Excessive Search in a Patent Race Game*

Kaustav Das†

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Abstract

There is both evidence and some literature on the fact that competing researchers tend to duplicate their efforts when the social planner would prefer they diversify and work on different approaches to the same research question. We address this question in a model where two firms compete to make the same invention, and the first one to invent gets all the surplus. The invention can either be made using a traditional *safe* method or an innovative *risky* method. Firms share a common belief about the likelihood of the risky method to be good. There is a unique Markov perfect equilibrium, and if firms differ in their ability to make the invention, using the risky method (conditional on the method being good), in equilibrium, there is always excessive use of the risky method. If early completion of project is promoted then the inefficiency in equilibrium goes down. The paper also falls in the literature on two-armed bandits; Unlike most papers in this area we have heterogeneous players and payoff externalities, and there is a unique markovian equilibrium.

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1 Introduction

Innovation constitutes an important part of the progress of society. Starting from the growth rate of an economy to the various aspects which affect the day to day life of individuals, R&D activities play an important role. Innovation is a *costly* and *uncertain* process. The uncertainty pertains to the fact that the exact path along which the R&D activities will

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†Department of Economics, University of Exeter; email: daskaustav84@gmail.com

bear success in the most efficient way is unknown. Therefore potential innovators go through trial-and-error experimentation along the available research avenues. Since experimentation along an avenue involves a cost, it is always desirable that at any point of time, resources are optimally (from the society's point of view) spread among the available methods. Use of a relatively less efficient research avenue will delay the invention. If the society discounts the future, then this delay imposes a cost.

In an environment of R&D competition when the patent mechanism is such that the first one to invent appropriates all the rent, a particular firm's decision about which research avenue to pursue is not only affected by the likelihood of the efficacy of the avenues, but also by the choices of other firms. It is worth exploring the efficient allocation of firms across research avenues and the distortions which can take place in a non-cooperative interaction. A possible distortion would be all firms engaged in R&D experimenting on the same approach, whilst the socially optimal allocation would involve diversifying effort on different approaches.

In this paper I analyse a highly stylised model to address the above issue. The model we consider is as follows. There are two firms who are trying to make the same discovery. The first firm to make the discovery gets a positive payoff and the other firm gets a payoff of zero. There are two potential avenues along which research can be conducted to make the discovery. One of them is the traditional way such that if research is conducted along this avenue, it is known with certainty that the expected time to make the discovery is finite. The other one is an innovative avenue. It can either be good or bad. In the former case, if research is conducted along this avenue, the expected time to make the discovery is strictly less than that using the traditional method. On the other hand, if the innovative avenue turns out to be bad then the discovery can never be made by conducting research along this avenue. We formally model this by supposing that if research is conducted along the traditional avenue by a firm then in continuous time the discovery is made according to a Poisson process with intensity $\pi_0 > 0$. If the innovative avenue is good then if firm i ($i = 1, 2$) conducts research along this avenue, in continuous time discovery is made according to a Poisson process with intensity π_i such that $\pi_1 \geq \pi_2 > \pi_0$. If the innovative avenue is bad, then no discovery can be made by conducting research along this. Firms have a common belief about the likelihood of the innovative avenue being good. Formally this is captured by supposing that the innovative avenue is good with probability $p \in (0, 1)$ and this belief p is commonly known. Each firm can observe the action of the other firm. This means as firms carry out research along the innovative avenue and do not get a breakthrough, they become pessimistic about the innovative method.

There are many real-world scenarios where the model of this paper can fit. Consider a manager of a firm who has two or more employees under his control. The manager needs to get an assignment done and would reward the employee in the form of a bonus to the one who does it first. The employees have to choose among several alternate avenues to get the assignment done, although they are not sure which avenue would finally lead to success. In this case, it is possible that one of the avenues will surely lead to success, but there is an alternate avenue which can either lead to success at a faster rate or can lead to failure. Here each of the employees competes with others to be the first one to do the assignment successfully. We can also find similar situations in the Pharmaceutical research industry where different firms compete to invent a new drug for a particular disease. It often happens that there is a status quo method and also a potentially more innovative method. The analysis by Pammolli et al. (2011) [19] looks at the investment in pharmaceutical research and development. They analyse a large database that contains information on R&D projects for more than $28,000$ and find that the decline in R&D productivity in pharmaceuticals in the past two decades is associated with an increasing concentration of R&D investments in areas in which is risky. In the current paper, we show how there can be inefficient duplicative search by competing firms when they are heterogeneous in their ability to get a breakthrough while searching along a good innovative avenue.

We first analyse the optimal solution of a benevolent social planner. The planner's solution involves making both firms to search along the innovative (safe) avenue if the belief is above (below) a particular threshold. There exists an interim range of beliefs where firm 1 is made to search along the innovative avenue and firm 2 is made to search along the safe avenue. To analyse the non-cooperative interactions between the firms, we restrict ourselves to stationary markovian strategies. We show that there is a unique markov perfect equilibrium which is qualitatively similar the planner's solution. In this equilibrium, both firms use a threshold type or cutoff strategy. That is, each firm chooses the innovative avenue only if the likelihood of it being good is above a particular threshold. Else, the safe avenue is chosen. For firm 1, the threshold belief above which the risky avenue is chosen is same as that in the planner's solution. However, for the firm 2, the threshold above which the risky avenue is chosen is strictly lower than the corresponding belief in the planner's solution. This implies that in the non-cooperative interactions between the firms, there is always inefficiency and it is in the form of excessive search along the risky avenue. That is, there exists a range of beliefs where the planner would have wanted the firms to search along different avenues but left on their own, firms search along the same avenue. Hence, in the non-cooperative equilibrium there is *too much* of *duplication* of R&D activities along the innovative avenue. Hence, this paper shows that the distortion in non-cooperative equilibrium in the form of too much of duplication is due to the difference in abilities of the firms in conducting research along the innovative avenue. We can also show that by increasing the discount rate r, we can shrink the range of beliefs over which there is too much duplication.

This paper can yield two policy prescriptions. First, from the equilibrium behaviour, we can infer that a funding agency while allocating research grants can increase efficiency by exogenously imposing the type of action in the grant award. That is, a funding agency while making the grant awards to different parties can reduce too much duplication by exogenously imposing actions. Secondly, we have shown that too much duplication can be reduced by increasing the discount rate r. This suggests that a funding agency by promoting early completion of a project can reduce the extent of the duplicative search.

We also consider the case when firms are homogeneous. This means $\pi_1 = \pi_2 > \pi_0$. First, we determine the optimal solution of a benevolent social planner. This is the solution which maximises the aggregate payoffs of the firms. For each belief, this solution assigns a unique action for each firm. The solution is of the threshold type. That is, as long as the likelihood of the innovative avenue being good is higher than a particular threshold, both firms are made to choose the innovative way. Else, they are made to choose the traditional way. Next, we move on to study the non-cooperative interactions between the firms. We restrict ourselves to stationary markovian strategies. These are strategies where action depends only on the current belief about the likelihood fo the innovative avenue being good. We show that there is a unique markov perfect equilibrium and it is efficient. Thus, the current model shows that if all firms are identical, then there is no distortion in the non-cooperative equilibrium.

Related Literature: This paper contributes to the relatively less explored area of the broad literature on R&D races. It shows that there is always a distortion in the choice of research avenue in a non-cooperative interaction. Bhattacharya and Mookerjee([3]), Dasgupta and Maskin([6]) are two of the early papers which explore this issue in a static framework. Chatterjee and Evans ([5]) analyse similar issues in a dynamic setting. The model of this paper is also related to [5]. However, we show that to have too much duplication in a patent race game, it is not necessary to have a perfect negative correlation between the potential avenues.

Two recent papers which look into the issue of duplicative search in a patent race game are [11] and [18]

Some other papers to look into similar issues are Fershtman and Rubinstein ([9]) and Akcigit and Liu([1]). ([9]) studies a two-stage model in which agents simultaneously rank a finite set of boxes. Exactly one of the boxes contains the prize. Players commit to open the boxes according to their ranked order. Inefficiency arises because the box which is most likely to have the prize is not opened first. Their model is static in nature.

This paper also contributes to the strategic bandit literature. Our model can be interpreted as a two-armed bandit model with extreme payoff externalities. [12] shows that with only informational externalities and homogeneous players, there is a multiplicity of equilibria and all equilibria have inefficiency in the form of free riding. This paper shows that by introducing payoff externalities, we can get rid of multiple equilibria. With homogeneous players, we have efficient equilibrium. However, with heterogeneous players, inefficiency is in the form of too much duplication.

