Voting Paradoxes in Four Candidate Elections

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Abstract

We try to find out a methodology for identifying Borda Paradox and Condorcet Paradox in four candidate elections in the absence of information on preference profile of citizens, and try to apply the same to find out existence of such paradoxes in the 2004, 2009 and 2014 Indian Parliamentary elections.

JEL classifications: $H11; H50$
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1 Introduction

This paper is an attempt to build upon the study of (Gupta, 2012)in which the author investigates into a methodology of identifying Borda Paradox and Condorcet Paradox in the absence of information on preference profiles of citizens in a three-candidate election contest. This paper tries to extend the same methodology over four candidate elections.

2 Methodology

2.1 Voting in Four Candidate Elections

In a four candidate election of A, B and C, D, there could be three kinds of voters each of whom rank A, B, C or D as their most preferred candidate followed by three other options. This study crucially assumes that voters with A, B or C as their first preference have identical choices for second, third and fourth places. We also assume that α_A, α_B, α_C and α_D proportion of the population have A, B, C and D as their first, second, third and fourth choice respectively. Therefore $\alpha_A + \alpha_B + \alpha_C + \alpha_D = 1$ We assume that $\alpha_A > \alpha_B > \alpha_C > \alpha_D$ and therefore candidate A is the winner in a first past the post system. We also assume $\alpha_A < 1$, implying that A is not a Strong Condorcet Winner, and therefore the final outcome in a pairwise voting amongst candidates will be known only after we know the full preference profile of all voters. Further, tabulating the possible voter profiles for the four kinds of voters gives the following:

Table 1: Possible preference profiles among 4 candidates in first past the post elections

Profile			
A	B	C	D
*	*	*	*
*	*	*	*
*	*	*	*

Given that the second, third and the fourth place for those who voted for candidate A, (denoted by * in Figure 1), can be filled in three, two and one way, there are $3 \times 2 \times 1 = 6$ possibilities and therefore Table 1 throws up $6^4 = 1296$ possibilities or preference profile of citizens. For each of these preference profiles, we can evaluate the pairwise ranking of any two candidates, A & B, A & C, A & D, B & C, B & D and finally C & D. Once each of these preference profiles have been evaluated, we can come up with the overall preference profile, and there can be five possible outcomes: either A, B, C, or D emerges as the Condorcet Winner, or a Condorcet Paradox. In our case, since A is the winner in the majority election and if A emerges as the Condorcet Winner, we call it a situation of "Winner Possible Condorcet Winner (WPCW). If B, C or D emerges as the Condorcet Winner, but given that A is the actual winner in a multi-candidate contest, it is a situation of "Possible Borda Paradox (PBP)". If it is a situation of a Condorcet Paradox, and given that A is the actual winner, we term it as a situation of "Outcomes with no clear mandate (OWNCM)". It should be noted that in four candidate elections there are two possibilities, namely:

$$\text{a)}\alpha_A + \alpha_D > \alpha_B + \alpha_C$$
$$\text{b)}\alpha_A + \alpha_D < \alpha_B + \alpha_C$$

Each of these cases will result in 1,296 preference profile matrices among the four candidates. Again the crucial assumption holds that people with one of the candidates as their top preference hold identical relative rankings among the the remaining three choices in arriving at their second, third and fourth preferences for a given voter kind or preference profile.

In the three candidate situation, Gupta (2011), had listed the results for all the $2 \times 2 \times 2 = 8$ outcomes. With four candidates, it becomes more difficult to directly arrive at the number of possible outcomes in the final societal rankings, as counted in the three-candidates. Therefore, it becomes necessary to use some computer programme to automate the process of counting using fundamental principles of combinatorial mathematics to arrive at the list of possible permutations and the exhaustive list of preference profiles. In this study, the authors have relied on Microsoft Excel to derive the two-tier permutation model of 6-nary combinations. This approach is an evolved application of binary generation of possible cases (seen more often) and has been explained in Appendix 1. The final results arrived at from these computations on all possible outcomes given the ordered ranking among the four candidates (as $\alpha_A > \alpha_B > \alpha_C > \alpha_D$) are summarized below:

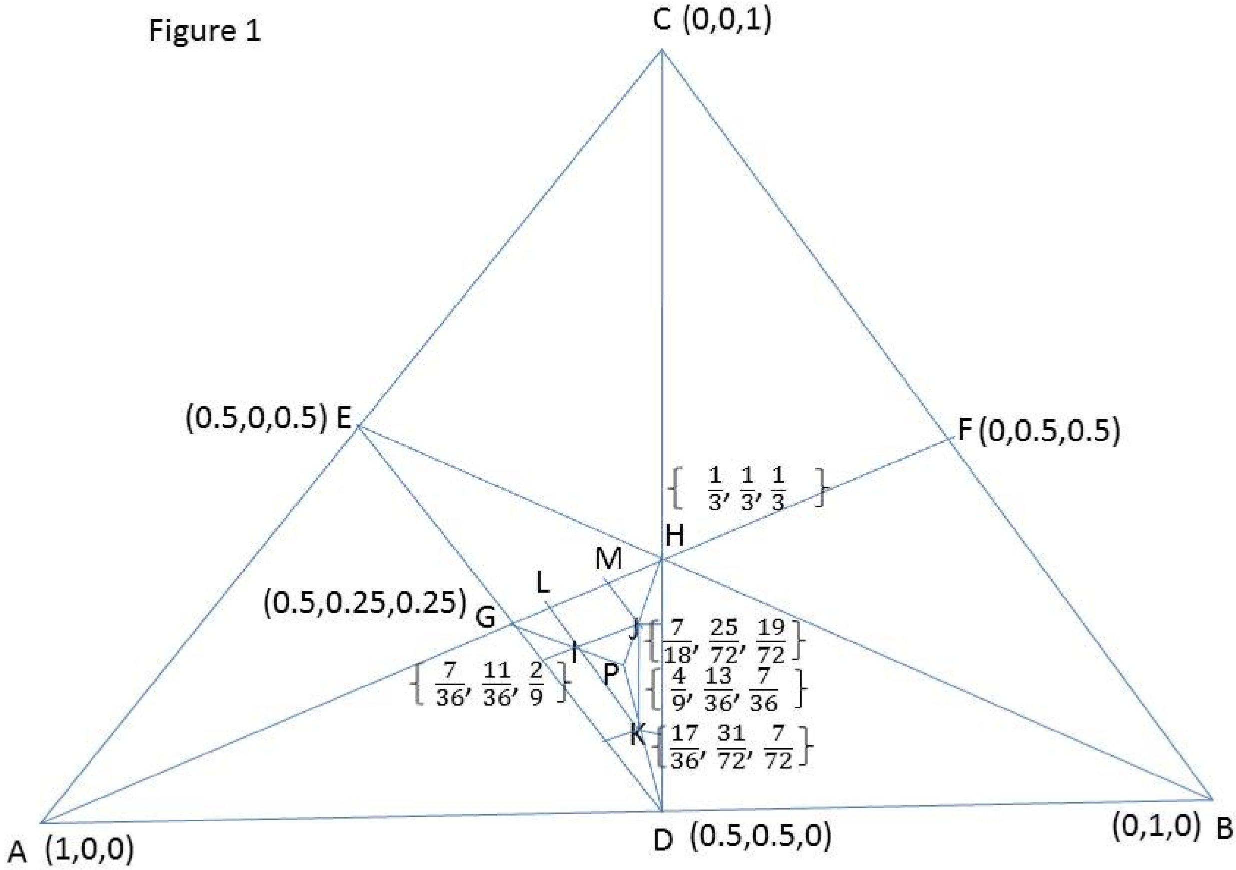
Table 2: Case a) $\alpha_A + \alpha_D > \alpha_B + \alpha_C$

