k^{th} Best Quasi-Transitive Rationalizability of Choice Functions*

Rupali Sharma†

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Abstract

We study rationalizability of choice functions where an agent selects the k^{th} best alternative. We propose two axioms which fully characterise a choice function that is rationalizable by a unique, reflexive, connected and quasi-transitive rationalization when k^{th} best elements are chosen under full domain. The standard literature on rationalizability of choice functions, where first best alternatives are chosen, takes a narrow and individualistic view of rationality. This paper studies behaviours that violate standard consistency axioms and yet are well-behaved.

Keywords: rationalizability • choice functions • rational choice • external reference

1 Introduction

Rationality of choice has been an essential part of discussions in social choice theory and is an extensively studied subject in the literature. What is meant by rational choice and what is a commonly acceptable notion of rationality of choice has been the questions that the voluminous literature on this subject has tried to answer. The fact that rationality of economic agents is fundamental for economic theory renders the debate over it inescapable. An important constituent of this debate is the rationalizability of choice functions. The theory of rationalizability of choice functions owes its existence to the inspiring work by Samuelson (1938) on the theory of revealed preference. He proposed that by observing the choice behaviour of a representative consumer, we can observe the preferences that she *reveals* in her actions. The pioneering work of Uzawa (1956) and Arrow

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†**Email:** rupalisharma23@gmail.com

(1959) augmented this theory thus laying the foundations of the choice theoretic framework of rationalizability theory. Subsequently, a large literature developed that examined choice functions to see if they are rationalizable, that is, the same choices as that of the choice function can be generated with some preference relation. Most of the literature in this area deals with the same kind of axiomatic treatment of choice behaviour wherein by satisfying certain conditions, we can get a preference relation that rationalizes the choice function. The nature of the rationalization, the properties that it satisfies, depends upon the underlying consistency condition of the axiom that is imposed. These conditions are devoid of any context or substantive principle outside of the choice function.

We examine rationalizability to confirm that the choices were made sensibly and say that if a choice function is rationalizable then choices are well-behaved. It, however, would be problematic to assume that for choices to be well-behaved or sensibly made a choice function needs to be rationalizable or that if a choice function is not rationalizable then there must be some inconsistency in the behaviour. Sen (1993) terms the *a priori* imposition of such conditions as the "internal consistency" of choice. He points out the difference between *imposed* and *entailed* internal consistency. While arguing against the former, Sen says that there is no internal way by which one can determine whether or not the choice behaviour is consistent. This issue arises because of the presumption that the *act* of choice can be viewed as a stand-alone action.

Sen (1993) draws attention to different circumstances under which an internally inconsistent choice behaviour would seem to be perfectly consistent.[1] Individuals might be operating under some norms which might be influencing their act of choice, benevolence might lead them to alter their choice behaviour or they could simply be following what society perceives as proper or good behaviour. To emphasise this we quote Sen's famous example. Suppose a person has been offered some cake and has to choose from the different pieces available. Let x, y and z be three pieces of different sizes such that z is bigger than y which is bigger than x. It is assumed that the individual has no difficulty assessing the difference between them. Suppose on being offered x and y, she chooses x and from x, y and z she decides to pick up y. If this person is fond of cakes and would like to choose the biggest piece, then her behaviour is indeed inconsistent. Alternatively, suppose she does not want to appear greedy and is operating under a norm of never choosing the largest piece. Keeping this in mind and that the size of z is the biggest and x the smallest, her behaviour is now perfectly rational.

In the above example, it was assumed that the individual has no difficulty assessing the difference between the size of the slices. However, this may not always hold. Suppose a person has been offered some cold drink and has to choose from different glasses of

[1] See Sen (1997) also.

the same drink. Say there are three glasses with 150ml, 140ml and 130ml of the same cold drink. Further suppose that the individual can differentiate in the volume of the drink only if there is a minimum difference of 20ml. Now, say on being offered 150ml and 140ml, she picks the glass with 150ml of cold drink; on being offered 140ml and 130ml, she picks 140ml. But when she is offered all three, she picks 130ml. If this person likes the drink and would like to pick the biggest glass, then her behaviour is inconsistent. However, if she does not want to appear greedy and, as before, is operating under a norm of never choosing the biggest glass, her behaviour now seems consistent. If she cannot differentiate between 150ml and 140ml, she could have randomly picked 150ml. Similarly, she randomly picked 140ml when offered 140ml and 130ml. But when all three were offered, she could differentiate between 150ml and 130ml and thus, picked 130ml. This inability to differentiate between the alternatives violates transitivity[2] and thus, other consistency conditions need to be explored.

Consider a different scenario. Suppose in a family there are five siblings of different ages. They might have been taught that whenever offered cake, they should leave the bigger pieces for younger siblings. So the eldest child always leaves the four biggest pieces on the plate and picks the fifth largest piece. The immediate younger sibling chooses the fourth largest piece and leaves three biggest pieces for the younger ones and so on. Here every sibling is again consistent in their choices although their choices fail to satisfy many of the standard rationalizability conditions. These examples perfectly depict how external references are important while rationalizing choice behaviour. As it is obvious, this kind of choice behaviour will violate previously stated axioms and will naturally be path dependent.

