RATIONALIZABLE IMPLEMENTATION OF SOCIAL CHOICE CORRESPONDENCES

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Abstract. We study the implementation of social choice correspondences (SCC), in a complete information setting, using rationalizability as our solution concept. We find a condition which we call r-monotonicity to be necessary for the rationalizable implementation of an SCC. r-monotonicity is strictly weaker than Maskin monotonicity, a condition introduced by Maskin (1999). If an SCC satisfies a no worst alternative condition and a condition which we call Θ_F-distinguishability, then we show that r-monotonicity is also sufficient for rationalizable implementation. We discuss the strength of these additional conditions. In particular, we find that a social choice correspondence which always selects at least two alternatives is rationalizably implementable if and only if it satisfies r-monotonicity. This paper, therefore, extends Bergemann et al. (2011) to the case of social choice correspondences.

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1. Introduction

The goal of implementation theory is to design a game form such that, for every preference profile, all the "equilibrium outcomes" of the game coincide with those recommended by the social choice rule. A central ingredient in this exercise is the choice of the solution concept. An extensive literature, starting from Maskin (1999), assumes Nash equilibrium as the solution concept. As is well known, the assumption of Nash equilibrium relies on the ability of the agents to correctly predict the strategy choice of their opponents. This can be problematic from the point of view of implementation theory. To see why consider the following example:

Example 1.1. Let $X = \{a, b, c\}$ be the set of alternatives, $N = \{1, 2\}$ be the set of agents and $\{\theta, \theta'\}$ be the set of states. The preference profile for each state is given in the table below. Numbers in the parentheses specify the vNM utilities.

Table 1. Preferences

θ		θ'	
1	2	1	2
$a(3)$	$b(3)$	$a(3)$	$a(3)$
$b(2)$	$a(2)$	$b(2)$	$c(2)$
$c(1)$	$c(1)$	$c(1)$	$b(1)$

Suppose the planner wants to select all the Pareto optimal alternatives in X. This objective can be represented by the following social choice correspondence, $F(\theta) = \{a, b\}$ and $F(\theta') = \{a\}$.

Let us consider the following game form which implements the SCC F in Nash equilibrium

	h	l
H	a	c
L	a	b

When the state of the world is θ we have a game of complete information among the agents. The Nash equilibria of this game are (H, h) and (L, l) with a and b as the equilibrium outcomes. This is consistent with the outcomes prescribed by F at θ. However, suppose agents fail to coordinate on which equilibrium to play. This coordination failure may lead to the strategy profile (H, l) with an undesirable outcome c.

To avoid the potential of miscoordination, we assume *(correlated) rationalizability* as our solution concept (Brandenburger and Dekel (1987)). A mechanism specifies a list of messages, one for each agent, and an outcome function. For each message profile, the outcome function specifies a lottery over alternatives. We say a social choice rule is rationalizably implementable if there exists a mechanism such that for every state of the world θ, (i) for every $a \in F(\theta)$ there exists a rationalizable strategy profile m such that a is selected with a strictly positive probability under m and; (ii) the support of the outcome of any rationalizable strategy profile is contained in $F(\theta)$.

The goal of this paper is to study which social choice correspondences are rationalizably implementable in the above sense. We provide a condition, which we call r-monotonicity and prove that it is necessary for rationalizable implementation of a SCC (Proposition 3.1). We show by example (see Example 3.2 and Remark 3.1) that r-monotonicity is strictly weaker than Maskin monotonicity, a condition central in the theory of Nash implementation. For social choice functions, r-monotonicity coincides with Maskin monotonicity. In general r-monotonicity is closely related to the 'set monotonicity' condition (Mezzetti and Renou (2012)) and 'extended monotonicity with respect to support F' condition (Bochet and Maniquet (2010)).

Under two additional conditions, *no worst alternative (NWA)* and Θ_F*-distinguishability*, we show that r-monotonicity is also sufficient for rationalizable implementation. The *no worst alternative (NWA)* condition requires that for every state θ, no full support lottery over $F(\theta)$ is worst for some agent. Assuming strict preferences, the NWA property places no restrictions at states where the SCC selects more than one alternative. Θ_F-*distinguishability*, a monotonicity type condition, applies only for the states where the SCC is single valued. In particular, if we consider an SCC F which always selects more than one alternative, then the conditions NWA and Θ_F-distinguishability place no restrictions.

Our main theorem (Theorem 5.1) states that for a society with atleast 3 agents if an SCC satisfies NWA, Θ_F-distinguishability and r-monotonicity then there exists a mechanism which rationalizably implements it.[1]An important corollary (Corollary 5.1) of

[1]To prove Theorem 5.1 we construct a canonical mechanism.

Theorem 5.1 is that an SCC which always selects at least two alternatives is rationalizably implementable *if and only if* it satisfies r-monotonicity.

The current work is motivated by a recent paper of Bergemann et al. (2011), who study the problem of rationalizable implementation of social choice functions[2]. We argue that focusing attention on social choice functions is restrictive for several reasons. First, there are important social choice correspondences within economics, such as the Pareto correspondence and the Walrasian correspondence, which have received considerable attention in the theory of implementation. Second, as shown by Bergemann et al. (2011), a stronger version of Maskin monotonicity is necessary for the rationalizable implementation of a social choice function.[3] However, *sometimes* Maskin monotonicity itself can be very restrictive for social choice functions.[4] In contrast, Maskin monotonicity is less restrictive for SCC's . This makes the study of SCC's very different than SCF's.

The results found in this paper are sharp in contrast to the case of social choice functions in two aspects. First, r-monotonicity is strictly weaker than Maskin monotonicity. Second, the "responsiveness" type of condition, which appears as a sufficient condition in Bergemann et al. (2011), is no longer required for states of the world where an SCC selects more than one alternative. This point can be seen clearly by Corollary 5.1 as described above. This paper, therefore, complements Bergemann et al. (2011) by extending their analysis to the important case of social choice correspondences.[5]

In a closely related paper Kunimoto and Serrano (2016) also study rationalizable implementation of correspondences. The key distinguishing feature between their work and the current work is in the notion of implementation. While our notion of implementation is based on Mezzetti and Renou (2012), the notion of implementation used in Kunimoto

[2]The main point where Bergemann et al. (2011) departs from the earlier literature is studying rationalizable implementation (i) without any domain restriction or allowing the use of transfers and (ii) allowing mechanisms which use integer games. Recently, Oury and Tercieux (2012) extend Bergemann et al. (2011) to environments with incomplete information.

[3]They show that a stronger version of Maskin monotonicity, strict Maskin monotonicity*, is necessary for the rationalizable implementation. In the Appendix 2, we present an example of an SCF which violates strict Maskin monotonicity* but satisfies Maskin monotonicity.

[4]See Muller and Satterthwaite (1977) and Saijo (1987). For example, Saijo (1987) shows that if the domain of an SCF includes the complete indifference profile then *any* Maskin monotonic SCF is constant.

[5]Unlike Nash implementation, rationalizable implementation of SCC's is not a trivial extension of that of SCF's.

and Serrano (2016) is based on Maskin (1999). Specifically, their definition of implementation requires that a social choice correspondence is rationalizably implementable if there exists a mechanism such that for every state of the world θ, (i) for every $a \in F(\theta)$ there exists a rationalizable strategy profile m such that $g(m) = a$ and; (ii) the outcome at any rationalizable strategy profile belongs to $F(\theta)$.

Even though the notion of implementation is different, the necessary condition is same in both the papers. There are however important differences in the sufficiency part of our results. First, the NWA condition we require is weaker than the one required in Kunimoto and Serrano (2016). Second, Kunimoto and Serrano (2016) do not require Θ_F- distinguishability which we require. Finally, the canonical mechanism constructed to prove sufficiency is different in both the papers. Therefore we consider that both the papers complement each other.

For social choice functions, rationalizable implementation has also been studied under a variety of settings. Abreu and Matsushima (1992) study virtual implementation as opposed to exact implementation. They show that in a complete information setting, any social choice function is virtually implementable in iterative elimination of strictly dominated strategies. Abreu and Matsushima (1994) strengthen virtual implementation to exact implementation but assume iterated elimination of weakly dominated strategies as the solution concept. By allowing small transfers they show that any SCF can be exactly implemented in iterated elimination of weakly dominated strategies. In economic environments, Sjöström (1994) studies a production economy and shows that any SCF can be exactly implemented in iterated elimination of weakly dominated strategies.

The rest of the paper is organized as follows. Section 2 presents the notation and definitions. In Section 3, we define r-monotonicity and prove that this is a necessary condition for rationalizable implementation (Proposition 3.1). In Section 4, we define two additional conditions. We state and prove our main theorem (Theorem 5.1) in Section 5. As an application of Theorem 5.1, in Section 6 we show that in a society with more than 3 agents Pareto correspondence can be rationalizably implementable. In Section 7, we study rationalizable implementation when a partially honest agent is present in the society. A partially honest agent strictly prefers to tell the truth if he is indifferent between lying and telling the truth. Finally, in Section 8 we discuss stronger notions of implementation.

2. Notation and Definitions

There is a set $N = \{1, 2, \ldots, n\}$ of $n \geq 2$ agents and a finite set of alternatives $X = \{a, b, \ldots, w\}$. Let $\Delta(X)$ denote the set of all lotteries over X with a generic element α. α_x denotes the probability assigned to alternative x under the lottery $\alpha \in \Delta(X)$. For every $\alpha \in \Delta(X)$, $supp\,(\alpha)$ denotes the set of alternatives which receive positive probability in α. Formally, $supp\,(\alpha) = \{x \in X \ s.t. \ \alpha_x > 0\}$. We assume that there is a finite set of states of the world which we denote by Θ. We assume that the state of the world is not known to the planner but is common knowledge among the agents. Thus we are in the setting of complete information as in e.g. Maskin (1999). State $\theta \in \Theta$ specifies the preferences of the players. Specifically, a state of the world $\theta \in \Theta$ specifies the vNM utility function $u_i : X \times \theta \mapsto \mathbb{R}$ for every agent i. We further assume that agents are expected utility maximizers. This allows us to extend the vNM utility function over X to $\Delta(X)$. With some abuse of notation we write $u_i(\alpha, \theta) = \sum \alpha_x u_i(x, \theta)$. For agent i denote by $L_i(\alpha, \theta) = \{\alpha' \in \Delta(X) \ s.t. \ u_i(\alpha, \theta) \geq u_i(\alpha', \theta)\}$ and $SL_i(\alpha, \theta) = \{\alpha' \in \Delta(X) \ s.t. \ u_i(\alpha, \theta) > u_i(\alpha', \theta)\}$ the lower contour set and strict lower contour set of lottery α at state θ respectively. For any set $X' \subseteq X$, let $U[X'] \in \Delta(X')$ denote the uniform lottery over the set X'. Furthermore, we assume that agents have strict preferences over pure outcomes. Formally, for any two distinct pure alternatives x and y, $u_i(x, \theta) \neq u_i(y, \theta)$ for every $i \in N$ and every $\theta \in \Theta$. We discuss this assumption in Section 9.

A mechanism is a game form $\Gamma = (M_1, ...M_n; g)$, where M_i is a countable strategy set for agent i and $g : M \mapsto \Delta(X)$ is the outcome function. For every strategy profile, the outcome function specifies a *lottery* over X.

Given a state of the world θ, a mechanism Γ induces a game of complete information among the agents. In this paper we assume *(correlated) rationalizability* as our solution concept in the sense of Brandenburger and Dekel (1987) (also see Chen et al. (2016), Bernheim (1984) and Pearce (1984)). Given a mechanism Γ and a state of the world θ, we denote by $R_i(\Gamma, \theta)$ the set of rationalizable strategies for agent i. Let $R(\Gamma, \theta) = R_1(\Gamma, \theta) \times R_2(\Gamma, \theta) \ldots \times R_n(\Gamma, \theta)$ denote the set of rationalizable strategy profiles. $R(\Gamma, \theta)$ can be described as an outcome of an iterative procedure. Each round in this iterative process eliminates strategies which are never best responses.

Let us now formally define the iterative process. Fix an agent i, let $R_i^0(\Gamma, \theta) = M_i$ and $R_{-i}^0(\Gamma, \theta) = \underset{j \neq i}{\times} R_j^0(\Gamma, \theta) = \underset{j \neq i}{\times} M_j$.

$$R_i^1(\Gamma,\theta) = \left\{ m_i \in R_i^0(\Gamma,\theta) \;\middle|\; \begin{array}{c} \exists \lambda \in \Delta(R_{-i}^0(\Gamma,\theta))\, s.t. \\ m_i \in \underset{m_i' \in M_i}{argmax} \displaystyle\sum_{m_{-i} \in R_{-i}^0(\Gamma,\theta)} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i}), \theta) \end{array} \right\}$$

For any $k \geq 1$, for every $i \in N$ we can recursively define $R_i^k(\Gamma,\theta)$ as follows.

$$R_i^k(\Gamma,\theta) = \left\{ m_i \in R_i^{k-1}(\Gamma,\theta) \;\middle|\; \begin{array}{c} \exists \lambda \in \Delta(R_{-i}^{k-1}(\Gamma,\theta))\; s.t. \\ m_i \in \underset{m_i' \in M_i}{argmax} \displaystyle\sum_{m_{-i} \in R_{-i}^{k-1}(\Gamma,\theta)} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i}), \theta) \end{array} \right\}$$

In the mechanism Γ and a state of the world θ, the set of rationalizable strategies for player i is the limit of the iterative process defined above. Formally, $R_i(\Gamma,\theta) = \bigcap_{k\geq 1} R_i^k(\Gamma,\theta)$. We denote the set of rationalizable strategy profiles by $R(\Gamma,\theta) = \times_{i\in N} R_i(\Gamma,\theta)$.[6]

Finally, we say that in a mechanism Γ a set of strategy profiles $T(\theta) \subseteq M$ satisfies the best response property at state θ, if for every $i \in N$ and for every $m_i \in T_i(\theta)$, there exists a belief $\lambda_i(m_i) \in \Delta(T_{-i}(\theta))$ such that m_i is a best response to $\lambda_i(m_i)$. It is easy to see that if a set $T(\theta)$ satisfies the best response property at state θ then $T(\theta) \subseteq R(\Gamma,\theta)$.

The goal of the social planner is summarized by a social choice correspondence $F : \Theta \mapsto 2^X \setminus \{\emptyset\}$. A special case of an SCC is a social choice function which is a mapping from the set of states to the set of pure alternatives i.e. $f : \Theta \mapsto X$. To achieve these goals, the planner needs to know the information about the true state of the world. The problem of the planner is to design a mechanism Γ such that when players are using rationalizable strategies, every alternative in $F(\theta)$ gets a positive chance to be selected and any alternative outside of $F(\theta)$ receives no chance of being selected[7]. The following definition captures this notion.

