On Wold's Sufficiency Approach to Representation of Preferences

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Abstract

This paper presents a simple set of sufficient conditions on sequence spaces [based on Wold (1943)] that guarantee representation of preference orders. It is shown that in the Wold approach weak monotonicity is not necessary for representation. Crucial to the approach is representation along the diagonal of the sequence space. Through a series of examples we show that our representation result is robust; it cannot be improved upon by dropping or weakening our assumptions. An example is also presented to show that existence of degenerate indifference classes is *not* a detriment to the representation of monotone preferences, thereby clarifying the extent to which substitution possibilities can be useful in representation.

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1 Introduction

There are two principal methods leading to the representability of a preference order by a numerical utility function. The first is due to Wold (1943) who proposed that if preferences were monotone, and if for every consumption bundle, there was a unique diagonal bundle to which it was indifferent, then the scalar associated with that diagonal bundle can be used as a numerical measure of the utility of the consumption bundle. The second is due to Debreu (1954), who showed that if the set of consumption bundles contained a *countable order-dense* subset, then the preference order can be represented by a numerical utility function.

The condition of Debreu turns out to be *necessary* as well for the numerical representation of a preference order [see Fishburn (1970), Kreps (1988) and Bridges and Mehta (1995)]. On the other hand, following Debreu (1954) non-representability crucially builds on the lexicographic preferences[1] he introduced. The inability to represent lexicographic preferences may be viewed as arising from the fact that there are not enough substitution possibilities — since each consumption bundle is indifferent only to itself.

The emphasis on substitution possibilities for the representation of preference orders appear in the writing of Georgescu-Roegen (1954). In his definitive study on consumer preferences, Chipman (1960, pp. 210) says that the countable order dense property "has little intuitive appeal". He suggests (see Chipman (1960, pp. 194)) an Axiom of Substitution as part of his formal axiomatic set up. The Wold approach can be seen as identifying a specific form of substitution which is sufficient to guarantee representability of monotone preferences (see Beardon and Mehta (1994) for an exposition of the Wold approach).

The conditions used by Wold are not necessary, even for the class of monotone preferences. Nevertheless, because of its transparent geometric intuition, the method of Wold has been widely used to establish numerical representation of preference orders under a variety of different assumptions on preferences [see, for example, Diamond (1965), Asheim, Mitra and Tungodden (2012), Mitra and Ozbek (2013), Banerjee (2014)]. In fact, the idea of Wold is so compelling that it is now even included in a basic text on Intermediate Microeconomics to illustrate how a utility function can be found to represent preferences [see Varian (2014)]. *We undertake an analysis of the Wold approach with the intention of exploring the role that substitution possibilities play as a sufficient condition for representation.*

Our first result (Theorem 1) shows that under a weak continuity assumption and existence of a weakly better and weakly worse off element along the diagonal, representation always obtains. In particular, preferences need not be monotone. Our version of the Wold approach consists

[1] This order is also called the "dictionary order" [see Munkres (2000)] and was known to set theorist and topologists alike, see Sierpinski (1965, pp. 221). In Sierpinski's masterful development of set theory, in Chapter 11 he deals with sets of order type λ — this is what we call representable orders. While his text was in development since 1909, its completion date is recorded in the preface as 1952. Our reference is to the second edition of the english translation which appeared in 1965.

of two steps: the first step establishes the existence of a utility function on the diagonal and in the second step, this function is extended to the entire space in a way that preserves the ranking of off-diagonal terms.

The continuity assumption we use, *Scalar Continuity*, is used in Mitra and Ozbek (2013). It states that the set of all profiles on the diagonal that weakly dominate and those that are dominated by x must be closed sets in the real line for every profile x. In other words, we require that the intersection of the upper and lower contour sets with the diagonal be closed in the real line (for a given x, these are denoted $A(x)$ and $B(x)$ respectively). Additionally we require that these sets are also non-empty; this is our specific way of imposing the condition that there are substitution possibilities.

Recent representation theorems by Mitra and Ozbek (2013) and Banerjee (2014) are established under fairly mild assumptions using variations of the Wold method. However, though there is considerable overlap of coverage in the two results, neither result follows from the other. Theorem 1 yields *both* of these representation theorems as corollaries.

In view of our sufficiency result (Theorem 1) we ask whether our result can be improved by dropping or weakening any of the conditions used. In Example 1 (section 4.1) we show that we cannot drop the condition on the non-emptiness of the sets $A(x)$ and $B(x)$ and still guarantee representation. On the other hand, we also show that if the non-emptiness condition fails to hold only on a countable subset of the space, a representation result can be recovered (Theorem 2).

A crucial step in the Wold approach is to show that the indifference set for each x has non-empty intersection with the diagonal. We introduce this as the *Wold Condition* (in section 4.2) and study whether our Theorem 1 can be improved by assuming this weaker sufficiency condition. Example 2 (in section 4.2) demonstrates the Wold condition *cannot* replace Scalar Continuity in Theorem 1. However, a weak monotonicity assumption guarantees representation in the presence of the Wold condition (Theorem 3). It is tempting to conjecture that this weak diagonal monotonicity is necessary for representation when the Wold condition is satisfied, we show that to be not the case in Example 3.

Finally, we study the possibility of representation when there are absolutely no substitution possibilities — indifference sets pertaining to each bundle is *degenerate*. We present an example of a preference order for which the indifference set for every x is a singleton but the order is still representable. This example shows in a very stark way that our Scalar Continuity condition is not necessary for representation. It demonstrates in general that *substitution conditions* (conditions that deal with properties of indifference classes associated with elements of the domain of preferences) *cannot* be *necessary for representation*, since representability can be obtained even when absolutely no substitution possibilities exist.

To understand the significance of our example let us provide a very rough paraphrase of the intuition for representability in the context of consumer demand theory associated with the extent of substitutability from any given consumption bundle. The problem with the lexicographic preference order is that it has no substitution possibilities at all: each point in the commodity space is indifferent only to itself. This entails that the preference order is extremely sensitive (to changes in any direction of the two-dimensional real space) and the set of real numbers is not large enough to capture this sensitivity. If substitution possibilities are present, so that one has non-degenerate "indifference curves", then "many" points can be assigned the same real number, thereby economizing on the use of real numbers and making it feasible to represent the preference order. Perhaps, the most explicit statement of the above intuition for non-representability appears in the standard text on microeconomic theory by Mas-colell, Whinston and Green (1995, pp.46), in their informal discussion of the lexicographic preference order. "With this preference ordering, no two distinct bundles are indifferent; indifference sets are singletons. Therefore, we have two dimensions of distinct indifference sets. Yet, each of these indifference sets must be assigned, in an order-preserving way, a different utility number from the one dimensional real line."

However, lest the reader gets carried away by this argument, the authors add a sentence of caution, "In fact, a somewhat subtle argument is actually required to establish this claim rigorously." Our example (in section 4.3) can also be seen as an elaboration on this sentence of caution, and particularly on the distinction between substitutability and the existence of non-degenerate indifference sets. Specifically, we present an example of a preference ordering (in two-dimensional real space), in which indifference sets are singletons, just like the lexicographic preference order, but which nevertheless can be represented by a real-valued function. Thereby, the example shows that non-degenerate indifference sets are not necessary for representability of a preference order. The example is not entirely straightforward to construct; it draws on methods which appear in the papers by Lindenbaum (1933) and Sierpinski (1934).

2 Preliminaries

2.1 Definitions and Axioms

2.1.1 Sequence Spaces and Binary Relations

Let $\mathbb{N}$ be the set of non-negative integers, and $\mathbb{R}$ the set of real numbers. Denote $\mathbb{R}^{\mathbb{N}}$ by Z. For $z, z' \in Z$, we write $z' \geq z$ if $z'_t \geq z_t$ for all $t \in \mathbb{N}$; we write $z' > z$ if $z' \geq z$ and $z' \neq z$; and we write $z' \gg z$ if $z'_t > z_t$ for all $t \in \mathbb{N}$.