From the above examples, we can see that willingly giving up the top alternative, when available, may not always be an unreasonable move on the part of an individual. Such choice behaviours merit further investigation. Inspired by Sen's example, Baigent and Gaertner (1996) have provided characterisations for choice behaviour that picks the second largest element if there is a uniquely largest; otherwise, the largest elements are picked. Gaertner and Xu (1999a) provide characterisation conditions for median rationalizability. These papers have also hinted at further possible developments towards the k^{th} best.[3] The characterisation for choosing the second best has been given by Banerjee (2008a,b). Banerjee (2009b) provides a characterisation for the k^{th} best ordering ratio-

[2]Transitivity is the most common consistency requirement imposed on the choice function. However, it was challenged on the grounds of being too stringent (Luce (1956)) and was also thought to be one of the culprits behind Arrow's impossibility result. Plott (1973) argued that if path independence is the desired property of social choice, as emphasised by Arrow (1963), the stronger rationality conditions need not be imposed. Much has not been achieved by solely easing the consistency requirements but it has helped in immediately circumventing the negative result.

[3]See Gaertner and Xu (1997, 1999b) also.

nalization in full domain. Needless to say, the area of k^{th} best offers a lot to be explored. Continuing with the framework developed by Banerjee (2009b), in this paper we would be looking at a choice behaviour where k^{th} best elements are chosen when they exist and we provide necessary and sufficient conditions for the choice function to be rationalizable by a reflexive, connected and quasi-transitive k-rationalization in full domain. We devote the next section to understanding what is meant by k^{th} best elements and k-rationalizability where k is a positive integer.

2 k-Rationalizability

We will begin the discussion on k^{th} best elements by first understanding what is meant by second best elements. Suppose we have a set S that consists of some elements and we assume that second best elements exist in this set. Let B be the set of best elements of the set S. If we remove the elements of the set B from set S, we are left with the reduced set $S - B$. Now, we find best elements of this set $S - B$. The best elements of the set $S - B$ are the second best elements of the set S as they are the best elements of the set in the absence of the elements of set B.

2-Rationalizability: A choice function is second best rationalizable (or *2-rationalizable*) if we can generate a binary relation such that if second best elements of that relation are chosen, whenever available, same choices are obtained. Whenever second best elements are available, they should be the ones chosen. If second best elements are not available, the first best elements must be chosen.

On similar lines we can obtain the third best elements of the set by eliminating the first best and second best elements of the set and finding the best elements from the remaining set. A more general form of this kind of choice behaviour would be where k^{th} best elements are chosen, whenever they exist, where k is a positive integer ($k \geq 2$). We can arrive at k^{th} best elements by picking the best elements after similarly eliminating consecutive sets of best elements up till $(k-1)^{th}$ best elements. That is to say, we remove the first best, second best, ..., $(k-1)^{th}$ best elements and find the best elements from the remaining set to obtain the k^{th} best elements of the set.

k-Rationalizability: A choice function is k^{th} best rationalizable (or *k-rationalizable*) if we can generate a binary relation such that if k^{th} best elements of that relation are chosen, whenever available, same choices are obtained. If, however, k^{th} best elements are not available, $(k-1)^{th}$ best elements are chosen and if $(k-1)^{th}$ best elements are also not available, $(k-2)^{th}$ best are chosen and so on. Note here that if k^{th} best elements are not available, the chosen elements are the worst elements of the set.

Consider a choice function which is k-rationalizable. If a choice function is k-rationalizable then the choice set must contain k^{th} best elements, if they exist. If there exists a k^{th} best element, to obtain this element we must eliminate $k-1$ consecutive and distinct sets of best elements. Thus, we are interested in finding the number of distinct consecutive preference levels present in the set. Let there be a set that contains a j^{th} best element in it, where $j > 1$. If this element is j^{th} best, it is obvious that there must be a $(j-1)^{th}$ best element that is preferred to this j^{th} best element and for this $(j-1)^{th}$ best element, there must exist a $(j-2)^{th}$ best element that is preferred to it and so on. Thus, for the existence of a j^{th} best element, there needs to be a sequence of distinct j number of elements in the set such that the element that is placed first in this sequence is preferred to the second element, second element is preferred to the third, and so on and lastly you have the $(j-1)^{th}$ element preferred to the j^{th} element where the j^{th} element is the j^{th} best element. Since we began by saying that a j^{th} best element exists, we know that the length of such a sequence is j. If, however, a j^{th} best element does not exist in the set then there cannot be any such sequence of distinct j number of elements present in the set. If at all there is a sequence, it must be of a length less than j.