Definition 2.1. *We say a mechanism $\Gamma(M;g)$ rationalizably implements an SCC F if for every $\theta \in \Theta$,*

(1) *For every $a \in F(\theta)$, there exists $m \in R(\Gamma,\theta)$ s.t. $a \in supp\,(g(m))$.*
(2) *For every $m' \in R(\Gamma,\theta)$, $supp\,(g(m')) \subseteq F(\theta)$.*

[6] If the game is infinite, transfinite induction is required to reach the fixed point of the iteration (see Lipman (1994) for a more formal treatment).

[7] In section 8 we discuss stronger notions of implementation. We demand that every alternative in $F(\theta)$ gets atleast an ϵ chance to be selected and any alternative outside of $F(\theta)$ receives no chance of being selected.

3. Necessary condition

In this section, we present a necessary condition for an SCC to be rationalizably implementable.

In a related model, using an implementability notion based on Maskin (1999), Kunimoto and Serrano (2016), working paper dated May 2016, independently prove a necessity result that is close to my result in this section. Furthermore, they come close in that draft to close the gap between necessity and sufficiency in their model. An earlier draft of my paper, dated June 2016, contained that necessity result, and also a sufficiency result that was far from closing the gap between the necessary and sufficient conditions. The current draft produces a sufficiency result that comes closer to closing that gap in my framework.

Definition 3.1. *For any pair (θ, θ'), we say that a lottery $\alpha \in \Delta(X)$* ***maintains position*** *if $L_i(\alpha, \theta) \subseteq L_i(\alpha, \theta')$ for every $i \in N$.*

In words, while moving from state θ to θ', a lottery α *maintains position* if it does not fall in any agent's preference ordering relative to any other lottery. Alternatively, while moving from state θ to θ', we say that a lottery α *does not maintain position* if there exists an agent i and a lottery β such that $\beta \in L_i(\alpha, \theta)$ and $\beta \notin L_i(\alpha, \theta')$.

Definition 3.2. *We say that an SCC F is r-monotonic if for any pair $(\theta, \theta') \in \Theta \times \Theta$ if every lottery $\alpha \in \Delta(F(\theta))$ maintains position, then $F(\theta) \subseteq F(\theta')$.*

r-monotonicity requires that while moving from state θ to θ', if *every* lottery over $F(\theta)$ maintains position, then the set of socially optimal alternatives in state θ as prescribed by F remain socially optimal in state θ' i.e. $F(\theta) \subseteq F(\theta')$.

To illustrate *r-monotonicity*, below we give an example of an SCC F which violates *r-monotonicity*. Then we study its relation with Maskin monotonicity (Maskin (1999)).

Example 3.1. Let $X = \{a, b, c\}$ and $N = \{1, 2\}$ be the set of agents. The preference profile for each state is given in the table below. Numbers in the parentheses specify the vNM utilities. Suppose the planner wants to implement the social choice correspondence $F(\theta) = \{b, c\}$ and $F(\theta') = \{a, b\}$.

We claim that for the pair (θ, θ') every lottery $\alpha \in \Delta(F(\theta)) = \Delta(\{b, c\})$ maintains position. To see this notice that for any $\alpha \in \Delta(X)$, $L_2(\alpha, \theta) = L_2(\alpha, \theta')$ and for any

Table 2. Preferences

θ		θ'	
1	2	1	2
$a(3)$	$b(3)$	$b(3)$	$b(3)$
$b(2)$	$a(2)$	$c(2)$	$a(2)$
$c(1)$	$c(1)$	$a(1)$	$c(1)$

$\alpha \in \Delta(F(\theta) = \{b, c\})$, $L_1(\alpha, \theta) \subseteq L_1(\alpha, \theta')$. Therefore for the pair (θ, θ'), the hypothesis of *r-monotonicity* is satisfied. However $F(\theta) \nsubseteq F(\theta')$. Thus F violates *r-monotonicity*.

Definition 3.3. *We say that an SCC F is Maskin monotonic if for any pair $(\theta, \theta') \in \Theta \times \Theta$, if $a \in F(\theta)$ maintains position, then $a \in F(\theta')$.*

Maskin monotonicity states that when moving from state θ to θ', if a socially optimal alternative maintains position, then it must remain socially optimal at state θ'.

Remark 3.1. If an SCC satisfies Maskin monotonicity then it satisfies r-monotonicity.

Proof. Consider a Maskin monotonic SCC F and assume that for the pair (θ, θ') every lottery $\alpha \in \Delta(F(\theta))$ maintains position. In particular, every degenerate lottery $\alpha \in \Delta(F(\theta))$ maintains position. Therefore, for the pair (θ, θ') every alternative $a \in F(\theta)$ maintains position. Since F satisfies Maskin monotonicity therefore $F(\theta) \subseteq F(\theta')$. □

r-monotonicity is strictly weaker than Maskin monotonicity. To see why, consider the following example.

Example 3.2. Let $X = \{a, b, c, d\}$ and $N = \{1, 2\}$ be the set of agents. The preference profile for each state is given in the table below. Numbers in the parentheses specify the vNM utilities. Suppose the planner wants to implement the following social choice correspondence $F(\theta) = \{a, b, c, d\}$ and $F(\theta') = \{a\}$.

Consider the pair (θ, θ'). For every $i \in N$, $L_i(b, \theta) \subseteq L_i(b, \theta')$ is true. Therefore b maintains position. Since $b \in F(\theta)$ Maskin monotonicity would require that $b \in F(\theta')$. However $b \notin F(\theta')$, thus F violates Maskin monotonicity at b. A similar argument holds for alternatives c and d. In contrast, F does not violate r-monotonicity. In fact, r-monotonicity is vacuously satisfied. To see this, note that while alternatives b, c and d maintain position, every lottery $\alpha \in \Delta(F(\theta))$ *does not* maintain position. In particular

Table 3. Preferences

θ		θ'	
1	2	1	2
$a(4)$	$a(4)$	$b(4)$	$a(4)$
$b(3)$	$d(3)$	$c(3)$	$d(3)$
$c(2)$	$b(2)$	$d(2)$	$b(2)$
$d(1)$	$c(1)$	$a(1)$	$c(1)$

consider alternative $a \in F(\theta)$. For agent 1, $L_1(a,\theta) \nsubseteq L_1(a,\theta')$ and thus a does not maintain position. Therefore for the pair (θ,θ'), r-monotonicity imposes no restriction on F. For pair (θ',θ), $F(\theta') \subseteq F(\theta)$ thus the SCC F satisfies r-monotonicity.

Proposition 3.1. *If a social choice correspondence F is rationalizably implementable then it satisfies r-monotonicity.*

Proof. Consider a mechanism $\Gamma = ((M_i)_i^n, g)$ which rationalizably implements the SCC F. Recall that under the mechanism Γ, the set of rationalizable strategies at state θ is denoted by $R(\Gamma,\theta) = (R_1(\Gamma,\theta),\ldots,R_n(\Gamma,\theta))$. We will assume that the hypothesis of r-monotonicity is true. For any pair $(\theta,\theta'), L_i(\alpha,\theta) \subseteq L_i(\alpha,\theta')$ is true for every i and every $\alpha \in \Delta(F(\theta))$. To prove proposition 3.1 we need to show that $F(\theta) \subseteq F(\theta')$.

Select an arbitrary agent i. For every rationalizable strategy $m_i \in R_i(\Gamma,\theta)$, there exists a belief $\lambda_i(m_i,\theta) \in \Delta(R_{-i}(\Gamma,\theta))$ such that m_i is a best response to $\lambda_i(m_i,\theta)$ in M_i. Select an arbitrary strategy $m_i \in R_i(\Gamma,\theta)$ with the associated belief $\lambda_i(m_i,\theta) \in \Delta(R_{-i}(\Gamma,\theta))$. For any strategy $m_i' \in M_i$ we denote by $\alpha_i(m_i',m_i)$ the lottery an agent gets when he selects strategy $m_i' \in M_i$ under the belief $\lambda_i(m_i,\theta)$.[8]

$$\alpha_i(m_i',m_i) = \sum_{m_{-i}\in R_{-i}(\Gamma,\theta)} \lambda_i(m_i,\theta)(m_{-i})g(m_i',m_{-i}) \tag{1}$$

Claim 3.1. *For $m_i \in R_i(\Gamma,\theta)$, supp $(\alpha_i(m_i,m_i)) \subseteq F(\theta)$.*

Proof. Consider *supp* $(\alpha_i(m_i,m_i))$. Denote the set of strategy profiles which can occur when agent i plays m_i having a belief $\lambda_i(m_i,\theta)$ by $R(m_i)$.

$$\mathit{supp}\,(\alpha_i(m_i,m_i)) = \bigcup_{m\in R(m_i)} \mathit{supp}\,(g(m)). \tag{2}$$

[8]Notice that for every strategy $m_i' \in M_i, \alpha_i(m_i',m_i)$ is a lottery under a fixed belief $\lambda_i(m_i,\theta)$.

Notice that by construction $R(m_i) \subseteq R(\Gamma, \theta)$. Therefore

$$supp\,(\alpha_i(m_i, m_i)) = \bigcup_{m \in R(m_i)} supp\,(g(m)) \subseteq \bigcup_{m \in R(\Gamma,\theta)} supp\,(g(m)). \tag{3}$$

Since $\Gamma = ((M_i)_i^n, g)$ rationalizably implements F we can conclude that

$$\bigcup_{m \in R(\Gamma,\theta)} supp\,(g(m)) = F(\theta). \tag{4}$$

Therefore,

$$supp\,(\alpha_i(m_i, m_i)) = \bigcup_{m \in R(m_i)} supp\,(g(m)) \subseteq \bigcup_{m \in R(\Gamma,\theta)} supp\,(g(m)) = F(\theta). \tag{5}$$

Hence,

$$supp\,(\alpha_i(m_i, m_i)) \subseteq F(\theta). \tag{6}$$

$\square$

We will use Claim 3.1 to show that the set $R(\Gamma, \theta)$ has the best response property at state θ'. Since $m_i \in R_i(\Gamma, \theta)$ is a best response to $\lambda_i(m_i, \theta) \in \Delta(R_{-i}(\Gamma, \theta))$ for every $m_i' \in M_i$,

$$u_i(\alpha_i(m_i, m_i), \theta) \geq u_i(\alpha_i(m_i', m_i), \theta). \tag{7}$$

Thus for every $m_i' \in M_i$, $\alpha_i(m_i', m_i) \in L_i(\alpha_i(m_i, m_i), \theta)$. By Claim 3.1 $\alpha_i(m_i, m_i) \in \Delta(F(\theta))$. Since the hypothesis of r-monotonicity is assumed to be true i.e. for any pair $(\theta, \theta'), L_i(\alpha, \theta) \subseteq L_i(\alpha, \theta')$ for every i and every $\alpha \in \Delta(F(\theta))$. In particular, $L_i(\alpha_i(m_i, m_i), \theta) \subseteq L_i(\alpha_i(m_i, m_i), \theta')$ for every i.

$$u_i(\alpha_i(m_i, m_i), \theta') \geq u_i(\alpha_i(m_i', m_i), \theta'),\ \textit{for every } m_i' \in M_i. \tag{8}$$

This means that at state θ', m_i is a best response to the belief $\lambda_i(m_i, \theta) \in \Delta(R_{-i}(\Gamma, \theta))$. Since $i \in N$ and $m_i \in R_i(\Gamma, \theta)$ were chosen arbitrarily, we can conclude that $R(\Gamma, \theta)$ has the best response property in state θ'. Thus, $R(\Gamma, \theta) \subseteq R(\Gamma, \theta')$.

To complete the proof we need to show that $F(\theta) \subseteq F(\theta')$. To proceed notice that the fact $R(\Gamma, \theta) \subseteq R(\Gamma, \theta')$ implies

$$\bigcup_{m\in R(\Gamma,\theta)} supp\,(g(m)) \subseteq \bigcup_{m\in R(\Gamma,\theta')} supp\,(g(m)). \tag{9}$$

Since Γ rationalizably implements F, we can conclude that $\bigcup_{m\in R(\Gamma,\theta)} supp\,(g(m)) = F(\theta)$ and $\bigcup_{m\in R(\Gamma,\theta')} supp\,(g(m)) = F(\theta')$. This together with equation (9) implies that $F(\theta) \subseteq F(\theta')$.

□

4. Sufficient Conditions

In this section we present two additional conditions under which r-monotonicity also becomes sufficient for rationalizable implementation. For any $X' \subseteq X$, let $\Delta^o(X') = \{\alpha \in \Delta(X') | supp\,(\alpha) = X'\}$ denote the set of lotteries which assigns a positive probability to every alternative in X'.

4.1. **No Worst Alternative.**

Definition 4.1. *An SCC F satisfies the no worst alternative condition if for every $i \in N$, every $\theta \in \Theta$, and every $\alpha \in \Delta^o(F(\theta))$ there exist $y_i(\theta) \in \Delta(X)$ s.t. $u_i(\alpha,\theta) > u_i(y_i(\theta),\theta)$.*

In our setting the NWA property is a natural extension of the NWA property used by Bergemann et al. (2011). Under the assumption of strict preferences, the NWA property places restrictions only for states where the SCC F is single valued.[9]

In implementation theory, the NWA property appears in various contexts. For example, see Cabrales and Serrano (2009) to guarantee full implementation in best response dynamics and in Tumennasan (2013) for guaranteeing full implementation in quantal response equilibrium. In many environments, this condition is easily satisfied. Below we give some examples.

Example 4.1. (Environments with money). Consider an environment where the mechanism designer is allowed to use arbitrarily small transfers. This allows the mechanism designer to select the alternatives from an extended outcome space $\Delta(X) \times \mathbb{R}^n_+$. In this setting a typical outcome is a $(n+1)$ tuple $(\alpha, t_1, \ldots, t_n)$, where $\alpha \in \Delta(X)$ and $(t_1, \ldots, t_n) \in \mathbb{R}^n_+$ s.t. $\sum_{i\in N} t_i = 0$. Furthermore, for every agent and every θ, $u_i(\alpha, t_i; \theta)$ is strictly decreasing in t_i. In this setting, every SCC F satisfies NWA. To

[9]Let θ be such that $|F(\theta)| \geq 2$. For every agent i, let $y_i(\theta) = \underset{y\in F(\theta)}{argmin}\ u_i(y,\theta)$.

see why this is true, for every $\alpha \in \Delta(F(\theta))$ we can define $y_i(\theta) = (\alpha, \epsilon_i)$ such that $u_i(\alpha, \theta) > u_i((\alpha, \epsilon_i), \theta)$.

