \in A(x_{-i})$, any $y'_i < y_i$ also belongs to $A(x_{-i})$. Hence $(y_i, x_{-i}) \in A$ for $0 \leq y_i < \alpha(x_{-i})$ and $(y_i, x_{-i}) \notin A$ for $y_i > \alpha(x_{-i})$. Finally note that any neighborhood of $(\alpha(x_{-i}), x_{-i})$ contains a point $(y_i, x_{-i})$ with $y_i < \alpha(x_{-i})$. So $(\alpha(x_{-i}), x_{-i})$ is a limit point of $A$. Since $A$ is a closed set, $(\alpha(x_{-i}), x_{-i}) \in A$. This proves that $A(x_{-i}) = [0, \alpha(x_{-i})]$.

(i)(b) Since $z \in A$, we have $z_i \in A(z_{-i})$, so $A(z_{-i})$ is non empty. First suppose $z_{-i} \hat{\neq} x_{-i}$. Since $A$ satisfies IMIA and $(\alpha(x_{-i}), x_{-i}) \in A$, if $(y_i, z_{-i}) \in A$ with $y_i > \alpha(x_{-i})$, then by Observation 2(ii), $\exists$ $\widetilde{x}_i > \alpha(x_{-i})$ such that $(\widetilde{x}_i, x_{-i}) \in A$, implying $\widetilde{x}_i \in A(x_{-i})$, contradicting (a). So $A(z_{-i})$ must be bounded above by $\alpha(x_{-i})$ and it has a least upper bound $\alpha(z_{-i}) \leq \alpha(x_{-i})$.

If $\alpha(z_{-i}) = 0$, it is the only point of $A(x_{-i})$ and $(\alpha(z_{-i}), z_{-i}) \in A$. If $\alpha(z_{-i}) > 0$, using strong monotonicity of the preference and closedness of the unhappy set $A$ as in (a) we again have $(\alpha(z_{-i}), z_{-i}) \in A$. Since $(\alpha(x_i), x_{-i}) \in A$, if $\alpha(z_{-i}) < \alpha(x_{-i})$, then by Observation 2(ii), $\exists$ $\widetilde{z}_i > \alpha(z_{-i})$ such that $(\widetilde{z}_i, z_{-i}) \in A$. This implies $\widetilde{z}_i \in A(z_{-i})$, a contradiction. So we must have $\alpha(z_{-i}) = \alpha(x_{-i}) > 0$. Denoting their common value by $\alpha_i$, it follows that $A(x_{-i}) = A(z_{-i}) = [0, \alpha_i]$ for any $z_{-i} \hat{\neq} x_{-i}$.

Now consider the case where $z_{-i}, x_{-i}$ are not totally different. Since $x > 0^N$, we can construct $\widetilde{x}$ such that for all $j \in N$: $0 < \widetilde{x}_j < x_j$ and $\widetilde{x}_j \neq z_j$. Since $x \in A$ and $x > \widetilde{x}$, by strong monotonicity, $\widetilde{x} \in A$. Since $\widetilde{x}_{-i} \hat{\neq} x_{-i}$, applying the result of the last paragraph we have $A(\widetilde{x}_{-i}) = [0, \alpha]$. Finally observing that $\widetilde{x} > 0^N$ and $z_{-i} \hat{\neq} \widetilde{x}_{-i}$, reapplying the result of the last paragraph we have $A(z_{-i}) = [0, \alpha_i]$.

As any neighborhood of $(\alpha_i, x_{-i})$ contains two points $y = (y_i, x_{-i}), y' = (y'_i, x_{-i})$ such that $y'_i < \alpha_i < y_i$, we have $y \in A$ and $y' \in X \setminus A$. This shows $(\alpha_i, x_{-i}) \in \partial A$. The same holds for $(\alpha_i, z_{-i})$.