Order of the set: The length of the longest sequence of the type where the first element is preferred to the immediate next element and the last element being the j^{th} best element is called as the *order of the set* . It gives us the largest number of distinct consecutive preference levels present in the set.

There may be many such sequences in a set. If the length of the longest sequence in a set S is l then the order of the set S is l, where l is a positive integer. Then we know there is an l^{th} best element present in the set and there does not exist any $(l+1)^{th}$ best element in the set. If we are considering k-rationalizability in full domain, we can observe all pair-wise choices and in every such pair-wise choice we know that the second best element will be chosen whenever there exists one. Otherwise both elements are chosen. Hence, we can easily form a sequence of the kind described above and the order of the set can be effortlessly determined.

3 Notations and Definitions

In this section we introduce various notations and definitions that will be used in the subsequent sections. We denote the non-empty and finite set of alternatives by X. Ω denotes the set of all non-empty subsets of X, i.e. $\Omega = 2^X - \{\emptyset\}$. $\mathbb{N}$ denotes the set of positive integers.

Let λ be a non-empty subset of Ω, $\emptyset \neq \lambda \subseteq \Omega$, which denotes the *domain* of the choice function. A *choice function* C is defined as a mapping $C : \lambda \mapsto \Omega$ such that the *choice set* (denoted by $C(A)$ for any set A in the domain) is non-empty and a subset of A, i.e. $\emptyset \neq C(A) \subseteq A$. Sometimes choice sets are restricted to be singletons but we impose no such restriction. In this paper, we have assumed that the domain of the choice function is *full* that is, $\lambda = \Omega$.

Let R be a binary relation over the set S, $R \subseteq S \times S$. The symmetric and asymmetric parts of R denoted by $I(R)$ (or simply I) and $P(R)$ (or P) respectively as $(\forall x, y \in S)[xIy \leftrightarrow xRy \wedge yRx]$ and $(\forall x, y \in S)[xPy \leftrightarrow xRy \wedge \sim yRx]$. We say R is an *ordering* iff R is reflexive, connected and transitive.

3.1 k^{th} Best Elements and k-Rationalization

Best element: An element $x \in S$ is called a *best or first best element* of set S according to a binary relation R *iff* $(\forall y \in S)(xRy)$. These elements are also referred to as *greatest elements* according to a binary relation R. The set of best or R-greatest elements in a set S is denoted by $G(S, R)$.

Second best element: An element $x \in S - G(S, R)$ is called as the *second best element* in S according to R *iff* $(\forall y \in S - G(S, R))(xRy)$.

kth **best element:** More generally, we obtain the k^{th} *best* elements according to R in S by removing the first best, second best, ... , and $(k-1)^{th}$ best elements from the set and then picking out best elements from the reduced set.
We define,

$$\begin{aligned} G_1(S, R) &= G(S, R) && \text{and} \\ G_i(S, R) &= G[S - \bigcup_{j=1}^{i-1} G_j(S, R), R] && \text{where } i \geq 2. \end{aligned}$$

Following the above notations, a set of first best elements in S according to R is denoted by $G_1(S, R)$, a set of second best elements in S according to R is denoted by $G_2(S, R)$ and so on. A set of k^{th} best elements in S according to R is denoted by $G_k(S, R)$.

Definition 1. *A binary relation R is said to be a* ***k-rationalization*** (k^{th} *best rationalization*) *of a choice function C iff for all $S \in \lambda$*

$$
\begin{aligned}
C(S) &= G_k(S,R) && \textit{if } G_k(S,R) \neq \emptyset \\
&= G_j(S,R) && \textit{if } G_j(S,R) \neq \emptyset \wedge G_{j+1}(S,R) = \emptyset \\
& && \textit{where } 1 \leq j < k;\ j,k \in \mathbb{N}.
\end{aligned}
$$

3.2 Order of a Set

Let R^* be a binary relation over X defined as follows:

$$(\forall x, y \in X)[xR^*y \leftrightarrow (\{x,y\} \in \lambda \wedge y \in C(\{x,y\}))]$$

P^* and I^* respectively are the asymmetric and symmetric parts of R^*.

For any $x \in X$ we define,

$$P_x = \{y \in X \mid yP^*x\}.$$

P_x is the set of all elements in X that are preferred to x with respect to R^*.

Definition 2. *The* ***order of any set*** $S \in \lambda$ *is defined as,*

$$
\begin{aligned}
O_S &= 0 && \textit{iff } S = \emptyset \\
&= 1 && \textit{iff } (\forall x, y \in S)(xI^*y) \\
&= n && \textit{iff otherwise}
\end{aligned}
$$

where n is the largest value of $m \in \mathbb{N}$ such that there exist distinct z_1, z_2, $\ldots$, z_m in S and $(\forall i \in \{1, 2, \ldots, m-1\})(z_iP^*z_{i+1})$.