Example 4.2. (Environments with time). Our second example is an environment where the mechanism designer is allowed to deliver the outcome with a delay. These type of environments are studied in Artemov (2015). This allows the mechanism designer to select the alternatives from an extended outcome space $\Delta(X) \times \mathbb{R}_+$. In this setting a typical outcome is (α, t), where $\alpha \in \Delta(X)$ and $t \in \mathbb{R}_+$. Furthermore, every agent has a preference for early delivery of outcome as opposed to late. Formally, $u_i(\alpha, t; \theta)$ is strictly decreasing in t. Similar to the example above, for every $\alpha \in \Delta(F(\theta))$, we can define $y_i(\theta) = (\alpha, \epsilon)$ such that $u_i(\alpha, \theta) > u_i((\alpha, \epsilon), \theta)$. In this setting, therefore, every SCC F satisfies NWA. Notice the distinction in environments with time and transfers is one between public outcomes and private outcomes.

The NWA property guarantees the existence of a set of allocations $\{y_i(\theta)\}_{i,\theta}$. A typical allocation, $y_i(\theta)$, may depend both on the agent and the state. In the construction of our canonical mechanism we require the existence two kinds of allocations. First, a state independent allocation which we call $\underline{y}_i$. Second, a state and agent independent allocation which we call $\underline{y}$. Furthermore, we require that at any state θ these allocations are not best for any agent. The NWA property guarantees the existence of such allocations. To see this consider the following average allocations:

$$\underline{y}_i = \frac{1}{\#\Theta} \sum_{\theta \in \Theta} y_i(\theta)$$

$$\underline{y} = \frac{1}{N} \sum_{i \in N} \underline{y_i}$$

Lemma 4.1. *For every $i \in N$ and every $\theta \in \Theta$, there exists a pair of allocation $y_i^*(\theta)$ and y_i^{**} s.t.*

(1) $u_i(y_i^*(\theta), \theta) > u_i(\underline{y}_i, \theta)$
(2) $u_i(y_i^{**}(\theta), \theta) > u_i(\underline{y}, \theta)$

Proof. First we prove (1) and show for any i and θ $u_i(y_i^*(\theta), \theta) > u_i(\underline{y}_i, \theta)$. Define $y_i^*(\theta)$ as

$$y_i^*(\theta) = \frac{1}{\#\Theta} \sum_{\theta' \in \Theta \backslash \{\theta\}} y_i(\theta') + \frac{1}{\#\Theta} \alpha(\theta), \text{ where } \alpha(\theta) \in \Delta^o(F(\theta)).$$

Consider the utility from lottery $y_i^*(\theta)$ at state θ,

$$u_i(y_i^*(\theta), \theta) = \frac{1}{\#\Theta} \sum_{\theta' \in \Theta \backslash \{\theta\}} u_i(y_i(\theta'), \theta) + \frac{1}{\#\Theta} u_i(\alpha(\theta), \theta).$$

By the NWA property we know that $u_i(\alpha(\theta), \theta) > u_i(y_i(\theta), \theta)$. Therefore

$$u_i(y_i^*(\theta), \theta) > \frac{1}{\#\Theta} \sum_{\theta' \in \Theta \backslash \{\theta\}} u_i(y_i(\theta'), \theta) + \frac{1}{\#\Theta} u_i(y_i(\theta), \theta) = u_i(\underline{y}_i, \theta).$$

Second we prove (2) and show that for any i and θ, $u_i(y_i^{**}(\theta), \theta) > u_i(\underline{y}, \theta)$. Define $y_i^{**}(\theta)$ as

$$y_i^{**}(\theta) = \frac{1}{N} \sum_{j \in N \backslash \{i\}} \underline{y_i} + \frac{1}{N} y_i^*(\theta).$$

Consider the utility from lottery $y_i^{**}(\theta)$ at state θ,

$$u_i(y_i^{**}(\theta), \theta) = \frac{1}{N} \sum_{j \in N \backslash \{i\}} u_i(\underline{y_i}, \theta) + \frac{1}{N} u_i(y_i^*(\theta), \theta). \tag{10}$$

We have already proved that $u_i(y_i^*(\theta), \theta) > u_i(\underline{y}_i, \theta)$ therefore

$$u_i(y_i^{**}(\theta), \theta) > \frac{1}{N} \sum_{j \in N \backslash \{i\}} u_i(\underline{y_i}, \theta) + \frac{1}{N} u_i(\underline{y}_i, \theta) = u_i(\underline{y}, \theta).$$

□

The allocation $\underline{y}$ appears directly in the construction of our canonical mechanism. Allocations $\underline{y_i}$ and $y_i^*(\theta)$ allows us to construct a set of allocations[10] $\{z_i(\theta, \theta')\}_{\theta, \theta'}$ for each agent i. We would like to think of a situation where agent i is playing a game with a group of people who always announce the same state. $\{z_i(\theta, \theta')\}_{\theta, \theta'}$ is the outcome function of this game. The following lemma establishes that for every agent i we can find allocations $\{z_i(\theta, \theta')\}_{\theta, \theta'}$ such that if the true state is θ then $z_i(\theta, \theta')$ is better than

[10]Here a typical allocation $z_i(\theta, \theta')$ is a lottery over X.

$z_i(\theta', \theta')$ and if the true state is θ' then $z_i(\theta, \theta')$ is not better than any $\alpha(\theta') \in \Delta^o(F(\theta'))$. This generalizes Lemma 2 in Bergemann et al. (2011) for social choice correspondences.

Lemma 4.2. *If an SCC F satisfies the NWA property then for every $i \in N$ and every θ, θ' there exists allocations $z_i\{(\theta, \theta')\}_{\{\theta,\theta'\}}$ for every $\alpha \in \Delta^o(F(\theta))$ such that*

$$u_i(\alpha(\theta'), \theta') > u_i(z_i(\theta, \theta'), \theta')$$
$$and\ for\ \theta \neq \theta',\ u_i(z_i(\theta, \theta'), \theta) > u_i(z_i(\theta', \theta'), \theta)$$

Proof. Define $z_i(\theta, \theta')$ as below

$$z_i(\theta', \theta') = (1 - \epsilon)y_i(\theta') + \epsilon \underline{y_i}$$
$$and\ for\ \theta \neq \theta', z_i(\theta, \theta') = (1 - \epsilon)y_i(\theta') + \epsilon y_i^*(\theta).$$

To prove the two conditions in the lemma above, first consider the utility of agent i from $z_i(\theta, \theta')$ at state θ assuming that $\theta' \neq \theta$.

$$u_i(z_i(\theta, \theta'), \theta) = (1 - \epsilon)u_i(y_i(\theta'), \theta) + \epsilon u_i(y_i^*(\theta), \theta) \tag{11}$$

From Lemma 4.1 we know that $u_i(y_i^*(\theta), \theta) > u_i(\underline{y}_i, \theta)$ therefore

$$u_i(z_i(\theta, \theta'), \theta) > (1 - \epsilon)u_i(y_i(\theta'), \theta) + \epsilon u_i(\underline{y}_i, \theta) = u_i(z_i(\theta', \theta'), \theta). \tag{12}$$

Notice that this is true irrespective of the value of ϵ.

Now by NWA we know that for every $\alpha(\theta') \in \Delta^o(F(\theta'))$, $u_i(\alpha(\theta'), \theta') > u(y_i(\theta'), \theta')$. Assume $\epsilon = 0$, then $z_i(\theta, \theta') = y_i(\theta')$. Therefore for $\epsilon = 0$, $u_i(\alpha(\theta'), \theta') > u(z_i(\theta, \theta'), \theta')$ for every $\alpha(\theta') \in \Delta^o(F(\theta'))$. By continuity this should also be true for some $\epsilon > 0$. Thus we can always choose a positive ϵ however small such that $u_i(\alpha(\theta'), \theta') > u(z_i(\theta, \theta'), \theta')$ is true for every $\alpha(\theta') \in \Delta^o(F(\theta'))$. □

In addition to the NWA condition, we require an additional condition which we call Θ_F - distinguishability.

4.2. Θ_F - Distinguishability.

Definition 4.2 (Distinguishability)**.** *We say that an SCC is distinguishable on a set* $\hat{\Theta} \subseteq \Theta$*, if for every ordered pair* $(\theta, \theta') \in \hat{\Theta} \times \Theta$ *there exists a lottery* $\alpha \in \Delta(F(\theta))$ *which does not maintain position.*

Our sufficient condition requires F to be distinguishable on the set of states where F is single valued. Define $\Theta_F = \{\theta \in \Theta \ s.t. \ |F(\theta)| = 1\}$.

Definition 4.3. *We say that an SCC F is Θ_F - distinguishable if it is distinguishable on $\Theta_F \subseteq \Theta$.*

Θ_F-distinguishability requires that whenever we move from a state θ where $F(\theta)$ is single valued to any other state θ', there exists an agent i and a lottery $\beta \in \Delta(X)$ such that $\beta \in L_i(F(\theta), \theta)$ and $\beta \notin L_i(F(\theta), \theta')$. For SCF's Θ_F-distinguishability is strictly stronger than Maskin monotonicity but for SCC's both the conditions are logically independent.

5. Main Theorem

In this section we present our main theorem. To prove this we construct a "canonical mechanism." We will show that, for a society with atleast 3 agents, our canonical mechanism rationalizably implements any SCC which satisfies r-monotonicity, NWA and Θ_F -distinguishability. The canonical mechanism we construct shares many basic features with the mechanism constructed by Maskin (1999) with important differences and modifications. In particular, our canonical mechanism is very closely related to the one used by Bergemann et al. (2011) and differs, non trivially, from theirs to accommodate the case of correspondences.

Theorem 5.1. *Assume $n \geq 3$. If an SCC F satisfies NWA, r-monotonicity and is Θ_F -distinguishable then it is rationalizably implementable .*

Proof. Below we describe the canonical mechanism which will be used to prove our main result. For each state of the world θ, a mechanism induces a game of complete information among the agents. We study the set of rationalizable strategy profiles for each θ which we denote by $R(\Gamma, \theta)$.

Message Space. $M^1 = \Theta$, $M^2 = Z_+$,$M^3 = \{\Delta(X)^\Theta | \ for \ every \ m^3 \in \Delta(X)^\Theta, \ m^4(\theta) \in \Delta(F(\theta))\}$, $M^4 = \{\Delta(X)^\Theta\}$, $M^5 = \Delta(X)$.

A typical message for a player i, $m_i = (m_i^1, m_i^2, m_i^3, m_i^4, m_i^5)$, can be described as follows:

- $m_i^1 \in M^1$: State of the world.
- $m_i^2 \in M^2$: An integer.
- $m_i^3 \in M^3$: A state contingent allocation plan such that $m_i^3(\theta) \in \Delta(F(\theta))$.
- $m_i^4 \in M^4$: A state contingent allocation plan such that $m_i^4(\theta) \in \Delta(X)$.
- $m_i^5 \in M^5$: A lottery over X.

Outcome Function.

(1) **Universal Agreement**: For every i, $m_i^1 = \theta'$, $m_i^2 = 1$.

$$g(m) = U[F(\theta')]$$

(2) **Unilateral Deviation**: There is an agent $i \in N$ s.t. $(m_i^1, m_i^2) \neq (m_j^1, m_j^2)$ and for every $j \neq \{i\}$,$(m_j^1, m_j^2) = (\theta', 1)$. Let $b(\theta') \in \underset{y \in F(\theta')}{argmin}\ u_i(y, \theta')$.

I **Coordination Failure**: $|F(\theta')| \geq 2$ and $m_i^2 = 1$.

$$g(m) = \left\{ \begin{array}{ll} U[F(\theta')] & \textit{with probability } \frac{1}{2} \\ b(\theta') & \textit{with probability } \frac{1}{2} \end{array} \right\}$$

II **Other Cases:** $|F(\theta')| = 1$ **or** $m_i^2 > 1$

(i) $u_i(U[F(\theta')], \theta') \geq u_i(m_i^4(\theta'), \theta')$**,**

$$g(m) = \left\{ \begin{array}{l} m_i^4(\theta') \textit{ with probability } (1 - \frac{1}{m_i^2+1}) \\ z_i(\theta', \theta') \textit{ with probability } \frac{1}{m_i^2+1} \end{array} \right\}$$

(ii) $u_i(U[F(\theta')], \theta') \ < \ u_i(m_i^4(\theta'), \theta')$

$$g(m) = z_i(\theta', \theta')$$

(3) **Disagreement**:

I **Coordination Failure**: For every $i \in N, m_i^2 = 1$ and $|F(m_i)| \geq 2$. Select an agent i at random.

$$g(m) = \left\{ \begin{array}{l} m_i^3(m_i^1) \textit{ with probability } \frac{1}{2} \\ U[F(m_i^1)] \textit{ with probability } \frac{1}{2} \end{array} \right\}$$

II **Other Cases**: Select an agent i announcing the highest integer m_i^2 and ties are broken arbitrarily.

$$g(m) = \left\{ \begin{array}{l} m_i^5 \textit{ with probability } (1 - \frac{1}{m_i^3+1}) \\ \underline{y} \textit{ with probability } \frac{1}{m_i^3+1} \end{array} \right\}$$

In Appendix A we show that the above mechanism rationalizably implements a SCC which satisfies NWA, r-monotonicity and Θ_F-Distinguishability. □

Remark 5.1. For SCF's our canonical mechanism is identical to the one constructed in Bergemann et al. (2011). Bergemann et al. (2011) show that a SCF is rationalizably implementable if it satisfies responsiveness[11], Maskin Monotonicity and NWA (See Proposition 2, Pg 1261). It can be easily checked that for SCF's responsiveness and Maskin monotoncity together imply Θ_F -distinguishability. For non-responsive SCF's Bergemann et al. (2011) discover a condition which they call strict Maskin monotonicity*. Under a weak restriction on the class of mechanisms, they prove that it is necessary for rationalizable implementation of an SCF. In Appendix 2 we show that strict Maskin monotonicity* is strictly stronger than Maskin monotonicity.

An important implication of Theorem 5 is that for states where F is multi-valued, NWA and Θ_F-distinguishability place no restrictions. This is sharp in contrast with respect to functions. To see this contrast clearly, consider an SCC F which always selects more than one alternative. Within this class we can use Theorem 5 to completely characterize the set of rationalizably implementable SCC's.

Corollary 5.1. Assume $N \geq 3$ and $|F(\theta)| \geq 2$, for every θ. An SCC F is rationalizably implementable if and only if it satisfies r-monotonicity.