(ii) As $A(x_{-i})$ is not bounded above, for any $k > 0$, there is $y_i > k$ such that $y_i \in A(x_{-i})$. As $A$ is an unhappy set, strong monotonicity of the preference implies any $y'_i < y_i$ also belongs to $A(x_{-i})$. This shows $A(x_{-i}) = \mathbb{R}_+$.

For $z \in A$, if the set $A(z_{-i})$ is bounded above, it has a least upper bound $\alpha(z_{-i}) \geq z_i$ and by the same reasoning as before, $(\alpha(z_{-i}), z_{-i}) \in A$. Consider any $k > \alpha(z_{-i})$. Since $A(x_{-i}) = \mathbb{R}_+$, $(k, x_{-i}) \in A$. As $x > 0^N$, we can construct $\widetilde{x}$ such that $\widetilde{x}_i = k$ and for all $j \in N \setminus \{i\}$: $0 < \widetilde{x}_j < x_j$ and $\widetilde{x}_j \neq z_j$. Since $(k, x_{-i}) \geq \widetilde{x}$, by strong monotonicity $\widetilde{x} \in A$. Since $(\alpha(z_{-i}), z_{-i}) \in A$, $\widetilde{x}_{-i} \hat{\not\geq} z_{-i}$ and $\widetilde{x}_i = k > \alpha(z_{-i})$, by Observation 2(ii) $\exists\ \widetilde{z}_i > \alpha(z_{-i})$ such that $(\widetilde{z}_i, z_{-i}) \in A$, a contradiction. This shows $A(z_{-i})$ must be also not bounded above and $A(z_{-i}) = \mathbb{R}_+$.

(iii) We have to show that there can be at most one $i \in N$ where the following hold:

$$\exists z \in A \text{ with } z > 0^N \text{ such that } A(z_{-i}) \text{ is bounded above} \tag{2}$$

If (2) does not hold, then there are $i, k \in N$ and $y, z \in A$ with $y > 0^N, z > 0^N$ such that $A(y_{-i})$ and $A(z_{-k})$ are both bounded above. By part (i), there are positive numbers $\alpha_i, \alpha_k$ such that $A(y_{-i}) = [0, \alpha_i]$ and $A(z_{-k}) = [0, \alpha_k]$.

Since $y > 0^N$ and $z > 0^N$, we can construct $\widetilde{x}, \widetilde{y}$ such that $0 < \widetilde{y}_j < \widetilde{x}_j < \min\{y_j, z_j\}$ for all $j \in N$ and $\widetilde{y}_k < \widetilde{x}_k < \alpha_k$. Then $y > \widetilde{x} > \widetilde{y}$. As $y \in A$, by strong monotonicity, $\widetilde{x}, \widetilde{y} \in A$. Then by part (1)(i), $A(\widetilde{x}_{-i}) = A(\widetilde{y}_{-i}) = [0, \alpha_i]$. Let $\overline{x} = (\alpha_i, \widetilde{x}_{-i}), \overline{y} = (\alpha_i, \widetilde{y}_{-i})$. Then $\overline{x}, \overline{y} \in \partial A$.

Since $z > 0^N$, we can construct $\widetilde{z}$ such that $0 < \widetilde{z}_i < \min\{z_i, \alpha_i\}$ and $\widetilde{z}_j = z_j$ for all $j \in N \setminus \{i\}$. Then $z \geq \widetilde{z}$. As $z \in A$, by strong monotonicity, $\widetilde{z} \in A$. Then by part (1)(i), $A(\widetilde{z}_{-k}) = [0, \alpha_k]$. Then $\overline{z} = (\alpha_k, \widetilde{z}_{-i}) \in \partial A$.

