Dynamics of 'Bundled' Aid-Debt Contracts: Progressive Lending

Dyotona Dasgupta*†and Dilip Mookherjee‡

September 18, 2017

1 Introduction

Progressive Lending, increasing loan size over time in case of successful repayment, is the most commonly observed dynamic incentive provided by the lenders in order to overcome ex post moral hazard or strategic default. This paper seeks to provide a theoretical explanation to this widely used mechanism in a 'bundled' aid-debt contract environment. We show that when the borrower's endowment is not very high, optimum loan amount increases over time so do her consumption and investment.

In particular, this paper pertains to the problem faced by a lender L (benevolent aid agency or MFI or lender in a competitive setting) that seeks to raise welfare of a poor borrower (aid recipient, borrower, entrepreneur) by providing loans to finance productive investments. The borrower is subject to an ex post moral hazard problem. To provide repayment incentives, L could start offering small loans and promise bigger loans in the future conditional on loan repayments. However, as Bulow and Rogoff (1989) and Rosenthal (1991) have shown, such strategies unravel completely if the agent's access to production or marketing opportunities is unaffected by default.

However in many circumstances L can provide access to superior technology and/or marketing opportunities. For instance, in the context of international aid provided by developed countries to a small developing country, access to developed country markets or technical assistance can be withdrawn via trade sanctions following any loan default. MFIs often bundle loans to poor borrowers with technical and marketing assistance for products produced by borrowers using the loans (e.g. Indian SHGs). In particular, this model is built on Thomas and Worrall (1994) where they assume that the lender not only provides capital but also technology and expertise not otherwise available to the borrower.

*ISI Delhi

†Dyotona Dasgupta acknowledges Robert S. Mc Namara Fellowship provided by the World Bank and Institute for Economic Developmet, Boston University for hosting.

‡Boston University

In this paper we study the design of such bundled aid-loan products. We show that welfare (and wealth) improving contracts do exist, and characterize the dynamics of optimal contracts. We consider a setting without any uncertainty where in autarky B has access only to a savings opportunity with a constant rate of return equal to her discount rate. With preferences for consumption smoothing, autarky results in maintaining initial wealth for ever. L provides B access to a technology represented by a neoclassical production function which is bundled with credit. We first study the scenario where the autarkic technology remains unaffected by the provision of L's technology. Under this circumstance, if the lender withdraws the access to this technology, which happens when the borrower defaults on the loan, the borrower can only thereafter invest in the savings technology. Later we allow for 'learning' in that even if the lender withdraws the access, the borrower can invest in an inferior technology represented by a neoclassical production technology. Return from this technology is lower than that from the technology provided by the lender but is higher, for small investment, than that from the savings technology.

Technology access in the absence of any loans would result in growing investment financed by own savings, ultimately converging to the efficient level (as represented by the standard Ramsey model). Providing loans in addition would potentially allow faster growth by lowering investment costs for B. We assume that L has unlimited access to external finance at a constant cost equal to B's discount rate. In the absence of any moral hazard, efficient investment can be sustained at every date, combined with perfect consumption smoothing.

With moral hazard, the first-best allocation cannot be sustained if B's initial wealth falls below a threshold. We characterize optimal contracts in this case. We show that the dynamics are qualitatively similar to the case where investments are self-financed: there is under-investment, but investments grow and eventually converge to the efficient level. In parallel, the agent's net wealth (value of production less inherited debt) and consumption grow and converge to stationary levels which correspond to the threshold where the moral hazard problem disappears. Loan sizes also grow; hence the optimal policy is characterized by 'progressive lending' policies.

Technology assistance by itself would also help raise B's welfare and generate the same long run outcome, bundling it with credit enables B's financing costs to be lowered and thereby generates additional welfare improvements. Credit provision alone would not generate any welfare improvements, owing to the moral hazard problem. The rest of the paper is organized as follows: Next we discuss the relevant literature, then we discuss the optimum contract where learning is not allowed, later we allow for learning. Finally we conclude.

1.1 Related Literature

This paper is related to different strands of literature. Like in sovereign debt models contractual enforcement of repayment is weak and hence default can only be prevented through dynamic incentives and threat of termination of relationships. On of the early papers is by Eaton and Gersovitz (1981) where repayment is ensured through reputation building in that if the sovereign does not repay it will not get any future loans. Grossman and van Huyck (1988), Atkeson (1991) and others argue in similar line. However, Bulow and Rogoff (1989) and more generally Rosenthal (1991) show that threat of termination of loan contract is not sufficient to ensure repayment. Cole and Kehoe (1997) show that *reputation spillovers* support debt. They argue that countries are involved in many different types of relationships and assume that when a country breaks trust of one of these relationships that adversely affects other trust relationships. So even when the country does not get directly affected from default but may get adversely affected from that spillover effect. In that case the country would repay if the loss in lifetime utility is higher than the short term gain from default.

We argue in similar spirit and adapt the framework of Thomas and Worrall (1994) in that, as stated above, lender provides loan bundled with superior technology and expertise not otherwise available to the borrower. However, unlike them we assume that the borrower's utility function is concave and also capital does not depriciate completely within a period, instead after deviation the borrower can invest the amount with which she deviates in the technology to which she has access on her own. And contrary to their assumption "... that the particular investment opportunity is specific to these parties, and whatever happens here does not affect any other activities undertaken..." we allow for learning in that when the borrower enters into this relationship she learns the technology (though not perfectly) which improves her post-deviation technology over the autarky technology. So outside option takes a very different form which changes her incentive to deviate and hence the optimum contract. In our model, the necessary condition for existence of loan contract where the borrower always chooses to repay is that the marginal return from deviation technology is weakly less than that from the lender's superior technology (strictly for some range of capital). Two other very closely related papers are Albuquerque and Hopenhayn (2004) and Ray (2002). While Albuquerque and Hopenhayn (2004) generalize outside option of the borrower than what she had in Thomas and Worrall (1994), they continue to assume that the borrower's utility function is linear.

Ghosh and Ray (2016) address adverse selection and ex post moral hazard together: there are two types of borrowers - good (patient) and bad (myopic), good borrowers always repay and bad borrowers never. So a lender provides small initial loans to screen out bad borrowers, so by the structure loan size becomes constant from second period onwards (in any particular borrower lender relationship). In our framework there is only one type of borrower and they are strategic in that they default whenever they have incentive to do so, so the lender has to design contract accordingly. We find at optimum, when the borrower is poor, loan size increases over time for more than one period which conforms to the empirical finding.

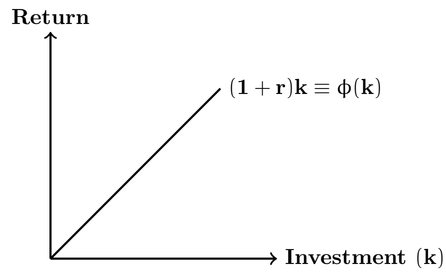
We can also relate this paper to Microfinance literature where legal enforcement is not possible. Though dynamic incentives especially that in the form of progressive lending is the most widely used institution to ensure repayment, there are very few papers which address this. One of the very first paper to address this is Aghion and Morduch (2000). They considered a simple two period model and show that how repayment incentive of the first period increases as loan amount in the second period increases. However, since this is a two period model, the borrower defaults at the second period with certainty and knowing that the lender has to set interest rate of the first period accordingly. Egli (2004) and Shapiro (2015) both unearth some limitations of the dynamic incentives involved in progressive lending. Egli (2004) develops a two period model with two types of borrowers - honest, and strategic. He shows that progressive lending may be counter-productive, in that strategic borrowers may be tempted to repay in the first period, so as to access larger loans later on. Shapiro (2015) examines a framework with uncertainty over borrower discount rates, showing that even in the efficient equilibrium all borrowers, except the most patient one, default. Note that both the papers involve default along the equilibrium path. However, in reality microfinance schemes appear to involve little or no default.

2 Framework

2.1 Payoffs and Technology

Consider an agent with endowment w. Her current payoff is given by $u(c)$ where c denotes consumption. We assume $u(\cdot)$ to be time-stationary, continuous, strictly increasing, concave and satisfy Inada conditions: $u'(0) = \infty$ and $u'(\infty) = 0$. We also impose limited liability: consumption must be non-negative, $u(0)$ is the floor consumption utility and we normalize it to zero. The agent's objective is to maximize present discounted value of her lifetime utility. We assume time to be discrete and goes on forever. We also assume future discount factor is δ. So the problem of the agent is to maximize $\sum_{t=0}^{\infty} \delta^t u(c_t)$.

Before the lender enters into the economy, the only way an agent (a potential borrower) can transfer wealth from one period to another is by investing in savings technology. We assume that net interest rate is r.



To abstract from any time trend created purely from time preference we make the following assumption

Assumption 1. $\delta = \dfrac{1}{1+r}$.