A wins	B wins	C wins	D wins	Condorcet Paradox	Total
600	108	108	108	372	1296

Table 3: Case b) $\alpha_A + \alpha_D < \alpha_B + \alpha_C$

A wins	B wins	C wins	D wins	Condorcet Paradox	Total
288	288	288	48	384	1296

Figure 1



3 Geometric Representation of Vote Shares

Once the listing of the profiles is complete, the next step is to vote shares, where none of the candidates is a Strong Condorcet Winner, that is none of the candidates have more than a 50% vote share. In a three candidate election, the Saari Triangle which is an equilateral triangle can be used to represent the vote share of candidates in a three candidate election. Figure 1 represents the figure used in Gupta (2012). Let l_A, l_B, l_C, be the length of the perpendiculars dropped from any point inside the triangle to lines BC, CA, and AB. Then the vote shares of A, B, C for any point in the triangle is given by $\frac{l_A}{\sum_i l_i}$, $\frac{l_B}{\sum_i l_i}$, $\frac{l_C}{\sum_i l_i}$, where

$i \in \{A, B, C\}$. In any first past the post system with three candidates, it is known that $\alpha_A > \alpha_B > \alpha_C$ and if A is not a Strong Condorcet Winner, $\alpha_A < 0.5$. Therefore the relevant vote shares lie in the region GHD in Figure 1. Again if all citizens who voted for A, B and C as their first preference have identical second and third preference, then of the 8 possible profiles generated, in two situations each, candidates A, B and C are Condorcet Winners and in two situations we have a Condorcet Paradox. The area GHD was divided into 4 equal parts, and the region IJK, the most distant from the profiles of strong Condorcet Winners were labelled as OWNCM, that is outcomes with no clear mandate. The Coordinates IJK were evaluated, using principles of geometry for areas of triangles, which entailed elaborate calculations. If the same approach were to be used for four candidate elections, calculations would be a lot messy. However, using coordinate geometry section formula rules, calculations become very simple and identical results are obtained in the three candidate case, which can be carried forward in the four candidate situation.

Based on the preference profiles' outcomes in the three candidate elections, it is thus established that:

$\frac{Area\triangle IJK}{Area\triangle GDH} = \frac{1}{4} = (side)^2$

The above equality holds true for ratio between any two corresponding sides of similar triangles for which ratio of areas is known. Separately, it is also known that for two concentric figures with sides/planes absolutely parallel and aligned to the corresponding sides, the two figures share a common centroid. This property has crucial implications in this study.

Centroid of a $\triangle$ is given by:

$$\frac{\sum x_i}{3}$$

where, x_i denotes coordinate of the vertices of the triangle.

It should be noted that the H is the centroid of the $\triangle ABC$, and therefore its coordinates are (1/3, 1/3,1/3). D and E are the midpoints of AB and AC respectively, so its coordinates re (0.5, 0.5, 0) and (0.5, 0, 0.5). G is the midpoint of D and E, so its coordinates are (0.5, 0.25, 0.25). Given the knowledge of coordinates of G,D and H, to arrive at the coordinates of I,J and K using position vectors, we exploit the property of geometric coincidence of the centroids of similar triangles GDH and IJK mentioned above. The coordinates of I, J, K are as follows (see Appendix 2):

