Strategy-Proof Random Social Choice Rules with Behavioral Agents[1]

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Extended Abstract

A celebrated result in random mechanism design theory is Gibbard (1977). It characterizes the structure of random strategy-proof social choice functions in voting environments. The main result shows that such social choice functions that satisfy an additional (mild) efficiency condition must be random dictatorships. The result rests on expected utility (or stochastic dominance) comparisons of lotteries. Much of the subsequent literature on this issue uses this criterion. However there is strong empirical evidence that shows that decision makers are not typically expected utility maximizers. In this paper we examine the same question with behavioral agents who have either "extreme liking" (Top-Lex) or "extreme disliking" (Bottom-Lex) preferences over lotteries. These may be regarded as expressing extreme risk loving and extreme risk aversion behaviors respectively.

We completely characterize efficient and strategy proof rules for Bottom-Lex rules and show that Gibbard's random dictatorship result continues to hold. However this is not true for Top-Lex preferences. We have shown that under Top-Lex preference any rule which is efficient and strategy proof has to be Top-Support only i.e. it gives positive probability only to the alternatives which are on the top of some agent's ordering. Moreover, the support of the lottery (alternatives which gets the positive probability) remain same if tops are same, but the magnitude of probabilities can (potentially) change with a change in any ordering. For 2 agents, we have completely characterized efficient and strategic proof rules. Also, we provide a class of rules, Top-Weight-Quota-Priority rules, satisfying these two conditions that depends on more than best alternatives, in-spite of the fact that such alternatives are the highest-valued under (Top) lexicographic preference, interestingly this shows that these rules are ***not*** *Top-only rules. A much richer class of random social choice functions are therefore strategy-proof with Top-Lex preferences.*

KeyWords: Random Social Choice Function, Strategy Proof, Top and Bottom Lexicographic Preferences, Tops Only Rules, Random Dictatorship, Behavioral Agents, TWQP rule.

JEL Classification: D71, D72

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1 Introduction

In the recent world political incidents, like result of U.S. election, Brexit event and strong religious sentiments on issues like the Hijab controversy, Cow Vigilante, Triple Talaq etc., it was observed that there are sections of the society which have very strong views in favour as well as against the agenda in question. But the true spirit of democracy lies in the freedom of expression, no matter how extreme the views maybe. To put it more eloquently, we quote an editorial article titled 'Democracy thrives on free speech' in *The Age* which reads

> *"The common oil of great and resilient democracies is the free flow of debate, the proposition and contest of ideas, both good and bad, from all and by all. The right for any person to express their ideas and their opinions freely is an abiding universal human right, one that has been enshrined within the International Covenant on Civil and Political Rights and made glorious within the constitutional documents of great modern states..."*

The above para captures the importance of listening to every voice in the society, no matter how different it is. So while dealing with any collective decision problem, which has wide applications in political-economy environments, we need to consider all opinions and behavior, their biases and extremes. In the standard social choice theory framework, the agents are assumed to be expected utility maximizers, but in real life usually it's not the case. There are many empirical evidences which go against expected utility maximization (see [1], [2], [8], [11] and [12]). Humans have peculiar behavior dimension to their decision making process and it cannot be simplified by the abstraction of expected utility maximization. This motivates us to introduce the extreme behavior in a collective decision problem and analyse its ramifications. To capture such extreme behavior the lexicographic preferences are best suited and lotteries are used to bring fairness property in the allocation of different alternatives. Therefore, this paper will analyse what happens when agents are behavioral and have lexicographic ordering in the random social choice framework.

The lexicographic ordering over lotteries on alternatives can be of two types. The extreme liking behavior is captured by Top-Lex (T-Lex) preferences and extreme disliking by Bottom-Lex (B-Lex) preferences. Out of two lotteries an agent with T-Lex preference will choose the lottery which has higher probability of the top alternative (irrespective of the probabilities to other alternatives). But, if probabilities are the same for top rank, then he will choose the one which gives higher probability to second rank alternative (irrespective of the probabilities to other alternatives). The same will go till the last rank. So, as the name goes in the T-Lex preference agents care only for the top ranked alternatives. Analogously, in B-Lex preferences agents want to minimize the probability of the worst alternative. Out of two lotteries an agent with B-Lex preference will choose the lottery which has lower probability of the worst (last ranked) alternative (irrespective of the probabilities to other alternatives). The economic interpretation of T-Lex and B-Lex are infinitely risk loving and infinitely risk averse individuals. They exactly capture the extreme behavioral parameter we want to capture in our model. The rest of the paper is organized as follows. section 2 highlights the contribution of the paper. Section 3 and 4 provide the literature review and preliminaries. Section 5 has main theorems. Last three sections give conclusion, references and appendix with detailed proofs.