The order of the set gives us the largest number of consecutive preference levels present in the set S. If any set S is empty then it is intuitively obvious that the order of the set will be zero. If S is non-empty, all elements are indifferent to each other then this set is a set with order one. That is to say, that no element is preferred to another and hence there is only one preference level. If neither of the above two cases is satisfied then we look for the longest chain of distinct elements that are consecutively joined by P^*. The number of elements present in this longest chain gives us the order of the set, O_S.

4 Axioms and Result

In this section we would be looking at characterisation conditions for a choice function when it is k^{th} best rationalizable for some positive integer k where $k \geq 3$.[4] We assume that

[4]If $k = 1$, we are back to the case of best rationalizability that we discussed in the beginning. A characterisation for $k = 2$, that is, second best rationalizability is provided in Banerjee (2008a) and Sharma (2017).

the domain of the choice function is *full* ($C : \Omega \mapsto \Omega$). Characterising results are available for an ordering k-rationalization and a reflexive, connected and acyclic k-rationalization.[5] In this paper we will be looking at characterising conditions for a reflexive, connected and quasi-transitive k-rationalization in full domain.

Here, we introduce the following two axioms:

A. 1. $(\forall S \in \Omega)[O_S \geq k \rightarrow C(S) = \{x \in S \mid O_{P_x \cap S} = k - 1\}]$

A. 2. $(\forall S \in \Omega)[O_S < k \rightarrow C(S) = \{x \in S \mid O_{P_x \cap S} = O_S - 1\}]$

Theorem. *If $k \geq 3$ then there exists a reflexive, connected and quasi-transitive k-rationalization of C if and only if C satisfies A.1 and A.2.*

Suppose the choice function is k-rationalizable with $k \geq 3$. The chosen elements then must be the k^{th} best elements, whenever they exist. Let us consider the case where k^{th} best elements are chosen. In that case there must exist some preferred elements in the set. Evidently this set of preferred elements cannot have any k^{th} best element in it. There must be a sequence of k consecutively preferred elements of which the last and the least preferred element is the k^{th} best element. For a quasi-transitive k-rationalization, ignoring the last element (that is the k^{th} best element), the remaining $k - 1$ elements must be there in the set of preferred elements constituting $k - 1$ distinct and consecutive preference levels. Hence, the order of this set of preferred elements must be $k - 1$. Therefore, whenever the order of a set is j and $j \geq k$, we know that k^{th} best elements exist, then for every chosen element the order of the preferred set must be exactly equal to $k - 1$. This requirement has been formalised as axiom A.1.[6]

Next we see what happens when a k^{th} best element does not exist. In the absence of k^{th} best elements, we search for the $(k-1)^{th}$ best elements and if they are also not available, we look for the $(k - 2)^{th}$ best elements, and so on. In such a scenario, irrespective of which elements we end up choosing, they are essentially the worst of the set. Therefore, whenever the order of a set is $j < k$, the order of the set of preferred elements must be $j - 1$ thereby ensuring that the worst elements of the set are chosen. This is our axiom A.2. If the set contains only best elements then, by definition, order of the set is one. In that case, all elements must be chosen and there cannot be any unchosen or preferred

[5]See Banerjee (2009b).

[6]This axiom was introduced by Banerjee (2009b) which uses this and another axiom to characterise a choice function that is k-rationalizable by an ordering rationalization R^*. Banerjee (2009a) provides charactersing results for a reflexive, connected and acyclic k-rationalization in full domain.

elements. According to A.2, the order of the unchosen and preferred set of elements in this case is zero, that is this set must be empty. Therefore, the choice set is the set itself.

Together axioms A.1 and A.2 fully characterise a choice function that is k-rationalizable under full domain by a quasi-transitive k-rationalization.[7] Full domain ensures that this k-rationalization is *unique*, reflexive and connected. This k-rationalization is R^* which we defined in section 3.2. We begin the proof by showing that if A.1 and A.2 are satisfied then R^* is quasi-transitive. Next we show that if R^* is quasi-transitive then an element x is the m^{th} best element of some set S according to R^* *iff* the order of the preferred set of elements of x in S or $O_{P_x \cap S}$ is $m-1$, where $m \in \mathbb{N}$. Using the above, we show that whenever the order of any set in the domain of the choice function is less than some given k, A.2 ensures that the worst elements according to R^* are chosen and whenever the order of the set is at least k, A.1 ensures that the k^{th} best elements according to R^* are chosen. Thus, R^* k-rationalizes the choice function. To show the necessity part of the theorem, we assume that a reflexive, connected and quasi-transitive k-rationalization of the choice function exists which is the same as R^* in full domain. We know that if for some set S, $O_S \geq k$ then k^{th} best elements are chosen. From the quasi-transitivity of R^*, x is a k^{th} best element *iff* the order of the preferred set of x in S is $k-1$. Thus, A.1 is satisfied. Similarly, if $O_S = j < k$ then j^{th} best elements are chosen. From the quasi-transitivity of R^*, the order of the preferred set of the chosen element in S is $j-1$. Thus, A.2 is also satisfied.