$$I = (17/36,\ 11/36,\ 2/9)$$

$$J = (7/18,\ 25/72,\ 19/72)$$

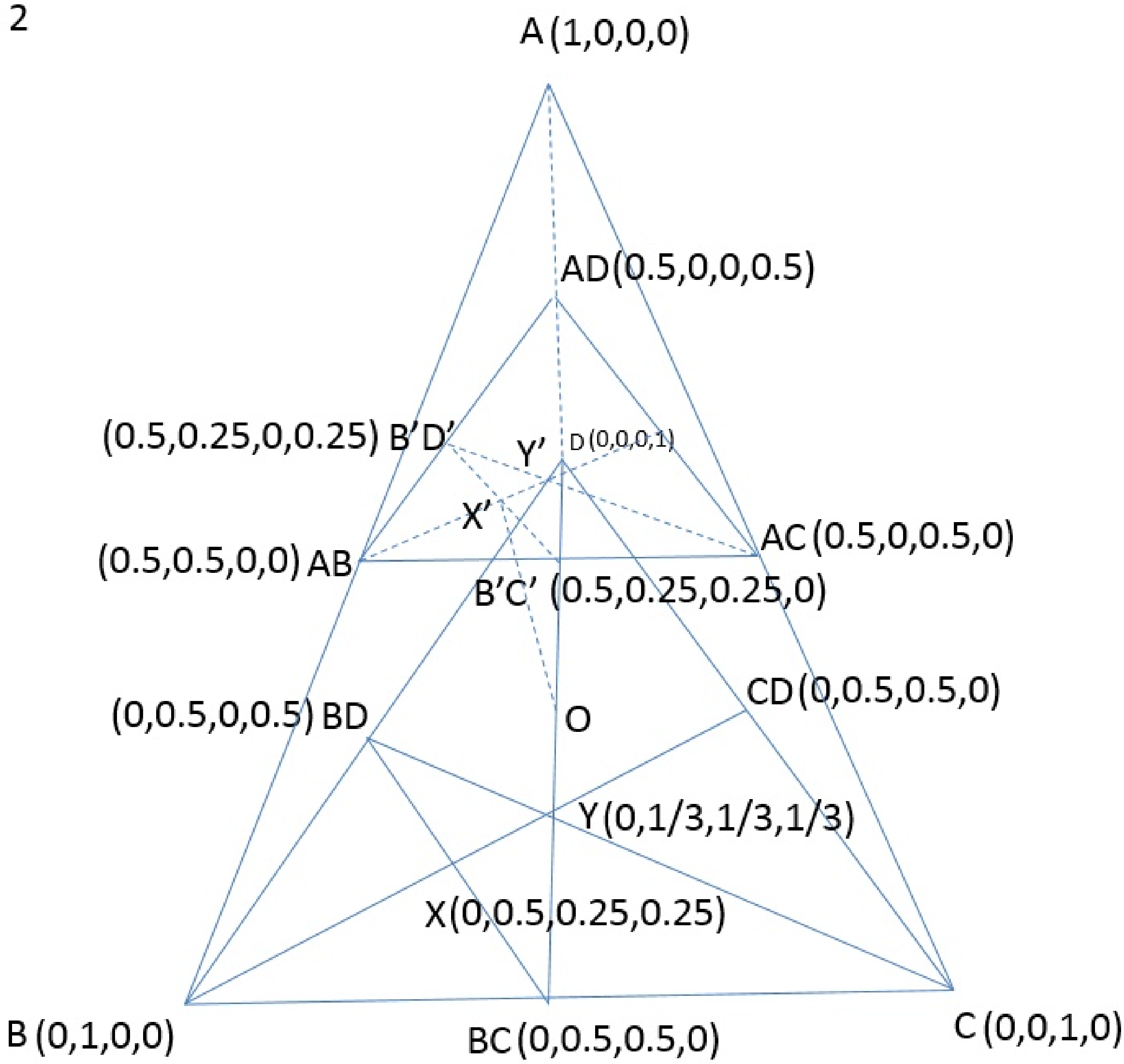
$$K = (17/36,\ 25/72,\ 7/72)$$

The coordinates of IJK evaluated exactly match that obtained by Gupta (2012) and therefore the same methodology can be used for the four candidate elections. Any point inside $\triangle IJK$ was termed outcomes with no clear mandate (OWNCM) and these profiles are possible cases of Condorcet Paradox. The highest and least vote shares of points I, J, and K, gives the lower and upper bounds for a point to lie in the area IJK. The lower and upper bounds any point to lie in IJK is given in the table below:

Table 4: Vote shares of candidates in situations of OWNCM

A's vote share lower limit	0.3889
A's vote share upper limit	0.4722
B's vote share lower limit	0.3056
B's vote share upper limit	0.4306
C's vote share lower limit	0.0972
C's vote share upper limit	0.2639

Figure 2



4 Geometry of Four Candidate Elections: Case a) $\alpha_A + \alpha_D > \alpha_B + \alpha_C$

We now try and apply Saari's triangle analogy in the context of four candidate elections with the help of a tetrahedron. Each point in the tetrahedron represents a unique vote share for four candidates A, B, C and D, with the sum of vote shares being equal to 1. Let l_A, l_B, l_C, l_D, be the length of the perpendiculars dropped from any point inside the tetrahedron to the planes $BCD\,ACD$, ABD and ABC. Then the vote shares of A, B, C and D for any point inside the tetrahedron is given by $\frac{l_A}{\sum_i l_i}$, $\frac{l_B}{\sum_i l_i}\frac{l_C}{\sum_i l_i}$, $\frac{l_C}{\sum_i l_i}$, where $i \in \{A,\, B,\, C, D\}$. Thus in Figure 2, the coordinates of the points A, B, C and D on the tetrahedron are given by (1,0,0,0), (0,1,0,0), (0,0,1,0) and (0,0,0,1).

Along the plane BCD of the tetrahedron, the vote share of A is zero. Let BC, BD and CD represent the midpoints of the points B and C, B and D, and C and D. Therefore the coordinates of BC, BD and CD are (0,0.5,0.5,0), (0,0.5,0,0.5) and (0,0,0.5,0.5) respectively. X is the midpoint of the line BC-BD and Y is the centroid of the triangle BCD. Both points X and Y lie on the plane BCD, so the vote share of A is zero for the points X and Y. Since point Y is equidistant from A, B and C, so the vote shares of A, B and C are the same at point Y. So, the coordinates of point Y is (0,1/3, 1/3, 1/3). Since X is the midpoint of BC-BD, the coordinates of X are (0, 1/2, 1/4, 1/4). Along the plane BCD, A's vote share is zero. As in the three candidate case, let α_i represent the vote share of candidate i at any point in the tetrahedron: $i \in \{A,\, B,\, C,\, D\}$.

In the triangle XYBC, $\alpha_A = 0$, $\alpha_B > \alpha_C > v_D$. We now look at the tetrahedron ABCD as a whole. The points AB, AC and AD are the midpoints of A and B, A and C, and A and D respectively. Their coordinates are therefore (0.5,0.5,0,0,), (0.5,0,0.5,0) and (0.5,0,0,0.5) respectively.

Therefore the plane AB-AC-AD represents all points in the tetrahedron where $\alpha_A = 0.5$.