The rest of the paper is organised as follows. Section 2 describes the environment. Subsections $2.1 - 2.2$ describes the social planner problem and the non-cooperative equilibrium. Subsection 2.3 discusses promotion of early completion of the project. Subsection 2.4 describes the case with homogeneous firms. Finally, section 3 concludes the paper.

2 The Environment

There are two firms who are simultaneously searching for a prize. The prize is of worth one unit. The first firm to discover the prize appropriates all the rent out of it. There are two potential avenues along which the search can be conducted. One of the avenues is *traditional* and *non-risky* in the sense that if a firm searches along this avenue, then in continuous time, the prize is discovered according to a Poisson process with intensity $\pi_0 > 0$. The other avenue is more *innovative* and there is a *risk* associated with it. If the other avenue turns out to be good then if firm i ($i = 1, 2$) searches along this avenue, in continuous time prize is discovered according to a Poisson process with intensity $\pi_i > \pi_0$. If this innovative avenue turns out to be bad then no discovery is possible if search is conducted along this avenue. We have $\pi_1 > \pi_2$. This implies that conditional on the innovative avenue being good, by searching along this, firm 1 is more able than firm 2 in discovering the prize. In all the subsequent analyses, the *traditional* avenue will be denoted as S and the *innovative* avenue will be denoted as R. Both firms discount the future using a common continuous time discounting rate $r > 0$. Firms do not incur any cost while searching for the prize. Firms have a common prior $p \in (0, 1)$, the probability with which the avenue R is good. Each firm can observe the action of the other firm.

2.1 Social Planner's Problem

Consider a benevolent social planner who wants to maximise the sum of the expected discounted payoffs of the firms. At each instant, the social planner chooses an avenue for a firm. The planner's action profile at an instant t is defined by $k^t = (k_1^t, k_2^t)$ such that $k_i^t \in \{0, 1\}$ ($i = 1, 2$). $k_i = 1(0)$ implies that firm i is asked to search along the avenue R (S). Let $v(p)$ be the optimal value function of the planner. It should satisfy the following Bellman equation

$$v(p) = \max_{k_i \in \{0,1\}(i=1,2)} \big\{(2-k_1-k_2)\pi_0\, dt + k_1\pi_1 p\, dt + k_2\pi_2 p\, dt$$

$$+(1-r\,dt)[1-(2-k_1-k_2)\pi_0\, dt - k_1\pi_1 p\, dt - k_2\pi_2 p\, dt][v(p) - v^{'}(p)p(1-p)(k_1\pi_1 + k_2\pi_2)\, dt]\big\} \quad (1)$$

For notational simplicity, from now on, we will do away with the argument of $v(p)$. Simplifying (1) and ignoring the terms of the order o- dt we get

$$rv = \max_{k_i \in \{0,1\}(i=1,2)} 2\pi_0(1-v) + k_1\big\{\pi_1 p[1-v-v^{'}(1-p)] - \pi_0[1-v]\big\} + k_2\big\{\pi_2 p[1-v-v^{'}(1-p)] - \pi_0[1-v]\big\} \quad (2)$$

It turns out that the planner's solution involves *specialisation* for extreme range of beliefs and *diversification* for interim range of beliefs. This means when it is very likely that the avenue R is good (bad) both firms are made to search along R (S). For interim range of beliefs, firm 1 is made to search along R and firm 2 is made to search along S. The following proposition formally states this result.

Proposition 1 *Planner's optimal solution is of the following type. There exists p_1^* and p_2^* satisfying*

$$0 < p_1^* < p_2^* < 1$$

such that for $p \in (p_2^*, 1]$ ($p \in (0, p_1^*]$) both firms are made to search along the avenue R (S). For $p \in (p_1^*, p_2^*]$, firm 1 is made to search along the avenue R and firm 2 is made to search along the avenue S. The optimal value function of the planner is given by

$$v(p) = \begin{cases} \frac{\pi_1+\pi_2}{r+\pi_1+\pi_2}p + C_{rr}(1-p)[\Lambda(p)]^{\frac{r}{\pi_1+\pi_2}} \equiv v_{rr} & : \quad \textit{If } p \in (p_2^*, 1], \\ & : \\ \frac{\pi_0}{r+\pi_0} + \frac{r\pi_1}{(r+\pi_0)(r+\pi_0+\pi_1)}p + C_{rs}(1-p)[\Lambda(p)]^{\frac{r+\pi_0}{\pi_1}} \equiv v_{rs} & : \quad \textit{if } p \in (p_1^*, p_2^*], \\ & : \\ \frac{2\pi_0}{r+2\pi_0} & : \quad \textit{if } p \in (0, p_1^*]. \end{cases} \quad (3)$$

where $p_1^* = \frac{\pi_0}{\pi_1}$ and p_2^* is such that it satisfies

$$v_{rr}(p_2^*) = v_{rs}(p_2^*) = \frac{\pi_0(\pi_1+\pi_2)}{r\pi_2 + \pi_0(\pi_1+\pi_2)}$$

C_{rr} is an integration constant and is given by $C_{rr} = \frac{\{\frac{r\pi_0}{(r+\pi_0)(r+2\pi_0)} - \frac{r\pi_1}{(r+\pi_0)(r+\pi_0+\pi_1)}p_1^*\}}{(1-p_1^*)[\Lambda(p_1^*)]^{\frac{r+\pi_0}{\pi_1}}}$.
The integration constant C_{rs} is determined from $v_{rr}(p_2^*) = v_{rs}(p_2^*)$.

Proof.

This proposition is proved in two steps. First, we derive the value function of the planner from the conjectured solution. Then, we show that the value function satisfies the Bellman equation of the planner (2). The formal proof is relegated to appendix (A). We will discuss an important characteristics of the planner's solution. Since at $p = p_2^*$, action of firm 2 is optimally switched, from (2) we can infer that

$$\pi_2 p_2^*[1 - v - v'(1-p)] = \pi_0[1 - v]$$

From the value function of the planner we know that $v'(p_2^*) > 0$. This implies

$$\pi_2 p_2^*[1 - v] > \pi_2 p_2^*[1 - v - v'(1-p)] = \pi_0[1 - v]$$

$$\Rightarrow p_2^* > \frac{\pi_0}{\pi_2}$$

If firm 2 is the only firm around, then he would have been shifted to search along the avenue S at a belief $p = \frac{\pi_0}{\pi_2}$. However, from the above result we can see that in presence of firm 1, although firm 1 is switched to avenue S at the Marshallian threshold, firm 2 is switched at a belief higher than the Marshallian threshold. The intuition behind this result is as follows. At $p = \frac{\pi_0}{\pi_2}$, the instantaneous rates of sucess of firm 2 at both the avenues are the same. However, if there is no sucess while searching along the avenue R, the belief is updated downwards. Since firm 1 searches along R till $p = \frac{\pi_0}{\pi_1}$, this reduces the time for which firm 1 searches along R. As the planner cares for the discovery only, we can infer that at $p = p_2^*$, the benefit of having firm 2 searching along R is outweighed by the cost. This explains why $p_2^* > \frac{\pi_0}{\pi_2}$

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2.2 Non-cooperative game

We restrict ourselves to Markovian strategies. These are strategies where action depends only on the likelihood of the risky avenue to be good. For player $i (i = 1, 2)$, $k_i(p)$ denotes the action of player i. $k_i(p) \in \{0, 1\}$. $k_i(p) = 0(1)$ means search is conducted along the risky (safe) avenue. Given k_2, if v_1 is the optimal value function of firm 1, then we have

$$v_1 = \max_{k_1 \in \{0,1\}} \{\pi_0(1 - k_1)\, dt + k_1 \pi_1 p\, dt$$

$$+(1-r\,dt)[1-\pi_0(1-k_1)\,dt-(1-k_2)\pi_0\,dt-(k_1\pi_1+k_2\pi_2)p\,dt][v_1(p)-(k_1\pi_1+k_2\pi_2)p(1-p)v^{'}(p)\,dt]\}$$

$$\Rightarrow rv_1 = \pi_0(1-v_1)+k_1\{\pi_1 p[1-v_1-v_1^{'}(1-p)]-\pi_0(1-v_1)\}-(1-k_2)\pi_0 v_1-k_2\{\pi_2 p[v_1+v_1^{'}(1-p)]\} \tag{4}$$

Similarly, for firm 2, if v_2 is his optimal value function, then given k_1 we have

$$rv_2 = \pi_0(1-v_2)+k_2\{\pi_2 p[1-v_2-v_2^{'}(1-p)]-\pi_0(1-v_2)\}-(1-k_1)\pi_0 v_2-k_1\{\pi_1 p[v_2+v_2^{'}(1-p)]\} \tag{5}$$

Best Responses:

We will now determine the best responses of the firms. First, we will consider the optimal behaviour of firm 1 under the contingency when firm 2 is searching along the safe avenue. From (4) we can see that searching along the risky avenue is optimal when $\pi_1 p[1 - v_1 - v_1^{'}(1-p)] > \pi_0(1-v_1)$. Putting $k_2 = 0$ in (4) we see that searching along the risky avenue is optimal as long as

$$rv_1 > \pi_0(1-v_1) - \pi_0 v_1 \Rightarrow v_1 > \frac{\pi_0}{r+2\pi_0}$$

Similarly, for firm 2 we can say that when firm 1 is searching along the safe avenue, it is optimal for firm 2 to search along the risky avenue as long as $v_2 > \frac{\pi_0}{r+2\pi_0}$.