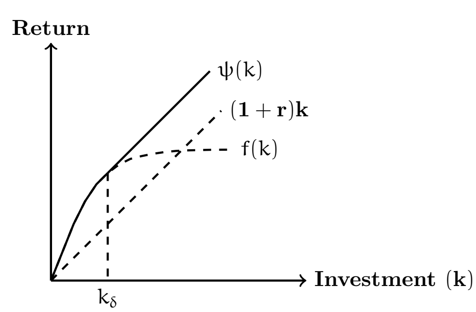
So, with preference for consumption smoothing, autarky results in maintaining endowment forever.

A lender L seeks to raise welfare of the agent by providing loans bundled with access to a technology represented by a neoclassical production function ($f(\cdot)$)[1] bundled with credit. As discussed above, given this $f(\cdot)$ and savings technology, it has to be decided how much to invest in which technology. We define k_δ, which is the maximum amount of investment such that the marginal return from investing in $f(\cdot)$ is higher than that from investing in the savings technology. Formally.

Definition 1. k_δ *solves argmax* $\delta f(k) - k$.

So the effective"transformation" technology ($\psi(k)$) is given by

$$\psi(k) = \begin{cases} f(k) & \text{when} \quad k \le k_\delta \\ f(k_\delta) + (1+r)(k - k_\delta) & \text{otherwise.} \end{cases}$$



[1] $f(\cdot)$ is strictly increasing, strictly concave and satisfies the Inada conditions: $f'(0) = \infty$ and $f'(\infty) = 0$.

2.2 Contracts and Timeline

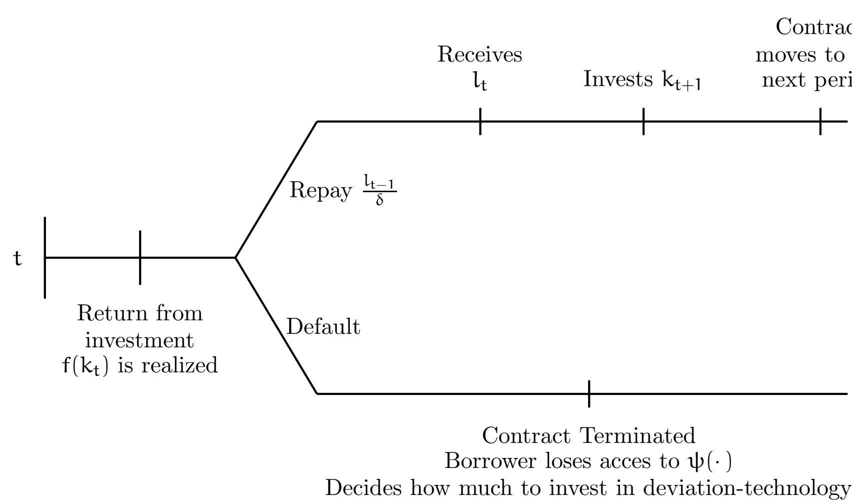
We assume that the lender can commit and the borrower cannot, in that she defaults on debt whenever she has incentive to do so. The lender provides a contract $\xi\langle\{p\},\{k\}\rangle$ where $p \equiv \{p_0, p_1, p_2, ..., p_T\}$ and $k \equiv \{k_1, k_2, ..., k_T\}$ denote net transfer and investment of each period respectively. Specifically, p_t denotes net transfer at t from the lender to the borrower and k_t denotes investment of $t-1^{th}$ period. Net interest rate on loan is assumed to be r which is equal to the net interest rate on savings. Two immediate implications of this assumption are:

1. $\{p_t\}_{t=0}^{\infty}$ can equivalently be written as a sequence of one-period debt contracts with loan l_t at date t which must be repaid at the following date at interest rate r, with $l_0 = p_0, p_t = l_t - \frac{l_{t-1}}{\delta}, t \geq 1.$

2. The entire k_t is invested in $f(\cdot)$, investing in savings technology can be adjusted through loan amount.

In case the borrower does not repay $\frac{l_{t-1}}{\delta}$ at any $t \geq 1$,[2] the lender terminates the contract and withdraws her access to $f(\cdot)$ technology. The borrower, then depending on her deviation-technology, optimally chooses amount to be consumed and invested. We do not allow for any renegotiation. The timeline of this game is as follows.

Timeline: If the agent accepts the contract, having there been no default till date, at any $t \geq 1$ the following things happen:



[2]Observe, partial repayment is not allowed in that even if the agent repays partially, the lender terminates the contract, so the borrower either repays the full amount or does not repay at all.

As it can be observed deviation-technology plays very important role in borrower's decision of repayment. We start with the case where improvement over autarky-technology is not allowed, that is deviation-technology is equivalent to autarky-technology ($\phi^D(\cdot) \equiv \phi(\cdot)$), this gives us a benchmark case. Then we allow for the provision of 'learning' which improves the deviation-technology over autarky-technology.

3 Benchmark Case: Model without Learning

Let us first concentrate on borrower's deviation payoff. Suppose the borrower defaults at period t (with $f(k_t)$). Consequently the lender terminates the credit contract and withdraws her access to $f(\cdot)$ technology. Hence, the problem of the agent is then to

$$\underset{\{k_{\tau+1}\}_{\tau=t}^{\infty}}{\text{Maximize}}\ u\big(f(k_t) - k_{t+1}\big) + \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} u\big(\phi^D(k_\tau - k_{\tau+1}\big)$$

Subject to,

$$k_{t+1} \leq f(k_t) \text{ and } \forall \tau \geq t+1 \quad k_{\tau+1} \leq \phi^D(k_\tau).$$

Since in this benchmark case $\phi^D(k) \equiv \phi(k) = (1+r)k = \frac{1}{\delta}k$, the borrower would maintain her wealth $f(k_t)$ forever. So she would consume $(1-\delta)f(k_t)$ which implies present discounted value of her lifetime utility at period t, from deviation with $f(k_t)$ is

$$V_A\big(f(k_t)\big) \equiv \frac{u\big((1-\delta)f(k_t)\big)}{1-\delta}.$$

Given this deviation payoff, the lender's problem is to

$$\underset{\langle\{p_t\}_{t=0}^{\infty},\{k_{t+1}\}_{t=0}^{\infty}}{\text{Maximize}} \Big[u(w + p_0 - k_1) + \sum_{t=1}^{\infty} \delta^t u\big(f(k_t) + p_t - k_{t+1}\big)\Big]$$

Subject to

Feasibility: $w + p_0 - k_1 \geq 0$ and $\forall t \geq 1 \quad f(k_t) + p_t - k_{t+1} \geq 0$

Sustainability: $p_0 + \sum_{t=1}^{\infty} \delta^t p_t \leq 0$

$$\text{DIC: } V_t \equiv \sum_{\tau=t}^{\infty} \delta^{\tau-t} u\big(f(k_\tau) + p_\tau - k_{\tau+1}\big) \geq \frac{u\big((1-\delta)f(k_t)\big)}{1-\delta} \equiv V_A\big(f(k_t)\big) \quad \forall t \geq 1. \tag{1}$$

Sustainability or break even condition ensures that the lender does not make any loss. DIC or dynamic incentive compatibility condition ensures that at any period t, the borrower does not have incentive to default. Or in other words, borrower's present discounted value of lifetime utility from repaymnet (on contract continuation payoff) is higher than that from default at any $t \geq 1$. Feasibility condition ensures that the borrower's consumption at any period is non negative. We ignore this condition as it is implied by DIC and our assumption of Inada condition in particular $u'(0) = \infty$. Given sustainability and DIC constraints the lender chooses net transfer and amount of investment of each period in order to maximize borrower's present discounted value of lifetime utility.

We denote the maximum value of the above problem by $V(w)$. It is evident that V is strictly increasing, since there is always the option of consuming the incremental wealth immediately which does not disturb any incentive constraints.

Note also at the outset that since p_0 does not enter any incentive constraint, sustainability condition always binds.

3.1 First-best Contracts

Consider the optimal contract when all the incentive constraints are dropped. It involves full consumption smoothing (via choice of transfers p_t) and efficient investment $k_t = k_\delta$. The constant consumption $c^*(w)$ is obtained by the requirement that the present value of consumption $\frac{c^*}{1-\delta}$ equals the present value of production minus investment, plus enodowment $w - k_\delta + \frac{\delta}{1-\delta}[f(k_\delta) - k_\delta]$, so

$$c^*(w) = (1-\delta)w + \delta f(k_\delta) - k_\delta \qquad \text{(1st Best Consumption)}$$

The borrower then attains welfare $V^*(w) = \frac{u(c^*(w))}{1-\delta}$.

As the first best contract is stationary, all the incentive constraints collapse to a single constraint $c^*(w) \geq (1-\delta)f(k_\delta)$, which reduces to the borrowers endowment exceeding a threshold w^*:

$$w \geq w^* \equiv f(k_\delta) - \frac{\delta f(k_\delta) - k_\delta}{1-\delta} \qquad \text{(Wealth Threshold without Learning)}$$

Intuitively, when $w < f(k_\delta)$, the borrower obtains a loan at the first date of $\delta[f(k_\delta) - w]$ and then repays $(1-\delta)[f(k_\delta) - w]$ at every subsequent date. If w falls below w^* the loan is too large, resulting in a repayment obligation that tempts the borrower to default. If $w \in [w^*, f(k_\delta))$, the repayment burden is positive but smaller than the value of access to the $f(\cdot)$ technology (which generates a per-period surplus of $\delta f(k_\delta) - k_\delta$) that is bundled with the loan.

