Repeated Trade with Two-sided Incomplete Information

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Abstract

This paper presents a finite-horizon bargaining model where each player has private valuation over the good for sale. The seller posts a price in each period, which the buyer has to accept, or reject. If he rejects, a new price is posted in the next period. Bargaining inefficiencies arise due to incomplete information. We show that the probability of trade over all periods decreases when we increase the horizon of the game.

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1 Introduction

We develop a model of bargaining between two players when both players have private valuations. They bargain over one unit of an indivisible good. It is a sequential game where the players discount future payoffs. In each period the seller posts a price, which the buyer has to either accept or reject. If he accepts, the trade takes place. If he rejects, the seller makes a subsequent offer. Since both of them do not know each other's private information, the actions they take in each period reveal their informations partially. The acceptance or rejection from the buyer makes the seller update his belief about the buyer's valuation, while the price offered by the seller acts as a signal of his own valuation.

In the static case, we know that efficient trade cannot take place due to incomplete information (**Chatterjee and Samuelson, 1983**). Our model is a finite horizon dynamic model, as we allow the players more opportunities for trade, if successful trade has not happened till then. Intuitively, we expect that as the players get more and more opportunities for trade, the total probability of trade over all the periods should increase. But we get a counter-intuitive result that as we increase the number of periods of the game, the total probability of trade decreases.

Our model is in the spirit of **Fudenberg and Tirole (1983)** and **Cramton (1983)**. The former deals with only a two-period model with only two-point distribution. This limits the horizon and information structure to analyze the information revelation. **Cramton (1983)** looks only for those equilibria where the seller completely reveals his information, so that the game becomes a

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one-sided incomplete information game. In contrast we construct an equilibrium where the private information for both parties are revealed gradually.

2 The model

A seller has an indivisible object for sale to a single buyer. A profile of values will be denoted as $v \equiv (v_b, v_s)$, where v_b is the value of the buyer and v_s is the value of the seller. The joint distribution of values (v_b, v_s) is given by a distribution μ over $[0,1]^2$.

The seller sells the good over T periods. We denote the T periods as $\{\Delta, 2\Delta, \ldots, T\Delta \equiv 1)$, where we assume that $\frac{1}{\Delta}$ is an integer equal to T. The payoff of agent $i \in \{b, s\}$ with value v_i is given as follows: if he gets a transfer of p_i in period t, then his payoff in period t is $(v_i x + p_i)\delta^{t-1}$, where $x \in \{0,1\}$ indicates whether agent i has the good or not in period t and $\delta \in (0,1)$ is the discount factor. [1]

For every $t \in \{1, \ldots, T\}$, if the good is not sold till period $(t-1)$, then she posts a price in period t to offer the good to the buyer. We denote this game as Γ^T. The **history** h^t at period $t \in \{1, \ldots, T\}$, consists of prices till period $(t-1)$: $\{p^1, \ldots, p^{t-1}\}$. The set of all possible histories at period t is H^t, and we assume $H^1 \equiv \emptyset$. So, a **strategy** of the seller is a collection of maps

$$\rho_s^t : [0,1] \times H^t \to \mathbb{R}_+ \quad \forall\, t \in \{1, \ldots, T\}.$$

The seller's (behavior) strategy describes the price that she will post given her valuation and the history so far. The strategy of the buyer is a collection of maps

$$\rho_b^t : [0,1] \times H^t \times \mathbb{R} \to \{0,1\}.$$

The strategy of the buyer describes his decision to accept or reject the price in every period t given his type and the history till period t.

We consider a *perfect Bayesian equilibrium (PBE)* of this game. So, the seller posts prices which are sequentially rational and the buyer makes decisions given his beliefs. In general, we have multiple equilibria of Γ^T. Let E^T be the set of all PBE of Γ^T. For every equilibrium $e \in E^T$, let $D_e^T(v_b, v_s) \in \{0,1\}$ denotes whether trade takes place between buyer of type v_b and seller of type v_s in equilibrium e.

The **efficiency** of equilibrium e is the probability measure of the event that trade takes place. Formally, efficiency of equilibrium e is given by

$$I_e^T := \int_{(v_b, v_s) \in [0,1]^2} D_e^T(v_b, v_s)\, d\mu(v_b, v_s).$$

[1] The value of the good can be enjoyed only once - either the seller enjoys it after T periods if no trade takes place or the buyer enjoys it whenever the trade takes place.