Note that (a) $\overline{y}_j < \overline{x}_j < \overline{z}_j$ for $j \in N \setminus \{i, k\}$, (b) $\overline{y}_i = \overline{x}_i = \alpha_i > \overline{z}_i$ and (c) $\overline{y}_k < \overline{x}_k < \alpha_k = \overline{z}_k$. This shows $\overline{x} \hat{\not\geq} \overline{z}$ and $\overline{y} \hat{\not\geq} \overline{z}$. Since $\overline{x}, \overline{y}, \overline{z}$ are all boundary points of the unhappy set $A$, by Proposition 1(ii), we conclude: $\overline{x} \sim \overline{z}$ and $\overline{y} \sim \overline{z}$. Then transitivity implies $\overline{x} \sim \overline{y}$. However, since $\overline{x}_i = \overline{y}_i$ and $\overline{x}_{-i} > \overline{y}_{-i}$, by strong monotonicity of the preference relation have $\overline{x} \succ \overline{y}$, which is a contradiction. This proves that there is at most one $i \in N$ where (2) holds. ■

# Appendix

**Proof of Proposition 1** (i) Suppose there are two unhappy sets $A, B$ and $x \in A, y \in B$ such that $x \notin B, y \notin A$. By definition of unhappy sets we must have $x \succ y$ and $y \succ x$, a contradiction.

For (ii), by the Result 1 we know there exists $x \in \partial A$.

(ii)(a) If the assertion is not true, then by completeness one of $x, y$ is strictly preferred to the other. Without loss of generality, let $x \succ y$. Then mild continuity implies $\exists\ \varepsilon > 0$ such that all points in $B_\varepsilon(x)$ is strictly preferred to all points in $B_\varepsilon(y)$. Since $x, y \in \partial A$, $\exists$ $\widetilde{x} \in B_\varepsilon(x)$, $\widetilde{y} \in B_\varepsilon(x)$ such that $\widetilde{x} \in A$, $\widetilde{y} \notin A$ and we have $\widetilde{x} \succ \widetilde{y}$. A contradiction since $A$ is an unhappy set.

(b) If the assertion is not true, then $y \succ x$ and mild continuity implies $\exists\ \varepsilon > 0$ such that $y$ is strictly preferred to all points in $B_\varepsilon(y)$. Since $x \in \partial A$, $\exists\ \widetilde{x} \in B_\varepsilon(x)$ such that $\widetilde{x} \notin A$ and we have $y \succ \widetilde{x}$, a contradiction since $y \in A$ and $A$ is an unhappy set.

(c) As $x \succ y$, mild continuity implies $\exists\ \varepsilon > 0$ such that all points in $B_\varepsilon(x)$ is strictly preferred to $y$. Since $x \in \partial A$, $\exists\ \widetilde{x} \in B_\varepsilon(x)$ such that $\widetilde{x} \in A$ and we have $\widetilde{x} \succ y$. Since $A$ is an unhappy set, we must have $y \in A$. ■

**Proof of Corollary 1** (i)-(ii) As the preference relation is continuous, strict preference orders are preserved around small neighborhoods of *any two* points rather than only totally different points. Thus for any $x \in \partial A$, conclusions of Proposition 1(iv) hold for *any* $y \in X$.

Part (i) of the corollary follows by applying Proposition 1(iv)(a) for any $y \in X$. Part (ii) follows by applying Proposition 1(iv)(b)-(c) for any $y \in X$.

(ii) By Result 1, the set $A$ cannot be both open and closed, so we can have either (a) $A$ is neither open nor closed, or (b) $A$ is either open or closed, but not both. First we rule out (a). To see this, suppose there is an unhappy set $A$ that is neither open nor closed. Since $A$ is not open, $\exists\ x \in A$ such that every neighborhood of $x$ contains a point outside $A$, so we have $x \in \partial A$. Since $A$ is not closed, $\exists\ y \notin A$ which is a limit point of $A$, that is, every neighborhood of $x$ contains a point in $A$, so we have $y \in \partial A$. Since $x, y \in \partial A$, by (i) we have $x \sim y$. However, since $x \in A$, $y \notin A$ and $A$ is an unhappy set, we must have $y \succ x$, a contradiction. This rules out (a). So the set $A$ is either open or closed, but not both.