Note that the incentive problem arises only for intermediate ranges of the discount factor. If δ approaches 1, the first best can be sustained for any initial w, as the threshold w^* goes to minus infinity.

While if δ approaches zero, the threshold approaches zero (as in this case, the efficient investment approaches zero), and the demand for loans vanishes.

When $w \geq w^*$, it follows the first-best contract is incentive compatible from the very first period, and $V(w) = V^*(w)$. And $V(w) < V^*(w)$ whenever $w < w^*$.

3.2 Second-best Contracts for Poor Borrowers

Now focus on poor borrowers, for whom $w < w^*$. We characterize features of the optimal contract. Let $c_t = f(k_t) + p_t - k_{t+1}$ denote consumption at date $t \geq 1$, and $c_0 = w + p_0 - k_1$.

Lemma 1. $c_t \geq c_{t-1}$ *for all* t.

Proof. Suppose otherwise, and $c_t < c_{t-1}$ for some t. Lower p_{t-1} slightly, and raise p_t correspondingly to keep $p_{t-1} + \delta p_t$ unchanged. This smooths consumption, raising V_l for every $l \leq t$, while leaving it unchanged for every $l > t$. Hence all incentive constraints are preserved, while raising $V(w) = V_0$. ∎

To make further progress as discussed above, we use the recursive formulation of the problem: $l_0 = p_0, p_t = l_t - \frac{l_{t-1}}{\delta}, t \geq 1$. In this notation, the problem of the lender becomes

$$\underset{\langle\{l_t\}_{t=0}^{\infty},\{k_{t+1}\}_{t=0}^{\infty}}{\text{Maximize}} \left[u(w + l_0 - k_1) + \sum_{t=1}^{\infty} \delta^t u\big(f(k_t) + l_t - \frac{l_{t-1}}{\delta} - k_{t+1}\big)\right]$$

Subject to

$$\text{DIC: } V_t \equiv \sum_{\tau=t}^{\infty} \delta^{\tau-t} u\big(f(k_\tau) + l_\tau - \frac{l_{\tau-1}}{\delta} - k_{\tau+1}\big) \geq \frac{u\big((1-\delta)f(k_t)\big)}{1-\delta} \equiv V_A\big(f(k_t)\big) \quad \forall t \geq 1. \tag{2}$$

Now observe that starting from any date t, the effect of past history is summarized in the single state variable $w_t \equiv f(k_t) - \dfrac{l_{t-1}}{\delta}$, the borrower's net wealth which is the value of current production less inherited debt. So the contracting problem has the following recursive representation.

Lemma 2. *The maximum attainable welfare* $V(w)$ *for a borrower with initial wealth* w *must satisfy*

$$V(w) = \max_{l,k}[u(w + l - k) + \delta V(f(k) - \frac{l}{\delta})] \quad \textit{subject to:} \quad V(f(k) - \frac{l}{\delta}) \geq \frac{u((1-\delta)f(k)}{1-\delta} \tag{3}$$

Proof. Let $\{l_0, k_{t+1}\}_{t\geq 0}$ be an optimal contract. The following conditions are necessary for optimality: (a) given l_0, k_1, the continuation contract $\{l_t, k_{t+1}\}_{t\geq 1}$ must be optimal for the borrower starting with $w_1 = f(k_1) - \frac{l_0}{\delta}$ (a problem in which constraints $IC_t, t \geq 2$ are incorporated), and (b) the initial loan and investment size l_0, k_1 satisfies condition (3) (a problem in which IC_1 is incorporated). ∎

Let $\Omega(l,k) \equiv f(k) - \frac{l}{\delta}$. Then condition (3) can be restated as:

$$V(w) = \max_{l,k}[u(w + l - k) + \delta V(\Omega(l,k)] \quad \text{subject to:} \quad V(\Omega(l,k)) \geq \frac{u((1-\delta)f(k)}{1-\delta} \tag{4}$$

This problem can be broken into two stages.

At the first stage, given any 'target' wealth Ω for the next date, select (l,k) to minimize the net investment cost, i.e, the sacrifice of current consumption $k - l$, subject to the incentive constraint $V(\Omega(l,k)) \geq \frac{u((1-\delta)f(k)}{1-\delta}$. Let the resulting minimized cost be denoted by $C(\Omega)$. Formally,

$$C(\Omega) = \min_{l,k}(k - l) \quad \text{subject to:} \quad f(k) - \frac{l}{\delta} = \Omega \quad \text{and} \quad V(\Omega) \geq \frac{u((1-\delta)f(k)}{1-\delta} \tag{5}$$

Then at the second stage, select the optimal target wealth $\Omega(w)$ for the next date, given current wealth w. We summarize this as follows.

Lemma 3. *The maximum attainable welfare* $V(w)$ *for a borrower with initial wealth* w *must satisfy*

$$V(w) = \max_{\Omega}[u(w - C(\Omega)) + \delta V(\Omega)] \tag{6}$$

Since V is strictly increasing, a higher target wealth is always valuable. The borrower must trade off a higher target wealth against the current cost. *Whether higher target wealth necessarily involves a higher cost is not evident, since the incentive constraint corresponding to a higher target wealth is less binding.* Nevertheless, the concavity of u implies the following.

Lemma 4. $\Omega(w)$ *is nondecreasing.*

Proof. If this is false, there exist $w_1 < w_2$ with $\Omega_1 \equiv \Omega(w_1) > \Omega_2 \equiv \Omega(w_2)$. Then $V(\Omega_1) > V(\Omega_2)$ and

$$u(w_2 - C(\Omega_2)) - u(w_2 - C(\Omega_1)) \geq \delta[V(\Omega_1) - V(\Omega_2)] > 0 \tag{7}$$

which implies $C(\Omega_1) > C(\Omega_2)$. On the other hand,

$$\delta[V(\Omega_1) - V(\Omega_2)] \geq u(w_1 - C(\Omega_2)) - u(w_1 - C(\Omega_1)) \tag{8}$$

which implies

$$u(w_2 - C(\Omega_2)) - u(w_2 - C(\Omega_1)) \geq u(w_1 - C(\Omega_2)) - u(w_1 - C(\Omega_1)) \tag{9}$$

This contradicts the concavity of u. ∎

Let us now focus on the first stage cost minimization problem. Given target wealth Ω and capital choice k, the associated current loan must be $l(\Omega, k) = \delta f(k) - \Omega$. Hence we can simplify (5) and reduce it to choice of investment alone as follows:

$$C(\Omega) = \Omega + \min_{k}(k - \delta f(k)) \quad \text{subject to:} \quad V(\Omega) \geq \frac{u((1-\delta)f(k)}{1-\delta} \tag{10}$$

The unconstrained solution to this problem involves selecting efficient investment k_δ, resulting in a cost of $\Omega - [\delta f(k_\delta) - k_\delta]$. This satisfies the constraint if and only if $V(\Omega) \geq \frac{u((1-\delta)f(k_\delta)}{1-\delta}$, i.e., if and only if $\Omega \geq \Omega^*$, defined by the property that $V(\Omega^*) = \frac{u((1-\delta)f(k_\delta)}{1-\delta}$.

Note that $\Omega^* = w^*$, the wealth threshold for implementability of the first-best. This is because w^* is defined by the property that the IC constraint just binds in the first-best contract, so $V(w^*) = V^*(w^*) = \frac{u((1-\delta)f(k_\delta)}{1-\delta}$. This implies $V(\Omega^*) = V(w^*)$. As V is strictly increasing, it follows that $\Omega^* = w^*$.

It follows that for $\Omega \geq w^*$, $C(\Omega) = \Omega - [\delta f(k_\delta) - k_\delta]$.

While if $\Omega < w^*$, the constraint binds:

$$V(\Omega) = \frac{u((1-\delta)f(k)}{1-\delta} \tag{11}$$

so the resulting investment size is $k(\Omega) = f^{-1}(\frac{u^{-1}((1-\delta)V(\Omega))}{1-\delta})$. Since $V(\Omega) < V(w^*) = \frac{u((1-\delta)f(k_\delta)}{1-\delta}$, it follows that there must be underinvestment: $k(\Omega) < k_\delta$. We summarize these observations below.

Lemma 5. *For target wealths Ω smaller than w^*, investment $k(\Omega)$ is smaller than the efficient level k_δ, and equal to the efficient level otherwise.*

This Lemma implies there is never any over-investment.

Next we turn to the determination of target wealths for the next period as a function of current wealth, which determines the wealth dynamic in the optimal contract. We already know that $\Omega(w) = w$ whenever $w \geq w^*$. So we focus on initial wealths below the threshold w^*. The next result shows that wealth must be non-decreasing over time.