Let Y be a non-empty connected set in $\mathbb{R}$ such that $[0, 1] \subset Y$[2] and $X = Y^m$ where $m \in \mathbb{M} = \mathbb{N} \cup \{\infty\}$ — such a space will be called a *sequence space*. The constant m-dimensional

[2] All of these conditions on Y amounts to saying that Y is an interval in $\mathbb{R}$. See Royden (1988), p.183.

vector $(1, 1, 1, ...)$ is denoted by e; clearly the vector $e \in X$. A particular subset of X, the diagonal D is of special significance in our paper. Define

$$D = \{x \in X : \text{there exists } \lambda \in Y \text{ such that } x = \lambda e\} \tag{1}$$

A binary relation $\succsim$ is called *representable* if there is some $u : X \to \mathbb{R}$ such that for all $x, x' \in X$

$$x' \succsim x \text{ if and only if } u(x') \geq u(x). \tag{2}$$

A real-valued function u that satisfies (2) will be called a *utility function* or a function that represents $\succsim$. We will consider a binary relation $\succsim$ on $X \times X$ satisfying a subset of the following conditions. The phrase "a binary relation on X" will often be used for a binary relation on $X \times X$. We deal with complete and transitive binary relations and formally state this as condition **O**. A binary relation satisfying condition **O** is called a *preference order*.

Order (O): The preference relation $\succsim$ is complete and transitive.

2.1.2 Scalar Continuity and Substitution Conditions

In addition to **O,** we document some conditions that have been studied in the context of representing monotone preferences on sequence spaces. We introduce a class of conditions that describe the extent of substitution possibilities allowed between points in X.

Define, for each $x \in X$, the sets

$$A(x) = \{\lambda \in Y : \lambda e \succsim x\}$$

and

$$B(x) = \{\lambda \in Y : x \succsim \lambda e\}.$$

We next define the concept of scalar continuity (**SC**) and the non-emptiness (**NE**) of the sets $A(x)$ and $B(x)$.

Scalar Continuity (SC): If for each $x \in X$, the sets $A(x)$ and $B(x)$ are closed subsets in Y.

Non-emptiness (NE): If for each $x \in X$, the set $A(x)$ and $B(x)$ are both non-empty.

The crux of the Wold argument (for instance in its application to consumer theory) is in establishing that each commodity bundle x is indifferent to at least one point along the diagonal — this is explicitly stated as condition **W** below. The nature of our query also necessitates the study of representability when there are *no* substitution possibilities. Under the reflexivity assumption, absent any substitution possibility, the set $IN(x)$ of y's that are indifferent to x would contain only x — this is stated explicitly as the *degenerate indifference set* condition. It will be clear from the analysis that follows that **SC** and **NE** is jointly a stronger requirement than **W**.

Wold Condition (W): The set $I(x) = A(x) \cap B(x)$ is a non-empty subset of Y.

Degenerate Indifference Set (DIS): The set $IN(x) = \{y \in X : y \sim x\} = \{x\}$.

2.1.3 Monotonicity Conditions

We introduce some conditions that relate the natural order of vectors in a sequence space ($\geq$) to the preference order ($\succsim$). Mitra and Ozbek (2013) study representability under an assumption of weak monotonicity (**M**) and Banerjee (2014) introduced diagonal Pareto (**DP**) in his study of representation following the Wold approach. Additionally we document two weak monotonicity conditions on the diagonal (**WID** and **WDD**).

Monotonicity (M): For $x, y \in X$ with $x \geq y$ we must have $x \succsim y$.

Diagonal Pareto (DP): For $\lambda, \mu \in Y$ with $\lambda > \mu$ we must have $\lambda e \succ \mu e$.

Weakly Increasing along the Diagonal (WID): For $\lambda, \mu \in Y$ and $\lambda > \mu$ we have $\lambda e \succsim \mu e$.

Weakly Decreasing along the Diagonal (WDD): For $\lambda, \mu \in Y$ and $\lambda > \mu$ we have $\mu e \succsim \lambda e$.

3 Representation

3.1 Wold Representation Theorem

In this section we spell out precisely the assumptions that make Wold's method work. It is shown here that assumptions **SC** and **NE** together imply representability of binary relations that satisfy **O**.

Theorem 1 *If a binary relation $\succsim$ defined on $X \times X$ satisfies **O**, **SC** and **NE**, then there is some $u : X \to \mathbb{R}$ representing $\succsim$ in the sense of (2).*

Remarks on Theorem 1

(a) Wold Technique: A crucial step of the Wold construction demonstrates that for each x there is some diagonal element to which x is indifferent. In the standard textbook treatment of this result (see, Mas-collel, Whinston and Green (1995) for instance) monotonicity is assumed along with continuity to achieve this step. In Theorem 1 condition **NE** plays a crucial role in establishing the indifference of each element to an element in the diagonal of the sequence space. Observe the proof of Theorem 1 is in two steps[3] — the first step shows the representability of $\succsim$ when restricted to the diagonal and the second step extends this function to the entire space. While the first step follows from **SC** (**NE** is not needed here and this part of the proof is similar to the proof of Theorem I in Debreu (1954)) the second step makes critical use of **NE**. The precise roles played by the conditions **NE** and **SC** is explored in section 4. While several applications of the result pertaining to monotone orders have appeared in Mitra and

[3] See the introduction in Beardon and Mehta (1994) for another clear discussion of the process that originates in Wold (1943).

Ozbek (2013) we present a simple example of an instance where representation follows, using Theorem 1, even when monotonicity fails.

Consider the following binary relation on $[0,1] \times [0,1]$ (denoted by X) for $x, y \in X$ we say

$$(x_1, x_2) \succsim (y_1, y_2) \text{ iff } f(\frac{x_1 + x_2}{2}) \geq f(\frac{y_1 + y_2}{2}) \tag{3}$$

where $f : [0,1] \to [0,1]$ is given by

$$f(t) = t - t^2.$$

Clearly from the definition (3) it follows that $\succsim$ satisfies **O**. To verify **SC** assume that $x \in X$ and $\lambda^n \in [0,1]$ for each natural number n and consider $x \succsim (\lambda^n, \lambda^n)$. Under the assumption that λ^n converges to some λ we need to show that $x \succsim (\lambda, \lambda)$. We obtain from $x \succsim (\lambda^n, \lambda^n)$ and (3) that $f((x_1 + x_2)/2) \geq f(\lambda^n)$ holds and as f is continuous on $[0,1]$ and $\lambda \in [0,1]$ (note that λ^n is a convergent sequence in a compact set, so the limit point, λ must belong to the set $[0,1]$ as well) it follows that $f(\lambda^n) \to f(\lambda)$. Finally as weak inequalities are preserved in the limit we must have $f(x_1 + x_2) \geq f(\lambda)$ implying $x \succsim (\lambda, \lambda)$ (from (3)) establishing that $B(x)$ is closed. A similar argument also shows that $A(x)$ is closed for every x in X, as needed for **SC**. Non-emptiness (**NE**) also easily follows since it can be verified that $(1/2, 1/2) \in A(x)$ and $(0,0) \in B(x)$ for all $x \in X$. Theorem 1 (or direct inspection of the definition itself) can be invoked to claim that the order is representable[4].

The very nature of the function f guarantees that this order satisfies neither **WID** or **WDD**. This follows from noting that f attains its (unique) maximum at $(1/2)$ and is strictly increasing in the sub-domain $[0, 1/2)$ and strictly decreasing in the sub-domain $(1/2, 1]$.