2.1 The one period model

In the one period model, the seller posts a price p and the buyer can either accept or reject the price. If the buyer accepts the price p, then he gets the object and realizes a payoff of $v_b - p$ and the seller realizes a payoff of p. If the buyer rejects the price p, the the seller keeps the object and realizes a payoff of v_s and the buyer realizes a payoff of 0.

Clearly, it is a dominant strategy for the buyer to accept the price p if $v_b > p$ and reject it if $v_b < p$. Given this, the seller gets a payoff of $p(1 - F(p)) + v_s F(p)$, by posting a price p. To maximize his payoff, we apply the first order condition (FOC), and get a necessary condition: $(1 - F(p)) - pf(p) + v_s f(p) = 0$. This is equivalent to stating that if p^* is an optimal solution then it must solve

$$p^* = v_s + \frac{1 - F(p^*)}{f(p^*)}.$$

Monotone hazard rate (MHR) condition implies that the second term is non-increasing in price. As a result, under MHR, there is a unique p^* which solves this equation. If F is the **uniform distribution**, then

$$p^* = \frac{v_s + 1}{2}.$$

This price means whenever $v_b < \frac{v_s+1}{2}$, there is no trade. But efficiency requires trade if $v_b > v_s$ and no-trade if $v_b < v_s$. This means there is inefficiency in equilibrium if $v_b \in (v_s, \frac{v_s+1}{2})$.

2.2 The multiple period model

In this model, there are T periods. For every $t < T$, if the seller fails to sell the object till period $(t - 1)$, then he offers it for sale in period t. However, he *does not commit* to a price for any of the periods. The question we ask is: *Does the efficiency of trade increase by allowing for multiple periods of trade?*

Here, the buyer uses a **cutoff strategy** such that in each period t there exists a cutoff v_t such that a buyer with valuation v greater than v_t would accept the t^{th} period price, and those below would reject it. The formal statement and proof are given later on in **Lemma 1**. Thus after period t, the seller's posteror about the buyer's valuation comes from the conditional distribution $\frac{F(v)}{F(v_{t+1})}$.

Throughout, we will assume that F is the **uniform** distribution. Under this assumption, we show that allowing multiple periods of trade *decreases* efficiency. This is formally stated in Theorem 2. Theorem 1 formally describes the perfect Bayesian equilibrium of the model.

Theorem 1 *Suppose values of the buyer and the seller are independently and identically distributed using uniform distribution in* $[0, 1]$*. Then, for every* T*, there is a unique equilibrium* e^*_T *and*

$$I^T_{e^*_T} < I^{T-1}_{e^*_{T-1}} \quad \forall\ T > 1.$$

Theorem 2 *Suppose the seller's period* t *posterior belief has support* $[o, v_{t+1}]$ *after some non-trivial history. Under* e^*_T*, the seller sets period-t price*

$$p_t = k_t v_t$$

where

$$k_t = \frac{(1-\delta+\delta k_{t-1}b_{t-1})}{(1-\frac{\delta k_{t-1}a_{t-1}}{2})}$$

and given an arbitray price $\widehat{p_t}$, *a buyer with valuation* $v > v(\widehat{p_t}, v_{t+1},, t)$ *accepts the price, and a buyer with valuation* $v < v(\widehat{p_t}, v_{t+1},, t)$ *rejects the price where* $v(\widehat{p_t}, v_{t+1},, t)$ *is the unique* v *solving*

$$1 + (\frac{k_{t-1}a_{t-1}}{2} - 1)\frac{\widehat{p_t}}{v} = \delta(1 - k_{t-1}b_{t-1})$$

Proof of Theorems 1 and 2 The proof is completed in *three steps*:

Step 1: *We solve for the equilibrium price and threshold valuations (p_t and v_t).*

We start from the last period. In the last period the seller's problem is

$$\begin{aligned} Max_{p_1}(1 - \frac{v_1}{v_2})(p_1 - v_s) \\ s.t. p_1 \quad &\leq \quad v_1. \end{aligned}$$

In the last period, the buyer accepts if and only if the price is not above his private valuation. We assume that buyer adopts a **cutoff strategy** from the last period till $(t-1)th$ period, such that in time period l, v_l is the cutoff valuation type such that all valuations above v_l would accept the current price while all valuations below v_l would reject the current price, $l = 1, 2, ...(t-1)$. In the last period $p_1 = v_1$. First order condition gives

$$v_1 = \frac{v_s + v_2}{2}.$$

Let v_t be the threshold valuation in period t, such that buyer with valuation greater than v_t accepts, and that with valuation lower than v_t rejects the t^{th} period price. We can show that at $t = 2$, $p_2 = k_2 v_2$, where $k_2 = \frac{1-\delta/2}{1-\delta/4}$, and

$$v_2 = \frac{v_s + 2k_2 v_3}{4k_2 - 1}.$$

We assume that in the $(t-1)$th period,

$$p_{t-1} = k_{t-1}v_{t-1}$$

and

$$v_{t-1} = a_{t-1}v_s + b_{t-1}v_t$$

$a_{t-1}, b_{t-1}, k_{t-1} > 0$.