Lemma 6. $\Omega(w) \geq w$ *for every* $w < w^*$.

Proof. Suppose this is false, and there is some $w < w^*$ such that $\Omega(w) < w$. Then combining with Lemma 4, it follows that wealth is non-increasing over time, so at every date it is smaller than w, and converges to some limit $w_\infty < w^*$. The corresponding sequence of investments will then be non-increasing, all strictly smaller than k_δ and converging to some $k_\infty < k_\delta$. At the same time, we have already established that consumption is nondecreasing, hence converging to some c_∞.

Now consider the earlier non-recursive formulation of the contracting problem. Consider a small increase in investment k_{T+1} at some date $T \geq 1$. Let S_t denote the surplus on IC_t, i.e., $S_t \equiv V_t - \frac{u((1-\delta)f(k_t))}{1-\delta}$. The effect on continuation payoffs and incentive surpluses at different dates are as follows. There is no effect on any of these at any date after $T+1$, so we focus on effects for $t \leq T+1$.

$$\frac{\partial S_{T+1}}{\partial k_{T+1}} = [u'(c_{T+1}) - u'((1-\delta)f(k_{T+1}))]f'(k_{T+1}) \tag{12}$$

If IC_{T+1} binds, we have $V_{T+1} \equiv \sum_{m=T+1}^{\infty} \delta^{m-T-1} u(c_m) = \frac{u((1-\delta)f(k_{T+1})}{1-\delta}$. As consumption is non-decreasing, it follows that $c_{T+1} \leq (1-\delta)f(k_{T+1})$, so by concavity of u we have $u'(c_{T+1}) - u'((1-\delta)f(k_{T+1}) \geq 0$. So IC_{T+1} is not jeopardized by the variation.

Next

$$\frac{\partial V_T}{\partial k_{T+1}} = \frac{\partial S_T}{\partial k_{T+1}} = u'(c_{T+1})\delta f'(k_{T+1}) - u'(c_T) \tag{13}$$

As $T \to \infty$, this converges to $u'(c_\infty)[\delta f'(k_\infty) - 1]$ which is strictly positive. Hence for all T sufficiently large, expression (13) is strictly positive. The same is true at all earlier dates. Hence the variation is feasible and welfare improving, so the contract cannot be optimal. ∎

We conclude that the wealth dynamic is monotone increasing, and so are investment levels. Our final result regards their long run limits.

Lemma 7. *If $w < w^*$, the sequence of net wealths w_t is nondecreasing and converges to the first best threshold w^*; the corresponding investment sequence k_t is nondecreasing and converges to k_δ, and consumption c_t is nondecreasing and converging to $c^*(w^*)$.*

Proof. Lemma 6 implies the wealth sequence is nondecreasing. Then the investment sequence is also non-decreasing (since it is a non-decreasing function of the target wealth). So both converge to limits w_∞, k_∞ respectively.

If $k_\infty < k_\delta$, the same argument as used in the proof of Lemma 6 generates a contradiction. Hence k_t converges to k_δ.

If $w_\infty < w^*$, the corresponding limit of investment $k_\infty < k_\delta$, which has been ruled out above. Hence $w_\infty \geq w^*$. Since $\Omega(w^*) = w^*$, Lemma 4 implies that $\Omega(w) \leq w^*$ for any $w < w^*$. Hence $w_\infty \leq w^*$, and it follows that $w_\infty = w^*$. The corresponding sequence of consumption must therefore converge to $c^*(w^*)$. ∎

The next Lemma shows that the limit wealth w^* cannot be reached in finite time from any initial $w < w^*$. Hence a credit constrained borrower must remain credit constrained for ever, with a level of investment that is lower than the efficient level. Such a borrower's investment level and wealth must grow perpetually as the credit constraint is gradually relaxed over time.

Lemma 8. $w < w^*$ *implies* $w < \Omega(w) < w^*$.

Proof. First we show that $\Omega(w) < w^*$. Since $\Omega(.)$ is non-decreasing, we have $\Omega(w) \leq \Omega(w^*) = w^*$. Suppose the claim is false and there exists $w < w^*$ such that $\Omega(w) = w^*$. In an optimal contract starting from $w_0 = w$, we must then achieve the first-best allocation from date 1 onwards, with $c_t = c^*(w^*) = (1-\delta)f(k_\delta), k_t = k_\delta$. The borrower's welfare then reduces to $u(w+p_0-k_\delta)+\frac{\delta u((1-\delta)f(k_\delta))}{1-\delta}$. Now budget balance implies $p_0 = \frac{\delta[\delta f(k_\delta)-k_\delta]}{1-\delta}$ (since B repays $\delta f(k_\delta) - k_\delta$ at every date $t \geq 1$ onwards). Hence date 0 consumption $c_0 = w + \frac{\delta^2 f(k_\delta)-k_\delta}{1-\delta} < (1-\delta)f(k_\delta)$, as $w < w^*$.

Now consider a variation on this contract where the production level continues to be stationary (denoted k). The loan at date 0 is modified to $\frac{\delta[\delta f(k)-k]}{1-\delta}$, while future per period repayments are modified to $\delta f(k) - k$. Welfare in this contract is $u(w + \frac{\delta^2 f(k)-k}{1-\delta}) + \frac{\delta u((1-\delta)f(k))}{1-\delta}$. Incentive compatibility are maintained and so is budget balance. The first order effect of a slight reduction in k below k_δ equals $u'(c_0) - u'(c^*(w^*)$ which is strictly positive since $c_0 < c^*(w^*)$. Hence a small reduction in k will improve welfare, and we obtain a contradiction.

Next, to show that $\Omega(w) > w$ for any $w < w^*$, note that otherwise there is some $w < w^*$ for which $\Omega(w) = w$. Then starting with initial wealth of w, wealth will remain stationary. This contradicts Lemma 7 since w_t must converge to w^*. ∎

The final result pertains to the dynamics of loan size: the optimal contract features progressive borrowing.

Lemma 9. *Starting with any* $w < w^*$, *the borrower obtains a loan* $l(w)$ *which is strictly positive and strictly increasing. Hence loan size* $l_t = l(w_t)$ *rises over time, eventually converging to* $\frac{\delta[\delta f(k_\delta)-k_\delta]}{1-\delta}$.

Proof. Let $y(w)$ denote date 1 output $f(k(w))$, where $k(w)$ is the investment of a borrower at date 0 with wealth w. Also let $V_A(w)$ denote the autarky payoff $\frac{u((1-\delta)w)}{1-\delta}$ of the borrower starting with wealth w. Then $l(w) = \delta[y(w) - \Omega(w)]$ and $w < w^*$ implies the incentive constraint in (5) binds: $V(\Omega(w)) = V_A(y(w))$.

We claim the following properties of the two value functions: (i) $w < w^*$ implies $V(w) > V_A(w)$, since the lender always has the option of making the borrower to maintain w for ever as in autarky, and the optimal contract starting with w is nonstationary and helps her attains higher limit wealth of w^* ; (ii) $V'(\Omega(w)) > V_A'(y(w))$. To see this observe that $V'(\Omega(w)) \geq u'(c_0(\Omega(w))$ where $c_t(w)$ denotes date t consumption starting with wealth w (as the lender always has the option to make the borrower consume that amount immediately). Next, note that $c_0(\Omega(w)) < (1-\delta)y(w)$, since $V(\Omega(w)) \equiv u(c_0(\Omega(w))) + \sum_{t=1}^{\infty} \delta^t u(c_t(\Omega(w))) = V_A(y(w))$, $c_0(\Omega(w)) \leq c_t(\Omega(w))$ for all $t \geq 1$ with strict inequality for some t as the optimal consumption sequence starting with wealth $\Omega(w)$ is not stationary (this in turn follows from noting that if consumption were stationary then V_t would be stationary, while we have shown above that k_t is strictly increasing, so IC_t could not bind at every date t). Hence $u'(c_0(\Omega(w))) > u'((1-\delta)y(w)) = V_A'(y(w))$, and (ii) now follows.

Claim (i) now implies that $l(w) = \delta[y(w) - \Omega(w)] > 0$ since $V_A(y(w)) = V(\Omega(w)) > V_A(\Omega(w))$. Moreover, $V_A'(y(w))y'(w) = V'(\Omega(w))\Omega'(w)$ so (ii) implies $y'(w) > \Omega'(w)$. Therefore $l'(w) = \delta[y'(w) - \Omega'(w)] > 0$. l_t must converge to $l(w^*) \equiv \frac{\delta[\delta f(k_\delta)-k_\delta]}{1-\delta}$. ∎

Summarizing the above discussion we write the following proposition

Proposition 1. *When the agent's endowment* $w \geq w^*$

i) Net wealth remains constant at w^*

ii) The agent invests k_δ *at every* $t \geq 0$

iii) At $t = 0$ *she receives* $\delta[f(k_\delta) - w]$ *from the lender*

and $\forall t \geq 1$ *she repays* $f(k_\delta) - w$ *and receives* $\delta[f(k_\delta) - w]$ *from the lender*

iv) at each $t \geq 0$ *she consumes* $c^*(w) \equiv (1-\delta)w + \delta f(k_\delta) - k_\delta$.