The points X', Y' are projections of the points X, Y on the plane AB-AC-AD. Since Y' is the centroid of the triangle AB-AC-AD, its coordinates are (1/2, 1/6, 1/6, 1/6). B'C' and B'D' are the midpoints of AB and AC and AB and AD; therefore their coordinates are (0.5, 0.25, 0.25, 0) and (0.5, 0.25,0,0.25). X' is the mid point of B'C' and B'D' and its coordinates are (1/2, 1/4, 1/8, 1/8). Therefore in the triangle X'Y'-B'C', $\alpha_A = 0.5$ and $\alpha_B > \alpha_C > \alpha_D$. We now need to identify the region where $\alpha_A \leq 0.5$ and $\alpha_A > \alpha_B > \alpha_C > v_D$. If O be the centroid of the tetrahedron, its coordinates are (1/4, 1/4, 1/4, 1/4). Therefore the region where $alpha_A \leq 0.5$ and $\alpha_A > \alpha_B > \alpha_C > \alpha_D$ is X'Y'-B'C'-O. This region lies within the following coordinates:

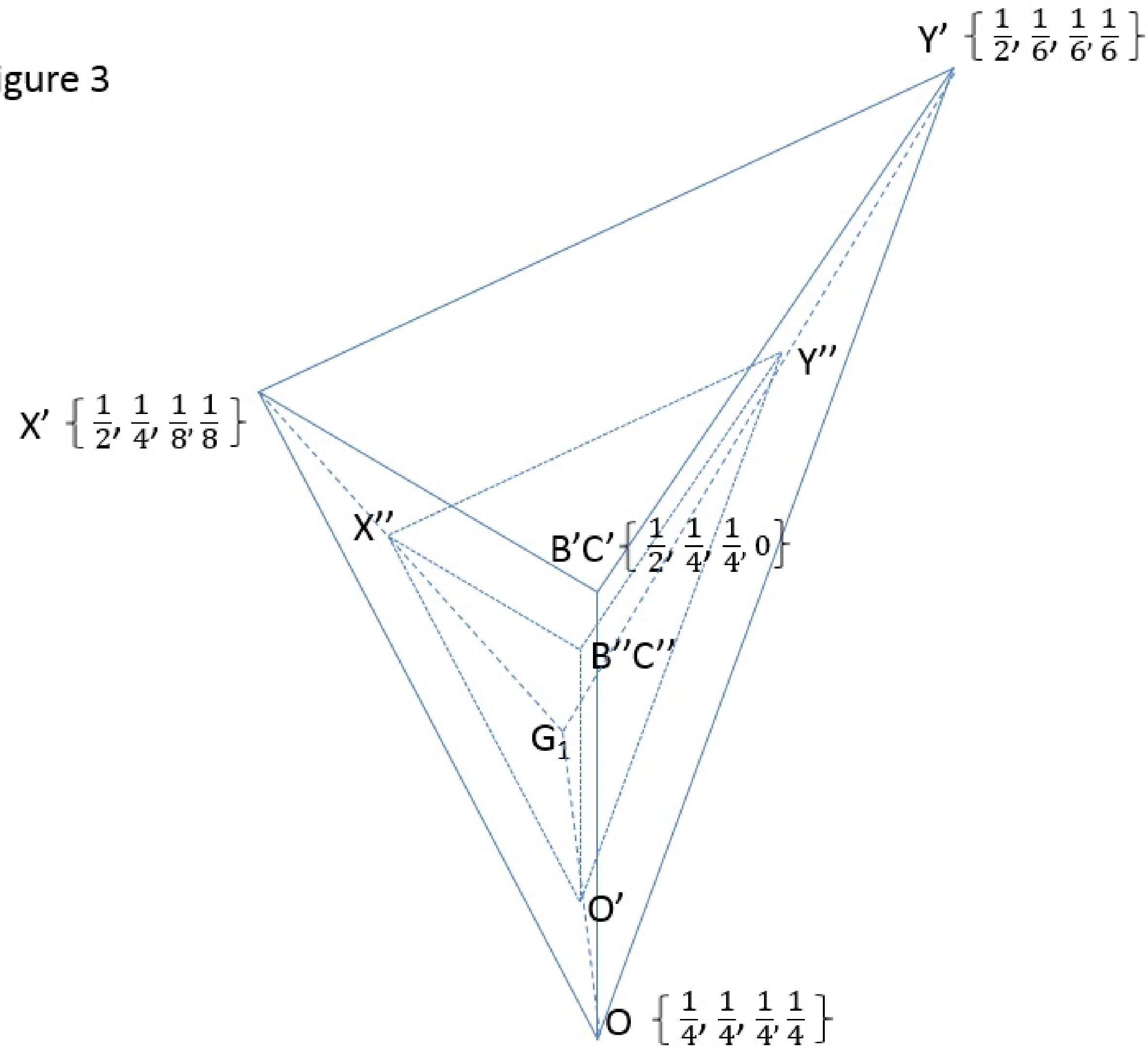
$X'Y' = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$

$B' = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$

$C' = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$

$O = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$

Figure 3



It should also be noted that the region enclosed by X’Y’-B’C’-O has the following characteristic, that is $\alpha_A + \alpha_D \geq \alpha_B + \alpha_C$. Figure 3 gives an expanded version of the tetrahedron X’Y’-B’C’-O. For these vote profiles, Table 2 gives us the chances of the winner being a Condorcet Winner, a Borda Paradox and a Condorcet Paradox being 600/1296, 324/1296 and 372/1296 respectively. Therefore, in the absence of second, third and fourth preference profile of voters, the region has to be appropriately partitioned, such that the ratio of the volumes matches the ratio of the probability of a winner being a Condorcet Winner, a Borda Paradox and a Condorcet Paradox. To do this, we fit a smaller tetrahedron X"Y"-B"C"-O', which is similar in volume to X'Y'-B'C'-O, both tetrahedrons sharing the same centroid G_1, the ratio of their volumes being 372/1296.

Centroid of a tetrahedron is given by:

$$\frac{\sum x_i}{4}$$

where, x_i denotes coordinate of the vertices of the tetrahedron.