We know that in the last period, $a_1 = \frac{1}{2}, b_1 = \frac{1}{2}$ and $k_1 = 1$. We will subsequently show that in the t^{th} period, for some $a_t, b_t, k_t > 0$

$$p_t = k_t v_t$$

and

$$v_t = a_t v_s + b_t v_{t+1}$$

Then we apply the logic of induction to derive the buyer's indifference condition in the t^{th} period. The buyer's indifference condition is given by

$$\begin{aligned}
v_t - p_t &= \delta E(v_t - p_{t-1} | v_s \leq \frac{1 - k_{t-1} b_{t-1}}{k_{t-1} a_{t-1}} v_t, v_s \leq p_t) \\
&= \delta E(v_t - p_{t-1} | v_s \leq \min(\frac{1 - k_{t-1} b_{t-1}}{k_{t-1} a_{t-1}} v_t, p_t)) \\
&= \delta E(v_t - p_{t-1} | v_s \leq p_t) \\
&= \delta E(v_t - k_{t-1}(a_{t-1} v_s + b_{t-1} v_t) | v_s \leq p_t) \\
&= \delta E((1 - k_{t-1} b_{t-1}) v_t - k_{t-1} a_{t-1} v_s | v_s \leq p_t) \\
&= \delta \frac{\int_0^{p_t} ((1 - k_{t-1} b_{t-1}) v_t - k_{t-1} a_{t-1} v_s) dv_s}{p_t} \\
&\Rightarrow \quad p_t = \frac{(1 - \delta + \delta k_{t-1} b_{t-1})}{(1 - \frac{\delta k_{t-1} a_{t-1}}{2})} v_t \\
&\Rightarrow \quad p_t = k_t v_t
\end{aligned}$$

The assumption we have made here is that $\min(\frac{1-k_{t-1}b_{t-1}}{k_{t-1}a_{t-1}} v_t, p_t) = p_t$. It can be easily checked that this holds true for $t = 1$. We will show subsequently that $\min(\frac{1-k_t b_t}{k_t a_t} v_{t+1}, p_{t+1}) = p_{t+1}$. Next we show that the buyer indeed uses a cutoff strategy in the t^{th} period. **Lemma 1 formally states it.**

Lemma 1: *If a buyer with valuation v_b accepts a price p_t, then a buyer with valuation $v > v_b$ will always accept p_t, for all $t > 1$.*

Proof. A buyer with valuation v_b, if he accepts a price p_t, gets a payoff $v_b - p_t$. If he rejects the price, he gets $\delta E(v_b - p_{t-1} | v_s \leq \frac{1-k_{t-1}b_{t-1}}{k_{t-1}a_{t-1}} v_t, v_s \leq p_t)$. The buyer prefers to accept the current price if

$$\begin{aligned}
v_b - p_t &> \delta E(v_b - p_{t-1} | v_s \leq \frac{1 - k_{t-1} b_{t-1}}{k_{t-1} a_{t-1}} v_t, v_s \leq p_t) \\
&= (1 - k_{t-1} b_{t-1}) v_b - k_{t-1} a_{t-1} \frac{p_t}{2}.
\end{aligned}$$

Since $k_{t-1}, b_{t-1} \epsilon (0, 1)$,

$$\frac{\partial(LHS)}{\partial v_b} > \frac{\partial(RHS)}{\partial v_b}.$$

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The seller's objective function, in the recursive form, is to maximize

$$\begin{aligned}
\pi_t &= (1 - \frac{v_t}{v_{t+1}})(p_t - v_s) + \frac{v_t}{v_{t+1}} \pi_{t-1} \\
&= (1 - \gamma_t)(k_t v_t - v_s) + \gamma_t \pi_{t-1}
\end{aligned}$$

where $\gamma_t = \frac{v_t}{v_{t+1}}$. Also let $\mu_t = \frac{\pi_t}{v_{t+1}}$. Therefore, we have

$$\begin{aligned}\mu_t &= \frac{\pi_t}{v_{t+1}} = (1-\gamma_t)(k_t\gamma_t - \frac{v_s}{v_{t+1}}) + \gamma_t \frac{\pi_{t-1}}{v_{t+1}} \\ &= (1-\gamma_t)(k_t\gamma_t - \beta_t) + \gamma_t^2 \mu_{t-1}\end{aligned}$$