Suppose $A$ is open. Let $x \in \partial A$. We must have $x \notin A$ since every neighborhood of $x$ contains a point outside $A$. As $A$ is an unhappy set, if $y \sim x$, we must have $y \notin \partial A$. This shows $A \cap I(x) = \varnothing$. Since $A \subseteq L(x) = \underline{L}(x) \cup I(x)$ (by (ii)), we must have $A \subseteq \underline{L}(x)$ and again by (ii) we have $A = \underline{L}(x)$.

Suppose $A$ is closed. Let $x \in \partial A$. Then either $x \in A$ or $x$ is a limit point of $A$. As $A$ is closed, we have $x \in A$. Since $A$ is an unhappy set, if $y \sim x$, we must have $y \in A$. This shows $I(x) \subseteq A$. Since $\underline{L}(x) \subseteq A$, (by (ii)), we conclude $L(x) = \underline{L}(x) \cup I(x) \subseteq A$. Again by (ii) it follows that $A = L(x)$. ■

**Proof of Proposition 2** Consider the induced preference $\succsim_S$ . Note by definition of lower contour set that any point in $X_S \setminus \overline{L}^*(x^S)$ is strictly preferred to any point in $L^*(x^S)$. To prove $\overline{L}^*(x^S)$ is an unhappy set, it remains to show that if $a \in X_S \setminus \overline{L}^*(x^S)$ and $b \in \partial L^*(x^S)$, then $a \succ_S b$.

If the result does not hold, there are $a \in X_S \setminus \overline{L}^*(x^S)$, $b \in \partial L^*(x^S)$ such that $b \succsim_S a$.

Denote

$$E = \{i \in S | a_i = b_i = 0\} \text{ and } T = S \setminus E$$

so that $T \cup E = S$. By monotonicity of the preference relation, $0^S \in L^*(x^S)$, so $a \neq 0^S$. Thus $T \neq \varnothing$ and there is $i \in S \setminus E$ with $a_i > 0$.

Since $a \in X_S \setminus L^*(x^S)$ and $a \notin \partial L^*(x^S)$, there is a neighborhood $B_\varepsilon(a) \subseteq X_S \setminus L^*(x^S)$. We can construct $\widetilde{a} \in B_\varepsilon(a)$ (so that $\widetilde{a} \in X_S \setminus L^*(x^S)$) such that (i) if $i \in E$, then $\widetilde{a}_i = a_i = 0$; (ii) if $i \in T$ and $a_i = 0$, then $\widetilde{a}_i = 0$; and (iii) if $i \in T$ and $a_i > 0$ then $0 < \widetilde{a}_i < a_i$ and $\widetilde{a}_i \neq b_i$. Note that $\widetilde{a}^T \hat{\neq} b^T$. By strong monotonicity $a \succ_S \widetilde{a}$ and transitivity implies $b \succ_S \widetilde{a}$.

If $E = \varnothing$, then $S = T$. So $\widetilde{a} = \widetilde{a}^T$, $b = b^T$ and we have $\widetilde{a} \hat{\neq} b$. Since $L^*(x^S)$ is an unhappy set, $b \in \partial L^*(x^S)$ and $b \succ_S \widetilde{a}$, by Proposition 1(ii)(c) we must have $\widetilde{a} \in L^*(x^S)$ which is a contradiction since $\widetilde{a} \in X_S \setminus L^*(x^S)$.

If $E \neq \varnothing$, then $\widetilde{a} = (\widetilde{a}^T, 0^E)$ and $b = (b^T, 0^E)$. Since $b \in \partial L^*(x^S)$, every neighborhood $B_\varepsilon(b)$ contains a point $\widetilde{b} \in L^*(x^S)$. From such an $\widetilde{b}$, construct $c = (\widetilde{b}^T, 0^E)$. By monotonicity, $\widetilde{b} \succsim_S c$, so we have $c \in L^*(x^S)$. Moreover $d(b^T, \widetilde{b}^T) = d(b, c) \leq d(b, \widetilde{b}) < \varepsilon$, so $\widetilde{b}^T \in B_\varepsilon(b^T)$. This shows every neighborhood of $b^T$ contains a point $\widetilde{b}^T \in X_T$ such that $(\widetilde{b}^T, 0^E) \in L^*(x^S)$.