We will now determine the optimal behaviour of the firms when the opponent firm is searching along the risky avenue. Suppose firm 2 is searching along the risky avenue. In that case putting $k_2 = 1$ in (2) we obtain that it is optimal for firm 1 to search along the risky avenue as long as

$$rv_1 > \pi_0(1-v_1) + \frac{\pi_0\pi_2}{\pi_1}(1-v_1) - \pi_2 p$$

$$\Rightarrow v_1 > \frac{\pi_0(\pi_1+\pi_2) - \pi_1\pi_2 p}{r\pi_1 + \pi_0(\pi_1+\pi_2)} \tag{6}$$

Similarly, when firm 1 searches along the risky avenue, it is optimal for firm 2 to search along the risky avenue as long as

$$v_2 > \frac{\pi_0(\pi_1+\pi_2) - \pi_1\pi_2 p}{r\pi_2 + \pi_0(\pi_1+\pi_2)} \tag{7}$$

Proposition 2 *There exists an equilibrium where firm 1 searches along the risky avenue (R) for beliefs p greater than* $\frac{\pi_o}{\pi_1}$ *and along the safe avenue (S) for* $p \leq \frac{\pi_0}{\pi_1}$*. Firm 2 searches along the risky avenue for beliefs p greater than* p_2^{*n} *and along the safe avenue for* $p \leq p_2^{*n}$ *where*

$$\frac{\pi_0}{\pi_1} < p_2^{*n} < \frac{\pi_o}{\pi_2}$$

Proof.

This proposition is proved in two steps. First, based on the conjectured equilibrium, the value functions of players are derived. We then show that the derived value functions satisfy

(4) and (5) for firms 1 and 2 respectively. That is, we show that no player has any incentive to deviate.

Let $p_1^* = \frac{\pi_0}{\pi_1}$. From the conjectured equilibrium behaviour, we derive the value functions $v_1(p)$ and $v_2(p)$ for firms 1 and 2 respectively.

$$v_1(p) = \begin{cases} v_1^{rr}(p) = \frac{\pi_1}{r+\pi_1+\pi_2}p + C_1^{rr}[\Lambda(p)]^{\frac{r}{\pi_1+\pi_2}} & : \text{ If } p \in (p_2^{*n}, 1], \\ & : \\ v_1^{rs}(p) = \frac{\pi_1}{r+\pi_0+\pi_1}p + C_1^{rs}[\Lambda(p)]^{\frac{r+\pi_0}{\pi_1}} & : \text{ if } p \in (\frac{\pi_0}{\pi_1}, p_2^*] \\ & : \\ \frac{\pi_0}{r+2\pi_0} & : \text{ if } p \in (0, \frac{\pi_0}{\pi_1}]. \end{cases} \tag{8}$$

$$v_2(p) = \begin{cases} v_2^{rr}(p) = \frac{\pi_2}{r+\pi_1+\pi_2}p + C_2^{rr}[\Lambda(p)]^{\frac{r}{\pi_1+\pi_2}} & : \text{ If } p \in (p_2^{*n}, 1], \\ & : \\ v_2^{rs}(p) = \frac{\pi_0}{r+\pi_0}(1 - \frac{\pi_1}{r+\pi_0+\pi_1}p) + C_2^{rs}[\Lambda(p)]^{\frac{r+\pi_0}{\pi_1}} & : \text{ if } p \in (\frac{\pi_0}{\pi_1}, p_2^*] \\ & : \\ \frac{\pi_0}{r+2\pi_0} & : \text{ if } p \in (0, \frac{\pi_0}{\pi_1}]. \end{cases} \tag{9}$$

C_1^{rs} is determined from $v_1^{rs}(p_1^*) = \frac{\pi_0}{r+2\pi_0}$ and is strictly positive. C_2^{rs} is determined from $v_2^{rs}(p_1^*) = \frac{\pi_0}{r+2\pi_0}$ and is strictly negative. p_2^{*n} is such that $v_2^{rs}(p_2^{*n}) = \frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p_2^{*n}}{r\pi_2+\pi_0(\pi_1+\pi_2)}$.

C_1^{rr} is determined from $v_1^{rr}(p_2^{*n}) = v_1^{rs}(p_2^{*n})$. It is strictly positive. Similarly, C_2^{rr} is determined from $v_2^{rr}(p_2^{*n}) = v_2^{rs}(p_2^{*n})$ and it is strictly positive.

Appendix (B) formally derives the value functions and shows that there exists a unique $p_2^{*n} \in (p_1^*, 1)$ such that $v_2(p_2^{*n}) = \frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p_2^{*n}}{r\pi_2+\pi_0(\pi_1+\pi_2)}$.

Appendix (C) formally shows that by standard verification arguments, the above value functions satisfy the Bellman equations (4) and (5) respectively.

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We will now discuss certain important aspects of the equilibrium constructed above. First, it is evident that in the equilibrium constructed above, both firms are using a threshold type strategy. That is they search along the risky avenue if the likelihood of the risky avenue being good is above a threshold. Else, they always search along the safe avenue.

Next, as shown in appendix (B), the slope of the value function of firm 2 at $p = p_2^{*n}$ is strictly negative. p_2^{*n} is the belief where firm 2 (given firm 1's equilibrium strategy) optimally shifts from searching along the risky avenue to search along the safe avenue. Incorporating this fact in the Bellman equation (5) we can infer that

$$\pi_2 p_2^{*n}[1 - v_2 - v_2'(1 - p_2^{*n})] = \pi_0(1 - v_2)$$

Since $v_2'(p_2^{*n}) < 0$, the above equality implies that $\pi_2 p_2^{*n} < \pi_0 \Rightarrow p_2^{*n} < \frac{\pi_0}{\pi_2}$. From the planner's solution we have seen that the optimality requires firm 2 to search along the safe avenue for all beliefs less than or equal to p_2^* and $p_2^* > \frac{\pi_0}{\pi_2}$. This implies that in the non-cooperatove equilibrium constructed in the above proposition involves inefficiency and the inefficiency is in form of too much search along the risky avenue. While social optimum requires firm 2 to search along the safe avenue for all beliefs less than or equal to p_2^*, firm 2 keeps searching along the risky avenue as long as the belief is greater than or equal to $p_2^(* n) < p_2^*$. This means for beliefs $p \in (p_2^{*n}), p_2^*]$, there is excessive search along the risky avenue. The fact that $p_2^{*n} < \frac{\pi_0}{\pi_2}$ can be intuitively explained as follows. In the current setting because of the winner takesd all structure, both firm want to be the first inventor. At the belief $p = \frac{\pi_0}{\pi_2}$, the myopic payoff to player 2 is same across the risky and the safe avenue. However, if firm 2 is searching along the risky avenue and there is no breakthrough, then the belief attributed to the risky arm being good goes down. Since firm 1 searches along the risky avenue till the belief is greater than or equal to $p_1^* = \frac{\pi_0}{\pi_1}$, as beief goes down, the duration spent by firm 1 in searching along the risky avenue goes down. This inceases the future expected payoff of firm 2. This explains why firm 2 has incentive to search along the risky arm for beliefs just below $\frac{\pi_0}{\pi_2}$.

The above proposition implies that in the non-cooperative equilibrium, we always have a range of beliefs over which there is excessive search along the risky avenue. For this range of beliefs, social optimality requires firm 2 to search aling the safe avenue but left on its own, it will search along the safe avenue.