5 Conclusion

In this paper we have argued that unlike what the mainstream literature on rationality of choice warrants, deviating from choosing one's first best alternative cannot be termed as unreasonable or senseless on the face of it. It is possible that these seemingly 'senseless' choices may prove to be perfectly well-reasoned if we were to add some context to our process of rationalizability. This body of work derives inspiration from Sen's argument of having an *external reference* where he argues that internal consistency conditions of choice cannot be applied in a context independent way. In order to see whether different parts of a choice function are consistent or not we need to specifically consider the context of that choice. Sen further argues that the conditions of internal consistency, however appealing, do not take us very far as an individual can be consistently "moronic" in her choices.[8] He illustrates by arguing that an individual who always chooses her worst alternative will have great consistency of behaviour but does it count as rational? He asserts that consistency should not be *a priori imposed*, rather it should *entail* from the

[7] Complete proof is provided in section 6.

[8] See Sen (2017), pp 302.

choice behaviour.

While Sen argues against internal consistency of choice, he also argues against the view of rationality as self-interest maximization.[9] This approach completely ignores the role of norms, social commitments, altruism or any need for cooperation. This narrow and limiting view of rationality has played a crucial role in the literature on rational choice and forms the basis of many economic theories. Fehr and Schmidt (2006) review and emphasise the importance of experimental evidence indicating that people are strongly motivated by *other-regarding preferences* and that altruism and fairness play important roles. They argue that if such preferences are overlooked, social scientists run the risk of providing incomplete or in some cases wrong explanations of the phenomena under study. A detailed discussion on such experimental evidence is beyond the scope of this paper, nonetheless, it provides an additional perspective to the debate. Following these arguments, we recognise that restricting rational choice to the first best is not a good idea. This paper looks at a case when choices have deviated from the first best and provides similar axiomatic framework for them. The framework advanced here is not rid of the problem of internal consistency posited by Sen and this forms the limitation of this work. Nonetheless, we argue that as long as economic theories rely on the conventional axiomatic structure, there is need to improve upon it to include more possibilities.

Including these results among the already existing ones, we now have necessary and sufficient conditions for choice functions under full domain to be k-rationalizable, for all $k \in \mathbb{N}$, by orderings as well as by binary relations that are quasi-transitive and acyclic. Further, similar characterisations for a choice function to be k-rationalizable with general domain remains largely unexplored. For $k = 1$, Houthakker's axiom is necessary and sufficient for a choice function to have an ordering k-rationalization. Banerjee (2008b) provides a characterisation result of an existential nature for choice functions with ordering 2-rationalization. Any further result in this context would be of great significance.

6 Appendix: Proofs

Lemma 1. *If $k \geq 3$ and A.1 and A.2 are satisfied then R^* is quasi-transitive.*

Proof: Let the choice function C satisfy A.1 and A.2. Suppose R^* is not quasi-transitive. Then there exist x, y, z in X such that xP^*y, yP^*z and $\sim xP^*z$.

Consider the set $T = \{x, y, z\} \in \Omega$.
$xP^*y \wedge yP^*z \rightarrow O_T = 3 \wedge P_y \cap T = \{x\}$
$\sim xP^*z \wedge xP^*y \rightarrow P_x \cap T = \emptyset \vee P_x \cap T = \{z\}$

[9] See Sen (2017), chapter A2.

$\sim xP^*z \wedge yP^*z \rightarrow P_z \cap T = \{y\}$
Hence, $O_{P_y \cap T} = O_{P_z \cap T} = 1$ and $O_{P_x \cap T} = 0$ or 1.

Suppose $k = 3$ and say $x \in C(T)$.
$\rightarrow O_{P_x \cap T} = k - 1 = 2$ [By A.1.]
This leads to a contradiction. Therefore, $x \notin C(T)$.
$y \in C(T) \rightarrow O_{P_y \cap T} = 2$ [By A.1.]
This leads to a contradiction. Therefore, $y \notin C(T)$.
Similarly, $z \in C(T)$ also violates A.1 as $O_{P_z \cap T} \neq k - 1$. Therefore, $C(T) = \emptyset$.
This leads to a contradiction. Hence, xP^*z and R^* is quasi-transitive.

Suppose $k > 3$ and say $x \in C(T)$.
$\rightarrow O_{P_x \cap T} = O_T - 1 = 2$ [By A.2.]
This leads to a contradiction. Therefore, $x \notin C(T)$.
$y \in C(T) \rightarrow O_{P_y \cap T} = 2$ [By A.2.]
This leads to a contradiction. Therefore, $y \notin C(T)$.
Similarly, $z \in C(T)$ also violates A.2 as $O_{P_z \cap T} \neq O_T - 1$. Therefore, $C(T) = \emptyset$.
This leads to a contradiction. Hence, xP^*z and R^* is quasi-transitive. This completes the proof. •

Lemma 2. *For any $S \in \Omega$, $O_S = j$ and R^* over S is acyclic if and only if $G_j(S, R^*) \neq \emptyset \wedge \bigcup_{i=1}^{j} G_i(S, R^*) = S$ where $j \in \mathbb{N}$.*

This lemma was introduced and proved by Banerjee (2009b). •

Lemma 3. *If R^* is quasi-transitive then $x \in G_m(S, R^*)$ if and only if $O_{P_x \cap S} = m - 1$.*

Proof: Let R^* be quasi-transitive.