2 Our Contribution

To the best of our knowledge this is the first attempt to characterize the efficient and strategy proof rules with lexicographic preferences. There are two main findings of this paper. First, if agents have B-Lex preferences then we completely characterize the efficient and strategy proof rules and show that Gibbard's Random Dictatorship result continues to hold. The other findings are on T-Lex. First we have shown that under Top-Lex preference any rule which is efficient and strategy proof has to be Top-Support only i.e. it gives positive probability to only alternatives which are on the top of some agent's ordering. Moreover, the alternatives which gets the positive probabilities remain same if tops are same, but the quantum of probabilities can (potentially) change with a change in any ordering. It means that for T-Lex preferences Gibbard's result does not hold, and we have a much larger class of rules are now possible, this is kind of possibility result. For 2 agents, we have completely characterized efficient and strategic proof rules. But for more than two agents it is difficult to exactly characterize the entire class of rules so we have provided a class of rules, Top-Weight-Quota-Priority Rule (TWQP rule), which is fairly general, it provides probability only to the top alternatives of agents and continue to do so if tops remain same, but exact probability an alternative gets, it depends on the full preference ordering of all agents. It includes random dictatorship as a very case. Because, the rule does not depend just on the top, so we have found a rule which is not a Tops-Only rule but falls under category of Top-Support rules. This rule is interesting because Top-support rules are rare in the literature and the fact that it is true in Top-Lex where tops are highly valued. We will explain the rule in detail later.

3 Literature Review

The seminal work in Deterministic Strategic Social Choice Theory is from Gibbard (1973), Satherwette (1975). Then, Gibbard (1997) provides the characterization in Random environment. Then, there are work on finding the dictatorial domains in deterministic and random social choice theory. Here the question is to find the structure of the domain for which strategy proofness along with some desirable properties like efficiency or unanimity give rise the dictatorial rule (see [3], [4]&[5]. But all such work is confined to stochastic dominance. The literature on lexicographic is limited and the closest paper to our work is by Cho (2016) where he provides the equivalence of different conditions of strategy-proof in T-Lex and B-Lex preferences. Cho(2012) is a good reference to understand properties of different lottery extensions.

4 Preliminaries

Let A be a finite set of $m \geq 3$ possible alternatives. The generic elements of this set will be shown by a, b, x, y. $N = \{1, 2, ..., n\}$ is the finite set of n individuals. The typical element is shown by $1, 2, i, j$. $\mathbb{P}$ is the set of all linear orders over A i.e. set of complete, transitive and anti-symmetric relations over A. We will write $P, \hat{P}, \bar{P}$ with and without subscript i to write its generic element. Some time we will write P_i^{ab} to highlight the fact that the first and second most preferred alternative are a and b respectively according to P_i. For any P_i

and a natural number k, $k(P_i)$ gives the rank k alternative in the ordering i.e. $k(P_i) = a$ iff $|\{b \in A \mid bP_i a\}| = k - 1$. Also $B(a, P_i) = \{b \in A \mid b = a \text{ } or \text{ } bP_i a\}$ i.e. $B(a, P_i)$ is set of (weakly) better alternatives than a in the ordering P_i.

$P^N = (P_1, P_2, ..., P_n) = (P_i, P_{-i}) = (P_i, P_j, P_{-ij})$ shows a preference profile, a preference ordering for each individual, i.e. $P^N \in \mathbb{P}^n$. $\triangle A$ shows set of all probability distributions over A i.e. $\triangle A = \{L \in \mathbb{R}^m_+ | \sum_{i=1}^m L_i = 1\}$. It is a simplex over m points. A typical element of $\triangle A$ is called lottery and if shown by L then L_a is the probability of alternative a assigned by the lottery L.

A random social choice function (RSCF) F is a mapping from a preference profile to a lottery i.e. $F : \mathbb{P} \to \triangle A$. We will write $F(P^N)$ for the lottery assigned by RSCF F for profile P^N and $F_a(P^N)$ for the probability of alternative a. A RSCF F is ex-post efficient if there is any Pareto dominated alternative for any preference profile P^N then it should be given zero probability at that profile i.e. if $\exists a \in A$ for which all $i \in N$ there is $b \in A$ such that $bP_i a$ then $F_a(P^N) = 0$. A RSCF $F : \mathbb{P}^n \to \triangle A$ is a random dictatorship if there exists weights $\beta_1, ..., \beta_n \in [0, 1]$ with $\sum\limits_{i=1}^{n} \beta_i = 1$ such that for all $P^N \in \mathbb{P}^n$,

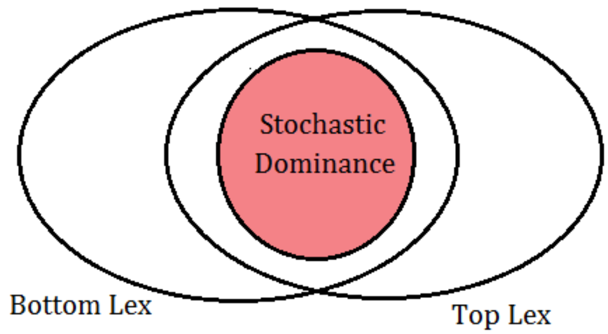
$$F_a(P^N) = \sum_{i \in N \,:\, P_i(1) = a} \beta_i$$

A social choice function being strategic proof means that telling truth is always (weakly) better. It simply means that no agent can ever manipulate his true preference (opinion). So a RSCF $F : \mathbb{P}^n \to \triangle A$ is strategy-proof if for all $i \in N$, all $P_{-i} \in \mathbb{P}^{n-1}$, for all $P_i \in \mathbb{P}$, we have

$$F(P_i, P_{-i}) \succeq^{\triangle A}_{P_i} F(P'_i, P_{-i}) \quad \forall \ P'_i \in \mathbb{P}$$

where $\succeq^{\triangle A}_{P_i}$ is some lottery extension over $\triangle A$ w.r.t. P_i.