(b) Relation to Representation Literature: The two step approach used in Theorem 1 is not new. Explicit use of this approach is made in Arrow and Hahn (1971), Monteiro (1987), Beardon and Mehta (1994) and Weibull and Voorneveld (2016) among others.

(i) Monteiro (1987): Condition **NE** appears in the literature as an order boundedness property of preference orders (see Monteiro (1987; pp. 148)). Condition **SC**, is also used in the proof of his main result but no explicit mention is made of the condition. To elaborate, in Theorem 1 [Monteiro (1987)], **NE** is used by considering a connected, separable subset of X, say F having the property that for every $x \in X$ there is some $a, b \in F$ such that $a \succsim x \succsim b$. This is used along with closed upper and lower contour sets to show the existence of a representation. **SC** is used in the first line of the proof of Theorem 1 (see Monteiro (1987), pp. 149), but the assumption of closed upper and lower contour sets need not translate to closed upper contour sets relative to a connected and separable subset, F which bounds $\succsim$ (condition **NE**). For instance on $X = [0,1]$ consider the order $\geq$ and observe that the the subset $F = (0,1)$ is connected and separable. However, $A(1/2)$ is *not* closed as it has a sequence that converges to 1 but 1 is *not* in F. Our

[4]One immediately observes that the representability issue here is trivial. Whether one can "construct" a non-representable binary relation on $\mathbb{R}$ is not currently known to us. See an elaboration of this issue in Remark (b) following Example 2.

approach makes this assumption (**SC** is precisely closedness of intersection of upper and lower contour sets with $F = D$) explicitly. Perhaps more importantly the two motivating issues for us are (a) emphasizing transparency and clarity over generality and (b) to show that the Wold (1943) approach is not intrinsically linked to monotonicity as is suggested in the literature and in the textbook treatment of his method — in other words, our focus is the general applicability of the Wold approach in showing existence of a representation even without an assumption of monotonicity.

It is worth pointing out that in the infinite dimensional case, that is when $X = Y^\infty$, order continuity (as assumed in Monteiro (1987)) would be equivalent to continuity defined using the sup-norm topology. Scalar continuity is known to be weaker than sup-norm continuity in this context (see, Mitra and Ozbek (2013)). This shows that in addition to the general transparency of stating the continuity on subsets of $\mathbb{R}$ our Theorem 1 does not directly follow from that of Monteiro (1987).

*(ii) Beardon and Mehta (1994):*There seems a general consensus that some form of monotonicity is needed for Wold approach to go through. In Beardon and Mehta (1994) this is very clearly stated — "...a close examination of Wold's paper reveals that, in fact, Wold *assumed only that the preference relation is weakly monotonic* and, using only this, he was able to sketch a proof (that is perhaps more subtle than he has been given credit for) of the existence of a continuous utility function." While this two steps approach has been used in more general settings we make the explicit argument that even weak monotonicity is *not* needed, it suffices to obtain representation on a rich enough sub-domain that can then be extended to the entire commodity space.

(iii) Mitra and Ozbek (2013), Banerjee (2014): Theorem 1 also unifies several related results in the literature. Consider the following corollary to Theorem 1.

Corollary 1 *Let $\succsim$ be a binary preference relation on $X \times X$ which satisfies conditions **O, M and SC**. Then, there is a function $u : X \to \mathbb{R}$ representing $\succsim$ in the sense of (2).*

The proof of Corollary 1 follows from Theorem 1 since it can be easily verified that **M** implies **NE**. Corollary 1 is the representation result stated as Proposition 2 in Mitra and Ozbek (2013). It is shown in their paper how various other results in the literature follow from this representation result, so those additional results are not discussed here. Banerjee (2014) presents a representation result comparable to Theorem 1 as well. He uses conditions **O**, **NE** and **SC**. However, instead of **M**, he uses condition **DP.** Our Theorem 1 shows that in his result (Theorem 1, pp. 499), the condition **DP** is *redundant.*

4 Examples and Generalizations

4.1 Non-Emptiness and Representation

A natural question to ask is whether we can strengthen Theorem 1 by dropping conditions **NE** or **SC**. In this section we provide an example of a binary relation that satisfies **O** and **SC** but **NE** *fails* and the order is *not* representable. This example demonstrates that one cannot strengthen the statement of Theorem 1 by considering a set of sufficient conditions including **O** and **SC** alone. Our example points to a particular weaker version of **NE** for which representation can be obtained under **O** and **SC**. These themes are explored in detail in this section.

Example 1 Consider the sequence space $X = [0,1] \times [0,1]$ and let $Y = [0,1]$. Consider the following subsets of X: $V = \{x \in X : x_1 = 1\}$, $D = \{x \in X :$ there is $\lambda \in Y$ such that $x = (\lambda, \lambda)\}$ and $R = X \setminus V \cup D$.

Define a binary relation $\succsim$ on $X \times X$ as follows: (i) For any $x, y \in V \cup R$ we say $x \succsim y$ iff $x \geq_L y$ (where, $\geq_L$ is the standard lexicographic ordering), (ii) for any $x \in D$ and $y \in R$ we let $x \succ y$, (iii) for $[x \in V$ and $y \in D]$ or $[x \in D$ and $y \in V]$ or $[x, y \in D]$ we declare $x \succsim y$ iff $\min\{x_1, x_2\} \geq \min\{y_1, y_2\}$.

$\succsim$ *is satisfies* ***O***: Completeness follows from observing that the sets V, D and R on which $\succsim$ is defined along with (i)-(iii) exhausts all possibilities of comparisons for ordered pairs in X. To verify transitivity first consider $x \in V$, $y \in D$ and $z \in R$ with $x \succsim y$ and $y \succsim z$. As $x \in V$ and $z \in R$ we must have $x \succ z$ (since $x_1 = 1 > z_1$ as $z \in X \setminus V \cup D$). The other cases follow from noting that the three components of the definition of $\succsim$ are transitive (for example, when attention is confined for triplets x, y, z in either (i) or (iii) transitivity readily follows). Thus, $\succsim$ is complete and transitive — showing that $\succsim$ *satisfies* **O**.

$A(x)$ and $B(x)$ for any $x \in X$ is a closed subset of Y: Consider $x \in R$. Observe that by (ii) for any $y \in D$ we have $y \succ x$, showing that for each $x \in R$ the set $A(x)$ must be $[0,1]$, hence closed. For $x \in D$ (with $x = (\lambda, \lambda)$) we must have $A(x) = [\lambda, 1]$ (using (iii)) when $\lambda < 1$ and $A(x) = \{1\}$ with $\lambda = 1$; in both cases $A(x)$ is closed. For every $x \in V$, using (iii) in the definition of $\succsim$ we must have $A(x) = [x_2, 1]$ when $x_2 < 1$ and $A(x) = \{1\}$ when $x_2 = 1$. This shows that $A(x)$ is closed when $x \in V$ as well. Thus, we have shown that $A(x)$ is closed for any $x \in X$.

Now consider $B(x)$ for $x \in R$. Observe that by (ii) for any $y \in D$ we have $y \succ x$, showing that for each $x \in R$, implying that $B(x)$ is empty, hence closed. For $x = (\lambda, \lambda) \in D$ we obtain $B(x) = [0, \lambda]$ whenever $\lambda > 0$ and $B(x) = \{0\}$ when $\lambda = 0$ showing that $B(x)$ is closed in Y. When $(x_1, x_2) \in V$ we get $B(x) = [0, x_2]$ when $x_2 > 0$ and $B(x) = \{0\}$ when $x_2 = 0$ (using (iii)). Hence in each case we have shown that $B(x)$ is a closed subset of Y — the condition **SC** is *satisfied* by $\succsim$.