[11]A social function is responsive if $\theta \neq \theta' \Rightarrow f(\theta) \neq f(\theta')$.

6. Application-Pareto Correspondence

In this section, using our main theorem we show that Pareto correspondences can is rationalizable implementable.

Example 6.1. Consider the Pareto correspondence

$$F(\theta) = \left\{ \ a \in X | \nexists \text{ an alternative } b \in X \text{ such that for every } i \in N \ , u_i(b,\theta) > u_i(a,\theta) \ \right\}$$

Corollary 6.1. Pareto correspondence is rationalizably implementable.

Proof. Let us partition the set of states by the following partition, $\{\Theta_F, \Theta \setminus \Theta_F\}$. For every $\theta \in \Theta_F$, $F(\theta)$ is singleton. This implies that for every $\theta \in \Theta_F$, $F(\theta)$ is the top alternative for every agent. Let us further partition Θ_F based on the alternatives which are top ranked in state θ. Let us denote this partition by $\{\Theta_F^a\}_{a\in X}$. For example, if $\theta \in \Theta_F^a$ then $F(\theta) = a$. Informationally the mechanism designer cares about the partition Θ_F^a rather than the actual state within this partition. Next we claim that the Pareto correspondence is distinguishable w.r.t. the partition $\{\Theta_F^a\}_{a\in X}$ of Θ_F.

Claim 6.1. *Pareto Correspondence is distinguishable w.r.t. $\{\Theta_F^a\}_{a\in X}$, i.e. for every pair (θ, θ') such that $\theta \in \Theta_F^a$, $F(\theta) \neq F(\theta')$, there exists an agent $i \in N$ and an alternative b*

$$u_i(a,\theta) > u_i(b,\theta) \ for \ every \ \theta \in \Theta_F^a$$
$$u_i(b,\theta') > u_i(a,\theta')$$

Proof. Let us select a state $\theta \in \Theta_F^a$. By definition we know that $F(\theta) = a$. Now consider a $\theta' \in \Theta$ such that $F(\theta) \neq F(\theta')$. This implies that there exists an alternative b such that $b \in F(\theta')$ and $b \neq a$. By the definition of Pareto correspondence we know that for every $i \in Nand\ for\ every\ \theta \in \Theta_F^a$, $u_i(a,\theta) > u_i(b,\theta)$. Also $b \in F(\theta')$, now let us assume that there is no i such that $u_i(b,\theta') > u_i(a,\theta')$ is true. But this would mean that $b \notin F(\theta')$. Hence we arrive at a contradiction. □

Claim 6.2. *Pareto correspondence satisfies r-monotonicity.*

Proof. We can indeed prove that Pareto correspondence is Maskin monotonic. To see this consider an alternative and a state θ such that $a \in F(\theta)$. By the definition of Pareto correspondence there does not exists an alternative $b \in X$ such that for every $i \in N$, $u_i(b,\theta) > u_i(a,\theta)$. Thus for every $b \in X$ there exists an agent i such that $u_i(a,\theta) >$

$u_i(b, \theta)$. Consider a θ' such that for every $i \in N$, $L_i(a, \theta) \subseteq L_i(a, \theta')$. Therefore at state θ' we can say that for every $b \in X$ there exists an agent i such that $u_i(a, \theta') > u_i(b, \theta')$. By the definition of Pareto correspondence, $a \in F(\theta')$. Since a was arbitrary we have shown that Pareto correspondence satisfies Maskin monotonicity. Finally we complete the proof by the fact that Maskin monotonicity implies r-monotonicity. □

Claim 6.3. *Pareto correspondence satisfies NWA property.*

Proof. First consider $\theta \in \Theta_F$. In these states, $F(\theta)$ is singleton and also the top alternative for everyone. Therefore any Pareto dominated alternative serves as a worst alternative. For states of the world θ' where F is multi-valued, for every $i \in N$ select $y_i(\theta') \in \underset{y \in F(\theta')}{argmin}\ u_i(y, \theta')$. Now for any lottery on $F(\theta')$ such that $\alpha \in \Delta^o(F(\theta'))$, $y_i(\theta')$ serves as a worst alternative for agent i. □

Using Claim 6.1, 6.2, 6.3 along with Theorem 5.1 we can now conclude that the Pareto correspondence is rationalizably implementable. □

7. Implementation with Partially Honest agents

Recently there has been a growing literature which allows for the possibility that there exists at least one agent who is partially honest. A partially honest agent, strictly prefers to tell the truth over lying, conditional on receiving the same outcome. Therefore this agent has a lexicographic preference for honesty. Moreover, the designer knows the existence of such an agent but need not necessarily know the identity of the agent. In this section we show that the set of implementable SCC's expand drastically, a result which is consistent with the existing literature on Nash implementation (see for example Matsushima (2008), Dutta and Sen (2012)). In this setting, the mechanism designer can ask everyone to announce a state of the world as "evidence". The utility of an agent now depends on the alternative chosen and also the evidence he/she announces i.e. utility from alternative a at state θ by announcing θ' is given by $u_i((a, \theta'); \theta)$.

Definition 7.1. *An agent i is partially honest if for every $\theta \in \Theta$, $u_i((a, \theta); \theta) > u_i((a, \theta'); \theta)$.*

The following claim follows immediately from the definition of partial honesty. The existential quantifier in the claim emphasizes that we just need the existence of at least one partially honest agent.

Claim 7.1. *Consider $\theta \neq \theta'$ then there exists an agent i such that for every $a \in X$.*

$$u_i((a,\theta);\theta) > u_i((a,\theta');\theta) \text{ and } u_i((a,\theta');\theta') > u_i((a,\theta);\theta').$$
$$u_i((a,\theta');\theta') > u_i((a,\theta);\theta') \text{ and } u_i((a,\theta);\theta) > u_i((a,\theta');\theta).$$

Theorem 7.1. *If there exists an agent i who is partially honest then an SCC which satisfies NWA is rationalizably implementable.*

Proof. To prove the above theorem we show that in the presence of a partially honest agent, every SCC satisfies the conditions of our main theorem. Interpreting an outcome as (a,θ), from Claim 7.1, it follows that the SCC is Θ_F-distinguishable and is also r monotonic. Since we have assumed that SCC satisfies NWA, the result follows directly from our main theorem (Theorem 5.1). □

8. Stronger Notions of Implementation

The notion of implementation studied in this paper is the most conservative. In particular, we assume that for any rationalizable strategy profile, at each state, the mechanism designer is completely agnostic about the probability received by each alternative. In this section, we relax this assumption and allow the mechanism designer to demand a minimal probability for each socially optimal alternative. More formally, we call this notion ϵ-rationalizable implementation.

Definition 8.1. *Let $\epsilon \in [0,1]$. We say a mechanism $\Gamma(M;g)$ ϵ-rationalizably implements an SCC F if for every $\theta \in \Theta$*

1. *For every $a \in F(\theta)$ there exists $m \in R(\Gamma,\theta)$ such that $g(m)(a) > \epsilon$.*
2. *For every $m' \in R(\Gamma,\theta)$, $supp\,(g(m')) \subseteq F(\theta)$.*

Proposition 8.1. *Let $\epsilon \in [0,1]$. If a social choice correspondence F is ϵ-rationalizably implementable then it satisfies r-monotonicity.*

Proof. Let us assume that F is ϵ-rationalizably implementable for any fixed $\epsilon \in [0,1]$. This implies that there exists a mechanism, say $\Gamma(M,g)$, which implements it. From the definition of ϵ-rationalizable implementation it follows that $\Gamma(M,g)$ also ϵ-rationalizably implements F for $\epsilon = 0$. Therefore by Proposition 3.1 we can conclude that F satisfies r-monotonicity. □

The necessity of r-monotonicity for ϵ-rationalizable implementation for any $\epsilon \in [0,1]$ is not surprising. What is surprising is that our main theorem (Theorem 5.1) remains true for any $\epsilon \in [0,1)$.

Theorem 8.1. *Let $\epsilon \in [0,1)$ and $n \geq 3$. If an SCC F satisfies NWA, r-monotonicity and is Θ_F -distinguishable then it is ϵ-rationalizably implementable.*

Proof. See Appendix 3 □

For $\epsilon = 1$ the above theorem is no longer valid. In particular we provide an example of an SCC (example 8.1) which cannot be $\epsilon = 1$-rationalizably implemented (Claim 8.1) but satisfies all the conditions of the above theorem and hence can be ϵ-rationalizably implemented for any $\epsilon \in [0,1)$ (Claim 8.2). Before moving to the example let us remind ourselves the definition of $\epsilon = 1$-rationalizably implementation.

Definition 8.2. *We say a mechanism $\Gamma(M;g)$, $\epsilon = 1$-rationalizably implements an SCC F if for every $\theta \in \Theta$*

(1) *For every $a \in F(\theta)$ there exists $m \in R(\Gamma,\theta)$ such that $a = g(m)$.*
(2) *For every $m' \in R(\Gamma,\theta)$, $supp\,(g(m')) \subseteq F(\theta)$.*

Example 8.1. Let $X = \{a,b,c,d\}$ be the set of alternatives and $N = \{1,2,3\}$ be the set of agents. There are two states of the world, θ and θ'. The preference profile for each state is given in the table below. Numbers in the parentheses specify the vNM utilities.

Table 4. Preferences

θ			θ'		
1	2	3	1	2	3
$d(3)$	$d(3)$	$d(3)$	$d(3)$	$d(3)$	$d(3)$
$a(2)$	$b(2)$	$a(2)$	$a(2)$	$a(2)$	$a(2)$
$c(1)$	$c(1)$	$c(1)$	$c(1)$	$b(1)$	$c(1)$
$b(0)$	$a(0)$	$b(0)$	$b(0)$	$c(0)$	$b(0)$

Suppose the planner's objective is described by the following social choice correspondence, $F(\theta) = \{a,b,c,d\}$ and $F(\theta') = \{a\}$.

Claim 8.1. *There is no mechanism which $\epsilon = 1$-rationalizably implements F.*

Proof. Let us assume by the way of contradiction that there exists a mechanism say $\Gamma = (M, g)$ which rationalizably implements F. Since we have assumed that Γ rationalizably implements F at state θ there exists a $m \in R(\Gamma, \theta)$ such that $g(m) = d$. For every agent d is the top alternative at state θ and θ'. Therefore m is a Nash equilibrium at state θ and θ'. This implies that $m \in R(\Gamma, \theta')$. Finally since Γ implements F, $d \in F(\theta')$ which is a contradiction. □

Claim 8.2. *For $\epsilon \in [0, 1)$, SCC F can be ϵ -rationalizably implemented.*

Proof. To prove the above claim we will check the conditions of Theorem 8.1. First consider r-monotonicity. For the pair (θ, θ'), we know that for $b \in F(\theta)$, $a \in L_2(b, \theta)$ and $a \notin L_2(b, \theta')$. This means that r-monotonicity is vacuously satisfied. For the pair (θ', θ) we know that $F(\theta') \subseteq F(\theta)$. Thus we have established that F satisfies r-monotonicity. Now consider Θ_F -distinguishability. We need to consider pair (θ', θ). From the preferences we know that $b \in L_2(a, \theta')$ and $b \notin L_2(a, \theta)$. Hence F satisfies Θ_F -distinguishability. Finally it is easy to see that the NWA property is satisfied. Again we only need to check NWA at state θ'. Since alternative $a = F(\theta')$ is second in ranking for everyone, we can find a lottery which is strictly lower than a for every agent. Since F satisfies all the conditions of Theorem 8.1 we can conclude that F can be ϵ -rationalizably implemented for any $\epsilon \in [0, 1)$.

□

9. Discussion

The canonical mechanism used to prove Theorem 5.1 has the property that it also implements the SCC in mixed Nash equilibrium, a notion of implementation introduced and studied in Mezzetti and Renou (2012).

Recently there is a growing literature which allows the agents to provide some costly evidence (See for e.g. Ben-Porath and Lipman (2012) and Kartik and Tercieux (2012)). This is in contrast to the standard literature where all the messages in a mechanism are cheap talk. We touched upon a specific case of this literature by studying the presence of a partially honest agent. A general study of rationalizable implementation with evidence is an interesting topic left for future research.

The assumption of strict preferences plays a crucial role in the proof of Theorem 5.1. However, the necessary condition still holds even if we allow for indifference. This centrality of strict preferences is reminiscent of the recent literature on dropping no veto power in characterizing Nash implementable SCC's. (See for e.g. Benoît and Ok (2008), Bochet (2007)).

Recently Artemov (2015) adds the possibility of a delay to the theory of Nash implementation. This allows a planner to deliver the F-optimal alternate with some delay. Furthermore, every agent discounts future. Therefore early delivery of outcome is better than the later one. If we allow for some ϵ delay, then we can always define the SCC such that it selects F -optimal alternatives along with an ϵ delay. This means that the modified SCC is never single-valued and, as a result, responsiveness or distinguishability type of conditions can be dropped altogether. A thorough study of rationalizable implementation with delay is an interesting topic which we leave for future research.

Appendix

Appendix A. Proof of Theorem 5

Message Space. $M^1 = \Theta$, $M^2 = Z_+$,$M^3 = \{\Delta(X)^\Theta |\ for\ every\ m^3 \in \Delta(X)^\Theta,\ m^4(\theta) \in \Delta(F(\theta))\}$, $M^4 = \{\Delta(X)^\Theta\}$, $M^5 = \Delta(X)$.

A typical message for a player i, $m_i = (m_i^1, m_i^2, m_i^3, m_i^4, m_i^5)$, can be described as follows:

- $m_i^1 \in M^1$: State of the world.
- $m_i^2 \in M^2$: An integer.
- $m_i^3 \in M^3$: A state contingent allocation plan such that $m_i^3(\theta) \in \Delta(F(\theta))$.
- $m_i^4 \in M^4$: A state contingent allocation plan such that $m_i^4(\theta) \in \Delta(X)$.
- $m_i^5 \in M^5$: A lottery over X.

Outcome Function.

(1) **Universal Agreement**: For every i, $m_i^1 = \theta'$, $m_i^2 = 1$.

$$g(m) = U[F(\theta')]$$

(2) **Unilateral Deviation**: There is an agent $i \in N$ s.t. $(m_i^1, m_i^2) \neq (m_j^1, m_j^2)$ and for every $j \neq \{i\}$,$(m_j^1, m_j^2) = (\theta', 1)$. Let $b(\theta') \in \underset{y \in F(\theta')}{argmin}\ u_i(y, \theta')$.