Consider the induced preference $\succsim_T$ on $X_T$. Note that for $y^T, z^T \in X_T$ :

$$y^T \succsim_T z^T \Leftrightarrow (y^T, 0^{N \setminus T}) \succsim (z^T, 0^{N \setminus T}) \Leftrightarrow (y^T, 0^E) \succsim_S (z^T, 0^E) \tag{3}$$

It is given $\succsim_T$ is mildly continuous on $X_T$. Since $b = (b^T, 0^E) \succ_S \widetilde{a} = (\widetilde{a}^T, 0^E)$, by (3) we have $b^T \succ_T \widetilde{a}^T$. Since $b^T \hat{\neq} \widetilde{a}^T$, by mild continuity of $\succsim_T$, there is a neighborhood $B_\varepsilon(b^T)$ such that for every $\widetilde{b}^T \in B_\varepsilon(b^T)$ we have $\widetilde{b}^T \succ_T \widetilde{a}^T$ so by (3)

$$(\widetilde{b}^T, 0^E) \succ_S (\widetilde{a}^T, 0^E) = \widetilde{a} \text{ for every } \widetilde{b}^T \in B_\varepsilon(b^T) \tag{4}$$

Since $\exists\ b^T \in B_\varepsilon(b^T)$ such that $(\widetilde{b}^T, 0^E) \in L^*(x^S)$, by (4) we have $\widetilde{a} \in L^*(x^S)$, which is a contradiction since $\widetilde{a} \in X_S \setminus L^*(x^S)$. ■

**Proof of Lemma 1** Let the set of all attributes be $N = \{1, \ldots, n\}$. Let $\succsim$ be a lexicographic preference on $X = \mathbb{R}^n_+$. Suppose 1 is the most important attribute of $\succsim$ .

Let $A \subseteq X$ be a closed unhappy set for $\succsim$ . Let $x, y \in A$ be such that $x \hat{\neq} y$ with $y^S > x^S$ and $x^{N \setminus S} > y^{N \setminus S}$. Then we have either $x \succ y$ or $y \succ x$.

If $x \succ y$, then we must have $1 \in N \setminus S$ and $x_1 > y_1 \geq 0$. Consider any $\widetilde{x}^S > x^S$ and let $\widetilde{x} = (\widetilde{x}^S, x^{N \setminus S})$. So in particular, $\widetilde{x}_1 = x_1$. Note that any neighborhood of $\widetilde{x}$ contains a point $z$ such that $z_1 < \widetilde{x}_1 = x_1$. So we have $x \succ z$. As $A$ is an unhappy set, we have $z \in A$. Thus any neighborhood of $\widetilde{x}$ contains a point $z \neq \widetilde{x}$ such that $z \in A$. So $\widetilde{x}$ is a limit point of $A$. As $A$ is closed, $\widetilde{x} \in A$. This shows $\widetilde{x} = (\widetilde{x}^S, x^{N \setminus S}) \in A$ for any $\widetilde{x}^S > x^S$.

If $y \succ x$, then $1 \in S$ and $y_1 > x_1 \geq 0$. Consider any $\widetilde{x}^S$ such that $y^S > \widetilde{x}^S > x^S$, so in particular $y_1 > \widetilde{x}_1 > x_1$. Let $\widetilde{x} = (\widetilde{x}^S, x^{N \setminus S})$. Then $y \succ \widetilde{x}$. As $A$ is an unhappy set, we have $\widetilde{x} \in A$. This shows $\widetilde{x} = (\widetilde{x}^S, x^{N \setminus S}) \in A$ for any $y^S > \widetilde{x}^S > x^S$. This proves that any closed unhappy set $A$ for $\succsim$ satisfies IMIA. ■