Observe that $B(x)$ is empty for any $x \in R$, so **NE** *fails* to hold for $\succsim$.

$\succsim$ *is **not** representable*: Suppose u represents $\succsim$. Consider the subset $W = \{(x_1, x_2) \in X : 0 < x_1 < 1 \text{ and } x_2 \in \{1, 0\}\}$. The set W is a subset of R — since W has empty intersection with D and V (since $0 < x_1 < 1$ and $x_2 \in \{1, 0\}$). The set W is also uncountable.

If $\succsim$ is representable, then the restriction of $\succsim$ to W is also representable. For each $\alpha \in (0, 1)$ we observe that $(\alpha, 1) \succ (\alpha, 0)$ (using $(\alpha, 1), (\alpha, 0) \in W \subset R$ and (i)) and $u((\alpha, 1)) > u((\alpha, 0))$ (as u represents $\succsim$ by assumption). Denote the interval $I(\alpha)$ is $(u((\alpha, 0)), u((\alpha, 1)))$. Let $\beta > \alpha$ (a similar argument can be made for the case $\beta < \alpha$) we have $(\beta, 0) \succ (\alpha, 1)$ (using, $W \subset R$ and (i)) showing that the interval $I(\beta)$ given by $(u((\beta, 0)), u((\beta, 1)))$ has a empty intersection with $I(\alpha)$ for any $\alpha \neq \beta$. This leads us to the familiar contradiction arising from a one-to-one correspondence between the countable rationals (each non-overlapping interval can be associated with a unique rational) and the uncountable reals. Thus, $\succsim$ *cannot* be representable. $\square$

It is to be noted that the incidence of failure of the non-emptiness requirement in Example 1 is on an uncountable subset of the space X. One is tempted to conjecture that if one can, in some way, restrict the set on which violation of **NE** occur we can possibly recover representability using weaker sufficient conditions than provided in Theorem 1.

Denote by $N_A = \{x \in X : A(x) \text{ is empty}\}$, $N_B = \{x \in X : B(x) \text{ is empty}\}$ and write Z for $X \setminus N_A \cup N_B$ We explore some properties of N_A and N_B defined using a binary relation satisfying **O.**

Property 1 *N_A and N_B cannot have a common point, that is, $N_A \cap N_B = \emptyset$*. This follows directly from the fact that $\succsim$ is a complete binary relation, consequently when either $A(x)$ ($B(x)$) is empty, $B(x)$ ($A(x)$) must be non-empty.

Property 2 *Assume $\succsim$ satisfies **SC**. If $x \in N_A$, $y \in N_B$ and $z \in Z$, then $x \succ z \succ y$ holds*. If $x \in N_A$, then $A(x)$ is empty and by completeness of $\succsim$ we must have $Y \subset B(x)$. So, for $\lambda \in Y$ we must have $x \succ \lambda e$ (since $x \sim \lambda e$ is ruled out for any λ in Y as $A(x)$ is assumed to be empty). A similar argument shows that for $y \in N_B$ we must have $\lambda e \succ y$. Now, transitivity of $\succsim$ implies $x \succ y$. As $z \in Z$, $A(z)$ and $B(z)$ are non-empty (by **NE**) and closed (by **SC**) subsets of Y. By the completeness of $\succsim$ we have $A(z) \cup B(z) = Y$. Since Y is connected it must imply that $A(z) \cap B(z)$ is non-empty so there is some $\mu \in Y$ such that $x \succ \mu e \sim z$ and $z \sim \mu e \succ y$, showing that $x \succ z \succ y$ is true as needed.

Property 3 *If $x \in N_A$ (or $x \in N_B$) and $y \sim x$, then $y \in N_A$ ($y \in N_B$)*. Suppose $x \in N_A$ ($x \in N_B$) and $y \sim x$, then $A(x) = A(y)$ ($B(x) = B(y)$) holds. This shows that $y \in N_A$ ($y \in N_B$) must be true as well.

We make a few additional observations true for each $i \in \{A, B\}$. Define for each $x \in N_i$ the set $E_i(x) = \{z \in X : z \sim x\}$. For $x, y \in N_i$ we must have $E_i(x) \cap E_i(y) = \emptyset$ or $E_i(x) = E_i(y)$ and

$$\cup_{x \in N_i} E_i(x) = N_i.$$

Denote the collection of sets $\{E_i(x) : x \in N_i\}$ by Ξ_i and observe that sets in Ξ_i is a partition of N_i. We impose a sufficient condition on the class of sets Ξ_i that weakens **NE** and in conjunction with **SC** implies representability.

Countable Emptiness (**CE**): The collections Ξ_A and Ξ_B are countable.

Observe that in Example 1, **CE** fails since $B(x)$ is empty for all $x \in R$. Furthermore, the order $\succsim$ in Example 1 restricted to the uncountable set R is lexicographic implying that the collection Ξ_B of N_B cannot be countable. We show in Theorem 2 that one can strengthen Theorem 1 by using **CE** in place of **NE**. The result is similar in spirit to Mitra and Ozbek (2013), they show that for monotone preferences if violations of **SC** occur only on a countable subset of X representability can still be recovered.

Theorem 2 *If $\succsim$ a binary relation defined on $X \times X$ satisfies **O**, **SC** and **CE**, then there is some $u : X \to \mathbb{R}$ representing $\succsim$ in the sense of (2).*

4.2 Scalar Continuity and Representation

This section is devoted to understanding the role of **SC** in Theorem 1. Dropping SC entirely will not yield a representation result. The standard lexicographic preference order of Debreu (1954) will suffice for this purpose. It is easily verified that for the lexicographic order on $\mathbb{R}^2_+$, the sets $A(x)$ and $B(x)$ are *not* closed for any $x \in D$. However, **NE** is satisfied for lexicographic preferences. These facts are routine and are not explicitly demonstrated here.

This however does not mean that Theorem 1 cannot be strengthened by using a weaker version of **SC**. We dedicate this section to showing that a particular natural weakening of **SC** (as suggested in the proof of Theorem 1) will not be sufficient to guarantee representation. Notice that in the proof of Theorem 1 we crucially used the fact that for each x one can find a point on the diagonal to which x is indifferent. **SC** is sufficient to guarantee that the indifference class of each x has a non-empty intersection with the diagonal. Condition **W** is a precise statement of this property and it is then logical to ask whether Theorem 1 holds true with this weaker condition. We provide the example of a binary relation on $\mathbb{R}^2$, that satisfies **O** and **W** but *fails* to be representable.

As a necessary background to our example we show that on *any* uncountable set we can define (i) a binary relation (satisfying **O**) that is representable and (ii) a binary relation that *fails* to be representable.

Let X be any set and $\succ$ be a binary relation defined on X. Following the terminology in Munkres (2000) we will call a binary relation $\succ$ a *simple order* if (a) it distinct elements are *comparable* (for any $x \neq y$ we have $x \succ y$ or $y \succ x$) (b) *transitive* (for $x, y, z \in X$ if $x \succ y$ and $y \succ z$ then $x \succ z$ must be true) and (c) irreflexive ($x \succ x$ does not hold for any x). A set X endowed with a linear order $\succ$ is called a *simple ordered set* and will be written as $(X, \succ)$. A well known example of a simple ordered set is $(\mathbb{R}, >)$ where, $>$ is the "greater than" order on real numbers.

We will show that on *any* uncountable set one can always define a preference order (that is, a binary relation satisfying condition **O**) which is not representable in the sense of (2).