For any lottery L and L' and an agent i with preference ordering P_i the lottery extension “Stochastic Dominance” compares as follows $L \succeq^{SD}_{P_i} L'$ iff for all $a \in A$ we have $\sum\limits_{b \in B(a, P_i)} L_b \geq \sum\limits_{b \in B(a, P_i)} L'_b$. Note that stochastic dominance is incomplete ordering over lotteries. Gibbard provides a characterization of strategic proof random rules which in turn gives characterization of random dictatorial rules. The result is as follows, Gibbard (1977): Suppose $|A| \geq 3$. A RSCF is ex-post efficient and strategy-proof if and only if it is a random dictatorship. Our paper is on lexicographic lottery extension. A lexicographic ordering over lotteries can be of two types, Top and Bottom. We have explained them in the introduction, the formal definition as follows. Top-Lex (T-Lex): For each P and each pair $L, L' \in \triangle A$, $L \succeq^{TL} L'$ if either (i) there is $k \in \{1, 2, ..., m\}$ such that for each $h \leq k - 1$, $L_{h(P)} = L'_{h(P)} \text{ } and \text{ } L_{k(P)} > L'_{k(P)}$ or (ii) $L = L'$. The other one is Bottom-Lex(B-Lex): For each P and each pair $L, L' \in \triangle A$, $L \succeq^{BL} L'$ if either (i) there is $k \in \{1, 2, ..., m\}$ such that for each $h \geq k + 1$, $L_{h(P)} = L'_{h(P)} \text{ } and \text{ } L_{k(P)} < L'_{k(P)}$ or (ii) $L = L'$.



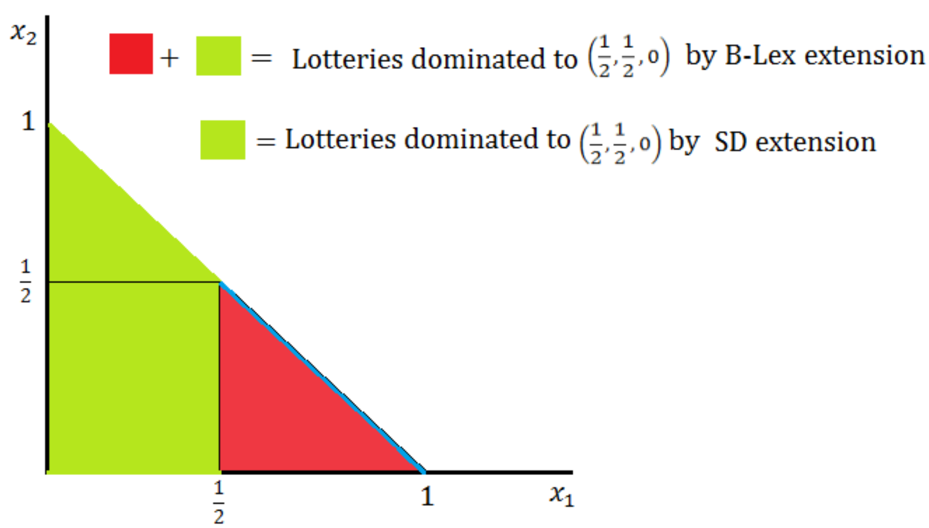
A graphical depiction of inclusion relation among SD, B-Lex and T-Lex.

Note both B-Lex and T-Lex are complete orderings and include SD(stochastic Dominance) which is an incomplete ordering. But B-lex and T-lex can give opposite results which is easy to see with a simple example suppose $x_1 \succ x_2 \succ x_3$ and lottery over three alternatives are $L = (\frac{1}{2}, 0, \frac{1}{2})$ & $L' = (0, 1, 0)$. So we have $L \succ^{T-Lex} L'$, $L' \succ^{B-Lex} L$ and according to SD they are not comparable. A figure above gives overall structure.

5 Main Result

We have two main results each on different types of lexicographic preference. The first theorem which is the main result of this paper is on Bottom-Lex preferences as follows.

Theorem 1: Suppose $|A| \geq 3$ *and* $\succeq^{\triangle A} = \succeq^{B-Lex}$ *. A RSCF is unanimous and strategy-proof if and only if it is a random dictatorship.*



A graphical depiction of lotteries dominated under SD and B-Lex

This theorem provides the full characterization of efficient and non-manipulable rules under B-Lex preferences. This result is in conformity with Gibbard's finding. The details of proof is referred to the appendix, here we give a intuitive explanation to highlight the importance of our result. Note that by replacing B-Lex with Stochastic Dominance(SD) and to get the same class of rules is non-trival infact it is actually a strong result[4] because B-Lex is a complete ordering and a super set of SD.