We will now show that the above described equilibrium is the unique markov perfect equilibrium of the model. The following proposition states this.

Proposition 3 *The equilibrium described in the preceding proposition is the unique equilibrium of the described game*

Proof.

Let p_l be the lowest belief such that in any equilibrium at p_l and at beliefs just above p_l at least one firm searches along the risky avenue.

First we will argue that $p_l > \frac{\pi_0}{\pi_2}$. Suppose this is possible. Then consider a belief $p \in (\frac{\pi_0}{\pi_2}, p_l)$. By the definition of p_l, the strategies of both firms in any equilibrium should entail them to search along the safe avenue. Suppose one of the firms (say firm i, $i = 1, 2$) deviates and searches along the risky avenue. In that case for all $p^{'} \in (p, p_l)$, we have $v_i(p^{'}) > \frac{\pi_0}{r+2\pi_0}$. This implies that this is a profitable deviation.

In similar manner we can argue that we cannot have $\frac{\pi_0}{\pi_1} < p_l \leq \frac{\pi_0}{\pi_2}$. In this case firm 1 can profitably deviate.

Finally, we will argue that we cannot have $p_l < \frac{\pi_0}{\pi_1}$. Suppose it is possible. There can be two situations.

$(i.)$ Both firms are searching along R at beliefs just above p_l. This means at $p = p_l$ both firms's payoffs are equal to $\frac{\pi_0}{r+2\pi_0}$. Since both firms search along the risky avenue at beliefs just above p_l, it should be the case that firm i's payoff is greater than or equal to $\frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p}{r\pi_2+\pi_0(\pi_1+\pi_2)}$. Consider firm 2. For any $p < \frac{\pi_0}{\pi_1}$ we have

$$\frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p}{r\pi_2+\pi_0(\pi_1+\pi_2)} > \frac{\pi_0\pi_1}{r\pi_2+\pi_0(\pi_1+\pi_2)}$$

We have

$$\frac{\pi_0\pi_1}{r\pi_2+\pi_0(\pi_1+\pi_2)} - \frac{\pi_0}{r+2\pi_0} = \pi_0[\frac{(r+\pi_0)(\pi_1-\pi_2)}{(r\pi_2+\pi_0(\pi_1+\pi_2))(r+2\pi_0)} > 0$$

The slope of the value function of firm 2 at p_l is strictly negative as $p_l < \frac{\pi_0}{\pi_2}$. This shows that at beliefs just above p_l, firm 2's value is strictly less than $\frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p}{r\pi_2+\pi_0(\pi_1+\pi_2)}$. Thus, firm 2 is not playing his best response.

$(ii.)$ One of the firms is searching along the safe avenue and the other is searching along the risky avenue at beliefs just above p_l. Similar to the previous case, we can show here as well that the player who is searching along the risky avenue is not playing a best response.

From the above arguments, we can infer that firm $p_l = \frac{\pi_0}{\pi_1}$.

We will now establish that for beliefs just above p_l, only mutual best responses are firm 1 searching along the risky avenue and firm 2 searching along the safe avenue.

To show this, we will first argue that both firms cannot search along the risky avenue for beliefs just above p_l. In that case for beliefs just above p_l, $v_2 < \frac{\pi_0}{r+2\pi_0}$. At $p = \frac{\pi_0}{\pi_1}$, $\frac{\pi_0}{r+2\pi_0} < \frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p}{r\pi_2+\pi_0(\pi_1+\pi_2)}$. Thus, firm 2 is not playing a best response.

Hence, we can conclude that in any markov perfect equilibrium of the game, for beliefs just above $\frac{\pi_0}{\pi_1}$, firm 1 searches along the risky avenue and firm 2 searches along the safe avenue.

Finally, we will argue that there does not exist any $p' \in (\frac{\pi_0}{\pi_1}, p_2^{*n})$ such that for beliefs right above p', firm 2 searching along the risky avenue and firm 1 searching along the safe avenue constitute mutual best responses. Suppose it is possible. Let p_l' be the lowest of such beliefs. Since for all $p \in (\frac{\pi_0}{\pi_1}, p']$, firm 2 searches along the safe avenue and firm 1 searches along the risky avenue, v_2 for beliefs right above p_l' is strictly less than $\frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p}{r\pi_2+\pi_0(\pi_1+\pi_2)}$. Thus firm 2 is not playing a best repsonse.

The above arguments now allow us to conclude that the equilibrium described in the previous proposition is the unique markov perfect equilibrium of the model.

■

We have shown that there is a unique markov perfect equilibrium of the patent race game and too much of duplication is a phenomenon in the non-cooperative equlibrium. The following subsection shows that by promoting early completion of a project, the range of beliefs over which there is duplicative search goes down.

2.3 Promoting early completion

In this subsection we show that by promoting early completion of a project, the extent of duplicative search in equilibrium can be reduced. Formally, early completion implies that the reward from the sucessful search is time dependant. We assume that the first firm to discover the prize gets a payoff of $e^{-\lambda\tau}$ where $\tau > 0$ is the time point at which the discovery takes place adn $\lambda > 0$. Thus, early completion of the project yields a higher payoff. Modifying the payoff from discovery results in an effective discount rate of $r + \lambda$. Thus by promoting early completion, we increase the effective discount rate. In the following proposition, we will argue that by increasing the the discount rate, we can reduce the extent of duplicative search.

Proposition 4 *Consider a $r_0 > 0$. Let $p_2^*(r_0)$ and $p_2^{*N}(r_0)$ be the thresholds where firm 2 switches to search along the safe avenue from the risky avenue in the planners problem and the equilibrium respectively. There exists a $\tilde{r} > 0$ with $r_0 < \tilde{r} < \infty$ such that for all $r > \tilde{r}$ we have $p_2^*(r) < p_2^*(r_0)$ and $p_2^{*N}(r) > p_2^{*N}(r_0)$.*

Proof.

We prove this proposition in two steps. First we show that by increasing the discount rate, the planner's threshold p_2^* can be reduced. This is done in the folloiwng lemma.

Lemma 1 *There exists a $\tilde{r}^p$ ($r_0 < \tilde{r}^p < \infty$) such that for all $r > \tilde{r}^p$, we have $p_2^*(r) < p_2^*(r_0)$.*

Proof. Let $\tilde{p}$ be such that

$$\frac{\pi_0}{r+\pi_0} + \frac{r\pi_1}{(r+\pi_0+\pi_1)(r+\pi_0)}\tilde{p} = \frac{\pi_0(\pi_1+\pi_2)}{r\pi_2+\pi_0(\pi_1+\pi_2)}$$

This gives us $\tilde{p} = \frac{\pi_0(r+\pi_0+\pi_1)}{r\pi_2+\pi_0(\pi_1+\pi_2)}$. From 3, we know that

$$\frac{\pi_0}{r+\pi_0} + \frac{r\pi_1}{(r+\pi_0+\pi_1)(r+\pi_0)}p_2^* + C_{rr}(1-p_2^*)[\Lambda(p_2^*)]^{\frac{r+\pi_0}{\pi_1}} = \frac{\pi_0(\pi_1+\pi_2)}{r\pi_2+\pi_0(\pi_1+\pi_2)}$$

Since $C_{rr} > 0$, we have $p_2^* < \tilde{p}$ for all $r > 0$. As $r \to \infty$, $\tilde{p} = \frac{\pi_0(1+\frac{(\pi_0+\pi_1)}{r})}{\pi_2+\frac{\pi_0(\pi_1+\pi_2)}{r}} \to \frac{\pi_0}{\pi_2}$. For all $r > 0$, we have proved that $p_2^* > \frac{\pi_0}{\pi_2}$. As $\frac{d\tilde{p}}{dr} < 0$ (please refer to appendix D.1 for a formal proof), we can find a $\tilde{r}^p$ with $r_0 < \tilde{r}^p < \infty$ such that for all $r > \tilde{r}^p$, we have $\tilde{p} < p_2^*(r_0)$. Since for all $r > 0$, $p_2^* < \tilde{r}^p$, for al $r > r_0$, $p_2^*(r) < p_2^*(r_0)$. This concludes the proof of the lemma.

■

Next we will show that by increasing the discount rate, the thereshold where firm 2 switches to search along the safe avenue from the risky avenue in the equilibrium can be increased. This is done in the following lemma

Lemma 2 *There exists a $\tilde{r}^n$ ($r_0 < \tilde{r}^n < \infty$) such that for all $r > \tilde{r}^n$, we have $p_2^{*N}(r) > p_2^{*N}(r_0)$.*

Proof.