Suppose $x \in G_m(S, R^*)$.
$\rightarrow x \notin G_{m-1}(S, R^*)$
$\rightarrow \big[\exists z_{m-1} \in S - \bigcup_{l=1}^{m-2} G_l(S, R^*)\big][z_{m-1}P^*x]$
$z_{m-1} \notin G_{m-2}(S, R^*) \rightarrow \big[\exists z_{m-2} \in S - \bigcup_{l=1}^{m-3} G_l(S, R^*)\big][z_{m-2}P^*z_{m-1}]$
Similarly, there exist $z_2, z_3, \ldots, z_{m-3} \in S$ such that $z_2P^*z_3 \wedge z_3P^*z_4 \wedge \cdots \wedge z_{m-3}P^*z_{m-2}$.
$z_2 \notin G_1(S, R^*) \rightarrow (\exists z_1 \in S)(z_1P^*z_2)$

Thus we have, $z_1 P^* z_2 \wedge z_2 P^* z_3 \wedge \cdots \wedge z_{m-2} P^* z_{m-1} \wedge z_{m-1} P^* x$.
Since R^* is quasi-transitive, $z_1, z_2 \ldots z_{m-1} \in P_x \cap S$ and therefore, $O_{P_x \cap S} = m - 1$.

We shall now show the converse.

Suppose $(\exists x \in S)[O_{P_x \cap S} = m - 1]$.
Let $P_x \cap S = S'$.
$S' \subset S \wedge x \in S \rightarrow O_S = j \geq m$
Let $O_S = j$. R^* is quasi-transitive, therefore, by lemma 2 we have $G_j(S, R^*) \neq \emptyset$ and $\bigcup\limits_{l=1}^{j} G_l(S, R^*) = S$. As $j \geq m$, we know $G_m(S, R^*) \neq \emptyset$. Thus, there exists $i \in \mathbb{N}$ such that $x \in G_i(S, R^*)$. Suppose $i \neq m$.

Say $i < m$.
$\rightarrow \big[\forall y \in S - \bigcup\limits_{l=1}^{i-1} G_l(S, R^*)\big][x R^* y]$
$\rightarrow \big[\forall y \in S - \bigcup\limits_{l=1}^{i-1} G_l(S, R^*)\big][y \notin P_x]$
$\rightarrow P_x \cap S \subseteq \bigcup\limits_{l=1}^{i-1} G_l(S, R^*)$
Let $T = \bigcup\limits_{l=1}^{i-1} G_l(S, R^*)$.
But $O_T = i - 1 < m - 1$ and $O_{P_x \cap S} = m - 1$.
This leads to a contradiction. Hence, $i \not< m$. (i)

Suppose $i > m$.
$x \in G_i(S, R^*) \rightarrow x \notin G_{i-1}(S, R^*)$
$\rightarrow \big[\exists z_{i-1} \in S - \bigcup\limits_{l=1}^{i-2} G_l(S, R^*)\big][z_{i-1} P^* x]$
$z_{i-1} \notin G_{i-2}(S, R^*) \rightarrow \big[\exists z_{i-2} \in S - \bigcup\limits_{l=1}^{i-3} G_l(S, R^*)\big][z_{i-2} P^* z_{i-1}]$
Similarly, there exist $z_2, z_3, \ldots, z_{i-3} \in S$ such that $z_2 P^* z_3 \wedge z_3 P^* z_4 \wedge \cdots \wedge z_{i-3} P^* z_{i-2}$.
$z_2 \notin G_1(S, R^*) \rightarrow (\exists z_1 \in S)(z_1 P^* z_2)$
We thus obtain, $z_1 P^* z_2 \wedge z_2 P^* z_3 \wedge \cdots \wedge z_{i-2} P^* z_{i-1} \wedge z_{i-1} P^* x$. Since R^* is quasi-transitive, $z_1, z_2, \ldots, z_{i-1} \in P_x \cap S$ and therefore, we have $O_{P_x \cap S} = i - 1 > m - 1$. This leads to a contradiction. Hence, $i \not> m$. (ii)

Thus, from (i) and (ii) it is clear that $i = m$ and $x \in G_m(S, R^*)$. This completes the proof. •

Theorem. *If $k \geq 3$ then there exists a reflexive, connected and quasi-transitive k-rationalization of C if and only if C satisfies A.1 and A.2.*

Proof: Let $k \geq 3$ and let A.1 and A.2 be satisfied by the choice function C. We have to show that R^* is a reflexive, connected and quasi-transitive k-rationalization of C. As the choice sets are non-empty and $\lambda = \Omega$, R^* is reflexive and connected. Since $k \geq 3$ and A.1 and A.2 hold, we know by lemma 1 that R^* is quasi-transitive. Hence, we are left to show that R^* is a k-rationalization of C.