Moreover, based on the preference profiles' outcomes (in Tables 5 and 6), it is thus established that:

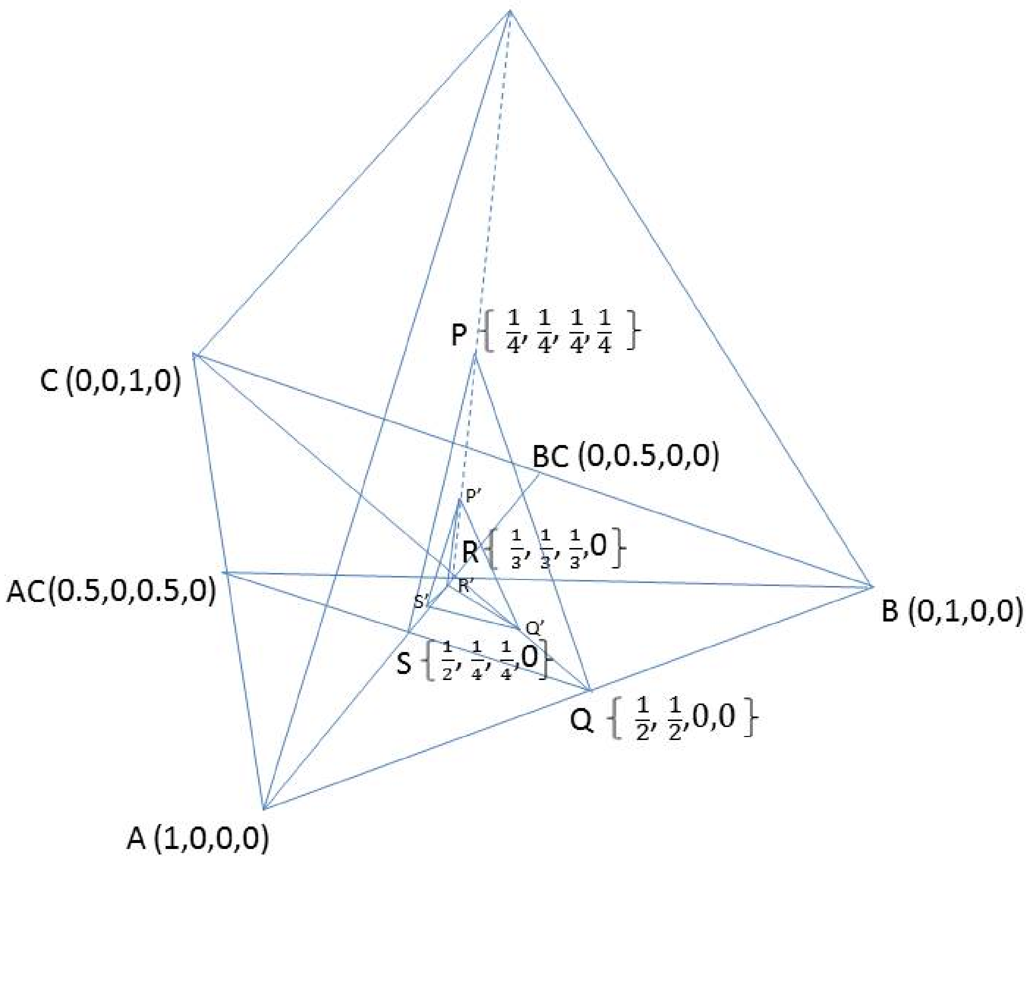
$$\frac{Volume\, of\, tetrahedron\, X''Y''Z''W''}{Volume\, of\, tetrahedron\, X'Y'Z'W'} = \frac{372}{1296} = (side)^3$$

Borrowing the application of section-formula used in the three-candidates case, it is similarly possible to use position vectors of the outer tetrahedron and centroid to determine the coordinates of the respective inner tetrahedrons which demarcate the space governing OWNCM. In figure 3, profiles lying inside the tetrahedron X"Y"B"C"O' correspond to OWNCM. In figure 3, coordinates of X", Y", B"C" and O' are evaluated. So any profile lying within the volume X"Y"B"C"O' will have their A, B, C and D candidate vote share lying between the maximum and minimum vote shares of A, B, C and D for points, X", Y", B"C" and O'. Therefore the vote shares of candidates in situations of OWNCM are given in the table below:

Table 5: Vote shares of candidates in situations of OWNCM
Case a) $\alpha_A + \alpha_D > \alpha_B + \alpha_C$

A's vote share lower limit	0.3138
A's vote share upper limit	0.4787
B's vote share lower limit	0.1879
B's vote share upper limit	0.2429
C's vote share lower limit	0.1498
C's vote share upper limit	0.2322
D's vote share lower limit	0.0461
D's vote share upper limit	0.2110

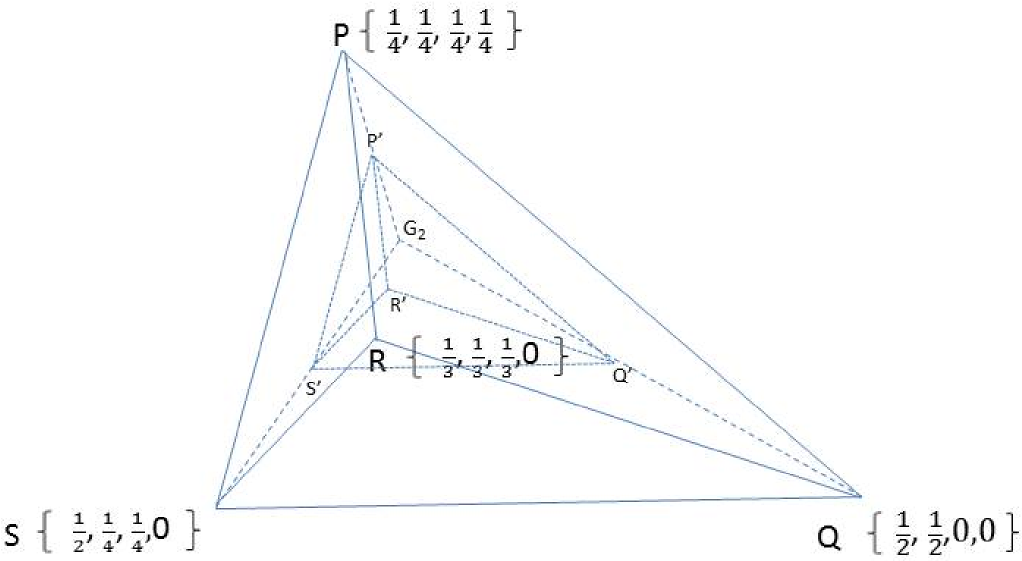
Figure 4



5 Geometry of Four Candidate Elections: Case b) $\alpha_A + \alpha_D < \alpha_B + \alpha_C$

Given that the identification of a four candidate election voting result in an equilateral tetrahedron has been explained in the last section, we now explain the process of identifying the region in the equilateral tetrahedron where $\alpha_A + \alpha_D < \alpha_B + \alpha_C$. To do this we turn the tetrahedron as in Figure 4, where point D is on the top and points A, B and C is on the base. Along any point on the plane ABC, $\alpha_D = 0$. Q is the midpoint of points A and B, and R is the centroid of the triangle ABC. S is the midpoint of AC and Q, in the $\triangle QRS$, $\alpha_D = 0$, $\alpha_A < 0.5$ and $\alpha_A > \alpha_B > \alpha_C$. P is the centroid of the tetrahedron ABCD. Since the maximum value of α_D can be 0.25, the region where $\alpha_A < 0.5$ and $\alpha_A > \alpha_B \, \alpha_C > \alpha_D$. Since for all the points P, Q, R and S, $\alpha_A + \alpha_D \leq \alpha_B + \alpha_C$, for any point inside the tetrahedron, $\alpha_A + \alpha_D < \alpha_B + \alpha_C$ will prevail with strict inequality.