Let X be an uncountable set and $\succ$ be a well-ordering of X (such a well-ordering[5] exists by the Axiom of Choice). Define $x \succeq y$ iff $x \succ y$ or $x = y$. We can verify easily that $\succeq$ satisfies **O**. We show that $\succeq$ is *not* representable in the sense of (2). If $\succeq$ was indeed representable, then there would be some $u : X \to \mathbb{R}$ such that whenever $x \succeq y$ is true we would obtain $u(x) \geq u(y)$. Non-representability of $\succeq$ is shown by demonstrating two contradictory implications of the definition $\succeq$ and its assumed representability.

Step 1: *$u(X)$ has the power of the continuum, and is a well-ordered subset of $(\mathbb{R}, >)$*: To see that $u(X)$ is well-ordered, consider a subset S of $u(X)$ and let $S' = \{x \in X : u(x) \in S\}$. As $S' \subset X$ and X is well-ordered by $\succ$ there is a smallest element $s' \in S$, hence for all $x \in S'$ we must have $x \succeq s'$ which implies (by representability) $u(s') \leq u(x)$ and $u(s') \in S$ proving that S has a smallest element in the natural ordering $\geq$. This shows that $u(X)$ must be well-ordered under the natural ordering of reals. By the completeness of $\succ$ and the fact that u represents $\succeq$, it follows that every $x \in X$ must be assigned a distinct number $u(x)$, showing that $u(X)$ must be uncountable, as X is uncountable. [More precisely, $\mathbb{R}$ has the same cardinality as X and X has the same cardinality as $u(X)$ which implies (by the Cantor–Schröder–Bernstein Theorem) that $\mathbb{R}$ has the same cardinality as $u(X)$.]

Step 2: *If $C \subset \mathbb{R}$ that is well-ordered (by $>$), then C must be countable:* Assume that C is uncountable. Since C is well-ordered and assumed uncountable, the collection of intervals (x, y) with $x < y$ where, y is the immediate successor[6] of x (under the assumption that C is uncountable, such an element exists for uncountably many x) must also be uncountable. However, with each non-overlapping interval we can associate a distinct rational (rational r that is between x and y), since the order intervals are non-overlapping the chosen rationals are distinct), thereby establishing a one-to-one correspondence between a uncountable class of non-overlapping intervals and the rationals, which are countable. This contradiction implies that C cannot be uncountable, hence it has to be countable as was required. This fact appears as an exercise in Kaplansky (1972, Exercise 5 pp. 54).

As Step 1 and 2 are in direct conflict, one that originates in the assumption that $\succeq$ is representable, it must be the case that $\succeq$ is *not* representable.

Example 2. We first define a non-representable order on $\mathbb{R}$ which we then extend to $\mathbb{R}^2$ and show that it satisfies **O** and **W** but is not representable.

[5] A formal textbook treatment of well-ordered sets can be found in Munkres (2000). A simple ordered set $(S, \succ)$ is well-ordered when $\succ$ is a order relation and every subset of S has a smallest element—an element $s^* \in S$ such that $s \succ s^*$ for all $s \in S$ and $s \neq s^*$.

[6] An element $y \in S$ is an immediate successor of $x \in S$ in the ordered set $(S, \succ)$ if $y \succ x$ and there is no $z \in S$ such that $y \succ z \succ x$ holds.

There is a simple order P on $\mathbb{R}$, such that the preference order xRy iff xPy or $x = y$ is *not* representable in the sense of (2). Now define the binary relation $\succsim$ on $\mathbb{R}^2$ by

$$(x_1, x_2) \succsim (y_1, y_2) \text{ iff } (x_1 + x_2)R(y_1 + y_2).$$

It is easy to see that $\succsim$ satisfies **O.** Observe that

$$(x_1, x_2) \sim (\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2})$$

and for any $\lambda \neq [(x_1 + x_2)/2]$ we have $(x_1, x_2) \not\sim (\lambda, \lambda)$. Thus, $I(x)$ for any $x \in \mathbb{R}^2$ is a singleton. As R is not representable it must be the case that $\succsim$ is *not* representable either. □

Remark on Example 2

(a) Long Chains: Our example shows that a non-representable binary relation exists in the one dimensional Euclidean space as well. The disadvantage of the construction is in the use of Axiom of Choice (which is equivalent to the statement that every set can be well-ordered) in defining the preference order. We are aware of non-representable preferences of two main types, they are (i) Debreu (1954) — the standard lexicographic preference in the two dimensional Euclidean space and (ii) Jech (1978) — the ordered set $(\omega_1, <)$ where ω_1 is the first uncountable ordinal and $<$ is the natural ordering of the ordinal numbers which is also known to be non-representable by a real-valued utility function. [In view of providing a readable account we omit the details of the definitions involved in (ii); the interested reader should Beardon, Candeal, Herden and Mehta (2002) for a comprehensive analysis of the issue of non-representability.] In Beardon, Candeal, Herden and Mehta (2002) it is shown that in sets with cardinality less than (or equal to) to $\mathbb{R}$, the only two possibilities of non-representability is when there is a completely ordered subset that is order-isomorphic (a function between two ordered set is order isomorphic when it is order preserving, see Beardon, Candeal, Herden and Mehta (2002)) to either an ordered set of type (i) or (ii). Our example is an explicit description and proof of how one can construct a non-representable order in $\mathbb{R}$ of type (ii)[7] (this is called a *long chain*, see Beardon, Candeal, Herden and Mehta (2002)).

It follows from Example 2 that condition **W** alone will not yield a representation result like Theorem 1. In fact an even more stringent requirement that $I(x) = A(x) \cap B(x)$ be a *singleton* will not yield a representation. This follows from noting that in Example 2, for $x \in \mathbb{R}^2$ there is only *one* point in D (the diagonal) to which x is indifferent, yet representation can fail.

Consider the possibility that an order is monotone when restricted to the diagonal, either in the sense of being weakly increasing (recall **WID**: for $\lambda, \mu \in Y$ and $\lambda > \mu$ we have $\lambda e \succsim \mu e$) or

[7] There are other "constructive" ways of defining a non-representable order on $\mathbb{R}$, however those may not suit our purpose as required in Example 2. For instance, since $\mathbb{R}$ and $\mathbb{R}^2$ are both uncountable, there is a bijection $g : \mathbb{R} \to \mathbb{R}^2$ and we can let xTy iff $g(x) \geq_L g(y)$. The binary relation T on $\mathbb{R}$ is an order but is not representable. The argument here does not rely on the Axiom of Choice (it uses the Cantor–Schröder–Bernstein Theorem) as the statement of our construction in Example 2 does.

weakly decreasing (recall **WDD**: for $\lambda, \mu \in Y$ and $\lambda > \mu$ we have $\mu e \succsim \lambda e$), then intuitively, condition **W** would allow us to represent each x by a point on the diagonal (precisely, the associated scalar to which x is indifferent) and hence recover representability. We confirm this intuition by explicitly considering preference orders that are weakly increasing along the diagonal. The verification of the result under an analogous "weakly decreasing along the diagonal condition" will trivially follow from the result stated in Theorem 3.

Theorem 3 *If $\succsim$ defined on $X \times X$ satisfies **WID** and **W**, then there is some $u : X \to \mathbb{R}$ that represents $\succsim$ in the sense of (2).*

One can replicate the proof of Theorem 3 under the assumption that the order is weakly decreasing on the diagonal, **WDD**.

It is tempting to explore whether every representable order that satisfies **W** must be either weakly increasing or weakly decreasing on the diagonal. This turns out to be not true — we show by means of an example that one can construct orders that are representable, satisfy **W** but fail to be either **WID** or **WDD**.