where $\beta_t = \frac{v_s}{v_{t+1}}$. The first order condition, with respect to γ_t, is

$$-(k_t\gamma_t - \beta_t) + (1-\gamma_t)k_t + 2\gamma_t\mu_{t-1} = 0$$

$$\begin{aligned}&\Rightarrow \quad 2(k_t - \mu_{t-1})\gamma_t = \beta_t + k_t \\ &\Rightarrow \quad \gamma_t^* = \frac{\beta_t + k_t}{2(k_t - \mu_{t-1})}\end{aligned}$$

Trade happens if

$$\begin{aligned} v_b &\geq v_t \text{ and } v_b \leq v_{t+1} \\ i.e. \ \frac{v_b}{v_{t+1}} &\geq \gamma_t \text{ and } \frac{v_b}{v_{t+1}} \leq 1 \\ i.e. \ \alpha_t &\geq \gamma_t^* = \frac{\beta_t + k_t}{2(k_t - \mu_{t-1})} \text{ and } \alpha_t \leq 1\end{aligned}$$

where $\alpha_t = \frac{v_b}{v_{t+1}}$. The **probability of trade** in the t^{th} period is

$$\begin{aligned} I_t &= \frac{1}{2}(\frac{1}{v_{t+1}} - \frac{k_t}{2(k_t - \mu_{t-1})})(\frac{2(k_t-\mu_{t-1})}{v_{t+1}} - k_t) \\ &= \frac{1}{v_{t+1}^2 k_t(k_t - \mu_{t-1})}(2(k_t - \mu_{t-1}) - k_t v_{t+1})^2\end{aligned}$$

Step 2: *We perturb the last period price p_1 from its equilibrium value by an amount Δ. We solve for the equilibrium in the perturbed model.* Therefore in the perturbed model, for $\Delta > 0$, we have,

$$p_1 = p_1^* + \Delta.$$

We show that *the probability of trade increases in the perturbed model, i.e.*

$$I_t^p < I_t \ \ \forall \ t.$$

In the perturbed model, the buyer's indifference condition becomes

$$\begin{aligned}
v_t - p_t &= \delta E(v_t - p_{t-1} | v_s \leq \frac{1 - k_{t-1}b_{t-1}}{k_{t-1}a_{t-1}} v_t, v_s \leq p_t) \\
&= \delta E(v_t - p_{t-1} | v_s \leq \min(\frac{1 - k_{t-1}b_{t-1}}{k_{t-1}a_{t-1}} v_t, p_t)) \\
&= \delta E(v_t - p_{t-1} | v_s \leq p_t) \\
&= \delta E(v_t - k_{t-1}(a_{t-1}v_s + b_{t-1}v_t + c_{t-1}\Delta) - \varepsilon_{t-1}\Delta | v_s \leq p_t) \\
&= \delta E((1 - k_{t-1}b_{t-1})v_t - k_{t-1}a_{t-1}v_s - (k_{t-1}c_{t-1} + \varepsilon_{t-1})\Delta | v_s \leq p_t) \\
&= \delta \frac{\int_0^{p_t}((1 - k_{t-1}b_{t-1})v_t - k_{t-1}a_{t-1}v_s - (k_{t-1}c_{t-1} + \varepsilon_{t-1})\Delta)dv_s}{p_t} \\
&\Rightarrow \quad p_t = \frac{(1 - \delta + \delta k_{t-1}b_{t-1})}{(1 - \frac{\delta k_{t-1}a_{t-1}}{2})} v_t + \frac{\delta(k_{t-1}c_{t-1} + \varepsilon_{t-1})\Delta}{(1 - \frac{\delta k_{t-1}a_{t-1}}{2})} \\
&\Rightarrow \quad p_t = k_t v_t + \varepsilon_t \Delta
\end{aligned}$$