Figure 5



In the profiles inside the tetrahedron PQRS, it is known from Table 3, there is a chance of 288/1296 of the winner being a Condorcet Winner, 384/1296 chance of a Condorcet Paradox and the rest 624/1296 chance of a Borda Paradox. So the region has to be partitioned such that regions closest to profiles of Strong Condorcet Winner are marked as profiles where a Winner Possible Condorcet Winner (WPCW). Profiles which are farthest from profiles of Strong Condorcet Winners, A, B, C,D are assumed to be possible cases of a Condorcet Paradox and labelled as OWNCM. To find out this region, a tetradedron similar in volume to PQRS is fitted inside the main tetrahedron PQRS, and all surfaces of the smaller tetrahedron are parallel to the surfaces of the bigger tetrahedron. In Figure 5, P'Q'R'S' is that smaller tetrahedron, and it is ensured that

$\frac{Volume\,of\,tetrahedron\,P'Q'R'S'}{Volume\,of\,tetrahdron\,PQRS} = \frac{384}{1296} = (side)^3$

Given that the coordinates of P, Q, R and S are known, and given that both PQRS and P'Q'R'S' have both the same centroid, the centroid G_2 can be estimated, and once that is done, the coordinates of P', Q', R'and S' can be evaluated (see Appendix 3). Once that is done, any point inside the tetrahedron P'Q'R'S', the vote shares of canididates A, B, C and D will lie between the highest and least vote shares of candidates A, B, C and D for the points P', Q', R' and S'. The results are given below.

Table 6: Vote shares of candidates in situations of OWNCM
Case b) $\alpha_A + \alpha_D < \alpha_B + \alpha_C$

A's vote share lower limit	0.2986
A's vote share upper limit	0.4653
B's vote share lower limit	0.2778
B's vote share upper limit	0.4444
C's vote share lower limit	0.0694
C's vote share upper limit	0.2917
D's vote share lower limit	0.0208
D's vote share upper limit	0.1875

6 Empirical Evidence

It will be of interest to see incidences of Borda Paradox and Condorcet Paradox in actual elections. We look for the same in Indian Parliamentary Elections of 2004, 2009, and 2014. As seen in the table below there are very few incidences of situations in 3 candidate or 4 candidate elections where the winner got less than 50% of the votes. There have just been one case of a possible Borda Paradox in 2009. Even situations of a Condorcet Paradox, termed as OWNCM, or outcomes with no clear mandates are not that common. So even in situations where the winner is not a Strong Condorcet Winner, it is most likely that the winner was a possible Condorcet Winner.

Table 7: Incidence of WPCW, PBP and OWNCM in Indian Parliamentary Elections

Three Candidate Elections				
Year	WPCW	PBP	OWNCM	Total
2004	3	0	2	5
2009	0	1	0	1
2014	0	0	0	0
Four Candidate Elections				
Year	WPCW	PBP	OWNCM	Total
2004	13	0	3	16
2009	2	0	2	4
2014	2	0	0	2

7 Conclusion

We have developed a methodology for evaluating chances of voting paradoxes in four candidate elections in the absence of information on the second, third and fourth preferences of voters. This has wide applications and can be used to analyze election results. The methodology is robust and has the promise of being extended to more than four candidate elections where geometry is not longer possible as a tool for analysis.

8 Appendix 1

Developing Preference Profile Matrices using Microsoft Excel

The exercise begins with the first step of listing out all possible preference profiles in a four candidate election, in which voters need to rank all the four contestants starting from the most preferred on top. In the previous study, this step of identifying all possible voter profiles was done manually, deriving upon analytical understanding. With only three candidates in the race, there are only 8 such preference profiles where one of the three candidates occupy the top ranking followed by permutations of the rest in the slots below.Subsequently, pair-wise comparisons evaluated among the three candidates in each voter profile to identify relative positioning as per Condorcet paradox requirements.

However, when the number of candidates in the contest increases to 4 and one each of A, B, C and D (say) occupy the first rank leaving open three slots for each voter kind, one encounters upon a large number of total possible preference profiles. More precisely, there are 6 kinds of rankings in which A is ranked on top; likewise 6 types in which B is ranked on top and so on. Thus, given that all of these possibilities are co-existent simultaneously for the four candidates among the entire set of voter preference profiles,there are 6x6x6x6 = 1,296 such profiles. In each profile, there are 4 kinds of voters who hold either of A, B, C or D on top with other candidates occupying the remaining three slots in their respective preference profiles. Now given the large number of preference profiles possible in the 4-candidate case study, it was considered prudent to generate the preference profiles using formulae built into Microsoft Excel (or any other programming software).

Given the premise of ranking of vote shares among A, B, C and D as $\alpha_A > \alpha_B > \alpha_C > \alpha_D$ there are two possible cases that are considered in this study. Fist, when the vote share of candidates A and D, is less than the vote share of candidates B and C taken together. Second, when the vote share of A and D is less than that of B and C. For each case, the same approach is followed in modelling out the required 1,296 permutation matrices (except for feeding in different boundary conditions for the relative vote share distribution among the 4 candidates). It is known that the first rank is respectively occupied by A, B, C and D for each kind of voter profiles. In each case, the remaining three rank slots can be filled in 3x2 = 6 possible ways across the four kinds of voters. Effectively, there are two tiers of nested permutations involved here. Firstly, the four kinds of voters can be permutated in 24 ways. Secondly, within each such permutation, each voter kind itself has 6 permutations possible given the 3 blank slots available for the second, third and fourth rankings. Using a simulation of binary arrangement of choices for the outer nest of permutations,this situation is analogous to a 4-nary arrangement of all possible combinations because there are 4 contestants. After running a Visual Basic code on Microsoft Excel to generate all possible numeric permutations in the 4-candidate, the authors replace the representative numerals with the electoral candidates where 1 represents A, 2 represents B and so on. Thereafter, for the second tier of nested permutations a similar representation of numeric possibilities is used. Here, numbers 0 to 5 indicate each of the 6 possible arrangements for each voter kind in a 6-nary generation of heximal arrangements, keeping one of A, B, C and D as first rank among the 4 voter kinds discussed above. In implementing this second nested permutation of 6x6x6x6 possibilities, the authors succeed in listing out the 1,296 possible preference profiles, with each unit being a 4x4 matrix - mapping the relative ranking among the four candidates (A,B,C and D) as indicated by the four kinds of voters.