Case 1: Let $G_j(S, R^*) \neq \emptyset \wedge \bigcup_{i=1}^{j} G_i(S, R^*) = S$ where $1 \leq j < k$.

From lemma 2, we know that $O_S = j < k$.
Say $x \in C(S)$.
$\rightarrow O_{P_x \cap S} = O_S - 1 = j - 1$ [By A.2.]
As R^* is quasi-transitive and $O_{P_x \cap S} = j - 1$, $x \in G_j(S, R^*)$. [Using lemma 3.]
Hence, we have $C(S) \subseteq G_j(S, R^*)$. (i)

Let $x \in G_j(S, R^*)$.
$\rightarrow O_{P_x \cap S} = j - 1 = O_S - 1$ [Using lemma 3.]
$\rightarrow x \in C(S)$ [By A.2.]
Hence, $G_j(S, R^*) \subseteq C(S)$. (ii)
Thus, using (i) and (ii) we have shown that $C(S) = G_j(S, R^*)$.

Case 2: Let $G_k(S, R^*) \neq \emptyset$.

Let $x \in G_k(S, R^*)$.
$\rightarrow O_{P_x \cap S} = k - 1$ [Using lemma 3.]
$\rightarrow x \in C(S)$ [By A.1.]
Hence, $G_k(S, R^*) \subseteq C(S)$. (iii)

Let $x \in C(S)$.
$\rightarrow O_{P_x \cap S} = k - 1$ [By A.1.]
As R^* is quasi-transitive and $O_{P_x \cap S} = k - 1$, $x \in G_k(S, R^*)$. [Using lemma 3.]
Hence, we have $C(S) \subseteq G_k(S, R^*)$. (iv)
Therefore, from (iii) and (iv) we have $C(S) = G_k(S, R^*)$.

Now we show the necessity part of the theorem.

Suppose that $k \geq 3$ and R is a reflexive, connected and quasi-transitive k-rationalization of the choice function C. As we have full domain, $R = R^*$. We shall now show that A.1

and A.2 hold.

Suppose, $O_S = j \geq k$. As R^* is quasi-transitive, using lemma 2, we know $G_j(S, R^*) \neq \emptyset$. Since $j \geq k$, thus $G_k(S, R^*) \neq \emptyset$.
Let $x \in C(S)$.
$\rightarrow x \in G_k(S, R^*)$ [As $C(S) = G_k(S, R^*)$.]
By lemma 3, $O_{P_x \cap S} = k - 1$. (1)

Now let's show the converse.
Let $(\exists x \in S)[O_{P_x \cap S} = k - 1]$.
As R^* is quasi-transitive, $x \in G_k(S, R^*)$. [By lemma 3.]
$\rightarrow x \in C(S)$ (2)

Using (1) and (2) we have established that $C(S) = \{x \in S \mid O_{P_x \cap S} = k - 1\}$ when $O_S \geq k$. Thus, A.1 holds.

Now suppose, $O_S = j < k$. As R^* is quasi-transitive, using lemma 2, $G_j(S, R^*) \neq \emptyset$. As R^* is a k-rationalization and $G_{j+1}(S, R^*) = \emptyset$, $C(S) = G_j(S, R^*)$.
Let $x \in C(S)$.
$\rightarrow x \in G_j(S, R^*)$
By lemma 3, $O_{P_x \cap S} = j - 1$. (3)

Now let's show the converse.
Let $(\exists x \in S)[O_{P_x \cap S} = j - 1]$.
As R^* is quasi-transitive, $x \in G_j(S, R^*)$. [By lemma 3.]
$\rightarrow x \in C(S)$ (4)

Using (3) and (4) we have shown that $C(S) = \{x \in S \mid O_{P_x \cap S} = O_S - 1\}$ when $O_S < k$. Thus, A.2 holds. Hence, the theorem is proved. $\blacksquare$

6.1 Independence of Axioms

Example 1. *$X = \{a, b, c, d\}$ and let $k \geq 3$.*

Consider the following choice function:

$C(\{a\}) = \{a\}$	$C(\{b\}) = \{b\}$	$C(\{c\}) = \{c\}$
$C(\{d\}) = \{d\}$	$C(\{a, b\}) = \{b\}$	$C(\{a, c\}) = \{c\}$
$C(\{a, d\}) = \{d\}$	$C(\{b, c\}) = \{b, c\}$	$C(\{b, d\}) = \{d\}$
$C(\{c, d\}) = \{d\}$	$C(\{a, b, c\}) = \{a, b, c\}$	$C(\{a, b, d\}) = \{d\}$
$C(\{a, c, d\}) = \{d\}$	$C(\{b, c, d\}) = \{b, c, d\}$	$C(\{a, b, c, d\}) = \{d\}$

From the above choice function we get the following R^*:

$$R^* = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(a,d),(b,c),(b,d),(c,b),(c,d)\}$$

Thus, $P_a = \emptyset$, $P_b = \{a\}$, $P_c = \{a\}$ and $P_d = \{a,b,c\}$.