Example 3 Define $f : \mathbb{R}^2_+ \to \mathbb{R}$ by

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{if } (x_1 + x_2) \in \mathbb{I} \\ (x_1 + x_2)/2 & \text{if } (x_1 + x_2) \in \mathbb{Q} \end{cases}$$

where, $\mathbb{I}$ is the set of irrational numbers and $\mathbb{Q}$ the set of rational numbers. Define the binary relation $\succsim$ on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ by

$$(x_1, x_2) \succsim (y_1, y_2) \text{ iff } f(x_1, x_2) \geq f(y_1, y_2).$$

Since $\succsim$ is defined using a real-valued function, it follows that $\succsim$ satisfies **O** and it is representable. It can be verified that for any $(x_1, x_2) \in \mathbb{R}^2_+$ we must have

$$(x_1, x_2) \sim (\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2})$$

showing that $\succsim$ satisfies **W**.

We conclude by showing that $\succsim$ fails to satisfy either **WID** or **WDD**. Let $\alpha > 0$ be irrational, and $r \in (\alpha, 2\alpha)$ be a rational number. Note that $(r/2, r/2) >> (\alpha/2, \alpha/2)$ but as $f(r/2, r/2) = (r/2) < \alpha = f(\alpha/2, \alpha/2)$ we have $(r/2, r/2) \prec (\alpha/2, \alpha/2)$, implying that $\succsim$ does *not* satisfy **WID**. For any two rational numbers $r < r'$ we must have $f(r, r) < f(r', r')$, showing that $\succsim$ does *not* satisfy **WDD** as well. Thus, we have defined a representable preference order (satisfying **O**) on a sequence space satisfying W that fails to meet either **WID** or **WDD**. There is no hope that either one of these conditions are necessary for representability under **W.** $\square$

4.3 Degenerate Indifference Set and Representation

This section provides an example of a monotone, representable order whose indifference classes are singleton sets. This example has two interesting properties. It runs counter to the intuition that non-degenerate indifference classes (sets having more than one point) are necessary for representation, since non-degeneracy intuitively provides one with enough maneuverability that the "numbering (of) various iso-utility classes, in such a way that a preferred class always has a higher number" [Banerjee (1964), pp. 160] is rendered possible. Secondly, it provides a stark example that the conditions of Theorem 1 are not necessary for representation, even for the class of binary relations satisfying monotonicity.

4.3.1 Definition of the Binary Relation

Consider the sequence space $X = Y^2$ with $Y = [0, 1]$. We will define a binary relation on $X \times X$ satisfying **O** and **M (NE** is satisfied by implication) but *not* **W (**hence, **SC** is not met either)**.** It will be representable and have singleton indifference sets, hence will satisfy **DIS**.

For any $p \in Y$ we can express p in its binary expansion form as follows:

$$p = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots + \frac{a_n}{2^n} + \cdots \tag{4}$$

where $a_i \in \{0, 1\}$ for each $i \in \mathbb{N}$. It is well known that every real in the interval $[0, 1]$ can be expressed in the form (4) and such an expansion is unique provided it is not of the form $\sum_{i \in S}(1/2^i)$ for some finite subset S of $\mathbb{N}$. Given some finite subset S of $\mathbb{N}$ let s denote the maximum element of S. For $x = \sum_{i \in S}(1/2^i)$ we will have exactly two binary representations: (a) $a_i = 1$ for all $i \in S$ and $a_i = 0$ whenever $i \neq S$ and (b) $a_i = 1$ for all $i < s$, $a_s = 0$, $a_i = 1$ for $i \geq (s + 1)$ and $a_i = 0$ for $i \in \mathbb{N} \backslash S \cup \{s + 2, s + 3, ...\}$. The analysis can now proceed by stating that to make the binary expansion unique we will use the expansion of the form (b) in the event of non-uniqueness[8]. Keeping this convention in mind, we will say that p has the *binary representation* $\{a_i\}$ when p can be expressed as (4) using the sequence $\{a_i\}$.

Now define $f : Y \to Y$ by

$$f(p) = \frac{a_1}{4} + \frac{a_2}{4^2} + \frac{a_3}{4^3} + \cdots + \frac{a_n}{4^n} + \cdots \tag{5}$$

and $h : Y \to Y$ by

$$h(q) = f(q)/2 \tag{6}$$

Finally define $u : X \to \mathbb{R}$ by

$$u(x_1, x_2) = f(x_1) + h(x_2). \tag{7}$$

Using u define the binary relation on $X \times X$ as:

$$\text{for all } x, y \in X \text{ we will say } x \succsim y \text{ iff } u(x_1, x_2) \geq u(y_1, y_2).$$

[8] An equivalent way of stating this, following Royden (1988), p. 40 would be to say that expression (4) is non-unique only in the case where p is of the form $q/2^n$ where $0 < q < 2^n$ is an integer.

4.3.2 Properties of the Binary Relation

We show that $\succsim$ satisfies **O**, **M**. Before verifying the properties we make a crucial observation that will be useful throughout the development of the example. For $(p, p') \in X$ let $m(p, p')$ denote the $\min\{i \in \mathbb{N} : a_i \neq a'_i\}$, where $\{a_i\}$and $\{a'_i\}$ are the standard binary expansions of p and p'.

Observation *Let* $1 \geq p' > p \geq 0$ *and* $\{a_i\}$ *and* $\{a'_i\}$ *be the standard binary expansions of* p *and* p' respectively. *Denote* $m(p, p')$ *by* r. *Then (a)* $a'_r = 1, a_r = 0$ *(b) there is some* $n \geq (r+1)$ *such that* $(a'_n - a_n) \in \{0, 1\}$ *(c)* $\frac{(4/3)}{4^r} \geq [f(p') - f(p)] \geq \frac{(2/3)}{4^r}$.

<u>$\succsim$ *satisfies* ***O***</u> The definition of $\succsim$ is made using a real-valued function u as defined in (7) which immediately implies that $\succsim$ satisfies **O**.

<u>$\succsim$ *satisfies* ***M***</u> Since (using (17)) f and h are strictly increasing on Y monotonicity on X follows immediately.

4.3.3 Degenerate Indifference Sets

We now turn to the demonstration that all indifference sets of $\succsim$ are degenerate. Suppose on the contrary, there is x, x' in X such that $x \neq x'$ and $u(x) = u(x')$. For convenience write $x = (p, q)$ and $x' = (p', q')$. As g, h are strictly increasing functions on Y, we can assume without any loss of generality that $p' > p$ and $q' < q$.

The fact that $u(x') = u(x)$ must yield applying (7)

$$f(p') - f(p) = h(q) - h(q'). \tag{8}$$

Using (5) and (8) we obtain

$$f(p') - f(p) = (1/2)[f(q) - f(q')]. \tag{9}$$

Denote $m(p, p')$ by r and $m(q, q')$ by s. Two exhaustive possibilities emerge: (i) $s \geq r$; (ii) $s < r$. In case (i) using (18)

$$(1/2)[f(q) - f(q')] \leq \frac{(1/2)(4/3)}{4^s} = \frac{(2/3)}{4^s} \leq \frac{(2/3)}{4^r} \tag{10}$$

and using (17) we get

$$f(p') - f(p) > \frac{(2/3)}{4^r}. \tag{11}$$

Now (10) and (11) contradicts (9).

In case (ii) we have $r \geq s + 1$ and so, using (18) we get

$$f(p') - f(p) \leq \frac{(4/3)}{4^r} \leq \frac{(4/3)}{4^{s+1}} = \frac{(1/3)}{4^s}. \tag{12}$$

On the other hand using (17) we have

$$(1/2)[f(q) - f(q')] > \frac{(1/2)(2/3)}{4^s} = \frac{(1/3)}{4^s}. \tag{13}$$

Clearly, (12) and (13) contradict (9). Thus, we must have that every indifference class associated with $\succsim$ must be a singleton or equivalently, degenerate. However, $\succsim$ is representable which follows directly from the definition of the binary relation.