When the agent's endowment $w < w^*$

i) Net wealth increases over time and converges to w^* *but the limit amount* w^* *is not reached within a finite time*

ii) Investment is uniquely determined as a function of next period's target wealth: $k(w) \equiv f^{-1}(\frac{u^{-1}((1-\delta)V(\Omega))}{1-\delta})$. *Hence investment is increasing over time and converges to* $k(w^*) \equiv k_\delta$

iii) The borrower obtains a loan $l(w)$ *which is strictly positive and strictly increasing. Hence loan size* $l_t = l(w_t)$ *rises over time, eventually converging to* $\frac{\delta[\delta f(k_\delta)-k_\delta]}{1-\delta}$

iv) Consumption is nondecreasing and converges to $c^*(w^*)$.

4 Model with the Provision of Learning

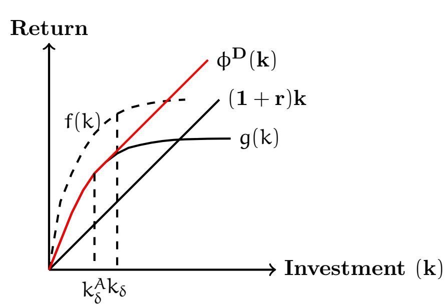
In this section we study the dynamics of aid-debt contract when the agent can acquire knowledge while using $f(\cdot)$ technology. Remember in our framework, the agent does not have access to $f(\cdot)$ technology on her own and the lender bundles access to this $f(\cdot)$ technology with credit contract. As in the benchmark case if the borrower defaults on her repayment, access to $f(\cdot)$ technology is withdrawn and credit contract is terminated. Hence from then onwards the agent has to invest in the deviation-technology $\phi^D(\cdot)$ in order to maximize present discounted value of her lifetime utility. But unlike the benchmark case $\phi^D(\cdot) \not\equiv \phi(\cdot)$ in the following way:

$$\text{Suppose } \psi(k) > \phi(k) \quad \forall k \in [\underline{k}, \bar{k}]$$
$$\text{Then } \psi(k) \geq \phi^D(k) > \phi(k) \quad \forall k \in [\underline{k}, \bar{k}].$$

More precisely, after losing access to $f(\cdot)$ technology, the borrower can now invest in $g(\cdot)$ to which the borrower did not have access originally. $g(\cdot)$ satisfies the following assumption: $g(\cdot)$ is strictly increasing, strictly concave and satisfies the Inada conditions: $g'(0) = \infty$ and $g'(\infty) = 0$. $g(k) < f(k)$ $\forall k$ and $g'(k) < f'(k)$ $\forall k$. And we define k_δ^A as the maximum amount of investment such that the marginal return from investing in $g(\cdot)$ is higher than that from investing in the savings technology. Formally.

Definition 2. k_δ^A *solves* $\max_k \delta g(k) - k$.

So, $\phi^D(\cdot)$ takes the following form:



$$\phi^D(k) = \begin{cases} g(k) & \text{if } k \leq k_\delta^A \\ g(k_\delta^A) + (1+r)(k - k_\delta^A) & \text{otherwise.} \end{cases}$$

Like before we first concentrate on borrower's deviation payoff. If the borrower deviates at period t with $f(k_t)$, thereafter she has to invest in $\phi^D(\cdot)$ in order to maximize present discounted value of her lifetime utility. Hence her problem is to

$$\begin{aligned}
&\underset{\{k_{\tau+1}\}_{\tau=t}^{\infty}}{\text{Maximize}}\ u\big(f(k_t)-k_{t+1}\big)+\sum_{\tau=t+1}^{\infty}\delta^{\tau-t}u\big(\phi^D(k_\tau-k_{\tau+1}\big)\\
&\text{Subject to,}\\
&k_{t+1}\le f(k_t)\text{ and }\forall\tau\ge t+1\ \ k_{\tau+1}\le\phi^D(k_\tau).
\end{aligned}$$

We denote the soultion to this problem by $V_A^L(f(k_t))$.[3] Now due to our assumption that $g'(k)<f'(k)\forall k$

$$\begin{aligned}
&\delta f'(k_\delta^A)>\delta g'(k_\delta^A)=f'(k_\delta)\\
&\Rightarrow k_\delta^A<k_\delta.
\end{aligned}$$

Give this observation we note that when $f(k_t)\ge g(k_\delta^A)$ the agent would maintain her wealth after deviation with $f(k_t)$. So in this case, she would invest k_δ^A in $g(\cdot)$ technology and $\delta[f(k_t)-g(k_\delta^A)]$ in the savings technology, so that her wealth in the next period becomes $g(k_\delta^A)+(1+r)\delta[f(k_t)-g(k_\delta^A)]=f(k_t)$. Hence,

$$\text{When } f(k_t)\ge g(k_\delta^A)\qquad V_A^L(f(k_t))\equiv\frac{u\big((1-\delta)f(k_t)+\delta g(k_\delta^A)-k_\delta^A\big)}{1-\delta}.$$

Finally observe that $V_A^L(w)$ is increasing in w.

Given this deviation payoff, the lender's problem is to

$$\begin{aligned}
&\underset{\langle\{p_t\}_{t=0}^{\infty},\{k_{t+1}\}_{t=0}^{\infty}}{\text{Maximize}}\left[u(w+p_0-k_1)+\sum_{t=1}^{\infty}\delta^t u\big(f(k_t)+p_t-k_{t+1}\big)\right]\\
&\text{Subject to}\\
&\text{Feasibility: } w+p_0-k_1\ge0\text{ and }\forall t\ge1\ \ f(k_t)+p_t-k_{t+1}\ge0\\
&\text{Sustainability: } p_0+\sum_{t=1}^{\infty}\delta^t p_t\le0\\
&\text{DIC: } V_t^L\equiv\sum_{\tau=t}^{\infty}\delta^{\tau-t}u\big(f(k_\tau)+p_\tau-k_{\tau+1}\big)\ge V_A^L\big(f(k_t)\big)\quad\forall t\ge1.
\end{aligned}\tag{14}$$

Like before we denote the maximum value of the above problem by $V^L(w)$. We further observe that $V^L(w)$ is increasing and the sustainability condition binds at optimum, as p_0 does not enter in any incentive constraint.

[3]To differentiate notations from the previous case we superscript equivalent notations by L (for learning).

4.1 First-best Contracts

First-best scenario involves consumption smoothing and investment of efficient amount (k_δ) in $f(\cdot)$ technology from the very first period. Since apart from deviation-technology nothing has changed values of the variable at the first-best do not change, given the wealth threshold for which first-best is incentive compatible. Observe since the agent's deviation payoff has increased vis-a-vis the benchmark case, wealth threshold would increase, we determine it later, but before that the first-best consumption. At the first-best, per period consumption of an agent with endowment with w is given by

$$c^{L^*}(w) = c^*(w) = (1-\delta)w + \delta f(k_\delta) - k_\delta \tag{1st Best Cons.}$$

The borrower then attains welfare $V^{L^*}(w) = V^*(w) = \dfrac{u(c^*(w))}{1-\delta}$.

Likewise we determine the threshold (w^{L^*}) for endowment beyond which first-best is implementable from the very first period. Since the borrower's deviation payoff has increased, as expected, this threshold would also increase. This threshold is given by

$$\begin{aligned} w^{L^*} &\equiv f(k_\delta) - \frac{\delta\big(f(k_\delta) - g(k_\delta^A)\big) - \big(k_\delta - k_\delta^A\big)}{1-\delta} \\ &> f(k_\delta) - \frac{\delta f(k_\delta) - k_\delta}{1-\delta} \equiv w^* \end{aligned} \tag{Wealth Threshold with Learning}$$

We get the last inequality from the fact that $\delta g(k_\delta^A) - k_\delta^A > 0$.

Like before note that the incentive problem arises only for intermediate ranges of the discount factor. Finally, when $w \geq w^{L^*}$ the first-best contract is incentive compatible from the very first period, and $V(w) = V^{L^*}(w)$. And $V(w) < V^{L^*}(w)$ whenever $w < w^{L^*}$.

4.2 Second-best Contracts for Poor Borrowers

First-best is implementable whenever the agent's wealth is at least w^{L^*}, so now we focus on poor borrowers, for whom $w < w^{L^*}$ and characterize features of the optimal contract. Let $c_t^L = f(k_t) + p_t - k_{t+1}$ and $c_0^L = w + p_0 - k_1$ denote cosumption at date $t \geq 0$. Following the argument of lemma 1, in this case also we get that optimum consumption is non-decreasing over time.

Lemma 10. $c_t^L \geq c_{t-1}^L$ *for all* t.