Let $\tilde{p}^n$ be the threshold such that

$$\frac{\pi_0(\pi_1 + \pi_2) - \pi_1\pi_2\tilde{p}^n}{r\pi_2 + \pi_0(\pi_1 + \pi_2)} = \frac{\pi_0}{r + 2\pi_0}$$

This gives us $\tilde{p}^n = \frac{\pi_0}{\pi_2}\frac{r\pi_1+\pi_0(\pi_1+\pi_2)}{\pi_1(r+2\pi_0)}$. From 9, we know that $v_2(p_2^{*N}) = \frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p_2^{*N}}{r\pi_2+\pi_0(\pi_1+\pi_2)}$. Since $v_2(p_2^{*N}) < \frac{\pi_0}{r+2\pi_0}$, we have $p_2^{*N} > \tilde{p}^n$ for all $r > 0$. As $r \to \infty$, $\tilde{p}^n \to \frac{\pi_0}{\pi_2}$. Since, $\frac{d\tilde{p}^n}{dr} > 0$ (please refer to appendix D.1 for a formal proof), we can find a $\tilde{r}^n$ ($r_0 < \tilde{r}^n < \infty$) such that for all $r > \tilde{r}^n$, we have $\tilde{p}^n > p_2^{*N}(r_0)$. Since for all $r > 0$, $p_2^{*N} > \tilde{p}^n$, for all $r > 0$, we have $p_2^{*N}(r) > p_2^{*N}(r_0)$. This concludes the proof of the lemma. ■

Let $\tilde{r} = \max\{\tilde{r}^p, \tilde{r}^n\}$. From the above two lemmas, we can infer that $p_2^*(r) < p_2^*(r_0)$ and $p_2^{*N}(r) > p_2^{*N}(r_0)$. In equilibrium, $[p_2^{*N}, p_2^*]$ is the range of beliefs for which there is duplicative search. Thus we have shown that by increasing r, we can shrink the range of beliefs over which there is duplicative search. This concludes teh proof of the proposition.

Figure 1 shows how p_2^* and p_2^{*N} changes with r for some particular values of the other parameters. The horizontal straight line is the belief $\frac{\pi_0}{\pi_1}$. The curve above it depicts the values of p_2^* and the curve below it depicts the values of p_2^{*N}.

■

We will now discuss the situation where players are homogeneous.

2.4 Homogeneous Firms

Suppose both firms are identical. This means $\pi_1 = \pi_2 > \pi_0$.

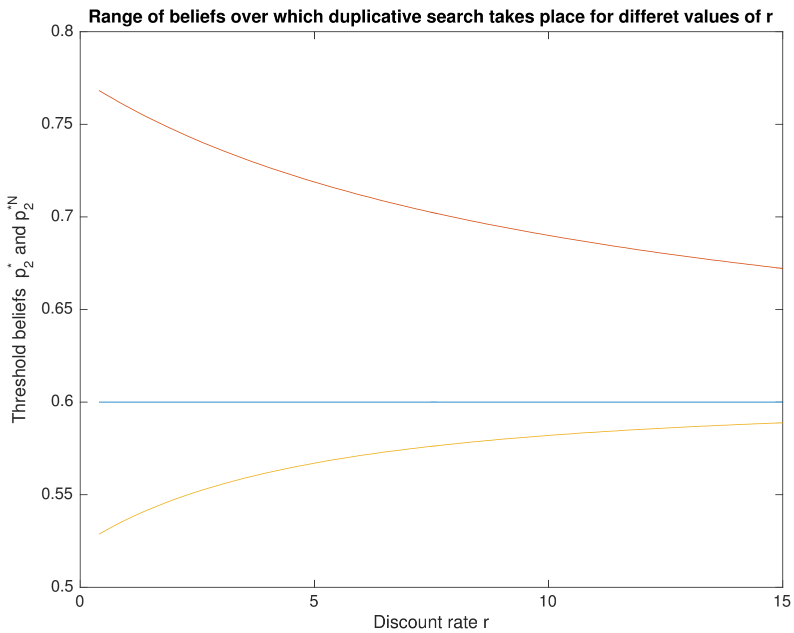

Figure 1: Range of beliefs over which duplicative search takes place.

2.4.1 Social Planner's problem

Like before, we consider a benevolent social planner who seeks to maximise the sum of the expected discounted payoffs of the firms. Let $v(p)$ be the optimal value function of the planner. Then we have

$$rv = \max_{k \in \{0,1,2\}} \{2\pi_0(1-v) + k[\pi_1 p(1 - v(p) - v^{'}(1-p)) - \pi_0(1-v)]\}$$

k denotes the number of firms the planner makes to search along the risky avenue.

The following proposition describes the optimal solution of the planner.

Proposition 5 *There exists a threshold* $p^* = \frac{\pi_0}{\pi_1}$ *such that for all* $p > p^*$*, both firms are made to search along the risky avenue and for all* $p \leq p^*$*, both firms are made to search along the safe avenue.*

Proof.

We first conjecture the optimal solution as described in the proposition and derive the value function. This gives us the folloiwng value function.

$$v(p) = \begin{cases} \frac{2\pi_1}{r+2\pi_1}p + C(1-p)[\Lambda(p)]^{\frac{r}{2\pi_1}} & : \quad \text{if } p > p^* \\ \frac{2\pi_0}{r+2\pi_0} & : \quad \text{if } p \leq p^* \end{cases}$$

where C is an integration constant. Imposing the smooth pasting condition at the belief $p = p^*$ we determine $p^* = \frac{\pi_0}{\pi_1}$. By standard verification arguments we can infer that the value function obtained satisfies the Bellman equation. ■

2.4.2 Non-cooperative game

As before, we restrict ourselves to stationary markovian strategies. Let v_i be the optimal payoff to firm $i(i = 1, 2)$ in an equilibrium. k_i denotes the strategy of player i. $k_i \in \{0, 1\}$. $k_i = 0(1)$ implies that firm i chooses to search along the safe (risky) avenue.

Given k_j $(j \neq i)$ v_i will satisfy

$$rv_i = \max_{k_i\{0,1\}} \pi_0[1-v]+k_i\{\pi p(1-v_i-v_i^{'}(1-p))-\pi_0(1-v_i)\}-(1-k_j)\pi_0 v_i-k_j[v_i p\pi_1+v_i^{'}p(1-p)\pi_1]$$

From the above Bellman equation we can derive the best responses of the firms.

Consider firm i. Given that the other firm is searching along the risky avenue, it is optimal for firm i to search along the risky avenue as long as

$$v_i \geq \frac{2\pi_0 - \pi_1 p}{r + 2\pi_0}$$

If the other firm is searching along the safe avenue, then firm i searches along the risky avenue as long as

$$v_i > \frac{\pi_0}{r + 2\pi_0}$$

It turns out that there exists a markov perfect equilibrium the outcome of which is identical to that in the planner's solution. The following proposition states this.

Proposition 6 *There exists an equilibrium such that both firms search along the risky avenue for $p > \frac{\pi_0}{\pi_1}$ and search along the safe avenue for all $p \leq \frac{\pi_0}{\pi_1}$.*

Proof.

We first derive the value functions based on the conjectured solution. Then we show that the obtained value function satisfies the Bellman equation.

■

We will now show that the equilibrium described in the previous proposition is the unique equilibrium of the model with homogeneous firms. The following proposition describes this.

Proposition 7 *The equilibrium described in the previous proposition is the unique equilibrium of the model*

Proof.

As with the heterogenoeus agents, we can show that $p_l = \frac{\pi_0}{\pi_1}$. At beliefs right above p_l if a firm is searching along the risky avenue, then the best response of the other firm is also to search along the risky avenue. This follows from the fact that at p_l each firm gets a payoff of $\frac{\pi_0}{r+2\pi_0}$. At $p = \frac{\pi_0}{\pi_1}$, we have

$$\frac{2\pi_0 - \pi_1 p}{r + 2\pi_0} = \frac{\pi_0}{r + 2\pi_0}$$

Since the value function of a firm for $p > \frac{\pi_0}{\pi_1}$ is strictly increasing and convex, for $p > \frac{\pi_0}{\pi_1}$ we have

$$v_i > \frac{2\pi_0 - \pi_1 p}{r + 2\pi_0}$$

Also, if the other firm is searching along the safe avenue, the best response of firm i is to search along the risky avenue. Hence, for $p > \frac{\pi_0}{\pi_1}$, it is a dominant action of each firm to choose to search along the risky avenue.