To explain it further we will take an example. Suppose there are only three alternatives and first agent has an ordering $x_1 \succ x_2 \succ x_3$ and RSCF f picks the lotter $f(P) = (\frac{1}{2}, \frac{1}{2}, 0)$ for some profile P, obviously, which includes the agent 1's ordering. Now we want to bring attention to the question, what will happen if agent 1 manipulates and reports a different ordering then his true preference as above. When we have SD extension for lottery comparison, for the rule to be strategy proof (for player 1), it should pick lotteries which are dominated by $(\frac{1}{2}, \frac{1}{2}, 0)$. See the diagram above. The RSCF has to pick a lottery in green region else agent 1 will manipulate but if we use B-Lex extension then RSCF can pick anything from green or red (excluding blue line). So you can see SD constrains the range of RSCF to a great extent which ultimately leads to random dictator along with efficiency. But B-Lex is a complete ordering so it doesn't restrict the range of RSCF and allows it to pick more lotteries which are dominated. This allows a much larger class of RSCF which can potentially satisfy these two conditions. But our result shows that finally we end up at the same class of rules, random dictator, which Gibbard had arrived but with much restriction.

The next few results is related to Top-Lex preferences. First we will show that strategy proofness and ex-post efficiency under T-lex imply that the only possible rules are Top-Support rules which we have explained earlier. Next theorem states the same.

Theorem 2: If agents have Top-Lex lottery extension then any rule which is efficient and strategy proof has to be Top-Support only rule.

The above result is departure from the existing standard results. Now the possible rules are much larger than random dictatorship, Top-Support rules include it a special case. The next result gives the complete characterization for two agents. The proof is in appendix.

Theorem 3: If $N = \{1, 2\}$, $|A| \geq 3$ then a RSCF f is efficient and strategy proof iff it is of the following form

1. $\forall\ (P_1, P_2),\ f(P_1, P_2) \in \triangle\{1(P_1), 1(P_2)\}$
2. *For any i, any (P_i', P_j') if we have $f_{1(P_i')}((P_i', P_j')) = 1$ then for all (P_i, P_j), $f_{1(P_i)}((P_i, P_j)) > 0$. Moreover if $1(P_i) = 1(P_i')$ then $f_{1(P_i)}((P_i, P_j)) = 1$.*

The above theorem states that for two agents case, the rule if it satisfies the above conditions then it has to pick a lottery over the tops of both agents. Along with the extra condition that if for a profile it pick a degenerate lottery where one agent gets the full probability then at all profiles he should get positive probability and if the top is also then he should get the full probability again.

If there are more than two agents then the complete characterization is hard particularly because of the case when some agents get zero probability. But if avoid those cases then then

[4] But we want to emphasis that neither of result imply the other. Both results seem to be independent.

we can something. We define a rule, Top-Weight-Quota-Priority Rule, which gives positive probability to top alternatives but the probabilities will depend on the whole preference ordering. To define the whole rule first we define two other rules. First we define *Quota-Priority (Q-P) Rule*. For n agents this rule first defines a priority matrix, $\mathcal{P}$, of size $n \times n$. Each number in the matrix is a probability priority so it has to be non-negative. Each column represents an agent and the total of each column is (probability) quota assigned to that respective agent. The sum of all quotas has be be 1. Each quota can be thought of power assigned to an agent to implement his priority. So for any entry i_k in the matrix, it represents probability agent i gives to priority(or rank) k. The quota is divided into priorities which has to satisfy two conditions. First it has to be in decreasing order and the second is the feasibility conditions. A typical priority matrix is given below.

$$\mathcal{P} = \begin{array}{cccccc} 1_1 & 2_1 & \cdot & \cdot & \cdot & n_1 \\ 1_2 & 2_2 & \cdot & \cdot & \cdot & n_2 \\ \vdots & \vdots & & \ddots & & \vdots \\ 1_n & 2_n & \cdot & \cdot & \cdot & n_n \\ & & & & & \\ \overline{\underline{q_1}} & \overline{\underline{q_2}} & \cdot & \cdot & \cdot & \overline{\underline{q_n}} \end{array}$$

This has to satisfy the following conditions :

1. $\sum_{k=1}^{n} i_k = q_i \; \forall i = 1, ..., n$ and $\sum_{i=1}^{n} q_i = 1$.
2. $i_k \geq i_{k+1} \geq 0 \; \forall i, k = 1, 2, ..., n.$ such that for all i either $i_2 = 0$ or $i_n > 0$.

The *Q-P Rule* works as follows. Given the preference profile $(P_1, P_2, ..., P_n)$ submitted by all agents, select an agent i if $q_i = 0$ then go to q_{i+1} otherwise distribute the probability q_i as follows. If $i_1 = q_i$ i.e. $i_k = 0$ for $k \geq 2$ then give i_1 weight to the alternative $1(P_1)$ i.e. the top alternative in the preference ordering submitted by agent i suppose its a and then move to next agent. But if $i_k > 0$ for $k \geq 2$ then comes the crucial step. Now check how many agents, in total, have alternative a as their top which is the same as top of agent i. Suppose m_1 such agents are there, which obviously includes agent i, then first m_1 priorities of agent i i.e. $i_1 + i_2 + ... + i_{m_1}$ goes to a. Next see what is the second rank alternative of agent i, lets say it is b. Now check how many agents have alternative b as their top. Suppose m_2 such agents are there then next m_2 priorities of agent i i.e. $i_{(m_1+1)} + i_{(m_i+2)} + ... + i_{(m_1+m_2)}$ goes to b.