Once the preference profile matrices are generated, the next step is to execute the six pair-wise comparisons among the four candidates for each voter kind to evaluate Condorcet Winners. At the same time, cases of 'No Condorcet Winners' are also simultaneously tabulated. It is fairly convenient to devise such a mechanism into Microsoft Excel using a nested-if approach. In a pairwise contest between any two candidates (say, A and B), the program looks for them in each voter ranking column to identify their relative rowwise positions. The candidate ranked higher wins the pair-wise contest (as if no other candidates are present like C or D) thereby aggregating the entire vote share into either A or B's portfolio. Likewise, the comparison is made across each of the four voter kinds to determine the winner of such pairwise contests. There are 6 such pair-wise contests among the 4 candidates for each of the 1,296 profile matrices. Whenever any one candidate wins at least half (3) of such 6 pair-wise battles, he/she is identified as the Condorcet Winner for that profile. If no such winner is identified, it gets grouped as a No-Condorcet-Winner (NCW) profile matrix. It is also worth noting that the limiting condition of segregating the two base cases of voter share combinations between A and D, versus B and C can be toggled among themselves in the Excel model and vote-share variables and manually entered in as input values which meet the required constraints.

9 Appendix 2

For $\triangle GDH$, coordinates of G, D, and H are (1/2, 1/4, 1/4), (1/2, 1,2, 0) and (1/3, 1/3, 1/3). Therefore, the centroid of the $\triangle GDH$, point P in Figure 1 will be given by

1/3{(1/2, 1/4, 1/4) + (1/2, 1,2, 0) + (1/3, 1/3, 1/3)}= (4/9, 13/36, 7/36).

Given that

$\frac{Area\triangle IJK}{Area\triangle GDH} = \frac{1}{4} = (side)^2$

Since P is the centriod of both $\triangle GDH$ and $\triangle IJK$,

$$\frac{PI}{PG} = \frac{PJ}{PH} = \frac{PK}{PD} = \frac{1}{2}$$

Therefore I, J, K are the mid-points of points P and G, P and H and finally P and D respectively. The coordinates of I, J, K will be:

$$I = \frac{1}{2}\{(4/9, 13/36, 7/36) + (1/2, 1/4, 1/4)\} = (17/36,\ 11/36,\ 2/9)$$

$$J = \frac{1}{2}\{(4/9, 13/36, 7/36) + (1/3, 1/3, 1/3)\} = (7/18,\ 25/72,\ 19/72)$$

$$K = \frac{1}{2}\{(4/9, 13/36, 7/36) + (1/2, 1/2, 0)\} = (17/36,\ 25/72,\ 7/72)$$

10 Appendix 3

Case a) $\alpha_A + \alpha_D > \alpha_B + \alpha_C$.

In Figure 3, coordinates of X', Y', B'C' and O are as follows:

$X' = \{1/2,\ 1/4,\ 1/8,\ 1/8\}$

$Y' = \{1/2,\ 1/6,\ 1/6,\ 1/6\}$

$B'C' = \{1/2,\ 1/4,\ 1/4,\ 1/4\}$

$O = \{1/4,\ 1/4,\ 1/4,\ 1/4\}$

Therefore coordinates of the centriod G' is as follows:

$$\begin{aligned} G_1 &= \frac{1}{4}[\{1/2,\ 1/4,\ 1/8,\ 1/8\} + \{1/2,\ 1/6,\ 1/6,\ 1/6\} \\ &+ \{1/2,\ 1/4,\ 1/4,\ 1/4\} + \{1/4,\ 1/4,\ 1/4,\ 1/4\}] \\ &= \{7/16,\ 11/48,\ 19/96,\ 13/96\} \end{aligned} \tag{1}$$

Given that $\frac{Volume\, of\, tetrahedron\, X''Y''Z''W''}{Volume\, of\, tetrahedron\, X'Y'Z'W'} = \frac{372}{1296} = (side)^3$

it implies that

$$\frac{G_1X''}{G_1X'} = \frac{G_1Y''}{G_1Y'} = \frac{G_1B''C''}{G_1B'C'} = \frac{G_1O''}{G_1O'} = (\frac{372}{1296})^{\frac{1}{3}} = 0.659649$$

Therefore the coordinates X", Y", B"C" and O' are evaluated as

$X'' = (0.4787,\ 0.2429,\ 0.1498,\ 0.1285)$

$Y'' = (0.4787,\ 0.1879,\ 0.1773,\ 0.1560)$

$B''C'' = (0.4787, 0.2429, 0.2323, 0.0461)$

$O' = (0.3138, 0.2429, 0.2323, 0.2110)$

Case b) $\alpha_A + \alpha_D < \alpha_B + \alpha_C$.