The seller's objective function in the perturbed model is

$$\begin{aligned}
\pi_t &= (1 - \frac{v_t}{v_{t+1}})(p_t - v_s) + \frac{v_t}{v_{t-1}} \pi_{t-1} \\
&= (1 - \gamma_t)(k_t v_t + \varepsilon_t \Delta - v_s) + \gamma_t \pi_{t-1}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mu_t &= \frac{\pi_t}{v_{t+1}} = (1 - \gamma_t)(k_t \gamma_t + \varepsilon_t \sigma_t - \frac{v_s}{v_{t+1}}) + \gamma_t \frac{\pi_{t-1}}{v_{t+1}} \\
&= (1 - \gamma_t)(k_t \gamma_t + \varepsilon_t \sigma_t - \beta_t) + \gamma_t^2 \mu_{t-1}
\end{aligned}$$

where $\sigma_t = \frac{\Delta}{v_{t+1}}$.
The first order condition, with respect to γ_t, is

$$-(k_t \gamma_t + \varepsilon_t \sigma_t - \beta_t) + (1 - \gamma_t)k_t + 2\gamma_t \mu_{t-1} = 0$$

$$\begin{aligned}
&\Rightarrow \quad 2(k_t - \mu_{t-1})\gamma_t = \beta_t + k_t - \varepsilon_t \sigma_t \\
&\Rightarrow \quad \gamma_t = \frac{\beta_t + k_t - \varepsilon_t \sigma_t}{2(k_t - \mu_{t-1})}
\end{aligned}$$

Step 3: *We compare the probability of trade in the benchmark model with the perturbed model.*

The **probability of trade** in the t^{th} period is

$$\begin{aligned} I_t^p &= \frac{1}{2}(\frac{1}{v_{t+1}} - \frac{k_t - \varepsilon_t\sigma_t}{2(k_t - \mu_{t-1})})(\frac{2(k_t - \mu_{t-1})}{v_{t+1}} - k_t + \varepsilon_t\sigma_t) \\ &= \frac{1}{v_{t+1}^2 k_t(k_t - \mu_{t-1})}(2(k_t - \mu_{t-1}) - k_t v_{t+1} + \varepsilon_t\sigma_t v_{t+1})^2 \end{aligned}$$

Clearly, $I_t^p > I_t, \forall t = 2, ...T$. Therefore the probability of trade at each time-period till $t = 2$ increases if p_1 is perturbed by an amount $\Delta > 0$. Let $D_t = I_t^p - I_t$. Therefore

$$D_t = \frac{1}{v_{t+1}^2 k_t(k_t - \mu_{t-1})}(4(k_t - \mu_{t-1}) - 2k_t v_{t+1} + \varepsilon_t\sigma_t)\varepsilon_t\sigma_t v_{t+1} > 0$$

$\forall t = 2, ...T$. In the last period at $t = 1$, $\gamma_1 = \frac{\beta_1+1}{2}$. In the last period the probability of trade is

$$I_1^p = -\frac{1}{4v_2^2}(2 - \sigma_1 v_2)^2$$

Therefore

$$D_1 = -\frac{1}{4v_2^2}(4 - \sigma_1 v_2)\sigma_1 v_2 < 0$$

Thus the probability of trade decreases in the last period, but increases in the rest of the periods from $t = 2$ to $t = T$. The total **efficiency** or the overall probability measure of the event that trade takes place, I_e^T can be expressed in terms of I_t if there exist $\phi_1, \phi_2, ..\phi_T \geq 0$, not all equal to zero such that

$$I_e^T := \sum_{t=1}^{T} \phi_t I_t$$

where $\phi_t = \phi^{T-t}$ is the weight allocated by the social planner to the event of trade in period t, $t = 1, 2, ...T$.

Similarly, in the pertubed model, the **total efficiency** can be expressed as

$$(I_e^T)^p := \sum_{t=1}^{T} \phi_t I_t^p$$

where $\phi_t = \phi^{T-t}$.

We need to show that

$$D_e^T = \sum_{t=1}^{T} \phi^{T-t} D_t = \sum_{t=2}^{T} \phi^{T-t} D_t - \phi^{T-1}|D_1| \; > 0$$

Dividing by ϕ^{T-1}, we can write

$$\frac{D_T}{\phi^{T-1}} + \frac{D_{T-1}}{\phi^{T-2}} + ... + \frac{D_2}{\phi} > |D_1|$$

For ϕ very close to zero, the inequality holds true. Also since LHS is continuously decreasing in ϕ, we can claim that, $\exists\ \overline{\phi} > 0$, such that for $\phi \epsilon (0, \overline{\phi}]$, $(I_e^T)^p > I_e^T$.

By monotonicity of the problem, I_t^p increases for increase in Δ. Therefore by monotonicity, $(I_e^T)^p$ increases for increase in Δ. For $\Delta \geq |1-p_1^*|$, $p_1 = 1$, and the T-period game becomes a $(T-1)$-period game, i.e. $\Gamma^T = \Gamma^{T-1}$. Thus

$$I_{e^*}^T < I_{e^*}^{T-1} \ \forall T > 1.$$

(Proved)

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