The recursive formulation of the problem in this case is given by: $l_0 = p_0, p_t = l_t - \frac{l_{t-1}}{\delta}, t \geq 1$. In this notation, the problem of the lender becomes

$$\begin{aligned}&\underset{\langle\{l_t\}_{t=0}^{\infty},\{k_{t+1}\}_{t=0}^{\infty}}{\text{Maximize}}\Big[u(w+l_0-k_1)+\sum_{t=1}^{\infty}\delta^t u\Big(f(k_t)-\frac{l_{t-1}}{\delta}+l_t-k_{t+1}\Big)\Big]\\&\text{Subject to}\\&\text{DIC: } V_t^L\equiv\sum_{\tau=t}^{\infty}\delta^{\tau-t}u\Big(f(k_\tau)-\frac{l_{\tau-1}}{\delta}+l_\tau-k_{\tau+1}\Big)\geq V_A^L\big(f(k_t)\big)\quad\forall t\geq 1.\end{aligned}\tag{15}$$

The recursive representation of the above problem, like before, is given by the following lemma. Since the proof is very similar to that of lemma 2, we keep it in Appendix.

Lemma 11. *The maximum attainable welfare* $V^L(w)$ *for a borrower with initial wealth* w *must satisfy*

$$V^L(w)=\max_{l,k}[u(w+l-k)+\delta V(f(k)-\frac{l}{\delta})]\quad\textit{subject to:}\quad V^L(f(k)-\frac{l}{\delta})\geq V_A^L\big(f(k)\big)\tag{16}$$

We denote target wealth by $\Omega^L(l,k)\equiv f(k)-\frac{l}{\delta}$. Then condition (16) can be restated as:

$$V^L(w)=\max_{l,k}[u(w+l-k)+\delta V^L(\Omega^L(l,k)]\quad\text{subject to:}\quad V^L(\Omega^L(l,k))\geq V_A^L(f(k))\tag{17}$$

So in this modified problem given any target wealth Ω^L, the first-stage problem is to minimize the investment cost $k-l$, subject to DIC. We denote the minimized cost of achieving Ω^L in this case by $C^L(\Omega^L)$. So formally,

$$C^L(\Omega^L)=\min_{l,k}(k-l)\quad\text{subject to:}\quad f(k)-\frac{l}{\delta}=\Omega^L\quad\text{and}\quad V^L(\Omega^L)\geq V_A^L(f(k))\tag{18}$$

Finally the job at the second stage is to optimally choose target wealth ($\Omega^L(w)$) for the next period, given current period wealth w. So we get the following lemma.

Lemma 12. *The maximum attainable welfare* $V^L(w)$ *for a borrower with initial wealth* w *must satisfy*

$$V^L(w)=\max_{\Omega^L}[u(w-C^L(\Omega^L))+\delta V^L(\Omega^L)]\tag{19}$$

As in the proof of lemma 4, using the concavity assumption of utility function it can be shown that target wealth for the next period is nondecreasing in current period wealth w. Again since the proof of this lemma is very similar to that of lemma 4 we keep it in the Appendix.

Lemma 13. $\Omega^L(w)$ *is nondecreasing.*

Now observe simplifying (18) we can write the following:

$$C^L(\Omega^L) = \Omega^L + \min_k (k - \delta f(k)) \quad \text{subject to:} \quad V^L(\Omega^L) \geq V_A^L(f(k)) \tag{20}$$

So when $\Omega^L \geq w^{L^*}$, the agent invests k_δ from the very first period and hence $C^L(\Omega^L)$ in that case becomes $\Omega^L - [\delta f(k_\delta) - k_\delta]$.

While if $\Omega^L < w^{L^*}$, the DIC constraint binds and in particular

$$V^L(\Omega^L) = V_A^L\big(f(k)\big) \tag{21}$$

so the resulting investment size is $k^L(\Omega^L) = f^{-1}\big(V_A^{L^{-1}}(V^L(\Omega^L))\big)$. Since $V^L(\Omega^L) < V^L(w^{L^*}) = V_A^L\big(f(k_\delta)\big)$, and V_A^L, V^L and f are increasing, hence when $\Omega^L < w^{L^*}$ there will be underinvestment: $k^L(\Omega^L) < k_\delta$. It is summarized in the following lemma.

Lemma 14. *For target wealths Ω^L smaller than w^{L^*}, investment $k^L(\Omega^L)$ is smaller than the efficient level k_δ, and equal to the efficient level otherwise.*

So given target wealth optimum investment is uniquely determined:

$$k^L \equiv \begin{cases} k_\delta & \text{if } \Omega^L \geq w^{L^*} \\ f^{-1}\big(V_A^{L^{-1}}(V^L(\Omega^L))\big) & \text{otherwise} \end{cases}$$

Further whenever agent's current wealth w is at least w^{L^*}, every period she gets $\delta[f(k_\delta) - w]$ as loan and repays $f(k_\delta) - w$, observe that this means when the borrower's wealth is higher than $f(k_\delta)$ she saves $\delta[f(k_\delta) - w]$ with the lender. Finally at this parametric condition target wealth and optimum consumption are constant at w and $c^{L^*}(w)$ (determined according to (1st Best Cons.)) respectively.

Hence now we turn to the case where initial wealth is below the threshold w^{L^*}. As in the benchmark case, that wealth dynamic is monotonically increasing and converges to w^{L^*}, but the limit wealth w^{L^*} cannot be reached in finite time from any initial wealth $w < w^{L^*}$. The borrower obtains a loan $l^L(w)$ which is strictly positive and strictly increasing. Hence loan size $l_t^L = l^L(w_t)$ rises over time, eventually converging to $\dfrac{\delta\Big[\delta\big(f(k_\delta) - g(k_\delta^A)\big) - \big(k_\delta - k_\delta^A\big)\Big]}{1-\delta}$. The corresponding investment sequence (k_t) and consumption (c_t) are nondecreasing and converge to k_δ and $c^{L^*}(w^{L^*})$ respectively. We summrize the above discussion in the following proposition and since the proof is also very similar to the that in the benchmark case, we keep it in the Appendix.

Proposition 2. *When the agent's endowment $w \geq w^{L^*}$*

i) Net wealth remains constant at w^{L^*}

ii) The agent invests k_δ *at every* $t \geq 0$

iii) At $t = 0$ *she receives* $\delta[f(k_\delta) - w]$ *from the lender*

and $\forall t \geq 1$ *she repays* $f(k_\delta) - w$ *and receives* $\delta[f(k_\delta) - w]$ *from the lender*

iv) at each $t \geq 0$ *she consumes* $c^{L^*}(w) \equiv (1-\delta)w + \delta f(k_\delta) - k_\delta$.

When the agent's endowment $w < w^{L^*}$

i) Net wealth increases over time and converges to w^{L^*} *but the limit amount* w^{L^*} *is not reached within a finite time*

ii) Investment is uniquely determined as a function of next period's target wealth: $k^L(w) \equiv f^{-1}\left(V_A^{L^{-1}}(V^L(\Omega^L))\right)$. *Hence investment is increasing over time and converges to* $k^L(w^{L^*}) \equiv k_\delta$

iii) The borrower obtains a loan $l^L(w)$ *which is strictly positive and strictly increasing, that is optimal contract features progressive lending. Hence loan size* $l_t^L = l^L(w_t)$ *rises over time, eventually converging to*
$$\frac{\delta\left[\delta\left(f(k_\delta) - g(k_\delta^A)\right) - \left(k_\delta - k_\delta^A\right)\right]}{1-\delta}$$

iv) Consumption is nondecreasing and converges to $c^{L^*}(w^{L^*})$.

5 Comparison

Now we compare these two scenarios: the benchmark case where deviation-technology is same as autarky-technology and the case where the agent can acquire knowledge while using the $f(\cdot)$ technology. First let us compare the endowment thresholds beyond which first-best is implementable. As we have already observe since $\delta g(k_\delta^A) - k_\delta^A > 0$

$$w^{L^*} \equiv f(k_\delta) - \frac{\delta\left(f(k_\delta) - g(k_\delta^A)\right) - \left(k_\delta - k_\delta^A\right)}{1-\delta} \qquad > f(k_\delta) - \frac{\delta f(k_\delta) - k_\delta}{1-\delta} \equiv w^*$$

Now we classify the borrowers according to their endowment in three categories $w \geq w^{L^*}$, $w \in [w^*, w^{L^*})$ and $w < w^*$. *Case 1.* $w \geq w^{L^*}$: In both the cases the borrower invests k_δ in $f(\cdot)$ from the very first period, at each $t \geq 0$ consumes $(1-\delta)w + \delta f(k_\delta) - k_\delta$. At $t = 0$ she receives $\delta[f(k_\delta) - w]$ from the lender and $\forall t \geq 1$ she repays $f(k_\delta) - w$ and receives $\delta[f(k_\delta) - w]$ from the lender.