I **Coordination Failure**: $|F(\theta')| \geq 2$ and $m_i^2 = 1$.

$$g(m) = \left\{ \begin{array}{ll} U[F(\theta')] & with\ probability\ \frac{1}{2} \\ b(\theta') & with\ probability\ \frac{1}{2} \end{array} \right\}$$

II **Other Cases:** $|F(\theta')| = 1$ **or** $m_i^2 > 1$**.**

(i) $u_i(U[F(\theta')], \theta') \geq u_i(m_i^4(\theta'), \theta')$**,**

$$g(m) = \left\{ \begin{array}{l} m_i^4(\theta')\ with\ probability\ (1 - \frac{1}{m_i^2+1}) \\ z_i(\theta', \theta')\ with\ probability\ \frac{1}{m_i^2+1} \end{array} \right\}$$

(ii) $u_i(U[F(\theta')], \theta') \ < \ u_i(m_i^4(\theta'), \theta')$

$$g(m) = z_i(\theta', \theta')$$

(3) **Disagreement**:

I **Coordination Failure**: For every $i \in N, m_i^2 = 1$ and $|F(m_i)| \geq 2$. Select an agent i at random.

$$g(m) = \left\{ \begin{array}{l} m_i^3(m_i^1) \ with \ probability \ \frac{1}{2} \\ U[F(m_i^1)] \ with \ probability \ \frac{1}{2} \end{array} \right\}$$

II **Other Cases**: Select an agent i announcing the highest integer m_i^2 and ties are broken arbitrarily.

$$g(m) = \left\{ \begin{array}{l} m_i^5 \ with \ probability \ (1 - \frac{1}{m_i^2+1}) \\ \underline{y} \ with \ probability \ \frac{1}{m_i^2+1} \end{array} \right\}$$

Throughout the proof we will assume that the true state of the world is $\theta \in \Theta$.

Lemma 1.1. *If an SCC F satisfies NWA, then for any $i \in N$, if $m_i \in R_i(\Gamma, \theta)$ then $m_i^2 = 1$.*

Proof. Assume by the way of contradiction that there exists an $i \in N$ such that $m_i \in R_i(\Gamma, \theta)$ and $m_i^2 > 1$. In the mechanism, the outcome is decided by either Rule $2(II)$ or Rule $3(II)$. For a strategy m_i, let us denote the set of strategy profiles of opponents which will lead to Rule $2(II)$ and Rule $3(II)$ as follows.

$$M_{-i}^{2(II)} = \{m_{-i} \in M_{-i} | m_j^1 = \theta' \ and \ m_j^2 = 1 \ for \ some \ \theta', for \ every \ j \neq i\}$$
$$M_{-i}^{3(II)} = M_{-i} \setminus M_{-i}^2$$

We will construct a strategy m_i^* for agent i and show that at state θ, for any belief $\lambda_i \in \Delta(M_{-i}^2 \cup M_{-i}^3)$ strategy m_i^* is better than strategy m_i. Notice that m_i^4 is relevant for Rule $2(II)$ and m_i^5 is relevant for Rule $3(II)$. This will allow us to break the argument into two separate cases. First case is $\lambda_i \in \Delta(M_{-i}^{2(II)})$ and second case $\lambda_i \in \Delta(M_{-i}^{3(II)})$.

Claim 1.1. *If there exists an* $i \in N$ *with a strategy* $m_i \in R_i(\Gamma, \theta)$ *with* $m_i^2 > 1$ *and beliefs such that* $\lambda_i \in \Delta(M_{-i}^2)$*, then*

(1) *At strategy profiles where rule* $2(II)(ii)$ *is applied,* m_i *is not a best response.*
(2) m_i *is not a best response at rule* $2(II)(i)$ *either.*

Proof. Let $m_i \in R_i(\Gamma, \theta)$ be a strategy of agent i with $m_i^2 > 1$ and consider that (m_i, m_{-i}) is a strategy profile such that rule $2(II)(ii)$ applies. This implies that $m_j^1 = \theta'$ for all $j \neq i$ and $u_i(U[F(m_{-i}^1)], m_{-i}^1) < u_i(m_i^4(m_{-i}^1), m_{-i}^1)$ and $g(m_i, m_{-i}) = z_i(m_{-i}^1, m_{-i}^1)$. By Lemma 4.2 we know that for $m_{-i}^1 \neq \theta$ there exist allocations $\{z_i(\theta', \theta)\}_{\theta, \theta' \in \Theta}$ such that

$$u_i(z_i(\theta, m_{-i}^1), \theta) > u_i(z_i(m_{-i}^1, m_{-i}^1), \theta)$$
$$u_i(U[F(m_{-i}^1)], m_{-i}^1) > u_i(z_i(\theta, m_{-i}^1), m_{-i}^1)$$

Therefore by announcing

$$m_i^{*4}(\hat{\theta}) = \left\{ \begin{array}{ll} U[F(\theta)] & if\ \hat{\theta} = \theta \\ z_i(\theta, \hat{\theta}) & if\ \hat{\theta} \neq \theta \end{array} \right\}$$

agent i is better off by inducing Rule $2(II)(i)$ as compared to Rule $2(II)(ii)$. Since the choice of strategy profile which induces Rule $2(II)(ii)$ was arbitrary, the claim holds for *any* such strategy profile.

Now consider the belief such that $\lambda_i \in \Delta(M_{-i}^2)$, utility from strategy m_i^*, where m_i^4 is replaced in m_i by m_i^{*4}, is given by:

$$\sum_{m_{-i} \in M_{-i}^2} \lambda_i(m_{-i})[(\frac{m_i^2}{m_i^2+1})u_i(m_i^{*4}(m_{-i}^1), \theta) + (\frac{1}{m_i^2+1})u_i(z_i(m_{-i}^1, m_{-i}^1), \theta)].$$

Since $u_i(m_i^{*4}(m_{-i}^1), \theta) > u_i(z_i(m_{-i}^1, m_{-i}^1), \theta)$ this utility is increasing in the choice of integer m_i^2. Therefore m_i is not a best response for any beliefs. □

Claim 1.2. *If there exists an* $i \in N$ *with a strategy* $m_i \in R_i(\Gamma, \theta)$ *with* $m_i^2 > 1$ *and a belief such that* $\lambda_i \in \Delta(M_{-i}^{3(II)})$ *then* m_i *is not a best response.*

Proof. Now define $m_i^{*5} = \underset{y \in \Delta(X)}{Max}\, u_i(y, \theta)$ and a very high integer m_i^{*2} such that agent i wins the integer game. Replace m_i by changing m_i^5 to m_i^{*5}. Now consider the belief such that $\lambda_i(m_{-i} \in M_{-i}^{3(II)}) = 1$, the utility from strategy m_i^*, where m_i^5 is replaced in m_i by m_i^{*5} and m_i^2 by m_i^{*2}, is given by:

$$\sum_{m_{-i}\in M_{-i}^{3(II)}} \lambda_i(m_{-i})[(\frac{m_i^{*2}}{m_i^{*2}+1})u_i(m_i^{*5},\theta)+(\frac{1}{m_i^{*2}+1})u_i(\underline{y},\theta)].$$

Since $u_i(m_i^{*5},\theta) > u_i(\underline{y},\theta)$, this utility is increasing in the choice of integer m_i^2. Therefore m_i is not a best response to any beliefs such that $\lambda_i \in \Delta(M_{-i}^3)$. □

Now consider the strategy m_i such that $m_i^2 > 1$ with any belief such that $\lambda_i \in \Delta(M_{-i}^{2(II)} \cup M_{-i}^{3(II)})$. We use Claim 1.1 and 1.2 to show that the strategy $m_i^* = (m_i^1, m_i^{*2}, m_i^3, m_i^{*4}, m_i^{*5})$ is better than strategy m_i. Consider the utility from m_i^*. From Claim 1.1 (1) we know that conditional on Rule 2, the outcome is determined by Rule $2(II)(i)$.

$$\sum_{m_{-i}\in M_{-i}^{2(II)(i)}} \lambda_i(m_{-i})[(\frac{m_i^{*2}}{m_i^{*2}+1})u_i(m_i^{*4}(m_{-i}^1),\theta)+(\frac{1}{m_i^{*2}+1})u_i(z_i(m_{-i}^1,m_{-i}^1),\theta)]$$
$$+\sum_{m_{-i}\in M_{-i}^{3(II)}} \lambda_i(m_{-i})[(\frac{m_i^{*2}}{m^{*2}+1})u_i(m_i^{*5},\theta)+(\frac{1}{m^{m_i^{*2}}+1})u_i(\underline{y},\theta)]$$

□

This is greater than the utility from the strategy m_i with $m_i^2 > 1$ under any beliefs such that $\lambda_i \in \Delta(M_{-i}^2 \cup M_{-i}^3)$.

$$\sum_{m_{-i}\in M_{-i}^{2(II)}} \lambda_i(m_{-i})[(\frac{m_i^{2}}{m_i^{2}+1})u_i(m_i^{4}(m_{-i}^1),\theta)+(\frac{1}{m_i^{2}+1})u_i(z_i(m_{-i}^1,m_{-i}^1),\theta)]$$
$$+\sum_{m_{-i}\in M_{-i}^{3(II)}} \lambda_i(m_{-i})[(\frac{m_i^{2}}{m_i^{2}+1})u_i(m_i^{5},\theta)+(\frac{1}{m_i^{2}+1})u_i(\underline{y},\theta)]$$

This immediately follows from Claim 1.1 and Claim 1.2, hence our assumption that $m_i \in R(\Gamma,\theta)$ together with $m_i^2 > 1$ leads to a contradiction. We have thus established that if m_i is a rationalizable strategy for agent i in state θ, then $m_i^2 = 1$.

Lemma 1.2. *If $m_i = (\theta', 1, m_i^3, m_i^4, m_i^5) \in R_i(\Gamma,\theta)$ then*

(1) $m_j = (\theta', 1, m_j^3, m_j^4, m_j^5) \in R_j(\Gamma,\theta)$ for every $j \in N$.
(2) m_i is a best response to beliefs $\lambda_i(m_{-i} \in M_{-i}^1) = 1$.

where

$$M^1_{-i} = \{m_{-i} \in M_{-i} | \forall j \in N \setminus \{i\},\, m_j = (\theta', 1, m^3_j, m^4_j, m^5_j)\}$$

Proof. First we show that if $m_i = (\theta', 1, m^3_i, m^4_i, m^5_i) \in R_i(\Gamma, \theta)$ then for every $j \in N$, $m_j = (\theta', 1, m^3_j, m^4_j, m^5_j) \in R_j(\Gamma, \theta)$. Let us assume that there exists a $j \in N$ s.t. $m_j \notin R_j(\Gamma, \theta)$. We will show that m_i is not a best response to any belief $\lambda_i \in \Delta(R_{-i}(\Gamma, \theta))$. To show this we construct a strategy $\hat{m}_i$ which is better than m_i for any belief $\lambda_i \in \Delta(R_{-i}(\Gamma, \theta))$. We replace m^4_i in m_i with $\hat{m}^4_i$ and m^5_i in m_i by m^{*5}_i[12].

$$\hat{m}^4_i(\hat{\theta}) = \left\{ \begin{array}{ll} U[F(\theta')] & if\ \hat{\theta} = \theta'. \\ U[F(\hat{\theta})] & if\ \hat{\theta}\ s.t.\ |F(\hat{\theta})| \geq 2\ and\ u_i(U[F(\hat{\theta})], \theta) > u_i(b(\hat{\theta}), \theta). \\ b(\hat{\theta}) & if\ \hat{\theta}\ s.t.\ |F(\hat{\theta})| \geq 2\ and\ u_i(U[F(\hat{\theta})], \theta) < u_i(b(\hat{\theta}), \theta). \\ m^{*4}_i(\hat{\theta}) & if\ Otherwise \end{array} \right\}$$

Under our assumption, that there exists a $j \in N$ such that $m_j \notin R_j(\Gamma, \theta)$, we know that the outcome is not decided by Rule (1). The strategy $\hat{m}_i$ is designed in such a way that for any $m_{-i} \in R_{-i}(\Gamma, \theta)$, $\hat{m}_i$ is better than m_i. There are four cases to be considered.

Case 1: $m_{-i} \in R_{-i}(\Gamma, \theta)$: Rule 2(I) is applied.
In this case everyone but i agrees. Let $m_j = (\tilde{\theta}, 1, m^3_i, m^4_j, m^5_j)$ for $j \neq i$. Under strategy m_i, i gets a uniform lottery over $U(F(\tilde{\theta}))$ and $b(\tilde{\theta})$. Under $\hat{m}_i$, i can ensure lottery $U(\tilde{\theta})$ or lottery $b(\tilde{\theta})$ by announcing a very high integer $\hat{m}^2_i$.

Case 2: $m_{-i} \in R_{-i}(\Gamma, \theta)$: Rule 2(II)(i) is applied.
As in the previous case, in this case everyone but i agrees. Let $m_j = (\tilde{\theta}, 1, m^3_i, m^4_j, m^5_j)$ for $j \neq i$. Under strategy m_i, i gets a uniform lottery over $m^{*4}_i(\tilde{\theta})$ and $z_i(\tilde{\theta}, \tilde{\theta})$. Under $\hat{m}_i$, i can ensure lottery $m^4_i(\tilde{\theta})$ by announcing a very high integer $\hat{m}^2_i$.

Case 3: $m_{-i} \in R_{-i}(\Gamma, \theta)$: Rule 3(I) is applied.
In this case i can choose $m^5_i = m^{*5}_i$. This allocation can be achieved with an arbitrarily high probability. This is better than a uniform lottery over $\hat{m}^3_i(\theta')$ and $U(F(\theta'))$.

Case 4: $m_{-i} \in R_{-i}(\Gamma, \theta)$: Rule 3(II) is applied.
In this case i can choose $m^5_i = m^{*5}_i$ and achieve this allocation with a arbitrarily high probability.

Thus we have shown that if $m_i = (\theta', 1, m^3_i, m^4_i, m^5_i) \in R_i(\Gamma, \theta)$ then, for every $j \in N$, $m_j = (\theta', 1, m^3_j, m^4_j, m^5_j) \in R_j(\Gamma, \theta)$.

[12] Where $m^{*5}_i = \underset{y \in \Delta(X)}{Max}\ u_i(y, \theta)$.

Now we will show that if $m_i = (\theta', 1, m_i^3, m_i^4, m_i^5) \in R_i(\Gamma, \theta)$ then, m_i is a best response to *only* beliefs such that $\lambda_i(m_{-i} \in M_{-i}^1) = 1$.

Let us assume that m_i is a best response to beliefs $\lambda_i(m_{-i} \in M_{-i}^1) \neq 1$. There are two cases to be considered.