**Proof of Lemma 2** Let $A \in \mathcal{A}_{\succsim}$. By Proposition 4 there can be at most one $i \in N$ where (2) holds. Suppose (2) fails to hold for all $i \in N$. We know $\exists\, x \in A$ with $x > 0^N$. Consider any point $z = (z_1, \ldots, z_n) \in X$. Since $A(x_{-1})$ is unbounded, $z^1 = (z_1, x_2, \ldots, x_n) \in A$. Since $A(z^1_{-2})$ is unbounded, $z^2 = (z_1, z_2, x_3, \ldots, x_n) \in A$. After $n$ steps we can show $z^n = (z_1, \ldots, z_n) \in A$. Using this reasoning for any $z \in X$, we have $X = A$, which cannot happen since $A$ is a proper subset of $X$. This shows there is exactly one $i \in N$ for which (2) holds. Denote this $i$ by $i^*$.

Then by Proposition 4(i) it follows that $\exists\, \alpha^A > 0$ such that for any $z \in A$: (a) $A(z_{-i^*}) = [0, \alpha^A]$ and (b) $A(z_{-j}) = \mathbb{R}_+$ for any $j \in N \setminus \{i^*\}$. Note from (a) that if $y \in X$ has $y_{i^*} > \alpha^A$, then $y \in X \setminus A$. Since $x \in A$, by (a), $(\alpha^A, x_{-i^*}) \in A$. Then by (b) it follows that $(\alpha^A, y_{-i^*}) \in A$ for any $y_{-i^*} \in X_{-i^*}$. Since $A$ is an unhappy set, strong monotonicity of the preference implies that $(z_{i^*}, y_{-i^*}) \in A$ for any $z_{i^*} \in [0, \alpha^A]$. This proves that $A = \{y \in X | 0 \leq y_{i^*} \leq \alpha^A\}$.

To complete the proof consider another set $B \in \mathcal{A}_{\succsim}$. Then there is $k \in N$ and $\alpha^B > 0$ such that $B = \{x \in X | 0 \leq x_k \leq \alpha^B\}$. It only remains to show that $k = i^*$. Recall by Proposition 1 that since $A, B$ are both unhappy sets, either $A \subseteq B$ or $B \subseteq A$. Without loss of generality let $A \subseteq B$. If $k \neq i^*$, then for any $z \in A$, $A(z_{-k}) = \mathbb{R}_+$, so there is a $z \in A$ such that $z_k > \alpha^B$, so that $z \notin B$, a contradiction. So we must have $k = i^*$. ■

**Proof of Proposition 3** A lexicographic preference on $X = \mathbb{R}^2_+$ is strong monotone and mildly continuous and by Lemma 1, any closed unhappy set of such a preference satisfies IMIA.

To prove the converse, let $\succsim$ be a rational, strong monotone and mildly continuous preference relation on $X = \mathbb{R}^2_+$ for which any closed unhappy set satisfies IMIA. Consider any $x = (x_1, x_2) \in \mathbb{R}^2_+$ such that $x_1, x_2$ are both positive. Let $L(x)$ be the lower contour set of $x$ and $\overline{L}(x) = L(x) \cup \partial L(x)$ be its closure. By Corollary 2, $\overline{L}(x)$ is a closed unhappy set, so it belongs to the family $\mathcal{A}_{\succsim}$ given in (1). By Lemma 2 we conclude there is $\kappa(x) > 0$ and a unique $i^* \in \{1, 2\}$ such that $\overline{L}(x) = \{y \in \mathbb{R}^2_+ | 0 \leq y_{i^*} \leq \kappa(x)\}$. Without loss of generality, let $i^* = 1$. Then for any $x$ where $x_1, x_2$ are both positive we have $\overline{L}(x) = \{y \in \mathbb{R}^2_+ | 0 \leq y_1 \leq \kappa(x)\}$.