Welfare in both the cases are equal: $\dfrac{u\left((1-\delta)w + \delta f(k_\delta) - k_\delta\right)}{1-\delta}$

Case 2. $w \in [w^*, w^{L^*})$: First-best is achievable only when learning is not possible, so investment at all t is higher when learning is not possible, moreover consumption is smooth, so welfare is higher when learning is not possible.

Case 3. $w < w^*$: In none of the cases first-best is implementable. However, observe $V_A(f(k)) < V_A^L(f(k))$, so for any given $w < w^*$ investment amount k which is incentive compatible in the benchmark case is not when we allow for learning. Hence borrower's wealth, starting from same endowment, increases at a faster rate which further relaxes the DIC.

Now observe borrower's welafre is strictly increasing in ivestment (since we are in a region where $k < k_\delta$ and $f'(k) > 1 + r$ which is the rate of interest on loan). Hence the borrower's welafre is higher in the benchmark case.

Proposition 3. *1.* $w \geq w^{L^*}$: *In both the cases the borrower invests* k_δ *from the very first period, at every period consumes* $(1 - \delta)w + \delta f(k_\delta) - k_\delta$*. Hence welfare is the same in both the cases.*

2. $w \in [w^*, w^{L^*})$: *In the benchmark case the borrower invests* k_δ *from the very first period and cosumption is smooth, that is first-best is implemented whereas in the other case first-best is not implementable. Hence, borrower's welfare is higher in the benchmark case.*

3. $w < w^*$: *First-best is not implementable in both the cases. However, investment, loan amount are higher in the benchmark case. Borrower's welfare is higher in the benchmark case.*

6 Appendix

Proof of Lemma 11. Let $\{l_0, k_{t+1}\}_{t \geq 0}$ be an optimal contract. The following conditions are necessary for optimality: (a) given l_0, k_1, the continuation contract $\{l_t, k_{t+1}\}_{t \geq 1}$ must be optimal for the borrower starting with $w_1 = f(k_1) - \frac{l_0}{\delta}$ (a problem in which constraints $IC_t, t \geq 2$ are incorporated), and (b) the initial loan and investment size l_0, k_1 satisfies condition (3) (a problem in which IC_1 is incorporated). ∎

Proof of Lemma 13. If this is false, there exist $w_1 < w_2$ with $\Omega_1^L \equiv \Omega^L(w_1) > \Omega_2^L \equiv \Omega^L(w_2)$. Then $V^L(\Omega_1^L) > V^L(\Omega_2^L)$ and

$$u(w_2 - C^L(\Omega_2^L)) - u(w_2 - C^L(\Omega_1^L)) \geq \delta[V^L(\Omega_1^L) - V^L(\Omega_2^L)] > 0 \quad (22)$$

which implies $C^L(\Omega_1^L) > C^L(\Omega_2^L)$. On the other hand,

$$\delta[V^L(\Omega_1^L) - V^L(\Omega_2^L)] \geq u(w_1 - C^L(\Omega_2^L)) - u(w_1 - C^L(\Omega_1^L)) \tag{23}$$

which implies

$$u(w_2 - C^L(\Omega_2^L)) - u(w_2 - C^L(\Omega_1^L)) \geq u(w_1 - C^L(\Omega_2^L)) - u(w_1 - C^L(\Omega_1^L)) \tag{24}$$

This contradicts the concavity of u. ■

In order to prove proposition 2, as in the benchmark case, we prove some lemmas.

Lemma 15. *If $w < w^{L^*}$, the sequence of net wealths w_t is nondecreasing and converges to the first best threshold w^{L^*}; the corresponding investment sequence k_t is nondecreasing and converges to k_δ, and consumption c_t is nondecreasing and converging to $c^{L^*}(w^{L^*})$.*

Proof. Suppose not and w_t is decreasing: there is some $w < w^{L^*}$ such that $\Omega^L(w) < w$. Then combining with Lemma 13, it follows that wealth is nonincreasing over time, so at every date it is smaller than w, and converges to some limit $w_\infty < w$. The corresponding sequence of investments will then be nonincreasing, all strictly smaller than k_δ and converging to some $k_\infty < k_\delta$. At the same time, we have already established that consumption is nondecreasing, hence converging to some c_∞.

Now consider a small increase in investment k_{T+1} at some date $T \geq 1$. Let S_t denote the surplus on IC$_t$, i.e., $S_t^L \equiv V_t^L - V_A^L(f(k_t))$. The effect on continuation payoffs and incentive surpluses at different dates are as follows. There is no effect on any of these at any date after $T + 1$, so we focus on effects for $t \leq T + 1$.

$$\frac{\partial S_{T+1}^L}{\partial k_{T+1}} = [u'(c_{T+1}) - u'(c_{T+1})]f'(k_{T+1}) = 0 \tag{25}$$

So IC$_{T+1}$ is not jeopardized by the variation. Next

$$\frac{\partial V_T^L}{\partial k_{T+1}} = \frac{\partial S_T^L}{\partial k_{T+1}} = u'(c_{T+1})\delta f'(k_{T+1}) - u'(c_T) \tag{26}$$

As $T \to \infty$, this converges to $u'(c_\infty)[\delta f'(k_\infty) - 1]$ which is strictly positive. Hence for all T sufficiently large, expression (26) is strictly positive. The same is true at all earlier dates. Hence the variation is feasible and welfare improving, so the contract cannot be optimal.

Now we show that w_t convergers to the first-best Wealth Threshold with Learning w^{L^*}, k_t is nondecreasing and converges to k_δ and c_t converges to $c^{L^*}(w^{L^*})$.

Since the wealth sequence is nondecreasing the investment sequence is also nondecreasing (since it is an increasing function of the target wealth). So both converge to limits w^L_∞, k^L_∞ respectively.

If $k^L_\infty < k_\delta$ we will get a contradiction (using the same argument as above). Hence k_t converges to k_δ.

If $w^L_\infty < w^{L^*}$, the corresponding limit of investment $k_\infty < k_\delta$, which has been ruled out above. Hence $w^L_\infty \geq w^{L^*}$. Since $\Omega^L(w^{L^*}) = w^{L^*}$, Lemma 13 implies that $\Omega^L(w) \leq w^{L^*}$ for any $w < w^{L^*}$. Hence $w^L_\infty \leq w^{L^*}$, and it follows that $w^L_\infty = w^{L^*}$. The corresponding sequence of consumption must therefore converge to $c^{L^*}(w^{L^*})$. ∎

Next we show that when $w < w^{L^*}$, target wealth is strictly increasing in current wealth, however it can never reach the first-best Wealth Threshold with Learning w^{L^*} within a finite time.

Lemma 16. $w < w^{L^*}$ *implies* $w < \Omega^L(w) < w^{L^*}$.

Proof. First we show that $\Omega^L(w) < w^{L^*}$. We have already seen that since $\Omega^L(\cdot)$ is nondecreasing, we have $\Omega^L(w) \leq \Omega^L(w^{L^*}) = w^{L^*}$. Suppose the claim is false and there exists $w < w^{L^*}$ such that $\Omega^L(w) = w^{L^*}$. In an optimal contract starting from $w_0 = w$, we must then achieve the first-best allocation from date 1 onwards, with $c_t = c^{L^*}(w^{L^*}) = (1-\delta)f(k_\delta)+\delta g(k^A_\delta)-k^A_\delta, k_t = k_\delta$. The borrower's welfare is then $u(w + p_0 - k_\delta) + \frac{\delta u((1-\delta)f(k_\delta)+\delta g(k^A_\delta)-k^A_\delta)}{1-\delta}$. Now budget balance implies $p_0 = \frac{\delta[\delta f(k_\delta)-k_\delta]}{1-\delta}$ (since the borrower repays $\delta f(k_\delta) - k_\delta$ at every date $t \geq 1$ onwards). Hence date 0 consumption $c_0 = w + \frac{\delta^2(f(k_\delta)-g(k^A_\delta))-(k_\delta-k^A_\delta)}{1-\delta} < (1-\delta)f(k_\delta) + \delta g(k^A_\delta) - k^A_\delta, k_t = k_\delta$, as $w < w^{L^*}$.

Now consider a variation on this contract where the production level continues to be stationary (denoted k such that $k \in (k^A_\delta, k_\delta)$). The loan at date 0 is modified to $\frac{\delta[\delta f(k)-k-\delta g(k^A_\delta)+k^A_\delta]}{1-\delta}$, while future per period repayments are modified to $\delta f(k) - k - \delta g(k^A_\delta) + k^A_\delta$. Welfare in this contract is $u(w + \frac{\delta^2(f(k_\delta)-g(k^A_\delta))-(k_\delta-k^A_\delta)}{1-\delta}) + \frac{\delta u((1-\delta)f(k)+\delta g(k^A_\delta)-k^A_\delta)}{1-\delta}$. Incentive compatibility are maintained and so is budget balance. The first order effect of a slight reduction in k below k_δ equals $u'(c_0) - u'(c^{L^*}(w^{L^*})$ which is strictly positive since $c_0 < c^{L^*}(w^{L^*})$. Hence a small reduction in k will improve welfare, and we obtain a contradiction.