Repeat this procedure till we exhaust the all priorities of agent i and then move to next agent and apply the same protocols. Note that for each agent we have at most n priories therefore the rule exhausts all priorities which add to q_i and the maximum step it will take is when all agents have different alternatives as their tops. Because all q_i's exactly add to 1 i.e. $(q_1, q_2, ..., q_n)$ forms a probability distribution, so when we are done with all agents we have effectively distributed total weight of 1 over alternatives. Therefore, *Q-P Rule* terminates at a lottery over alternatives. We want to point out that suppose $i_{k+1} = 0$ i.e. positive priority ends at k^{th} priority/level. Then the above procedure ends even early and it will end at the alternative which gets the last priority i_k.

Remember that tops-only rule give the same allocation if tops remain the same i.e. changing the ordering keeping the top same has no effect. The *Q-P Rule* gives probability only to the top alternatives of agents. But it is not a tops-only rule because if an agent changes his preference ordering but keeping the top same it can affect the probability of alternatives other than his top. It includes random dictatorship as a special case when $i_1 = q_i$ for all i.

Next rule we want to define is $Top - Weight$ (TW) *Rule*. Lets first define few terms. Given any profile P^N, let $T(P^N) = \{1(P_i) \mid i \in N\}$ contains all the top alternatives in this profile. Next we define the rule by weight function $\mathcal{W}$ over all preference profiles as follows, for any profile, $\mathcal{W}(P^N) \in \triangle T(P^N)^+$ such that [5]
1. It gives strict positive probability to all and only top alternatives.
2. For any two profiles P^N and $\hat{P}^N$ if there is an alternative (say) a such that $\{i \in N \mid 1(P_i) = a\} \subsetneq \{i \in N \mid 1(\hat{P}_i) = a\}$ then $\mathcal{W}_a(P^N) < \mathcal{W}_a(\hat{P}^N)$ i.e. if from one profile to the other the set of agents who have a as their top increases then the probability of a should also increase.

The above rule is simple and it gives probability only to the tops and if even a single agent changes its top the probability of all the tops can change but subject to (population) monotonicity condition. It simply means if more agents have an alternative as their tops then its probability should also increase. Here increasing in agents just don't mean to increase in numbers rather they should include the previous one and include more. One can observe that it is a Top-only rule but not a random dictator though includes it as a special case. It is easy to see that this rule is strategy-proof and ex-post efficient. Now we can define our main rule, Top-Weight-Quota-Priority Rule. This is combination of the above two rule where Top-Weight probabilities should add to (say) $\alpha \leq 1$ and Quota-Priority should add to $\beta \leq 1$ and $\alpha + \beta = 1$.

Theorem 4 : The Top-Weight-Quota-Priority Rule satisfy ex-post efficiency and strategy proof when agents have Top-Lex Preferences. Moreover it is ***not*** *a Tops-Only rule.*

6 Conclusion

We conclude this paper by mentioning in brief what we have done and what is left for the future research. With ex-post efficiency and strategy proof, under B-Lex preferences we have given the complete characterization. Under T-lex preferences, such rules should fall under the category of Top-Support only rules. For two agents we have complete characterization. For more than two agents we have given a (very) general class of rules, $TWQP$ rules, which includes random dictator as very special case and this rule is not a Top-Only rule. For future research, the complete characterization for more than two agents in T-Lex can be done. Also it will be interesting to see why Gibbard's result hold for B-Lex and not for T-Lex. One can use other lottery extensions in this set up and see its implication. This results will provide further insights to the famous "Impossibility Results".

[5] It means set of lotteries over set $T(P^N)$ such that each component is strictly positive.

7 References

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8 Appendix

8.1 Proof of Theorem 1

Our proof is on the lines of Sen(2011) paper. First we will show it for two agents and then we will prove the result for n agents by induction.

Lemma 0: Monotonicity Condition for B-Lex Strategy proof RSCF
Take any $P_{-i},\ P_i,\ P_i'$ and if $\exists k$ such that $l(P_i) = l(P_i')\ \forall l \leq k$ then

$$F_{l(P_i)}(P_i, P_{-i}) = F_{l(P_i)}(P_i, P_{-i})\ \forall l \leq k.$$

This also imply that $\sum_{r<k} F_{r(P_i)}(P_i, P_{-i}) = \sum_{r<k} F_{r(P_i)}(P_i, P_{-i})$. This can be shown very easily. If the above claim is not true then we can construct a profile where an agent can manipulate. A contradiction to to the hypothesis that it is a strategy proof RRSCF

Lemma 1: $F(P^a, P^{ba}) = F(P^{ab}, P^b) = (\lambda^{ab}, 1 - \lambda^{ab})$ for all $a, b \in A,\ P$
Proof:

$$F\begin{pmatrix} a & b \\ b & a \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} = F\begin{pmatrix} a & b \\ \cdot & a \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} \quad \forall\ P^{ab},\ P^{ba},\ P^a$$

If not then agent 1 can manipulate either of preferences depending on which gives the better lottery.