In what follows we show $\kappa(x) = x_1$. Note that if $\kappa(x) < x_1$, then $x$ will be outside $\overline{L}(x)$, so we must have $\kappa(x) \geq x_1$. If $\kappa(x) > x_1$, we can construct $\widetilde{x} \in \mathbb{R}^2_+$ such that $\widetilde{x}_1 = \kappa(x) > x_1$ and $\widetilde{x}_2 > x_2$. Observe that $\widetilde{x} \in \overline{L}(x)$ and any neighborhood of $\widetilde{x}$ contains a point $y$ such that $y_1 > \kappa(x)$, so that $y \notin \overline{L}(x)$. This shows $\widetilde{x} \in \partial \overline{L}(x)$.

Next observe that $\widetilde{x} > x$ so by monotonicity $\widetilde{x} \succ x$. Since $\succsim$ is mildly continuous and $\widetilde{x} \hat{\not\sim} x$, there exists a neighborhood $B_\varepsilon(\widetilde{x})$ such that all points there is strictly preferred to $x$. So we have $\widetilde{x} \notin \partial L(x)$. But we know $\widetilde{x} \in \partial \overline{L}(x)$. This is a contradiction since for any set $A$, the boundary of its closure is a subset of $\partial A$ (see, e.g., Chapter 3 of Mendelson, 1990). This shows we must have $\kappa(x) = x_1$. So for any $x$ where $x_1, x_2$ are both positive:

$$\overline{L}(x) = \{y \in \mathbb{R}^2_+ | 0 \leq y_1 \leq x_1\} \tag{5}$$

To show that $\succsim$ is lexicographic, consider any $y, z \in \mathbb{R}^2_+$ such that $y \neq z$. If $z_1 > y_1$, then $\exists$

$x$ with $x_1, x_2$ both positive such that $z_1 > x_1 > y_1$. Then by (5) it follows that $y \in \overline{L}(x)$ and $z \notin \overline{L}(x)$. Since $\overline{L}(x)$ is an unhappy set, we must have $z \succ y$. This shows whenever $z_1 > y_1$, we must have $z \succ y$. Finally let $y \neq z$ such that $z_1 = y_1$. Then by strong monotonicity, $z \succ y$ if $z_2 > y_2$ and $y \succ z$ if $y_2 > z_2$. This shows that $\succsim$ is a lexicographic preference with linear order $1 <_0 2$. ∎

**Proof of Lemma 3** (i)(a) Take any $S \subseteq N$ with $|S| = 2$. Consider the induced preference $\succsim_S$ defined on $X_S = \mathbb{R}^2_+$. Since $\succsim$ is strong monotone (Axiom 1) and rational, so is $\succsim_S$ . By Axiom 3, $\succsim_S$ is mildly continuous. By Axiom 2, any closed unhappy subset for $\succsim_S$ satisfies IMIA. Then by Corollary 3 we conclude that $\succsim_S$ is a lexicographic preference on $X_S = \mathbb{R}^2_+$. Since $|S| = 2$, $\exists\ i, j \in S$ such that $i \succ^* j$, that is, the linear order of the lexicographic preference $\succsim_S$ is $i <_0 j$.

(i)(b) Take any $x^S, y^S \in X_S = \mathbb{R}^2_+$ where $x^S \neq y^S$. Then either (a) $x_i > y_i$ or (b) $y_i > x_i$ or (c) ($x_i = y_i$ and $x_j > y_j$) or (d) ($x_i = y_i$ and $y_j > x_j$).

Consider any $z^{N\setminus S} \in X_{N\setminus S}$. Since $i \succ^* j$, if (a) or (c) hold, then $(x^S, 0^{N\setminus S}) \succ (y^S, 0^{N\setminus S})$ and by Axiom 4 we have $(x^S, z^{N\setminus S}) \succ (y^S, z^{N\setminus S})$. If (b) or (d) hold, then $(y^S, 0^{N\setminus S}) \succ (x^S, 0^{N\setminus S})$ and by Axiom 4 we have $(y^S, z^{N\setminus S}) \succ (x^S, z^{N\setminus S})$.