Next, to show that $\Omega^L(w) > w$ for any $w < w^{L^*}$, note that otherwise there is some $w < w^{L^*}$ for which $\Omega^L(w) = w$. Then starting with initial wealth of w, wealth will remain stationary. This contradicts Lemma 15 since w_t must converge to w^*. ∎

Finally we show that when $w < w^{L^*}$ optimum loan size increases over time.

Lemma 17. *Starting with any* $w < w^*$, *the borrower obtains a loan* $l^L(w)$ *which is strictly positive and strictly increasing. Hence loan size* $l_t^L = l^L(w_t)$ *rises over time, eventually converging to* $\dfrac{\delta\Big[\delta\big(f(k_\delta) - g(k_\delta^A)\big) - \big(k_\delta - k_\delta^A\big)\Big]}{1-\delta}$.

Proof. Let $x(w)$ denote date 1 output $f(k^L(w))$, where $k^L(w)$ is the investment of at date 0 with wealth w. As we have observed $l^L(w) = \delta[x(w) - \Omega^L(w)]$ and since $w < w^{L^*}$ $V^L(\Omega^L(w)) = V_A^L(x(w))$. Like before we establish the lemma using the following two properties

i) $V^L(w) > V_A^L(w)$

Observe, $f(k) > g(k)$ $\forall k$, so even if the lender does not provide any loan borrower's welfare would be higher than that from autarky where she access to $g(\cdot)$ and the savings technology: $V^L(w)|_{\text{without loan}} > V_A^L(w)$. Now we are considering the case where $w < w^{L^*}$, that is even after getting loan the borrower cannot invest k_δ from the very first period, so her welfare when she gets loan is higher than that when she does not get loan (and can invest in $f(\cdot)$), hence $V^L(w) > V^L(w)|_{\text{without loan}} > V_A^L(w)$.

ii) $V^{L'}(\Omega^L(w)) > V_A^{L'}(x(w))$ since $w < w^{L^*}$ $k < k_\delta$. So we consider two cases

a) $k \in [k_\delta^A, k_\delta)$

Observe here $V_A^L(x(w)) \equiv \dfrac{u\big((1-\delta)x(w) + \delta g(k_\delta^A) - k_\delta^A\big)}{1-\delta}$ and $V_A^{L'}(x(w)) = u'\big((1-\delta)x(w) + \delta g(k_\delta^A) - k_\delta^A\big)$.

Now, $V^{L'}(\Omega^L(w)) \geq u'(c_0^L(\Omega^L(w))$ where $c_t^L(w)$ denotes date t consumption starting with wealth w (as the borrower always has the option of immediately consuming any incremental wealth). Next, note that $c_0^L(\Omega^L(w)) < (1-\delta)x(w) + \delta g(k_\delta^A) - k_\delta^A$, since $V^L(\Omega^L(w)) \equiv u(c_0^L(\Omega^L(w))) + \sum_{t=1}^{\infty} \delta^t u(c_t^L(\Omega^L(w))) = V_A^L(x(w))$, $c_0^L(\Omega^L(w)) \leq c_t^L(\Omega^L(w))$ for all $t \geq 1$ with strict inequality for some t as the optimal consumption sequence starting with wealth $\Omega^L(w)$ is not stationary (this in turn follows from noting that if consumption were stationary then V_t would be stationary, while we have shown above that k_t is strictly increasing, so IC_t could not bind at every date t). Hence $u'(c_0^L(\Omega^L(w))) > u'((1-\delta)x(w) + \delta g(k_\delta^A) - k_\delta^A) = V_A^{L'}(x(w))$, so $V^{L'}(\Omega^L(w)) > V_A^{L'}(x(w))$.

b) $k < k_\delta^A$

Recall $V^L(\Omega^L(w)) = V_A^L(x(w))$, we also know that on contract consumption converges to $c^{L^*}(w^{L^*})$ and post-deviation consumption converges to $g(k_\delta^A) - k_\delta^A$ and in both the cases consumption is (weakly) increasing over time, which implies that after a finite point of time on contract consumption becomes higher than that from post-deviation. So to have $V^L(\Omega^L(w)) = V^L_A(x(w))$, $c^L_0(\Omega^L(w)) < c^{L^A}_0(x(w))$ where $c^L_0(\Omega^L(w))$ and $c^{L^A}_0(x(w))$ are defined as above: $c^L_t(w)$ and $c^{L^A}_0(x(w))$ denote on-contract and post-deviation consumption starting with wealth w at date t, respectively. So we have

$$V^{L'}(\Omega^L(w)) \geq u'(c^L_0(\Omega^L(w)) > u'(c^{L^A}_0(x(w))) = V^{L'}_A(x(w))$$

where the first inequality is coming from the fact we discussed above, second inequality is coming from the discussion above: $c^L_0(\Omega^L(w)) < c^{L^A}_0(x(w))$, finally, the last equality is coming from Envelope theorem.

Claim (i) now implies that $l^L(w) = \delta[x(w) - \Omega^L(w)] > 0$ since $V^L_A(x(w)) = V^L(\Omega^L(w)) > V^L_A(\Omega^L(w))$. Moreover, $V^{L'}_A(x(w))x'(w) = V^{L'}(\Omega^L(w))\Omega^{L'}(w)$ so (ii) implies $x'(w) > \Omega^{L'}(w)$. Therefore $l^{L'}(w) = \delta[x'(w) - \Omega^{L'}(w)] > 0$. l^L_t must converge to $l^L(w^{L^*}) \equiv \dfrac{\delta\left[\delta\left(f(k_\delta) - g(k^A_\delta)\right) - \left(k_\delta - k^A_\delta\right)\right]}{1-\delta}$.
∎

7 Conclusion

This paper is in a very nascent stage and we plan to extend it in various direction like how far our result is driven by the fact that production technology is convex? What happens if we allow for another technology which is more productive but requires a minimum initial investment? We have studied the time path of loan, but it is interestig to study that of the net transfer, if that becomes negative after a certain point of time that means the borrower initially gets loan and in the later stage in life she repays those, the net sense. We have assumed that investment amount is observable and contractible, given that repayment is not enforceable this might seem to be a strong assumption, so we want to relax this assumption. We have assumed that the lender is benevolent, what happens if the lender maximizes weighted sum of the borrower and his own profit? We also want to study the case where $g(\cdot)$ technology changes with time and converges to $f(\cdot)$ technology, this will capture the fact that as time passes the borrower learns more and that can also be dependent on investment amount in that a borrower which invests more learn more, given other things. Finally, we want to address borrowers with self-control problem. What happens if the lender bundles commitment product with credit service? Since the borrower knows either she will not be able to invest the entire amount which she wants to invest when self-control problem is very high or even if she invests that amount, she has to incur some self-control cost, to avoid this she may choose to repay the loan which she woruldn't have repaid if she had no such self-control problem.

References

Aghion, B. and J. Morduch, "Microfinance Beyond Group Lending," *The Economics of Transition*, 2000, *8* (2), 401–420.

Albuquerque, R. and H. A. Hopenhayn, "Optimal Lending Contracts and Firm Dynamics," *Review of Economic Studies*, 2004, *71*, 285–315.

Atkeson, A., "International Lending with Moral Hazard and Risk of Repudiation," *Econometrica*, 1991, *59* (4), 1069–1089.

Bulow, J. and K.S. Rogoff, "Sovereign Debt: Is to Forgive to Forget?," *American Economic Review*, 1989, *79* (1), 43–50.

Cole, H. L. and P. J. Kehoe, "Reviving Reputation Model of International Debt," *Federal Reserve Bank of Minneapolis Quarterly Review*, Winter 1997, *21* (1), 21–30.

Eaton, J. and M. Gersovitz, "Debt with potential repudiation: theoretical and empirical analysis," *Review of Economics Studies*, 1981, *48* (2), 289–309.

Egli, D., "Progressive Lending as an Enforcement Mechanism in Microfinance Programs," *Review of Development Economics*, 2004, *8* (4), 505–520.

Ghosh, P. and D Ray, "Information and Enforcement in Informal Credit Markets," *Economica*, 2016, *83* (329), 59–90.

Grossman, H. and J. van Huyck, "Sovereign Debt as a Contingent Claim: Excusable Default, Repudiation, and Reputation," *American Economic Review*, 1988, *78* (5), 1088–97.

Ray, D., "The Time Structure of Self-Enforcing Agreements," *Econometrica*, 2002, *70* (2), 547–582.

Rosenthal, R. W., "On the incentives associated with sovereign debt," *Journal of International Economics*, 1991, *30*, 167–176.

Shapiro, D. A., "Microfinance and Dynamic Incentives," *Journal of Development Economics*, 2015, *115*, 73–84.

Thomas, J. and T. Worrall, "Foreign Direct Investment and Risk of Expropriation," *Review of Economic Studies*, 1994, *61* (1), 81–108.