Banerjee (1964, pp. 160-161) provides a lucid exposition of how existence of non-degenerate indifference sets is ensured if one postulates that preferences are representable by a *continuous* utility function[9]. Note that while the order defined in this section is representable, even when the underlying binary relation exhibits a serious dearth of substitution possibilities such a representation cannot be continuous. Mathematically, this fact has nothing to do with indifference sets, preference orders or utility functions. This is contained in Sierpinski (1965), pp. 70-71, so we state this as a maxim without proof.

There exists no continuous function $f(x, y)$ of two real variables (even continuous only with respect to each variable separately) on $X = Y^2$ with $Y = [0, 1]$ which for different pairs of real numbers (x, y) would always assume different values.

5 Conclusion

This paper has produced results on two aspects of representation. The first set of results (Theorem 1, 2 and 3) distills the Wold approach and clarifies that monotonicity plays no role in obtaining representation. We provide several examples to show that our results cannot be improved upon by either dropping or weakening the conditions scalar continuity and non-emptiness.

Our second contribution addresses the role of substitution possibilities for an order to be representable — we show, by means of an example, that a monotone order can be constructed with no substitution possibilities (the indifference class for each x is a singleton) however, the order is representable. In view of our Theorem 1 and Chipman (1960; pp. 214) Theorem 3.3 (which he proves using an axiom of substitution), the present example provides a clear limit to the intuitive appeal that implications of substitution possibilities can have for representability.

[9] Actually, Banerjee (1964, p.161) goes on to say,"Thus iso-utility points (i.e., indifference classes) necessarily exist if we start from a real-valued utility function." But, clearly, he means continuous real-valued utility function, in view of the argument he has provided

6 Appendix

6.1 Proof of Theorem 1, 2 and 3

This appendix collects the proofs of the main results (Theorem 1, 2 and 3).

Theorem 1 *If a binary relation $\succsim$ defined on $X \times X$ satisfies **O**, **SC** and **NE**, then there is some $u : X \to \mathbb{R}$ representing $\succsim$ in the sense of (2).*

Proof of Theorem 1: We will demonstrate that $\succsim$ is representable by showing the existence of a countable order dense subset of X. Let $Z = \{\lambda e : \lambda \in Y \cap \mathbb{Q}\}$, where $\mathbb{Q}$ denotes the set of rationals in $\mathbb{R}$. Then Z is a countable set.

Consider any $x, y \in X$ with $x \succ y$. The sets $A(x)$ and $B(y)$ are non-empty closed subsets of Y by condition **SC**. Further, they are disjoint. [For if there is some $\lambda \in A(x) \cap B(y)$, then $\lambda e \succsim x \succ y \succsim \lambda e$, which contradicts condition **O**]. Since Y is an interval of $\mathbb{R}$, and hence a connected set, we cannot have $A(x) \cup B(y) = Y$. Thus, there is some $\lambda' \in Y$, such that $\lambda' \notin A(x)$ and $\lambda' \notin B(y)$. That is, $x \succ \lambda' e$ and $\lambda' e \succ y$. Since $\lambda' \in Y$, we can find a sequence $\{\lambda^n\}$ of rationals in Y, such that $\lambda^n \to \lambda'$ as $n \to \infty$.

We claim that there is $N(x) \in \mathbb{N}$, such that for all $n \geq N(x)$, we have $x \succ \lambda^n e$. For, if this does not hold, then there is a subsequence $\{\lambda^{n_r}\}$ of $\{\lambda^n\}$, such that $\lambda^{n_r} \to \lambda'$ as $n_r \to \infty$, and $\lambda^{n_r} e \succsim x$ for all n_r. Since $A(x)$ is a closed subset of Y by condition **SC**, we must therefore have $\lambda' e \succsim x$, which contradicts the fact that $x \succ \lambda' e$. This establishes the claim. We can similarly establish the claim that there is $N(y) \in \mathbb{N}$, such that for all $n \geq N(y)$, we have $\lambda^n e \succ y$.

Pick any $n \geq \max\{N(x), N(y)\}$. Then, we have $x \succ \lambda^n e \succ y$. Since $\lambda^n e \in Z$, we have now found an element $z \in Z$ such that $x \succ z \succ y$. That is, the preference order $\succsim$ on X has the countable order dense property on X and is therefore representable in the sense of (2) by Lemma II in Debreu (1954). ■

Theorem 2 *If $\succsim$ a binary relation defined on $X \times X$ satisfies **O**, **SC** and **CE**, then there is some $u : X \to \mathbb{R}$ representing $\succsim$ in the sense of (2).*

Proof of Theorem 2: Recall that the subset $X \setminus N_A \cup N_B$ is denoted by Z. The first step of the argument is to verify that there is some $v : Z \to (-1, 1)$ such that v represents the restriction of $\succsim$ to Z. Observe that the diagonal D of X is a subset of Z–this follows from noting that for any $\lambda e \in D$, reflexivity implies that **NE** holds. By Theorem 1 (notice that on Z the binary relation satisfies both **SC** and **NE** in addition to **O** and as $D \subset Z$, the arguments from Theorem 1 apply) there is a function $\hat{v} : Z \to \mathbb{R}$ that represents $\succsim$ restricted to Z. A suitable transformation[10] of $\hat{v}$ provides a $v : Z \to (-1, 1)$ that represents $\succsim$ on $Z \times Z$.

[10] For any $a < b$, we can define $v : Z \to (a, b)$ by

$$v(z) = \frac{(b-a)}{\pi} \arctan \hat{v}(z) + \frac{a+b}{2}$$

We will now show that restriction of $\succsim$ to the set N_A and N_B are representable. Consider the class of sets Ξ_A; by **CE**, Ξ_A is countable and can be written as $\{E_1, E_2, ...\}$. Define a binary relation $\succ^*$on $\Xi_A \times \Xi_A$ by $E \succ^* E'$ iff $x \succ y$ for all $x \in E$ and $y \in E'$. We will show that $\succ^*$ is a asymmetric and negatively transitive binary relation on $\Xi_A \times \Xi_A$. Asymmetry of $\succ^*$ follows from the asymmetry of $\succ$. To show that $\succ^*$ satisfies negative transitivity let $\neg(E \succ^* E')$ and $\neg(E' \succ^* E'')$ for distinct sets E, E' and E'' in Ξ_A. We will verify that $\neg(E \succ^* E'')$ holds. Since $\neg(E \succ^* E')$ and $\neg(E' \succ^* E'')$ holds, there exists $x \in E$, $x'_1, x'_2 \in E'$ and $x'' \in E''$ such that $x'_1 \succ x$ and $x'' \succ x'_2$ holds (note that since E and E' are distinct, any pair of elements from $E \times E'$ must be strictly ordered). Since $x'_1, x'_2 \in E'$ we must have $x'_1 \sim x'_2$ implying $x'' \succ x$ by transitivity. This shows that $\neg(E \succ^* E'')$ holds. By Theorem 1.3.1 in Bridges and Mehta (1995), there is $w : \Xi_A \to (1, 2)$ that represents $\succ^*$ (the range can be chosen to be any interval by Footnote 8). Define $v_A : N_A \to (1, 2)$ by $v_A(x) = w(E(x))$, where, $E(x)$ is the unique indifference class to which x belongs to (Property 1). A similar argument establishes the existence of a function $v_B : N_B \to (-2, -1)$ such that v_B represents $\succsim$ restricted to N_B.