(ii) Suppose $i \succ^* j$ and $j \succ^* k$. Let $S = \{i, k\}$. First observe that if $x^S, y^S$ are such that $x_i = y_i$ and $x_k > y_k$, then by strong monotonicity we have $(x^S, 0^{N\setminus S}) \succ (y^S, 0^{N\setminus S})$.

So let $x^S, y^S$ are such that $x_i > y_i$. Consider non negative numbers $x_k, y_k, \widetilde{x}_j$ such that $\widetilde{x}_j > 0$. Denote $T = \{i, j, k\}$ and let

$$x^T = (x_i, 0, x_k), \widetilde{x}^T = (y_i, \widetilde{x}_j, x_k), y^T = (y_i, 0, y_k) \text{ and}$$

$$x = (x^T, 0^{N\setminus T}), \widetilde{x} = (\widetilde{x}^T, 0^{N\setminus T}), y = (y^T, 0^{N\setminus T}).$$

Note that $x_i > \widetilde{x}_i = y_i$ and $x_\ell = \widetilde{x}_\ell$ for any $\ell \neq i, j$. Since $i \succ^* j$, by (i)(b), we have $x \succ \widetilde{x}$. Next observe that $\widetilde{x}_j > y_j = 0$ and $\widetilde{x}_\ell = y_\ell$ for any $\ell \neq j, k$. Since $j \succ^* k$, by (i)(b) we have $\widetilde{x} \succ y$. By transitivity, we have $x \succ y$. Noting that $x = (x^S, 0^{N\setminus S})$ and $y = (y^S, 0^{N\setminus S})$ we conclude that if $x_i > y_i$, then $(x^S, 0^{N\setminus S}) \succ (y^S, 0^{N\setminus S})$. This shows that $i \succ^* k$.

(iii) Note from (i)-(ii) for any two different $i, j \in N$, either $i \succ^* j$ or $j \succ^* i$. Moreover for $i, j, k \in N$, if $i \succ^* j$ and $j \succ^* k$, then $i \succ^* k$. We prove the result by induction on $n$. The result clearly holds for $n = 2$. For $m \geq 3$, assume the result holds for all $n \leq m - 1$ and let $M = \{1, \ldots, m\}$. Let $S = \{i \in M | i \succ^* m\}$ and $T = \{i \in M | m \succ^* i\}$. Then $S, T$ are disjoint and $S \cup T = M$. Let $|S| = s$, $|T| = t$. Then $0 \leq s, t \leq m - 1$. So by induction hypothesis, we have $S = \{i_1, \ldots, i_s\}$ such that $i_1 \succ^* \ldots \succ^* i_s$ and $T = \{j_1, \ldots, j_t\}$ such that $j_1 \succ^* \ldots \succ^* j_t$. Since $i_s \succ^* m$ and $m \succ^* j_1$, we have $i_1 \succ^* \ldots \succ^* i_s \succ^* m \succ^* j_1 \ldots \succ^* j_t$. ∎

## References


Bossert W., Pattanaik, P.K., Xu, Y. 1994. Ranking opportunity sets: An axiomatic approach. Journal of Economic Theory 63, 326-345

Fishburn, P.C. 1975. Axioms for lexicographic preferences. Review of Economic Studies 42, 415-419

Goswami, M.P., Mitra, M., Sen, D. 2017. Naive reasoning and the Lexicographic preference order. Working paper

Mendelson, B. 1990. Introduction to Topology. Dover Publications (Third Edition)

Mitra, M., Sen, D. 2017. Subsistence, saturation and irrelevance in preferences. Working paper

Petri, H., Voorneveld, M. 2016. Characterizing lexicographic preferences. Journal of Mathematical Economics, 63, 54-61

Rudin, W. 1976. Principles of Mathematical Analysis. McGraw-Hill Inc.

Tversky, A. 1972. Elimination by aspects: A theory of choice. Psychological Review 79, 281-299