Lemma 2: $\lambda^{ab} = \lambda^{cd}$
Proof :

$$1 - \lambda^{ab} = F_b\begin{pmatrix} a & b \\ c & a \\ b & \cdot \\ \vdots & \vdots \end{pmatrix} \leq F_b\begin{pmatrix} c & b \\ b & a \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} = 1 - \lambda^{cb}$$

$$1 - \lambda^{ab} = F_b\begin{pmatrix} a & b \\ b & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} \geq F_b\begin{pmatrix} c & b \\ a & c \\ b & \cdot \\ \vdots & \vdots \end{pmatrix} = 1 - \lambda^{cb}$$

$$\Rightarrow \lambda^{ab} = \lambda^{cb}$$

A similar argument shows that $\lambda^{cb} = \lambda^{cd}$, which imply $\Rightarrow \lambda^{ab} = \lambda^{cd}$.

$$\lambda^{cb} = F_c \begin{pmatrix} c & b \\ b & d \\ \cdot & c \\ \vdots & \vdots \end{pmatrix} \leq F_c \begin{pmatrix} c & d \\ b & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} = \lambda^{cd}$$

$$\lambda^{cb} = F_c \begin{pmatrix} c & b \\ d & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} \geq F_c \begin{pmatrix} c & d \\ d & b \\ \cdot & c \\ \vdots & \vdots \end{pmatrix} = \lambda^{cd}$$

Lemma 3: $F(P^{ax}, P^{bx}) = (\lambda^{ab}, 1 - \lambda^{ab})$ for all $a, b, x \in A,\ P$
Proof: We know that

$$F \begin{pmatrix} x & b \\ a & x \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} = (\lambda^{xb}, 1 - \lambda^{xb})$$

Now swap x and a. Note because of previous claim, probability of alternative b should remain the same. So we get

$$F \begin{pmatrix} a & b \\ x & a \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} = L = \begin{pmatrix} L_a \\ L_x \\ L_b = 1 - \lambda^{xb} \end{pmatrix}$$

Because probability of b is same, if $L_a \neq \lambda^{xb}$ i.e. $L_x > 0$. Then agent 1 will manipulate here by $F \begin{pmatrix} a & b \\ b & x \\ \vdots & \vdots \end{pmatrix}$.

Lemma 4: $F(P_1^a, P_2^b) = (\lambda^{ab}, 1 - \lambda^{ab})\ \ \forall\, a, b \in A\ \ \forall P_1^a, P_2^b$
Proof: Take any profile (P_1^{ax}, P_2^{by}). Case 1: $x = y$ then because of previous lemma we are done. Case 2: $x \neq y$. We have following possibilities to consider.

If $F_c(P_1^{ax}, P_2^{by}) > 0$
for any $b \succ_1 c$ then agent 1 will manipulate here by P_1^{ab}.

If $F_b(P_1^{ax}, P_2^{by}) > 1 - \lambda^{ab}$ then again 1 will manipulate here by $\hat{P}_1^{ay}$.

If $F_b(P_1^{ax}, P_2^{by}) < 1 - \lambda^{ab}$ then again agent 1 will manipulate at $(\hat{P}_1^{ay}, P_2^{by})$ by P_1^{ax}.

A similar argument follows for agent 2.

If $F_c(P_1^{ax}, P_2^{by}) > 0$
for any $a \succ_2 c$ then agent 2 will manipulate here by P_2^{ba}.

If $F_a(P_1^{ax}, P_2^{by}) > \lambda^{ab}$ then again 2 will manipulate here by $\hat{P}_2^{bx}$.

If $F_a(P_1^{ax}, P_2^{by}) < \lambda^{ab}$ then again agent 2 will manipulate at $(\hat{P}_1^{ax}, P_2^{bx})$ by P_2^{ay}.

Therefore, we have established that for two agents the claim of random dictatorship is true. Now we use the induction argument. Assume that for all integers $k < n$, the following statement is true:
Induction Hypothesis (IH): Assume $m \geq 3$. If $F : \mathbb{P}^k \longrightarrow \triangle A$ satisfies ex-post efficiency and strategic proof, then it is a random dictatorship.
Let $\hat{N} = \{I, 3, ..., n\}$ be a set of voters where $3, ..., n \in N$. Define a RSCF $g : \mathbb{P}^{n-1} \longrightarrow \triangle A$ for the set of voters $\hat{N}$ as follows: For all $(P_I, P_3, ..., P_n) \in \mathbb{P}^{n-1}$

$$g(P_I, P_3, ..., P_n) = F(P, P, P_3, ..., P_n) \;\; \forall P_I = P, \quad \text{alternatively} \quad g(P, P_{-12}) = F(P, P, P_{-12})$$

Lemma 5 : RSCF g is ex-post efficiency and strategic-proof.
Proof: g is ex-post efficient and strategic-proof, for all agents except I, because F is. To see for I, note that

$$\begin{aligned} g(\hat{P}, P_{-12}) = F(\hat{P}, \hat{P}, P_{-12}) \;{}^{2}\!\underset{P}{\preceq}\; F(\hat{P}, P, P_{-12}) \;{}^{1}\!\underset{P}{\preceq}\; & F(P, P, P_{-12}) \\ = & \, g(P, P_{-12}) \end{aligned}$$

So we have shown $g(P_I, P_{-12}) \succeq_{P_I} g(\hat{P}_I, P_{-12})$ for preference profiles.