To complete the argument define $u : X \to \mathbb{R}$ by

$$u(x) = \begin{cases} v_A(x) & x \in N_A \\ v_B(x) & x \in N_B \\ v(x) & x \in X \setminus N_A \cup N_B \end{cases}$$

We are left to verify that u represents $\succsim$. For $x \sim y$, can arise only when x, y both belong to a particular indifference class of either N_A or N_B or they are indifferent element of Z. In either case the definitions of v_A, v_B and v can be used to conclude $u(x) = u(y)$. Let $x \succ y$. If x, y are both in either N_A, N_B or Z, then the fact that v_A, v_B and v are representations of $\succsim$ restricted to N_A, N_B and Z respectively allows us to conclude $u(x) > u(y)$. The only other possibilities for $x \succ y$ are $x \in N_A$ and $y \in Z \cup N_B$ or $x \in Z$ and $y \in N_B$ (by Property 3). In each of these instances of strict ordering, the choice of range of v_A, v_B and v yields the required strict relation $u(x) > u(y)$. ■

Theorem 3 *If $\succsim$ defined on $X \times X$ satisfies **WID** and **W**, then there is some $u : X \to \mathbb{R}$ that represents $\succsim$ in the sense of (2).*

Proof of Theorem 3: The result is obvious if for each x, there is a unique λ satisfying the Wold condition **W**. However, under assumption **W** there is no guarantee that the λe on the diagonal to which x is indifferent be unique. The crux of the argument rests on the following observation, which we state and use without proof for now.

Observation 1. For each $x \in X$, we must have $(\inf I(x), \sup I(x)) \subseteq I(x)$.

Define $u : X \to \mathbb{R}$ by $u(x) = (1/2)(\inf I(x) + \sup I(x))$. We demonstrate that u indeed represents $\succsim$. Let x, x' be such that $x \sim x'$. In this case, $I(x) = I(x')$, which immediately implies that $u(x) = u(x')$ as was needed. Suppose now for $x, x' \in X$ we have $x \succ x'$. By

for all $z \in Z$. It can be checked easily that v also represents $\succsim$ on $Z \times Z$.

construction $u(x) \in (\inf I(x), \sup I(x))$, which yields $x \sim u(x)e$ (using Observation 1) and $x' \sim u(x')e$. Now $x \succ x'$ implies

$$u(x)e \succ u(x')e. \tag{14}$$

Using **WID** we must have

$$u(x) \geq u(x'). \tag{15}$$

We now need to rule out the possibility $u(x) = u(x')$. If $u(x) = u(x')$ holds, then $u(x') \in (\inf I(x), \sup I(x))$ which yields (from Observation 1) $u(x') \in I(x)$ implying $u(x')e \sim x \sim u(x)e$, a contradiction to (14). Thus, $u(x') \neq u(x)$ must hold. Now using (15) we conclude that whenever $x \succ x'$ we must have $u(x) > u(x')$.

We conclude by providing a short proof of Observation 1.

(i) *If $\mu_1 < \mu_2$ be such that $\mu_1, \mu_2 \in I(x)$, then $(\mu_1, \mu_2) \in I(x)$.* Assume $\alpha \in (\mu_1, \mu_2)$, then $\mu_1 e \precsim \alpha e \precsim \mu_2 e$ (follows from **WID**) and $\mu_2 e \precsim \mu_1 e$ (as $\mu_1, \mu_2 \in I(x)$) implies (using transitivity) $\alpha e \sim \mu_1 e \sim x$. Thus, $\alpha \in I(x)$.

As $I(x)$ is non-empty subset of the real line (by **W)** it has a well defined (unique) supremum and infimum. If $\sup I(x) = \inf I(x)$, then **W** implies that the intersection of the indifference class $I(x)$ with the diagonal is unique. When $\inf I(x) < \sup I(x)$, given $\alpha \in (\inf I(x), \sup(I(x))$ there exists $\alpha_1 < \alpha < \alpha_2$ such that $\alpha_1, \alpha_2 \in I(x)$. Observe that in this case $[\alpha_1, \alpha_2] \in I(x)$ (by (i)) and hence $\alpha \in I(x)$, as was needed. This concludes the proof of Observation 1 and hence of the main result. ■

6.2 Proof of Observation

We present in this section the proof of Observation from section 4.3.2.

Observation *Let $1 \geq p' > p \geq 0$ and $\{a_i\}$ and $\{a_i'\}$ be the standard binary expansions of p and p'* respectively. *Denote $m(p, p')$ by r. Then (a) $a_r' = 1, a_r = 0$ (b) there is some $n \geq (r+1)$ such that $(a_n' - a_n) \in \{0, 1\}$ (c) $\frac{(4/3)}{4^r} \geq [f(p') - f(p)] \geq \frac{(2/3)}{4^r}$.*

Proof: Let a and a' denote the sequences $\{a_i\}$and $\{a_i'\}$, the binary representations of p, p' respectively. Since $p \neq p'$ there is some i for which $a_i \neq a_i'$ holds, which guarantees that $m(p, p')$ is well defined. There are two possibilities: (i) $a_r' = 0$ and $a_r = 1$; (ii) $a_r' = 1$ and $a_r = 0$.

Suppose (i) is true then,

$$\begin{aligned} p' - p &= \sum_{n=r}^{\infty} \frac{(a_n' - a_n)}{2^n} = -\frac{1}{2^r} + \sum_{n=r+1}^{\infty} \frac{(a_n' - a_n)}{2^n} \\ &\leq -\frac{1}{2^r} + \sum_{n=r+1}^{\infty} \frac{1}{2^n} = 0 \end{aligned}$$

which contradicts the fact that we are given $p' > p$. Thus, (i) cannot be true, implying (ii) must hold verifying (a). For ready reference let us make note part (a) explicitly.

$$\text{If } 1 \geq p' > p \geq 0, \text{ then } a'_r = 1 \text{ and } a_r = 0, \text{ where } r = m(p, p'). \tag{16}$$

Now consider the possibility that $(a'_n - a_n) = -1$ for all $n \geq r+1$, then

$$\begin{aligned} p' - p &= \sum_{n=r}^{\infty} \frac{(a'_n - a_n)}{2^n} = \frac{1}{2^r} + \sum_{n=r+1}^{\infty} \frac{(a'_n - a_n)}{2^n} \\ &= -\frac{1}{2^r} - \sum_{n=r+1}^{\infty} \frac{1}{2^n} = 0 \end{aligned}$$

would again be in violation of the assumed ordering $p' > p$. This shows that (b) must also be true.

To show (c) evaluate the difference $[f(p') - f(p)]$ as follows

$$\begin{aligned} [f(p') - f(p)] &= \frac{(a'_r - a_r)}{2^r} + \sum_{n=r+1}^{\infty} \frac{(a'_n - a_n)}{4^n} \\ &= \frac{1}{4^r} + \sum_{n=r+1}^{\infty} \frac{(a'_n - a_n)}{4^n} \\ &> \frac{1}{4^r} + \sum_{n=r+1}^{\infty} \frac{(-1)}{4^n} \\ &= \frac{1}{4^r} - \frac{1}{4^{r+1}} \frac{4}{3} = \frac{1}{4^r} \frac{2}{3} > 0. \end{aligned} \tag{17}$$

The second line of (17) follows from the first line using (16) and strict inequality in the the third line follows from noting that $(a'_n - a_n) > -1$ for at least some $n \geq r+1$. This shows that $[f(p') - f(p)] \geq \frac{(2/3)}{4^r}$. From the second line of (17) using $(a'_n - a_n) \leq 1$ for all $n \geq r+1$ we obtain

$$\begin{aligned} [f(p') - f(p)] &= \frac{1}{4^r} + \sum_{n=r+1}^{\infty} \frac{(a'_n - a_n)}{4^n} \\ &\leq \frac{1}{4^r} + \sum_{n=r+1}^{\infty} \frac{1}{4^n} = \frac{1}{4^r} \frac{4}{3} \end{aligned} \tag{18}$$

proving (c). ■

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