Case 1: $\lambda_i(m_{-i} \in M_{-i}^1) = 0$.
In this case by assumption the outcome is not decided by Rule 1. This case then becomes similar to part (1) of this Lemma. The interesting case is then the following.

Case 2: $0 < \lambda_i(m_{-i} \in M_{-i}^1) \leq 1$.

In this case we can show that $\hat{m}_i$ is better than m_i. The argument is simple. When every agrees with i, i can ensure the lottery $U[F(\theta')]$ with a very high probability. There is however some loss as compared to the $U[F(\theta')]$ with probability one probability. In all other cases agent i strictly gains by ensuring a lottery $\hat{m}_i^4(\hat{\theta})$ and m_i^{*5} with an arbitrarily high probability. By a suitable choice of integer the loss in the event where everyone agrees is overcome by the gain in all other events. □

Now we are ready to characterize the set $R(\Gamma, \theta)$. Let the set of Nash equilibrium at state θ be denoted by $NE(\Gamma, \theta)$

Lemma 1.3. *If* $m_i = (\theta', 1, m_i^3, m_i^4, m_i^5) \in R_i(\Gamma, \theta)$ *then the strategy profile* $m = (m_1, \ldots, m_n)$ *where* $m_1 = ... = m_n = (\theta', 1, m_i^3, m_i^4, m_i^5)$ *is a Nash equilibrium.*

Proof. Let $m_i = (\theta', 1, m_i^3, m_i^4, m_i^5) \in R_i(\Gamma, \theta)$ then by Lemma 1.2 m_i is a best response to $\lambda_i(m_{-i} \in M_{-i}^1, \theta) = 1$. This is true for everyone since by part (1), $m_j = (\theta', 1, m_j^3, m_j^4, m_j^5) \in R_j(\Gamma, \theta)$ for every $j \in N$ and i was chosen arbitrarily in the proof of Lemma 1.2. Hence $m_1 = ... = m_n = (\theta', 1, m_i^3, m_i^4, m_i^5)$ is a Nash equilibrium of Γ. □

With some abuse of notation let us denote a Nash equilibrium at state θ by θ'. That is if $m_1 = ... = m_n = (\theta', 1, m_i^3, m_i^4, m_i^5)$ is a Nash equilibrium of Γ at state θ then we write θ'.

Let $NE(\Gamma, \theta) = \{\theta' \in \Theta | m_1 = ... = m_n = (\theta', 1, m_i^3, m_i^4, m_i^5)$ is a Nash equilibrium$\}$.

Lemma 1.4. *If* $\theta' \in NE(\Gamma, \theta)$ *then* $F(\theta') \subseteq F(\theta)$.

Proof. Let us assume that $\theta' \in NE(\Gamma, \theta)$ but there exists an alternative 'a' such that $a \in F(\theta')$ and $a \notin F(\theta)$. Since $\theta' \in NE(\Gamma, \theta)$, for every $i \in N$, $m_i = (\theta', 1, ., ., .) \in R_i(\Gamma, \theta)$. Since $a \in F(\theta')$ and $a \notin F(\theta)$, by the assumption that F satisfies r-monotonicity there exists an agent $i \in N$ and a pair of lotteries $\alpha \in \Delta(F(\theta'))$ and $\alpha' \in \Delta(X)$ such that

$$u_i(\alpha, \theta') \geq u_i(\alpha', \theta') \textit{ and } u_i(\alpha', \theta) > u_i(\alpha, \theta).$$

Under the assumption of expected utility theory for agent i we can say that for lottery $U[F(\theta')]$ there exists a lottery $\gamma \in \Delta(X)$ such that

$$u_i(U([F(\theta')], \theta') \geq u_i(\gamma, \theta') \textit{ and } u_i(\gamma, \theta) > u_i(U([F(\theta')], \theta).$$

To see why this true, we can always express $U[F(\theta')]$ as a linear combination of two lotteries with support in $F(\theta')$. Formally, we say that for every $\beta \in \Delta(F(\theta'))$ there exists a $\delta \in \Delta(F(\theta'))$ and $p \in (0, 1)$ such that

$$U([F(\theta')] = p\beta + (1 - p)\delta$$

In particular we can express the uniform lottery as a linear combination of α and δ. Therefore $u_i(U([F(\theta')], \theta')$ can be written as

$$u_i(U([F(\theta')], \theta') = pu_i(\alpha, \theta') + (1 - p)u_i(\delta, \theta')$$

Define $\gamma = p\alpha' + (1 - p)\delta$, we know that $u_i(\alpha, \theta') \geq u_i(\alpha', \theta')$ and $u_i(\alpha', \theta) > u_i(\alpha, \theta)$. Therefore it follows that

$$u_i(U([F(\theta')], \theta') = pu_i(\alpha, \theta') + (1 - p)u_i(\delta, \theta') \geq pu_i(\alpha', \theta') + (1 - p)u_i(\delta, \theta') = u_i(\gamma, \theta')$$
$$u_i(\gamma, \theta) = pu_i(\alpha', \theta) + (1 - p)u_i(\delta, \theta) > pu_i(\alpha, \theta) + (1 - p)u_i(\delta, \theta) = u_i(U([F(\theta')], \theta)$$

Now consider a deviation of this agent to a strategy $m_i^* = (m_i^1, m_i^2, m_i^3, m_i^{*4}, m_i^{*5})$ where $m_i^{*2} > 1$ and $m_i^{*4}(\hat{\theta})$ is γ if $\hat{\theta} = \theta'$ and m_i^4 otherwise. In this case the agent is sure that Rule $2(II)$ is triggered. Utility of i with strategy m_i^* is given by

$$(\frac{m_i^{*2}}{m_i^{*2} + 1})u_i(\gamma, \theta) + (\frac{1}{m_i^{*2} + 1})u_i(z_i(\theta', \theta'), \theta) \tag{21}$$

Since we know that $u_i(\gamma, \theta) > u_i(U[F(\theta')], \theta)$, strategy m_i^* can be made better than m_i by a choice of very high integer m_i^{*2}. This however contradicts the assumption that $\theta' \in NE(\Gamma, \theta)$. $\square$

Lemma 1.5. *If $\theta' \in NE(\Gamma, \theta)$ then either $\theta' = \theta$ or $|F(\theta')| \geq 2$.*

Proof. Let us assume that $\theta' \in NE(\Gamma, \theta)$, $\theta' \neq \theta$ and $|F(\theta')| = 1$. Let us $F(\theta') = a$. Since $|F(\theta)|$, by the assumption of Θ_F - distinguishability, we know that there exists an agent $i \in N$ and a lottery $\alpha \in \Delta(X)$ such that

$$u_i(a, \theta') \geq u_i(\alpha, \theta') \textit{ and } u_i(\alpha, \theta) > u_i(a, \theta).$$

By a very similar argument as in Lemma 1.4 we can show that $\theta' \notin NE(\Gamma, \theta)$. This leads to contradiction. Hence either $\theta' = \theta$ *or* $|F(\theta')| \geq 2$ must be true. □

Lemma 1.6. *For every $\theta \in \Theta$ and $m \in R(\Gamma, \theta), g(m) \in \Delta(F(\theta))$.*

Proof. Let $R(\Gamma, \theta)$ be the set of rationalizable strategies at state θ, and select an arbitrary strategy profile $m \in R(\Gamma, \theta)$. By Lemma 1.1 we know that for every $i \in N, m_i$ is of the following form $m_i = (\theta', 1, m_i^3, m_i^4, m_i^5)$.

First consider the case where the true state $\theta \in \Theta_F$. In this case, the set $R(\Gamma, \theta)$ can be described by a strategy profile where for every $i \in N$, $m_i = (\theta, 1, m_i^3, m_i^4, m_i^5)$ which also forms a Nash equilibrium. To see why this is true, consider a strategy profile $m \in R(\Gamma, \theta)$ such that there exists an agent $i \in N$ with $m_i = (\theta', 1, m_i^3, m_i^4, m_i^5)$ and $\theta' \neq \theta$. By Lemma 1.3 we know that $\theta' \in NE(\Gamma, \theta)$. By Lemma 1.4 we know that $F(\theta') \subseteq F(\theta)$. Since $F(\theta)$ is singleton, $F(\theta')$ must be singleton. Furthermore we have assumed that $\theta \neq \theta'$. The fact that $\theta \neq \theta'$ and $F(\theta')$ is singleton together contradict Lemma 1.5.

Now consider the case where the true state $\theta \in \Theta \setminus \Theta_F$. Notice that in this case $|F(\theta)| \geq 2$. Select an arbitrary rationalizable strategy profile $m \in R(\Gamma, \theta)$. Using Lemma 1.3 we know that for every $i \in N$, m_i^1 is a Nash equilibrium at state θ and Lemma 1.4 $F(m_i^1) \subseteq F(\theta)$. Finally using Lemma 1.5 we know that $|F(m_i^1)| \geq 2$.

We have thus established that for any rationalizable strategy profile m, $|F(m_i^1)| \geq 2$. This means that the outcome is decided by Rule 1, $2(I)$ or $3(I)$. In all these cases $g(m) \in \Delta(F(\theta))$. This completes the proof.

□

Lemma 1.7. *For every $\theta \in \Theta$, for every i, $m_i = (\theta, 1, m_i^3, m_i^4, m_i^5) \in R_i(\Gamma, \theta)$.*

Proof. This follows from the fact that $\theta \in NE(\Gamma, \theta)$. To see this, by the construction of the mechanism the payoff from any unilateral deviation is bounded above by $U[F(\theta)]$. This verifies part (1) of the definition of rationalizable implementation. □

2. Example

Bergemann et al. (2011) introduce a condition which they call strict Maskin monotonicity* and show that, under a weak restriction on the class of mechanisms, is also necessary for the rationalizable implementation of an SCF f. The goal of this section is to provide an example demonstrating that strict Maskin monotonicity* is strictly stronger than Maskin monotonicity. Since we assume strict preferences, our example does not rely on the indifference in the preferences of the agents. To the best of our knowledge this is the first example in the literature. Before going to the example we define Strict Maskin monotonicity*.

Given a SCF f, let us consider the unique partition of $\Theta : P_f = \{\Theta_z\}_{z=f(\theta)}$. In other words, $\Theta_z = \{\theta \in \Theta | f(\theta) = z\}$. We are now ready to define the notion of strict Maskin monotonicity*

Definition 2.1. *A social choice function f satisfies strict Maskin monotonicity* if there exists a partition P of Θ which is finer than P_f such that for any θ:*
$\theta' \in P(\theta)$ whenever for all i and y,

$$[\forall \hat{\theta} \in P(\theta) : u_i(f(\theta), \hat{\theta}) > u_i(y, \hat{\theta})] \Rightarrow [u_i(f(\theta), \theta') \geq u_i(y, \theta')].$$

Example 2.1. Let $X = \{a, b, c, d\}$, $N = \{1, 2, 3\}$ and $\Theta = \{\theta, \theta', \theta'', \theta'''\}$. The preference profile for each state is given in the table below. Numbers in the parentheses specify the vNM utilities. Consider the following SCF $f(\theta) = f(\theta') = f(\theta') = a$ and $f(\theta''') = b$.

Table 5. Preferences

θ			θ'			θ''			θ'''		
1	2	3	1	2	3	1	2	3	1	2	3
$a(3)$	$a(3)$	$a(3)$	$b(3)$	$c(3)$	$b(3)$	$c(3)$	$b(3)$	$c(3)$	$b(3)$	$c(3)$	$c(3)$
$b(2)$	$c(2)$	$b(2)$	$a(2)$	$a(2)$	$a(2)$	$a(2)$	$a(2)$	$a(2)$	$a(2)$	$a(2)$	$a(2)$
$c(1)$	$b(1)$	$c(1)$	$c(1)$	$d(1)$	$c(1)$	$d(1)$	$c(1)$	$b(1)$	$c(1)$	$b(1)$	$b(1)$
$d(0)$	$d(0)$	$d(0)$	$d(0)$	$b(0)$	$d(0)$	$b(0)$	$d(0)$	$d(0)$	$d(0)$	$d(0)$	$d(0)$
$f(\theta) = a$			$f(\theta') = a$			$f(\theta'') = a$			$f(\theta''') = b$		

Claim 2.1. *The SCF f violates strict Maskin monotonicity*.*

Proof. Let us consider the partition $P_f = \{\{\theta, \theta', \theta''\}, \{\theta'''\}\}$. Let us assume that the SCF f satisfies strict Maskin monotonicity* with respect to P_f. With the help of following claim we will show that this will lead to a contradiction.

Claim 2.2. *For every* $i \in N$, $\bigcap_{\hat{\theta} \in P_f(\theta)} L_i(a, \hat{\theta}) \subseteq L_i(a, \theta''')$.

Proof. Consider agent 1. Let $\alpha \in \bigcap_{\hat{\theta} \in P_f(\theta)} L_1(a, \hat{\theta})$. This implies $\alpha \in L_1(a, \theta')$. Since $L_1(a, \theta') = L_1(a, \theta''')$ therefore $\alpha \in L_1(a, \theta''')$. Now consider agent 2. Let $\alpha \in \bigcap_{\hat{\theta} \in P_f(\theta)} L_3(a, \hat{\theta})$. This implies $\alpha \in L_2(a, \theta')$. Since $L_2(a, \theta') = L_2(a, \theta''')$ therefore $\alpha \in L_2(a, \theta''')$. Finally consider agent 3. Let $\alpha \in \bigcap_{\hat{\theta} \in P_f(\theta)} L_3(a, \hat{\theta})$. This implies $\alpha \in L_3(a, \theta'')$. Since $L_3(a, \theta'') = L_2(a, \theta''')$ therefore $\alpha \in L_2(a, \theta''')$. □

Using the above claim and the definition of strict Maskin monotonicity* we can conclude that $\theta''' \in P(\theta)$, which is a contradiction. With the help of following remark we can show a similar argument holds true for any partition of Θ which is finer than P_f.

Remark 2.1. For every i, $L_i(a, \theta') \subseteq L_i(a, \theta)$ and $L_i(a, \theta'') \subseteq L_i(a, \theta)$.