Similarly we can show that for $p \leq \frac{\pi_0}{\pi_1}$, it is a dominant action of each firm to choose to search along the safe avenue.

This shows that the equilibrium described in the previous proposition is the unique MPE of the model with homogeneous firms.

■

The implication of the above proposition is that with homogeneous firms, there is a unique equilibrium and the equilibrium implements the efficient outcome.

3 Conclusion

We have shown how in a patent race model with dynamic learning and optimal readjustment of project selection heterogeneous firms tend to use the innovative method excessively. The unique equilibrium of the model with heterogeneous firms implies some important policy prescriptions for research funding agencies. Finally, the model also identifies the condition under which we can get rid of the multiplicity of equilibria in a strategic experimentation model with two armed bandits.

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APPENDIX

A Proof of proposition (1)

Consider the range of beliefs $(0, p_1^*]$. According to the conjectured solution, both firms are made to search along the avenue S. This implies for this range of beliefs, $v(p) = \frac{2\pi_0}{r+2\pi_0}$. This is obtained by putting $k_1 = k_2 = 0$ in (2).

For $p \in (p_1^*, p_2^*]$, according to the conjectured solution, firm 1 searches along the avenue R and 2 searches along the avenue S. This means for this range of beliefs, we have $k_1 = 1$ and $k_2 = 0$. Substituting this in (2), we obtain the following O.D.E

$$v^{'}\pi_1 p(1-p) + v[r + \pi_0 + \pi_1 p] = \pi_0 + \pi_1 p$$

Solving this we get

$$v(p) = \frac{\pi_0}{r+\pi_0} + \frac{r\pi_1}{(r+\pi_0)(r+\pi_0+\pi_1)}p + C_{rs}(1-p)[\Lambda(p)]^{\frac{r+\pi_0}{\pi_1}}$$

where C_{rr} is an integration constant. Since the value function obtained for this range of beliefs is continuous, we have $v(p_1^*) = \frac{2\pi_0}{r+2\pi_0}$. This gives us $C_{rr} = \frac{\{\frac{r\pi_0}{(r+\pi_0)(r+2\pi_0)} - \frac{r\pi_1}{(r+\pi_0)(r+\pi_0+\pi_1)}p_1^*\}}{(1-p_1^*)[\Lambda(p_1^*)]^{\frac{r+\pi_0}{\pi_1}}}$

Since the planner optimally switches the action of firm 1 at $p = p_1^*$, we have $v^{'}(p_1^*) = 0$, the manifestaion of the smooth pasting condition. This gives us

$$\frac{r\pi_1}{(r+\pi_0)(r+\pi_0+\pi_1)} - C_{rr}[\Lambda(p_1^*)]^{\frac{(r+\pi_0)}{\pi_1}}[1 + \frac{r+\pi_0}{\pi_1}\frac{1}{p_1^*}]$$

Substituting the value of C_{rr} we get $p_1^* = \frac{\pi_0}{\pi_1}$.

Consider the range of beliefs $(p_2^*, 1]$. According to the conjectured solution, both firms are made to search along the avenue R. Substituting $k_1 = k_2 = 1$ in (2) we get the following O.D.E

$$v^{'}(\pi_1+\pi_2)p(1-p) + v[r + (\pi_1+\pi_2)p] = (\pi_1+\pi_2)p$$

Solving this we obtain

$$v(p) = \frac{\pi_1+\pi_2}{r+\pi_1+\pi_2}p + C_{rr}(1-p)[\Lambda(p)]^{\frac{r}{\pi_1+\pi_2}}$$

Where C_{rr} is an integration constant. At $p = p_2^*$, planner optimally switches the action of firm 2. From (2) it must be the case that at $p = p_2^*$ we have

$$\pi_2 p_2^*(1 - v - v^{'}(1-p)) = \pi_0(1-v)$$

Since smooth-pasting condition is satisfied at $p = p_2^*$, we have

$$\pi_2 p_2^*[1 - v - v^{'}(1-p)] = r\frac{\pi_2}{\pi_1+\pi_2}v$$

Thus,

$$\pi_2 p_2^*(1 - v - v^{'}(1-p)) = r\frac{\pi_2}{\pi_1+\pi_2}v = \pi_0(1-v)$$

$$\Rightarrow v(p_2^*) = \frac{\pi_0(\pi_1+\pi_2)}{\pi_0(\pi_1+\pi_2)+r\pi_2}$$

Since the value function is continuous, we must have $v_{rr}(p_2^*) = v_{rs}(p_2^*) = \frac{\pi_0(\pi_1+\pi_2)}{\pi_0(\pi_1+\pi_2)+r\pi_2}$. We will now show that there does exist a $p_2^* \in (p_1^*, 1)$ such that $v_{rs}(p_2^*) = \frac{\pi_0(\pi_1+\pi_2)}{\pi_0(\pi_1+\pi_2)+r\pi_2}$.

At $p_2^* = p_1^*$, we have $v_{rs} = \frac{2\pi_0}{r+2\pi_0} < \frac{\pi_0(\pi_1+\pi_2)}{\pi_0(\pi_1+\pi_2)+r\pi_2}$. At $p_2^* = 1$, we have $v_{rs} = \frac{\pi_0}{r+\pi_0} + \frac{r\pi_1}{(r+\pi_0)(r+\pi_0+\pi_1)} > \frac{\pi_0(\pi_1+\pi_2)}{\pi_0(\pi_1+\pi_2)+r\pi_2}$. Since v_{rs} is strictly convex and increasing, $\exists$ a unique $p_2^* \in (p_1^*, 1)$ such that $v_{rs}(p_2^*) = \frac{\pi_0(\pi_1+\pi_2)}{\pi_0(\pi_1+\pi_2)+r\pi_2}$. C_{rr} is determined from $v_{rs}(p_2^*) = v_{rr}(p_2^*)$. This completes the derivation fo the value function. We will now verify that the obtained value function satisfies (2).

Consider $p \in (0, p_1^*]$. $v^{'} = 0$. Since $p < \frac{\pi_0}{\pi_1} < \frac{\pi_0}{\pi_2}$,

$$\pi_i p(1 - v - v^{'}) < \pi_0(1 - v) \text{ for } i = 1, 2$$

This implies $k_1 = k_2 = 0$ is optimal.

For $p \in (p_1^*, p_2^*]$, we have

$$\pi_1 p[1 - v - v^{'}(1 - p)] = (r + \pi_0)\{v - \frac{\pi_0}{r + \pi_0}\}$$

To have $\pi_1 p[1 - v - v^{'}(1 - p)] \geq \pi_0[1 - v]$ we need $v(p) \geq \frac{2\pi_0}{r+2\pi_0}$. Since this is true for this range of beliefs, $k_1 = 1$ is optimal. Similarly, as $v(p) \leq \frac{\pi_0(\pi_1+\pi_2)}{\pi_0(\pi_1+\pi_2)+r\pi_2}$, we have $\pi_2 p[1 - v - v^{'}(1 - p)] \leq \pi_0[1 - v]$. This implies $k_2 = 0$ is optimal. Finally, consider the range of beliefs $(p_2^*, 1]$. Since $v(p) \geq \frac{\pi_0(\pi_1+\pi_2)}{\pi_0(\pi_1+\pi_2)+r\pi_2} > \frac{\pi_0(\pi_1+\pi_2)}{\pi_0(\pi_1+\pi_2)+r\pi_1}$, we have

$$\pi_i p[1 - v - v^{'}(1 - p)] \geq \pi_0[1 - v]$$

Thus $k_1 = k_2 = 1$ is optimal. This shows that the obtained value function satisfies the Bellman equation of the planner.