Let $\beta, \beta_3, ..., \beta_n$ be the weights associated with the random dictatorship g; i.e. β_i, is the weight associated with voter $i = 3, ..., n$ and β is the weight associated with voter I. Let $T^x = \sum_{\{i \neq 1,2 \,:\, 1(P_i) = a\}} \beta_i$. So $g_x(P_I^a, P_{-12}) = T^x$ for all $x \neq a$ and $g_a(P_I^a, P_{-12}) = T^a + \beta$.

Lemma 6: $F(P^a, P^a, P_{-12}) = F(P^a, \hat{P}^a, P_{-12})$ for all $a \in A$ and all such orderings.
Proof: Because F is strategic-proof we have

$$F(P^a, P^a, P_{-12}) \underset{P^a}{\succeq} F(P^a, \hat{P}^a, P_{-12}) \underset{\hat{P}^a}{\succeq} F(\hat{P}^a, \hat{P}^a, P_{-12})$$

Because Lexico-Down is linear order then above relation can hold only if they if the middle lottery is also same. This lemma also imply that $F_x(P^a, \hat{P}^a, P_{-12}) = F_x(P^b, \hat{P}^b, P_{-12}) \;\; \forall\, x \neq a, b$.

To simplify the notation now onwards we will drop P_{-12} with the understanding that $F(P, \hat{P}, P_{-12}) \equiv F(P, \hat{P})$.

Lemma 7: $F(P^{ab}, P^b) = F(P^a, P^{ba}) =$ (lets say)[6] $L^{(ab)}$
Proof: Because $F_x(P^{ab}, P^b) = F_x(P^a, P^{ba}) = F_x(P^{ab}, P^{ba}) = T^x \quad \forall x \neq a, b$. This can be

[6] This is not exactly equivalent to just top alternatives of 1 & 2. This means that at-latest one of them has top of other agent as his second rank.

argued from previous lemma. One just needs to put b and a on rank 1 for agent 1 and 2 (while keeping b on top of agent 2) and then flip it.

So if for any profile $F\begin{pmatrix} a & b \\ b & \cdot \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} \neq F\begin{pmatrix} a & b \\ b & a \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix}$ then 2 will manipulate at either of the case depending on which is better lottery.

Similarly if $F(P^a, P^{ba}) \neq F(P^{ab}, P^{ba})$ then agent 1 will manipulate at either of the case depending on which is better lottery. So we will write, for all $1(P_1) = a \neq b = 1(P_2)$, $L_a^{(ab)} = \lambda^{ab}\beta + T^a$. The $\lambda^{ab} \in [0,1]$ is the proportion given to agent 1 from the weight β when tops of both agents are in the exact pattern consisting a & b. And remaining $1 - \lambda^{ab}$ of β goes to agent 2. Note that order is important, so potentially $\lambda^{ab} \neq \lambda^{ba}$.

Lemma 8: $\lambda^{ab} = \lambda^{cd} \;\; \forall\, a, b, c, d, \in A$.
Proof: First note that $L_x^{(ab)} = L_x^{(cd)} = T^x \;\; \forall\, x \neq a, b, c, d$. So we just need to focus on these four alternatives.

$$(1-\lambda^{ab})\beta + T^b = F_b\begin{pmatrix} a & b \\ c & a \\ b & \cdot \\ \vdots & \vdots \end{pmatrix} \leq F_b\begin{pmatrix} c & b \\ b & a \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} = (1-\lambda^{cb})\beta + T^b$$

$$(1-\lambda^{ab})\beta + T^b = F_b\begin{pmatrix} a & b \\ b & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} \geq F_b\begin{pmatrix} c & b \\ a & c \\ b & \cdot \\ \vdots & \vdots \end{pmatrix} = (1-\lambda^{cb})\beta + T^b$$

$$\Rightarrow \lambda^{ab} = \lambda^{cb}$$

A similar argument below shows that $\lambda^{cb} = \lambda^{cd}$, which imply $\Rightarrow \lambda^{ab} = \lambda^{cd}$.

$$\lambda^{cb}\beta + T^c = F_c\begin{pmatrix} c & b \\ b & d \\ \cdot & c \\ \vdots & \vdots \end{pmatrix} \leq F_c\begin{pmatrix} c & d \\ b & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} = \lambda^{cd}\beta + T^c$$

$$\lambda^{cb}\beta + T^c = F_c\begin{pmatrix} c & b \\ d & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} \geq F_c\begin{pmatrix} c & d \\ d & b \\ \cdot & c \\ \vdots & \vdots \end{pmatrix} = \lambda^{cd}\beta + T^c$$