Let us assume that this SCF f satisfies strict Maskin monotonicity* with a possibly finer partition than P_f. Consider an arbitrary partition P of Θ. Since this partition is finer than P_f, there exists θ' or θ'' such that $\theta \notin P(\theta')$ or $\theta \notin P(\theta'')$. There are four mutually exclusive and exhaustive cases which we need to consider

First, consider the partition $P = \{\{\theta\}, \{\theta'\}, \{\theta''\}, \{\theta'''\}\}$. Remark 2.1 together with the assumption that SCF f satisfies strict Maskin monotonicity* implies that $\theta \in P(\theta'')$, which is a contradiction since $P(\theta'') = \{\theta''\}$. We can use a similar argument for partition $P = \{\{\theta, \theta'\}, \{\theta''\}, \{\theta'''\}\}$ and $P = \{\{\theta, \theta''\}, \{\theta'\}, \{\theta'''\}\}$. Finally consider $P = \{\{\theta'', \theta'\}, \{\theta\}, \{\theta'''\}\}$. Remark 2.1 together with the assumption that SCF f satisfies strict Maskin monotonicity* implies that $\theta \in P(\theta'')$, which is a contradiction since $P(\theta'') = \{\theta'', \theta'\}$

□

Claim 2.3. *SCF f, satisfies Maskin monotonicity.*

Proof. To prove that f satisfies Maskin monotonicity we need to show that for every (θ, θ') pair such that $f(\theta) \neq f(\theta')$, there exists $i, j \in N$ such that $L_i(f(\theta), \theta) \nsubseteq L_i(f(\theta), \theta')$ and $L_j(f(\theta'), \theta') \nsubseteq L_j(f(\theta'), \theta)$. There are three pairs to be considered, $(\theta, \theta'''), (\theta', \theta''')$ and (θ'', θ''').

- (θ, θ'''): $b \in L_1(f(\theta), \theta)$ and $b \notin L_1(f(\theta), \theta''')$.
- (θ', θ'''): $c \in L_3(f(\theta'), \theta')$ and $c \notin L_3(f(\theta'), \theta''')$.
- (θ'', θ'''): $b \in L_1(f(\theta''), \theta'')$ and $b \notin L_1(f(\theta''), \theta''')$.
- (θ''', θ): $a \in L_1(f(\theta'''), \theta''')$ and $a \notin L_1(f(\theta'''), \theta)$.
- (θ''', θ'): $d \in L_2(f(\theta'''), \theta''')$ and $d \notin L_2(f(\theta'''), \theta')$.
- (θ''', θ''): $a \in L_1(f(\theta'''), \theta''')$ and $a \notin L_1(f(\theta'''), \theta'')$.

$\square$

3. Proof of Theorem 8.1

Theorem 3.1. *Theorem 8.1 Let $\epsilon \in [0, 1)$ and $n \geq 3$. If an SCC F satisfies NWA, r-monotonicity and is Θ_F -distinguishable then it is ϵ-rationalizably implementable.*

The proof of Theorem 8.1 follows very closely the proof of Theorem 5. This extension is reminiscent of the extension of Maskin's theorem from SCF's to SCC's. Before we describe the canonical mechanism we denote by $\alpha(\theta, a)$ a lottery which assigns ϵ probability to an alternative $a \in F(\theta)$ and rest of the probability to $U[F(\theta)]$. Formally,

$$\alpha(\theta', a) = \epsilon a + (1 - \epsilon)U[F(\theta')]$$

Message Space. $M^{1(i)} = \Theta$, $M^{1(ii)} = \{X^\Theta | \; for \; every \; m^{1(ii)} \in X^\Theta, \, m^{1(ii)}(\theta) \in F(\theta)\}$, $M^2 = Z_+$, $M^3 = \{\Delta(X)^\Theta | \; for \; every \; m^3 \in \Delta(X)^\Theta, \, m^3(\theta) \in \Delta(F(\theta))\}$, $M^4 = \{\Delta(X)^{\Theta \times X}\}$, $M^5 = \Delta(X)$.

A typical message for a player i, $m_i = (m_i^1, m_i^2, m_i^3, m_i^4, m_i^5)$, can be described as follows:

- $m_i^1 = (m_i^{1(i)}, m_i^{1(ii)})$, where
  - $m_i^{1(i)} \in M^{1(i)}$: State of the world.
  - $m_i^{1(ii)} \in M^{1(ii)}$: A state contingent recommendation such that $m_i^{1(ii)}(\theta) \in F(\theta)$.
- $m_i^2 \in M^2$: An integer.
- $m_i^3 \in M^3$: A state contingent allocation plan such that $m_i^3(\theta) \in \Delta(F(\theta))$.
- $m_i^4 \in M^4$: A state and an alternative contingent allocation plan such that $m_i^4(\theta, a) \in \Delta(X)$.
- $m_i^5 \in M^5$: A lottery over X.

Outcome Function.

(1) **Universal Agreement**: For every i, $m_i^1 = (m_i^{1(i)}, m_i^{1(i)}) = (\theta', a')$, and $m_i^2 = 1$.

$$g(m) = \alpha(\theta', a')$$

(2) **Unilateral Deviation**: There is an agent $i \in N$ s.t. $(m_i^1, m_i^2) \neq (m_j^1, m_j^2)$ and for every $j \neq \{i\}$,$(m_j^1, m_j^2) = ((\theta', a'), 1)$. Let $b(\theta') \in \underset{y \in F(\theta')}{argmin}\ u_i(y, \theta')$.

I **Coordination Failure**: $|F(\theta')| \geq 2$ and $m_i^2 = 1$.

$$g(m) = \left\{ \begin{array}{ll} \alpha(\theta', a') & with\ probability\ \frac{1}{2} \\ b(\theta') & with\ probability\ \frac{1}{2} \end{array} \right\}$$

II **Other Cases:** $|F(\theta')| = 1$ **or** $m_i^2 > 1$

(i) $u_i(\alpha(\theta', a'), \theta') \geq u_i(m_i^4(\theta', a'), \theta')$**,**

$$g(m) = \left\{ \begin{array}{l} m_i^4(\theta', a')\ with\ probability\ (1 - \frac{1}{m_i^2+1}) \\ z_i(\theta', \theta')\ with\ probability\ \frac{1}{m_i^2+1} \end{array} \right\}$$

(ii) $u_i(\alpha(\theta', a'), \theta') \ < \ u_i(m_i^4(\theta', a'), \theta')$

$$g(m) = z_i(\theta', \theta')$$

(3) **Disagreement**:

I **Coordination Failure**: For every $i \in N, m_i^2 = 1$ and $|F(m_i)| \geq 2$. Select an agent i at random.

$$g(m) = \left\{ \begin{array}{l} m_i^3(m_i^1)\ with\ probability\ \frac{1}{2} \\ U[F(m_i^1)]\ with\ probability\ \frac{1}{2} \end{array} \right\}$$

II **Other Cases**: Select an agent i announcing the highest integer m_i^2 and ties are broken arbitrarily.

$$g(m) = \left\{ \begin{array}{l} m_i^5\ with\ probability\ (1 - \frac{1}{m_i^3+1}) \\ \underline{y}\ with\ probability\ \frac{1}{m_i^3+1} \end{array} \right\}$$

Throughout the proof we will assume that the true state of the world is $\theta \in \Theta$.

Lemma 3.1. *If an SCC F satisfies NWA, then for any $i \in N$, if $m_i \in R_i(\Gamma, \theta)$ then $m_i^2 = 1$.*

Proof. Assume by the way of contradiction that there exists an $i \in N$ such that $m_i \in R_i(\Gamma, \theta)$ and $m_i^2 > 1$. Given the outcome function, the outcome is decided by either Rule $2(II)$ or Rule $3(II)$. For a strategy m_i, let us denote the set of strategy profiles of opponents which will lead to Rule $2(II)$ and Rule $3(II)$ as follows.

$$M_{-i}^{2(II)} = \{m_{-i} \in M_{-i} | m_j^1 = (\theta', a) \textit{ and } m_j^2 = 1 \textit{ for some } (\theta', a), \textit{for every } j \neq i\}$$
$$M_{-i}^{3(II)} = M_{-i} \setminus M_{-i}^2$$

We will construct a strategy m_i^* for agent i and show that at state θ, for any belief $\lambda_i \in \Delta(M_{-i}^2 \cup M_{-i}^3)$ strategy m_i^* is better than strategy m_i. Notice that m_i^4 is relevant for Rule $2(II)$ and m_i^5 is relevant for Rule $3(II)$. This will allow us to break the argument into two separate cases. First case is $\lambda_i \in \Delta(M_{-i}^{2(II)})$ and second case $\lambda_i \in \Delta(M_{-i}^{3(II)})$.

Claim 3.1. *If there exists an $i \in N$ with a strategy $m_i \in R_i(\Gamma, \theta)$ with $m_i^2 > 1$ and beliefs such that $\lambda_i \in \Delta(M_{-i}^2)$, then*

(1) *At strategy profiles where rule $2(II)(ii)$ is applied, m_i is not a best response.*
(2) *m_i is not a best response at rule $2(II)(i)$ either.*

Proof. Let $m_i \in R_i(\Gamma, \theta)$ be a strategy of agent i with $m_i^2 > 1$ and consider that (m_i, m_{-i}) is a strategy profile such that rule $2(II)(ii)$ applies. This implies that $m_j^1 = (\theta', a)$ for all $j \neq i$ and $u_i(\alpha(\theta', a), m_{-i}^{1(i)}) < u_i(m_i^4(m_{-i}^{1(i)}, m_{-i}^{1(ii)}), m_{-i}^{1(i)})$ and $g(m_i, m_{-i}) = z_i(m_{-i}^1, m_{-i}^1)$. By Lemma 4.2 we know that for $\theta \neq m_{-i}^1$ there exist allocations $z_i(\theta', \theta)$ such that

$$u_i(z_i(\theta, m_{-i}^1), \theta) > u_i(z_i(m_{-i}^1, m_{-i}^1), \theta)$$
$$u_i(\alpha(a, \theta'), m_{-i}^1) > u_i(z_i(\theta, m_{-i}^1), m_{-i}^1)$$

Therefore by announcing

$$m_i^{*4}(\hat{\theta}, x) = \left\{ \begin{array}{ll} \alpha(\theta, x) & if\ \hat{\theta} = \theta \\ z_i(\theta, \hat{\theta}) & if\ \hat{\theta} \neq \theta \end{array} \right\}$$

agent i is better off by inducing Rule $2(II)(i)$ as compared to Rule $2(II)(ii)$. Since the choice of strategy profile which induces Rule $2(II)(ii)$ was arbitrary, the claim holds for *any* such strategy profile.

Now consider the belief such that $\lambda_i \in \Delta(M^2_{-i})$, utility from strategy m_i^*, where m_i^4 is replaced in m_i by m_i^{*4}, is given by:

$$\sum_{m_{-i} \in M^2_{-i}} \lambda_i(m_{-i})[(\frac{m_i^2}{m_i^2+1})u_i(m_i^{*4}(m^1_{-i}), \theta) + (\frac{1}{m_i^2+1})u_i(z_i(m^1_{-i}, m^1_{-i}), \theta)].$$

Since $u_i(m_i^{*4}(m^1_{-i}), \theta) > u_i(z_i(m^1_{-i}, m^1_{-i}), \theta)$ this utility is increasing in the choice of integer m_i^2. Therefore m_i is not a best response for any beliefs. □

Claim 3.2. *If there exists an $i \in N$ with a strategy $m_i \in R_i(\Gamma, \theta)$ with $m_i^2 > 1$ and a belief such that $\lambda_i \in \Delta(M^{3(II)}_{-i})$ then m_i is not a best response.*

Proof. Now define $m_i^{*5} = \underset{y \in \Delta(X)}{Max}\, u_i(y, \theta)$ and a very high integer m_i^{*2} such that agent i wins the integer game. Replace m_i by changing m_i^5 to m_i^{*5}. Now consider the belief such that $\lambda_i(m_{-i} \in M^{3(II)}_{-i}) = 1$, the utility from strategy m_i^*, where m_i^5 is replaced in m_i by m_i^{*5} and m_i^2 by m_i^{*2}, is given by:

$$\sum_{m_{-i} \in M^{3(II)}_{-i}} \lambda_i(m_{-i})[(\frac{m_i^{*2}}{m_i^{*2}+1})u_i(m_i^{*5}, \theta) + (\frac{1}{m_i^{*2}+1})u_i(\underline{y}, \theta)].$$

Since $u_i(m_i^{*5}, \theta) > u_i(\underline{y}, \theta)$, this utility is increasing in the choice of integer m_i^2. Therefore m_i is not a best response to any beliefs such that $\lambda_i \in \Delta(M^3_{-i})$. □

Now consider the strategy m_i such that $m_i^2 > 1$ with any belief such that $\lambda_i \in \Delta(M^{2(II)}_{-i} \cup M^{3(II)}_{-i})$. We use Claim 3.1 and 3.2 to show that the strategy $m_i^* = (m_i^1, m_i^{*2}, m_i^3, m_i^{*4}, m_i^{*5})$ is better than strategy m_i. Consider the utility from m_i^*. From Claim 3.1 (1) we know that conditional on Rule 2, the outcome is determined by Rule $2(II)(i)$.

$$\sum_{m_{-i}\in M^{2(II)(i)}_{-i}} \lambda_i(m_{-i})[(\frac{m^{*2}_i}{m^{*2}_i+1})u_i(m^{*4}_i(m^1_{-i}),\theta)+(\frac{1}{m^{*2}_i+1})u_i(z_i(m^1_{-i},m^1_{-i}),\theta)]$$
$$+\sum_{m_{-i}\in M^{3(II)}_{-i}} \lambda_i(m_{-i})[(\frac{m^{*2}_i}{m^{*2}+1})u_i(m^{*5}_i,\theta)+(\frac{1}{m^{m^{*2}_i}+1})u_i(\underline{y},\theta)]$$

□

This is greater than the utility from the strategy m_i such that $m_i^2 > 1$ with beliefs such that $\lambda_i \in \Delta(M^2_{-i}\cup M^3_{-i})$

$$\sum_{m_{-i}\in M^{2(II)}_{-i}} \lambda_i(m_{-i})[(\frac{m^{2}_i}{m^{2}_i+1})u_i(m^{4}_i(m^1_{-i}),\theta)+(\frac{1}{m^{2}_i+1})u_i(z_i(m^1_{-i},m^1_{-i}),\theta)]$$
$$+\sum_{m_{-i}\in M^{3(II)}_{-i}} \lambda_i(m_{-i})[(\frac{m^{2}_i}{m^{2}_i+1})u_i(m^{5}_i,\theta)+(\frac{1}{m^{2}_i+1})u_i(\underline{y},\theta)]$$

This immediately follows from Claim 3.1 and Claim 3.2, hence our assumption that $m_i \in R(\Gamma,\theta)$ together with $m_i^2 > 1$ leads to a contradiction. We have thus established that if m_i is a rationalizable strategy for agent i in state θ, then $m_i^2 = 1$.