In Figure 5, coordinates of P, Q, R and S are as follows:

$P = \{1/4, 1/4, 1/4, 1/4\}$

$Q = \{1/2, 1/2, 0, 0\}$

$R = \{1/3, 1/3, 1/3, 0\}$

$S = \{1/2, 1/4, 1/4, 0\}$

Therefore coordinates of the centriod G_2 is as follows:

$$\begin{aligned} G_2 &= \frac{1}{4}[\{1/4, 1/4, 1/4, 1/4\} + \{1/2, 1/2, 0, 0\} \\ &+ \{1/3, 1/3, 1/3, 0\} + \{1/2, 1/4, 1/4, 0\}] \\ &= \{19/48, 1/3, 5/24, 1/16\} \end{aligned} \tag{2}$$

Given that $\frac{Volume\, of\, tetrahedron\, P'Q'R'S'}{Volume\, of\, tetrahdron\, PQRS} = \frac{384}{1296} = (side)^3$

it implies that

$$\frac{G_2P'}{G_2P} = \frac{G_2Q'}{G_2Q} = \frac{G_2R'}{G_2R} = \frac{G_2S'}{G_2S} = (\frac{384}{1296})^{\frac{1}{3}} = \frac{2}{3}$$

Therefore the coordinates X", Y", B"C" and O' are evaluated as

$P' = \{\frac{43}{144}, \frac{10}{36}, \frac{17}{72}, \frac{3}{16}\}$

$Q' = \{\frac{67}{144}, \frac{16}{36}, \frac{5}{72}, \frac{1}{48}\}$

$R' = \{\frac{17}{48}, \frac{1}{3}, \frac{7}{24}, \frac{1}{48}\}$

$S' = \{\frac{67}{144}, \frac{10}{36}, \frac{17}{72}, \frac{1}{48}\}$

11 Appendix 4

In Table 8, we report the vote shares in the 5 electoral districts, from the 2004, Indian Parliamentary Elections with three candidate elections with no Strong Condorcet Winner, and only two electoral districts, namely Bijapur and Nowrangpur can be identified as OWNCM.

Table 8: Vote shares of candidates with most and least votes in the five electoral districts with no Strong Condorcet Winner with Three Candidate Elections: Indian Parliamentary Elections 2004

No.	Electoral District	state	share first	share third	Classification
1	Anakapalli	Andhra Pradesh	0.4928	0.0341	WPCW
2	<u>Bijapur</u>	Karnataka	<u>0.4367</u>	<u>0.1741</u>	OWNCM
3	<u>Nowrangpur</u>	Orissa	<u>0.4611</u>	<u>0.1097</u>	OWNCM
4	Sambalpur	Orissa	0.4818	0.0546	WPCW
5	Daman and Diu	Daman and Diu	0.4951	0.0207	WPCW
Note: The underlined electoral districts are situations with OWNCM					

Likewise, in Indian Lok Sabha elections of 2009 there were two constituencies with 3-candidates' contests. However, none of them resulted in the OWNCM situation. Instead, we further evaluate the data extending our methodology to predict if the constituencies represent a scenario of Winner Possible Condorcet Winner (WPCW) and Possible Borda Paradox (PBP) depending on the relative vote shares of the three candidates versus corresponding limits. In our understanding, the region represented by GIKD in Figure 4 indicates those instances where the Winner is a probable Condorcet Winner. However, if a particular instance of non-plurality winner nether fits into the OWNCM nor the WPCW classification, then it possibly indicates PBP scenario. On this basis, results and classification of constituencies from Lok Sabha 2009 and Lok Sabha 2014 elections for 3-candidates' constituencies are summarized in Table 13 below:

In 2014, there was only one Lok Sabha constituency of three candidates and it had a plurality winner with a vote share of approximately 0.5224.

Table 9: Vote shares of candidates with most and least votes in the electoral district with no Strong Condorcet Winner with Three Candidate Elections: Indian Parliamentary Elections 2009

No.	Electoral District	state	share first	share third	Classification
1	Kokrajhar	Assam	0.4880	0.2116	PBP

Similarly, the study was replicated for the case of 4-candidates' constituencies for the Lok Sabha elections held in India during 2004, 2009 and 2014. In this case, the classification followed a two-step criteria: for both cases of relative votes' share between A and B versus C and D. Drawing from the analogy of the region represented by GIKD in the 3-candidates' Saari triangle, we identify the relevant volume region within the tetrahedron that suitably captures the "Winner Possible Condorcet Winner" (WPCW)and "Possible Borda Paradox" (PBP) instances respectively.

The findings are summarized in the subsequent tables:

Table 10: Vote shares of candidates with most and least votes in the electoral district with no Strong Condorcet Winner with Four Candidate Elections: Indian Parliamentary Elections 2004

No.	Electoral District	state	share first	share fourth	Classification
1	Parvathipuram	Andhra Pradesh	0.4869	0.0164	WPCW
2	Hindupur	Andhra Pradesh	0.4835	0.0156	WPCW
3	Adilabad	Andhra Pradesh	0.4997	0.0219	WPCW
4	Mehsana	Gujarat	0.4884	0.0147	WPCW
5	Dohad	Gujarat	0.4406	0.0315	OWNCM
6	Raichur	Karnataka	0.3508	0.0581	OWNCM
7	Mangalore	Karnataka	0.4861	0.0198	WPCW
8	Udipi	Karnataka	0.4737	0.0314	WPCW
9	Dhule	Maharashtra	0.4625	0.0398	WPCW
10	Kolhapur	Maharashtra	0.4942	0.0135	WPCW
11	Mayurbhanj	Orissa	0.3743	0.0571	OWNCM
12	Berhampur	Orissa	0.4948	0.0348	WPCW
13	Koraput	Orissa	0.4550	0.0657	WPCW
14	Kalahandi	Orissa	0.4735	0.0474	WPCW
15	Jalore	Rajasthan	0.4898	0.0373	WPCW
16	Lakshadweep	Lakshadweep	0.4902	0.0072	WPCW

References

Gupta S. (2012), "Looking for Voting Paradoxes in Indian Elections", Quality & Quantity, Volume 46(3), page 949-958.

Table 11: Vote shares of candidates with most and least votes in the electoral district with no Strong Condorcet Winner with Four Candidate Elections: Indian Parliamentary Elections 2009

No.	Electoral District	state	share first	share fourth	Classification
1	Arunachal West	Arunachal Pradesh	0.4916	0.0104	WPCW
2	Chhota Udaipur	Gujarat	0.4620	0.0538	WPCW
3	Tura	Meghalaya	0.4514	0.0321	OWNCM
4	Nabarangpur	Orissa	0.3893	0.0613	OWNCM

Table 12: Vote shares of candidates with most and least votes in the electoral district with no Strong Condorcet Winner with Four Candidate Elections: Indian Parliamentary Elections 2014

No.	Electoral District	state	share first	share fourth	Classification
1	Arunachal East	Arunachal Pradesh	0.4533	0.0172	WPCW
2	Mizoram	Mizoram	0.4859	0.0150	WPCW