B Derivation of the value functions in equilibrium

First, we derive $v_1(p)$. Consider the range of beliefs $p \in (0, \frac{\pi_0}{\pi_1}]$. According to the conjectured equilibrium, both firms search along the safe avenue for these beliefs. Hence, $k_1 = k_2 = 0$ for the considered range of beliefs. This gives us

$$rv_1 = \pi_0(1 - v_1) - \pi_0 v_1 \Rightarrow v_1 = \frac{\pi_0}{r + 2\pi_0}$$

Consider the range of beliefs $p \in (\frac{\pi_0}{\pi_1}, p_2^{*n}]$. Putting $k_2 = 0$ and $k_1 = 1$ in (4) we get

$$rv_1 = \pi_1[1 - v_1 - v_1^{'}(1 - p)] - \pi_0 v_1$$

$$\Rightarrow v_1^{'} + v_1 \frac{[r + \pi_0 + \pi_1 p]}{\pi_1 p(1 - p)} = \frac{\pi_1}{\pi_1(1 - p)}$$

The solution to the above differential equation is

$$v_1(p) = v_1^{rs}(p) = \frac{\pi_1}{r + \pi_0 + \pi_1} p + C_1^{rs}(1 - p)[\Lambda(p)]^{\frac{r+\pi_0}{\pi_1}}$$

where C_1^{rs} is an integration constant and $\Lambda(p) = \frac{1-p}{p}$. Since the value function is continuous, we have

$$v_1^{rs}(p_1^*) = \frac{\pi_1}{r + \pi_0 + \pi_1} p_1^* + C_1^{rs} [\Lambda(p_1^*)]^{\frac{r+\pi_0}{\pi_1}} = \frac{\pi_0}{r + 2\pi_0}$$

where $p_1^* = \frac{\pi_0}{\pi_1}$. This gives us $C_1^{rs} = \frac{\frac{\pi_o}{r+2\pi_o} - \frac{\pi_0}{r+\pi_1+\pi_0}}{(1-p_1^)[\Lambda(p_1^*)]^{\frac{r+\pi_0}{\pi_1}}} > 0$. Substituting this value of C_1^{rs} we get

$$v_1^{'}(p_1^*) = \frac{\pi_1}{r + \pi_0 + \pi_1} - C_1^{rs} [\Lambda(p_1^*)]^{\frac{r+\pi_0}{\pi_1}} (1 + \frac{r}{\pi_1 p}) = 0$$

This implies that $v_1(p)$ for the range $p \in (\frac{\pi_0}{\pi_1}, p_2^{*n}]$ is strictly increasing and convex.

Finally consider the range of beliefs $p \in (p_2^{*n}, 1]$. Putting $k_1 = k_2 = 1$ in (4) we get

$$v_1^{'} + v_1 \frac{[r + (\pi_1 + \pi_2)p]}{(\pi_1 + \pi_2)p(1-p)} = \frac{\pi_1}{(\pi_1 + \pi_2)(1-p)}$$

The solution to the above differential equation is

$$v_1(p) = v_1^{rr}(p) = \frac{\pi_1}{r + \pi_1 + \pi_2} p + C_1^{rr}(1-p)[\Lambda(p)]^{\frac{r}{(\pi_1+\pi_2)}}$$

where C_1^{rr} is an integration constant. This integration constant is determined from

$$v_1^{rr}(p_2^{*n}) = \frac{\pi_1}{r + \pi_1 + \pi_2} p_2^{*n} + C_1^{rr}(1 - p_2^{*n})[\Lambda(p_2^{*n})]^{\frac{r}{(\pi_1+\pi_2)}} = v_1^{rs}(p_2^{*n})$$

Next, we will derive the value function of player 2 in the conjectured equilibrium. Like the derivation of $v_1(p)$, it is easy to see that for $p \in (0, \frac{\pi_0}{\pi_1}]$, $v_2(p) = \frac{\pi_0}{r+2\pi_0}$. For $p \in (\frac{\pi_0}{\pi_1}, p_2^{*n}]$, according to the conjectured equilibrium, we have $k_2 = 0$ and $k_1 = 1$. Substituting these values in (5) we get

$$v_2^{'} + v_2 \frac{r + \pi_0 + \pi_1 p}{\pi_1 p(1-p)} = \pi_0$$

The solution to the above differential equation is

$$v_2(p) = v_2^{rs}(p) = \frac{\pi_0}{r + \pi_0}(1 - \frac{\pi_1}{r + \pi_0 + \pi_1} p) + C_2^{rs}(1-p)[\Lambda(p)]^{\frac{r+\pi_0}{\pi_1}}$$

Since v_2 is continuous, we have

$$\frac{\pi_0}{r + \pi_0}(1 - \frac{\pi_1}{r + \pi_0 + \pi_1} p_1^*) + C_2^{rs} [\Lambda(p_1^*)]^{\frac{r+\pi_0}{\pi_1}} = \frac{\pi_0}{r + 2\pi_0}$$

This gives us

$$C_2^{rs}(1 - p_1^*)[\Lambda(p_1^*)]^{\frac{r+\pi_0}{\pi_1}} = \frac{\pi_0}{r + 2\pi_0} - \frac{\pi_0}{r + \pi_0}(1 - \frac{\pi_1}{r + \pi_0 + \pi_1} p_1^*)$$

$$\Rightarrow C_2^{rs} = \frac{\frac{\pi_0^2(\pi_0-\pi_1)}{(r+2\pi_0)(r+\pi_0+\pi_1)(r+\pi_0)}}{(1-p_1^*)[\Lambda(p_1^*]^{\frac{r+\pi_0}{\pi_1}}} < 0$$

Using this value of C_2^{rs} we get

$$v_2^{rs'}(p_1^*) = -\frac{\pi_0\pi_1}{(r+\pi_0)(r+\pi_0+\pi_1)} - C_2^{rs}[\Lambda(p_1^*)]^{\frac{r+\pi_0}{\pi_1}}(1+\frac{r+\pi_0}{\pi_1 p_1^*}) = 0$$

This implies for $p \in (\frac{\pi_0}{\pi_1}, p_2^{*n}]$, $v_2(p)$ is strictly decreasing and strictly concave.

Finally, consider the range of beliefs $p > p_2^{*n}$. According to the conjectured equilibrium, $k_2 = k_1 = 1$. Substituting this in (5) we obtain

$$v_2^{'} + v_2\frac{[r+(\pi_1+\pi_2)p]}{(\pi_1+\pi_2)p(1-p)} = \frac{\pi_2}{(\pi_1+\pi_2)(1-p)}$$

The solution to the above differential equation gives us

$$v_2(p) = v_2^{rr}(p) = \frac{\pi_2}{r+\pi_1+\pi_2}p + C_2^{rr}(1-p)[\Lambda(p)]^{\frac{r}{\pi_1+\pi_2}}$$

where C_2^{rr} is the integration constant. Since the value function is continuous, we have

$$\frac{\pi_2}{r+\pi_1+\pi_2}p_2^{*n} + C_2^{rr}(1-p_2^{*n})[\Lambda(p_2^*)]^{\frac{r}{\pi_1+\pi_2}} = v_2^{rs}(p_2^{*n})$$

To determine the sign of the integration constant C_2^{rr} we first show that the left and right derivative of $v_2(p)$ at $p = p_2^{*n}$ are equal. This is shown in the following lemma.

Lemma 3 *In the conjectured equilibrium, the right hand derivative and the left hand derivative of v_2 at $p = p_2^{*n}$ are equal.*

Proof.

We need to show that at p_2^{*n} we must have

$$v_2^{'}(p_2^{*n-}) = v_2^{'}(p_2^{*n+})$$

$$\Rightarrow v_2^{'rs}(p_2^{*n}) = v_2^{'rr}(p_2^{*n})$$

First observe that C_2^{rr} is determined from

$$\frac{\pi_2}{r+\pi_1+\pi_2}p_2^{*n} + C_2^{rr}(1-p_2^{*n})[\Lambda(p_2^*)]^{\frac{r}{\pi_1+\pi_2}} = v_2^{rs}(p_2^{*n})$$

Since, v_2^{rs} is decreasing in p. Thus higher (lower) is the value of p_2^{*n}, lower (higher) is the value of of C_2^{rr} and hence, higher (lower) is the value of $v_2^{'rr}(p_2^{*n})$.

Suppose the slopes are not equal. Then either $v_2^{'rs}(p_2^{*n}) > v_2^{'rr}(p_2^{*n})$ or $v_2^{'rs}(p_2^{*n}) < v_2^{'rr}(p_2^{*n})$. In the former case, if firm 2 searches along the safe avenue for belifs just above p_2^{*n}, then he gets higher payoff. In the latter case if firm 2 shifts to search along the safe avenue at a lower belief $p_2^{*n'}$ ($p_2^{*n'} < p_2^{*n}$, then C_2^{rr} will be higher and hence, firm 2 will get a higher payoff for all $p \in (p_2^{*n'}, p_2^{*n}]$. Since at $p = p_2^{*n}$ firm 2 optimally shifts from searching along the risky avenue to search along the safe avenue, we must have $v_2'(p_2^{*n-}) = v_2'(p_2^{*n+}) \Rightarrow v_2^{'rs}(p_2^{*n}) = v_2^{'rr}(p_2^{*n})$. This is formally called the *smooth pasting* condition.