Lemma 3.2. *If $m_i = ((\theta',a),1,m_i^3,m_i^4,m_i^5) \in R_i(\Gamma,\theta)$ then*

(1) $m_j = (\theta',1,m_j^3,m_j^4,m_j^5) \in R_j(\Gamma,\theta)$ for every $j\in N$.
(2) m_i is a best response to beliefs $\lambda_i(m_{-i}\in M^1_{-i}) = 1$.

where

$$M^1_{-i} = \{m_{-i}\in M_{-i}|\forall j\in N\setminus\{i\},\, m_j = ((\theta',a),1,m_j^3,m_j^4,m_j^5)\}$$

Proof. First we show that if $m_i = ((\theta',a),1,m_i^3,m_i^4,m_i^5) \in R_i(\Gamma,\theta)$ then for every $j\in N$, $m_j = ((\theta',a),1,m_j^3,m_j^4,m_j^5) \in R_j(\Gamma,\theta)$. Let us assume that there exists a $j\in N$ s.t. $m_j \notin R_j(\Gamma,\theta)$. We will show that m_i is not a best response to any belief $\lambda_i \in \Delta(R_{-i}(\Gamma,\theta))$. To show this we construct a strategy $\hat{m}_i$ which is better than m_i for any belief $\lambda_i \in \Delta(R_{-i}(\Gamma,\theta))$. We replace m_i^4 in m_i with $\hat{m}_i^4$ and m_i^5 in m_i by m_i^{*5}.

$$\hat{m}_i^4(\hat{\theta},x) = \left\{ \begin{array}{ll} \alpha(\theta',a) & if\ \hat{\theta}=\theta'\ and\ x=a. \\ \alpha(\hat{\theta},a) & if\ \hat{\theta}\ s.t.\ |F(\hat{\theta})|\geq 2\ and\ u_i(\alpha(\hat{\theta},x),\theta) > u_i(b(\hat{\theta}),\theta). \\ b(\hat{\theta}) & if\ \hat{\theta}\ s.t.\ |F(\hat{\theta})|\geq 2\ and\ u_i(\alpha(\hat{\theta},x),\theta) < u_i(b(\hat{\theta}),\theta). \\ m_i^{*4}(\hat{\theta},x) & if\ Otherwise \end{array} \right\}$$

Under our assumption, that there exists a $j \in N$ such that $m_j \notin R_j(\Gamma,\theta)$, we know that the outcome is not decided by Rule (1). The strategy $\hat{m}_i$ is designed in such a way that for any $m_{-i} \in R_{-i}(\Gamma,\theta)$, $\hat{m}_i$ is better than m_i. There are four cases to be considered.

Case 1: $m_{-i} \in R_{-i}(\Gamma,\theta)$: Rule 2(I) is applied.
In this case everyone but i agrees. Let $m_j = ((\tilde{\theta},\tilde{a}),1,m_i^3,m_j^4,m_j^5)$ for $j \neq i$. Under strategy m_i, i gets a uniform lottery over $\alpha(\tilde{\theta},\tilde{a})$ and $b(\tilde{\theta})$. Under $\hat{m}_i$, i can ensure lottery $\alpha(\tilde{\theta},\tilde{a})$ or lottery $b(\tilde{\theta})$ by announcing a very high integer $\hat{m}_i^2$.

Case 2: $m_{-i} \in R_{-i}(\Gamma,\theta)$: Rule 2(II)(i) is applied.
As in the previous case, in this case everyone but i agrees. Let $m_j = ((\tilde{\theta},\tilde{a}),1,m_i^3,m_j^4,m_j^5)$ for $j \neq i$. Under strategy m_i, i gets a uniform lottery over $m_i^{*4}(\tilde{\theta},\tilde{a})$ and $z_i(\tilde{\theta},\tilde{\theta})$. Under $\hat{m}_i$, i can ensure lottery $m_i^4(\tilde{\theta},\tilde{a})$ by announcing a very high integer $\hat{m}_i^2$.

Case 3: $m_{-i} \in R_{-i}(\Gamma,\theta)$: Rule 3(I) is applied.
In this case i can choose $m_i^5 = m_i^{*5}$. This allocation can be achieved with an arbitrarily high probability. This is better than a uniform lottery over $\hat{m}_i^3(\theta')$ and $U(F(\theta'))$.

Case 4: $m_{-i} \in R_{-i}(\Gamma,\theta)$: Rule 3(II) is applied.
In this case i can choose $m_i^5 = m_i^{*5}$ and achieve this allocation with a arbitrarily high probability.

Thus we have shown that if $m_i = ((\theta',a),1,m_i^3,m_i^4,m_i^5) \in R_i(\Gamma,\theta)$ then, for every $j \in N$, $m_j = ((\theta',a),1,m_j^3,m_j^4,m_j^5) \in R_j(\Gamma,\theta)$.

Now we will show that if $m_i = ((\theta',a),1,m_i^3,m_i^4,m_i^5) \in R_i(\Gamma,\theta)$ then, m_i is a best response to *only* beliefs such that $\lambda_i(m_{-i} \in M_{-i}^1) = 1$.

Let us assume that m_i is a best response to beliefs $\lambda_i(m_{-i} \in M_{-i}^1) \neq 1$. There are two cases to be considered.

Case 1: $\lambda_i(m_{-i} \in M_{-i}^1) = 0$.
In this case by assumption the outcome is not decided by Rule 1. This case then becomes similar to part (1) of this Lemma. The interesting case is then the following.

Case 2: $0 < \lambda_i(m_{-i} \in M_{-i}^1) \leq 1$.

In this case we can show that $\hat{m}_i$ is better than m_i. The argument is simple. When every agrees with i, i can ensure the lottery $\alpha(\theta', a)$ with a very high probability. There is however some loss as compared to the $\alpha(\theta', a)$ with probability one probability. In all other cases agent i strictly gains by ensuring a lottery $\hat{m}_i^4(\hat{\theta}, x)$ and m_i^{*5} with an arbitrarily high probability. By a suitable choice of integer, m_i^2, the loss in the event where everyone agrees is overcome by the gain in all other events. □

Now we are ready to characterize the set $R(\Gamma, \theta)$. Let the set of Nash equilibrium at state θ be denoted by $NE(\Gamma, \theta)$

Now we are ready to characterize the set $R(\Gamma, \theta)$. Let the set of Nash equilibrium at state θ be denoted by $NE(\Gamma, \theta)$

Lemma 3.3. *If $m_i = ((\theta', a'), 1, m_i^3, m_i^4, m_i^5) \in R_i(\Gamma, \theta)$ then the strategy profile $m = (m_1, \ldots, m_n)$ where $m_1 = ... = m_n = ((\theta', a'), 1, m_i^3, m_i^4, m_i^5)$ is a Nash equilibrium.*

Proof. Let $m_i = ((\theta', a'), 1, m_i^3, m_i^4, m_i^5) \in R_i(\Gamma, \theta)$ then by Lemma 3.2 m_i is a best response to $\lambda_i(m_{-i} \in M_{-i}^1, \theta) = 1$. This is true for everyone since by part (1), $m_j = ((\theta', a'), 1, m_j^3, m_j^4, m_j^5) \in R_j(\Gamma, \theta)$ for every $j \in N$ and i was chosen arbitrarily in the proof of Lemma 3.2. Hence $m_1 = ... = m_n = ((\theta', a'), 1, m_i^3, m_i^4, m_i^5)$ is a Nash equilibrium of Γ. □

With some abuse of notation let us denote a Nash equilibrium at a state θ by (θ', a'). That is if $m_1 = ... = m_n = ((\theta', a'), 1, m_i^3, m_i^4, m_i^5)$ is a Nash equilibrium of Γ at state θ then we write (θ', a').

Let $NE(\Gamma, \theta) = \{(\theta', a') \in \Theta \times X | m_1 = ... = m_n = ((\theta', a'), 1, m_i^3, m_i^4, m_i^5)$ is a Nash equilibrium$\}$.

Lemma 3.4. *If $(\theta', a') \in NE(\Gamma, \theta)$ then $F(\theta') \subseteq F(\theta)$.*

Proof. Let us assume that $(\theta', a') \in NE(\Gamma, \theta)$ but there exists an alternative 'a' such that $a \in F(\theta')$ and $a \notin F(\theta)$. Using the assumption that F satisfies r-monotonicity there exists an agent $i \in N$ and a pair of lotteries $\alpha \in \Delta(F(\theta'))$ and $\alpha' \in \Delta(X)$ such that

$$u_i(\alpha, \theta') \geq u_i(\alpha', \theta') \textit{ and } u_i(\alpha', \theta) > u_i(\alpha, \theta).$$

Under the assumption of expected utility theory, for agent i we can say that for lottery $\alpha(\theta', a')$ there exists a lottery $\gamma \in \Delta(X)$ such that

$$u_i(\alpha(\theta', a'), \theta') \geq u_i(\gamma, \theta') \textit{ and } u_i(\gamma, \theta) > u_i(\alpha(\theta', a'), \theta)$$

To see why this true, we can always express $U[\alpha(\theta', a')]$ as a linear combination of two lotteries with support in $F(\theta')$. Formally, we say that for every $\beta \in \Delta(F(\theta'))$ there exists a $\delta \in \Delta(F(\theta'))$ and $p \in (0,1)$ such that

$$\alpha(\theta', a') = p\beta + (1-p)\delta$$

In particular we can express the lottery $\alpha(\theta', a')$ as a linear combination of α and δ. Therefore $u_i(U(\alpha(\theta', a'), \theta')$ can be written as

$$u_i(\alpha(\theta', a'), \theta') = pu_i(\alpha, \theta') + (1-p)u_i(\delta, \theta')$$

Define $\gamma = p\alpha' + (1-p)\delta$, we know that $u_i(\alpha, \theta') \geq u_i(\alpha', \theta')$ and $u_i(\alpha', \theta) > u_i(\alpha, \theta)$. Therefore it follows that

$$u_i(\alpha(\theta', a'), \theta') = pu_i(\alpha, \theta') + (1-p)u_i(\delta, \theta') \geq pu_i(\alpha', \theta') + (1-p)u_i(\delta, \theta') = u_i(\gamma, \theta')$$
$$u_i(\gamma, \theta) = pu_i(\alpha', \theta) + (1-p)u_i(\delta, \theta) > pu_i(\alpha, \theta) + (1-p)u_i(\delta, \theta) = u_i(\alpha(\theta', a'), \theta)$$

Now consider a deviation of this agent to a strategy $m_i^* = (m_i^1, m_i^2, m_i^{*3}, m_i^4, m_i^{*5})$ where $m_i^{*2} > 1$ and $m_i^{*4}(\hat{\theta})$ is γ if $\hat{\theta} = \theta'$ and m_i^4 otherwise. In this case the agent is sure that Rule $2(II)$ is triggered. Utility of i with strategy m_i^* is given by

$$(\frac{m_i^{*2}}{m_i^{*2}+1})u_i(\gamma, \theta) + (\frac{1}{m_i^{*2}+1})u_i(z_i(\theta', \theta'), \theta) \tag{26}$$

Since we know that $u_i(\gamma, \theta) > u_i(\alpha(\theta', a'), \theta)$, strategy m_i^* can be made better than m_i by a choice of very high integer m_i^{*2}. This however contradicts the assumption that $(\theta', a') \in NE(\Gamma, \theta)$. □

Lemma 3.5. *If $(\theta', a') \in NE(\Gamma, \theta)$ then either $\theta' = \theta$ or $|F(\theta')| \geq 2$.*

Proof. Let us assume that $(\theta', a') \in NE(\Gamma, \theta)$, $\theta' \neq \theta$ and $|F(\theta')| = 1$. Let us denote $F(\theta') = a$. By the assumption of Θ_F - distinguishability, we know that there exists an agent $i \in N$ and a lottery $\alpha \in \Delta(X)$ such that

$$u_i(a,\theta') \geq u_i(\alpha,\theta') \text{ and } u_i(\alpha,\theta) > u_i(a,\theta)$$

By a very similar argument as in Lemma 3.4 we can show that $(\theta', a') \notin NE(\Gamma, \theta)$. This leads to contradiction. Hence either $\theta' = \theta$ *or* $|F(\theta')| \geq 2$ must be true. ☐

Lemma 3.6. *For every $\theta \in \Theta$ and $m \in R(\Gamma,\theta), g(m) \in \Delta(F(\theta))$.*

Proof. Let $R(\Gamma,\theta)$ be the set of rationalizable strategies at state θ, and select an arbitrary $m \in R(\Gamma,\theta)$. By Lemma 1.1 we know that for every $i \in N, m_i$ is of the following form $m_i = ((\theta',a'),1,m_i^3,m_i^4,m_i^5)$.

First consider the case where $\theta \in \Theta_F$. In this case, the set $R(\Gamma,\theta)$ can be described by a strategy profile where for every $i \in N$, $m_i = ((\theta',a'),1,m_i^3,m_i^4,m_i^5)$, which also forms a Nash equilibrium. To see why this is true, consider a strategy profile $m \in R(\Gamma,\theta)$ such that there exists an agent $i \in N$ with $m_i = ((\theta',a'),1,m_i^3,m_i^4,m_i^5)$ and $\theta' \neq \theta$. By Lemma 3.3 we know that $(\theta',a') \in NE(\Gamma,\theta)$. By Lemma 3.4 we know that $F(\theta') \subseteq F(\theta)$. Since $F(\theta)$ is singleton, $F(\theta')$ must be singleton. Furthermore we have assumed that $\theta \neq \theta'$. The fact that $\theta \neq \theta'$ and $F(\theta')$ is singleton together contradict Lemma 3.5.

Now consider the case where $\theta \in \Theta \setminus \Theta_F$. Notice that in this case $|F(\theta)| > 1$. Select an arbitrary rationalizable strategy profile $m \in R(\Gamma,\theta)$. Using Lemma 3.3 we know that for every $i \in N$, m_i^1 is a Nash equilibrium at state θ and Lemma 3.4 $F(m_i^1) \subseteq F(\theta)$. Finally using Lemma 3.5 we know that $|F(m_i^1)| \geq 2$.

We have thus established that for any rationalizable strategy profile m, $|F(m_i^1)| \geq 2$. This means that the outcome is decided by Rule 1, $2(I)$ or $3(I)$. In all these cases $g(m) \in \Delta(F(\theta))$. This completes the proof.

☐

Lemma 3.7. *For every $\theta \in \Theta$ and every $a \in F(\theta)$, $m_i = ((\theta,a),1,m_i^3,m_i^4,m_i^5) \in R_i(\Gamma,\theta)$ for every i.*

Proof. This follows from the fact that $(\theta,a) \in NE(\Gamma,\theta)$. To see this, by the construction of the mechanism the payoff from any unilateral deviation is bounded above by $\alpha(\theta,a)$. This verifies part (1) of the definition of ϵ rationalizable implementation. ☐

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