Here, A.1 is satisfied but A.2 is violated. Consider the set $\{a,b,c\}$. We have $P_a \cap \{a,b,c\} = \emptyset$ and hence, $O_{P_a \cap \{a,b,c\}} = 0$. Thus, $C(\{a,b,c\}) = \{a,b,c\} \neq \{x \in \{a,b,c\} \mid O_{P_x \cap \{a,b,c\}} = O_{\{a,b,c\}} - 1 = 1\}$ and A.2 is violated for any $k \geq 3$. $\diamond$

Example 2. *$X = \{a,b,c,d\}$ and let $k = 4$.*

Consider the following choice function:

$$\begin{array}{lll}
C(\{a\}) = \{a\} & C(\{b\}) = \{b\} & C(\{c\}) = \{c\} \\
C(\{d\}) = \{d\} & C(\{a,b\}) = \{b\} & C(\{a,c\}) = \{c\} \\
C(\{a,d\}) = \{d\} & C(\{b,c\}) = \{c\} & C(\{b,d\}) = \{d\} \\
C(\{c,d\}) = \{d\} & C(\{a,b,c\}) = \{c\} & C(\{a,b,d\}) = \{d\} \\
C(\{a,c,d\}) = \{d\} & C(\{b,c,d\}) = \{d\} & C(\{a,b,c,d\}) = \{a,b,c,d\}
\end{array}$$

From the above choice function we get the following R^*:

$$R^* = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(a,d),(b,c),(b,d),(c,d)\}$$

Thus, $P_a = \emptyset$, $P_b = \{a\}$, $P_c = \{a,b\}$ and $P_d = \{a,b,c\}$.

Here, A.2 is satisfied but A.1 is violated. Consider the set $\{a,b,c,d\}$. We have $P_a \cap \{a,b,c,d\} = \emptyset$ and hence, $O_{P_a \cap \{a,b,c,d\}} = 0$. Thus, $C(\{a,b,c,d\}) = \{a,b,c,d\} \neq \{x \in \{a,b,c,d\} \mid O_{P_x \cap \{a,b,c,d\}} = k-1\}$ and A.1 is violated. $\diamond$

References

Arrow, K. J. (1959). Rational choice functions and orderings. *Economica*, 26(102):121–127.

Arrow, K. J. (1963). *Social Choice and Individual Values*. Wiley, New York.

Baigent, N. and Gaertner, W. (1996). Never choose the uniquely largest: A characterization. *Economic Theory*, 8:239–249.

Banerjee, T. (2008a). Characterization of a second-best rationalizable choice function. *Available at SSRN 1697452*.

Banerjee, T. (2008b). Second best element rationalizability of choice functions with general domain. *Available at SSRN 1447769.*

Banerjee, T. (2009a). *Freedom, Rationality and Well-Being.* PhD thesis, Jawaharlal Nehru University, New Delhi.

Banerjee, T. (2009b). Necessary and sufficient condition for a choice function to be k-rationalizable. *Available at SSRN 1335061.*

Fehr, E. and Schmidt, K. M. (2006). The economics of fairness, reciprocity and altruism–experimental evidence and new theories. *Handbook of the Economics of Giving, Altruism and Reciprocity*, 1:615–691.

Gaertner, W. and Xu, Y. (1997). Optimization and external reference; a comparison of three axiomatic systems. *Economics Letters*, 57:57–62.

Gaertner, W. and Xu, Y. (1999a). On rationalizability of choice functions: A characterization of the median. *Social Choice and Welfare*, 16:629–638.

Gaertner, W. and Xu, Y. (1999b). On the structure of choice under external references. *Economic Theory*, 14:609–620.

Luce, R. D. (1956). Semiorders and a theory of utility discrimination. *Econometrica*, 24(2):178–191.

Plott, C. R. (1973). Path independence, rationality, and social choice. *Econometrica*, 41(6):1075–1091.

Samuelson, P. A. (1938). A note on the pure theory of consumer's behaviour. *Economica*, 5(17):61–71.

Sen, A. K. (1993). Internal consistency of choice. *Econometrica*, 61(3):495–521.

Sen, A. K. (1997). Maximization and the act of choice. *Econometrica*, 65(4):745–779.

Sen, A. K. (2017). *Collective Choice and Social Welfare: Expanded Edition.* Penguin Books Limited.

Sharma, R. (2017). *Consistency Conditions for Rational Choice.* PhD thesis, Jawaharlal Nehru University, New Delhi.

Uzawa, H. (1956). Note on preference and axioms of choice. *Annals of the Institute of Statistical Mathematics*, 8(1):35–40.