Lemma 9: $F(P^{ac}, P^{bc}, P_{-12}) = F(P^{ac}, P^{bc}, P_{-12})$
Proof: Start with any profile $(P^{ca}, P^{bc}, P_{-12})$. Then let agent 1 swaps a and c. This makes probability of any $x \neq a, c$ same in both the lotteries (pre and post swap). Now we need to

check only for a & c.

$$\left(L_x^{(cb)} =\right) F_x \begin{pmatrix} c & b \\ a & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} = F_x \begin{pmatrix} a & b \\ c & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} \ \forall x \neq a, c$$

$$\text{If } \ F_c \begin{pmatrix} a & b \\ c & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} > T^c \left(= L_c^{(ab)}\right) \text{ Then agent 1 will manipulate here by } P^{ab}$$

$$\text{If } \ F_a \begin{pmatrix} a & b \\ c & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} > \lambda^{ab}\beta + T^a \left(= L_a^{(ab)}\right) \text{ Then agent 2 will manipulate here by } P^{ba}$$

. Since $F_a + F_c$ should add to the exactly those two terms, which completes this lemma.

Lemma 10: $F(P^a, P^b, P_{-12}) = L^{(ab)}$ for all such profiles. Alternatively $F(P^N) = L^{(1(P_1),1(P_2))}$ for all profiles. Proof: Because B-Lex is a complete ordering and the previous lemmas imply that for each possible case the claim is true means the probabilities remain same as tops remain the same. And we started with an arbitrary profile this completes the proof of our theorem.

8.2 Proof of Theorem 2

First we highlight few implications of strategy proofness in T-Lex extension. If a RSCF f is strategy proof under T-Lex then following is true.
1. Take any two profiles where only agent i is changing, (P_i, P_{-i}) *and* (P_i', P_{-i}), if first k ranks are same for i then so the probabilities are i.e. $l(P_i) = l(P_i')$ for all $l = 1, 2, ..., k$ imply $f_{l(P_i)}(P_i, P_{-i}) = f_{l(P_i)}(P_i', P_{-i})$ for all $l = 1, 2, ..., k$. It is easy to see why it should be true because if it is not then agent i will manipulate either of the preference, which is true since T-Lex is a complete ordering.
2. Weak Monotonicity condition. If an alternative (say a (b)) is moved to higher (lower) rank, but keeping the ranks above it exactly same then its probability can only increase (decrease). Formally, if moving from (P_i, P_{-i}) to (P_i', P_{-i}) and $\exists$ k such that $l(P_i) = l(P_i')$ for all $l < k$ and $k(P_i) = b = (k+1)(P_i')$ & $(k+1)(P_i) = a = k(P_i')$ then $f_a(P_i', P_{-i}) \geq f_a(P_i, P_{-i})$ and $f_b(P_i', P_{-i}) \leq f_b(P_i, P_{-i})$. Note this is a weaker from the usual monotonicity condition because, if all alternatives which are above [7] are not ranked exactly same then potentially the probability of a can go down despite its improvement.

To prove the main claim, fix a profile P^N where $x \in T(P^N)$ and $f_x(P^N) > 0$. If no such profile exists then we are done. Now we start with a different profile each agent's top

[7] All the alternatives above a

$|T(P^N)|$ ranks contain alternatives only from $T(P^N)$ subject to top same as in P^N. Now any alternative $x \notin T(P^N)$ should get zero probability because of ex-post efficiency. Now start moving x up to rank 2 for all agents. We can argue with different cases that the probability of x remain 0. Now using the above two fact reconstruct the profile P^N from here and we will get $f_x(P^N) = 0$ which is contradiction to the starting hypothesis. Therefore, $\forall\, x \notin T(P^N) \;\Rightarrow\; f_x(P^N) = 0$. So we are done.

8.3 Proof of Theorem 3 and Theorem 4

Q-P rule provides positive probability only to the tops of agents which straight forward makes it ex-post efficient. Mainly we have to check for the other condition. Lets see strategy proofness first. Fix any agent i and a true profile $P^N = (P_i, P_{-i})$. Now consider a deviation by this agent. If he keeps the top same then probability of top remain same. And by changing his top he can only reduce its probability which can only make him worse. So he has no incentive to change in top. Now suppose he change his ordering other than top, then if there is any change in the probability then it will make him worse off only. Because by increasing the rank of an alternative it will only increase its probability. And this increase will be compensated with the alternative which is going down. Therefore my manipulating he is increasing the probability of a less preferred alternative and decreasing the probability of a more preferred alternative which certainly makes him worse off. Since our arguments are valid of any agent any profile, this makes the rule strategy proof. In the previous lines we have shown that by changing the ordering other than top can change the probability. So this is not a tops-only rule. And because both T-W and Q-P rules are strategy-proof and ex-post efficient then so TWQP rule. And in the case of two agents TWQP rule gives use the results of theorem 2 for positive probabilities part. When there is any degenerated lottery, strategy proofness imply those conditions.