■

One implication of the above lemma is that $v_2^{'rr}(p_2^{*n}) < 0 \Rightarrow C_2^{rr} > 0$. Hence, $v_2(p)$ for $p > p_2^{*n}$ is strictly convex.

Finally, we will show that there exists a $p_2^{*n} \in (p_1^*, 1)$ such that

$$v_2^{rs}(p_2^{*n}) = \frac{\pi_0(\pi_1 + \pi_2) - \pi_1\pi_2 p_2^{*n}}{r\pi_2 + \pi_0(\pi_1 + \pi_2)}$$

At $p_2^{*n} = p_1^*$, $v_2^{rs}(p_2^{*n}) = \frac{\pi_0}{r+2\pi_0}$ and $\frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p_2^{*n}}{r\pi_2+\pi_0(\pi_1+\pi_2)} = \frac{\pi_0\pi_1}{r\pi_2+\pi_0(\pi_1+\pi_2)}$. Thus, we have

$$\frac{\pi_0(\pi_1 + \pi_2) - \pi_1\pi_2 p_2^{*n}}{r\pi_2 + \pi_0(\pi_1 + \pi_2)} - v_2^{rs}(p_2^{*n}) = \frac{\pi_0(r + \pi_0)(\pi_1 - \pi_2)}{(r\pi_2 + \pi_0(\pi_1 + \pi_2))(r + 2\pi_0)} > 0$$

At $p_2^{*n} = 1$, we have $v_2^{rs}(p_2^{*n}) = \frac{\pi_0}{r+\pi_0+\pi_1}$ and $\frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p_2^{*n}}{r\pi_2+\pi_0(\pi_1+\pi_2)} = \frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2}{r\pi_2+\pi_0(\pi_1+\pi_2)}$. Thus, we have

$$\frac{\pi_0(\pi_1 + \pi_2) - \pi_1\pi_2 p_2^{*n}}{r\pi_2 + \pi_0(\pi_1 + \pi_2)} - v_2^{rs}(p_2^{*n})$$

$$= \frac{r\pi_1(\pi_0 - \pi_2) + \pi_1(\pi_0 - \pi_2)}{(r + \pi_1 + \pi_0)(r\pi_2 + \pi_0(\pi_1 + \pi_2)} < 0$$

For $p > p_1^*$, both v_2^{rs} and $\frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p_2^{*n}}{r\pi_2+\pi_0(\pi_1+\pi_2)}$ are decreasing in p. The maximum magnitude of the slope of v_2^{rs} is $\frac{\pi_0\pi_1}{(r+\pi_0)(r+\pi_1+\pi_0)}$. The magnitude of the slope of $\frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p_2^{*n}}{r\pi_2+\pi_0(\pi_1+\pi_2)}$ is $\frac{\pi_1\pi_2}{r\pi_2+\pi_0(\pi_1+\pi_2)}$. Since we have

$$\frac{\pi_1\pi_2}{r\pi_2 + \pi_0(\pi_1 + \pi_2)} - \frac{\pi_0\pi_1}{(r + \pi_0)(r + \pi_1 + \pi_0)}$$

$$= \frac{\pi_1[r\pi_2(r + \pi_0 + \pi_1) + \pi_0\pi_1(\pi_2 - \pi_0)]}{(r\pi_2 + \pi_0(\pi_1 + \pi_2))[(r + \pi_0)(r + \pi_0 + \pi_2)} > 0$$

Hence, we can infer that there exists a unique $p_2^{*n} \in (p_1^*, 1)$ such that

$$v_2^{rs}(p_2^{*n}) = \frac{\pi_0(\pi_1 + \pi_2) - \pi_1\pi_2 p_2^{*n}}{r\pi_2 + \pi_0(\pi_1 + \pi_2)}$$

This concludes the formal derivation of the value functions.

C Verification arguments for the non-cooperative equilibrium

First we show that the value function v_1 in (8) satisfies the Bellman equation (4). This means given the action profile of firm 2, firm 1 is taking optimal actions at all beliefs. Consider the range $p \in (0, \frac{\pi_0}{\pi_1}]$ first. From our best response analysis we know that when the other firm is choosing to search along the safe avenue, it is optimal for firm 1 to search along the safe avenue as long as $v_1 \leq \frac{\pi_0}{r+2\pi_0}$. Since $v_1 = \frac{\pi_0}{r+2\pi_0}$ for the considered range of beliefs, firm 1 is taking optimal actions in this range.

Now, consider the beliefs $p \in (\frac{\pi_0}{\pi_1}, p_2^{*n}]$. From the conjectured equilibrium we know that for this range of beliefs, firm 2 is searching along the safe avenue. Firm 1's value function for this range of beliefs is strictly convex and increasing. Since $v_1(p_1^*) = \frac{\pi_o}{r+2\pi_0}$, we know that for all $p \in (\frac{\pi_0}{\pi_1}, p_2^{*n}]$, $v_1(p) > \frac{\pi_o}{r+2\pi_0}$. This shows that firm 1 is playing his best response. Next, consider firm 2. From the derivation of the value function we know that $v_2(p) \leq \frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p}{r\pi_2+\pi_0(\pi_1+\pi_2)}$ for all $p \in (\frac{\pi_0}{\pi_1}, p_2^{*n}]$. This shows that firm 2 is playing his best response for this range of beliefs.

Finally, we consider the range of beliefs $p \in (p_2^{*n}, 1]$. From the derived value function of firm 2 we know that v_2 for this range of beliefs is strictly increasing and convex. Since at $p = p_2^{*n}$ we have $v_2(p) = \frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p}{r\pi_2+\pi_0(\pi_1+\pi_2)}$, for all $p > p_2^{*n}$, we have $v_2(p) > \frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p}{r\pi_2+\pi_0(\pi_1+\pi_2)}$. This implies that player 2 is playing his best response.

Firm1's value at $p = p_2^{*n}$ is strictly higher than firm 2's value. For $p \in (p_2^{*n}, 1]$, v_1 is strictly increasing and convex. As $\pi_1 > \pi_2$, we have $\frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p}{r\pi_2+\pi_0(\pi_1+\pi_2)} > \frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p}{r\pi_1+\pi_0(\pi_1+\pi_2)}$. Thus, for all $p \in (p_2^{*n}, 1]$, $v_1(p) > \frac{\pi_0(\pi_1+\pi_2)-\pi_1\pi_2 p}{r\pi_1+\pi_0(\pi_1+\pi_2)}$. This shows that firm 1 is playing his best response for this range of beliefs.

D Auxillary results

D.1 $\frac{d\tilde{p}}{dr} < 0$

We have $\tilde{p} = \frac{\pi_0(r+\pi_0+\pi_1)}{[r\pi_2+\pi_0(\pi_1+\pi_2)]}$.

$$\frac{d\tilde{p}}{dr} = \frac{\pi_0}{[r\pi_2 + \pi_0(\pi_1 + \pi_2)]} - \frac{\pi_0\pi_2(r + \pi_0 + \pi_1)}{[r\pi_2 + \pi_0(\pi_1 + \pi_2)]^2}$$

$$= \frac{\pi_0}{[r\pi_2 + \pi_0(\pi_1 + \pi_2)]}\{1 - \frac{\pi_0(r + \pi_0 + \pi_1)}{[r\pi_2 + \pi_0(\pi_1 + \pi_2)]}$$

$$= \frac{\pi_0}{[r\pi_2 + \pi_0(\pi_1 + \pi_2)]}\{\frac{\pi_1(\pi_0 - \pi_2)}{[r\pi_2 + \pi_0(\pi_1 + \pi_2)]} < 0$$

D.2 $\frac{d\tilde{p}^n}{dr} > 0$

We have $\tilde{p}^n = \frac{\pi_0}{\pi_2}\frac{(r\pi + \pi_0\pi_1 + \pi_0\pi_2)}{(r + 2\pi_0)\pi_1}$

$$\frac{d\tilde{p}^n}{dr} = \frac{\pi_0}{\pi_2}[\frac{\pi_1}{(r + 2\pi_0)\pi_1} - \frac{\pi_1[r\pi_1 + \pi_0(\pi_1 + \pi_2)]}{(r + 2\pi_0)^2\pi_1^2}]$$

$$= \frac{\pi_0}{\pi_2(r + 2\pi_0)}[\frac{\pi_0(\pi_1 - \pi_2)}{\pi_1(r + 2\pi_0)}] > 0$$