A Unified Characterization of Randomized Strategy-proof Rules*

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Abstract

We present a unified characterization of the unanimous and strategy-proof random rules on a class of domains that are based on some prior ordering over the alternatives. We identify a property called top-richness so that if a domain satisfies this property, then an RSCF on that domain is unanimous and strategy-proof if and only if it is a convex combination of tops-restricted min-max rules. Many well-known domains such as single-crossing, single-peaked, single-dipped etc., and some domains of practical importance such as distance-based domains etc. satisfy top-richness. We also offer a characterization of the tops-only and strategy-proof random rules on top-rich domains satisfying top-connectedness. Next, we provide a characterization of the domains where an RSCF is unanimous and strategy-proof if and only if it is a random min-max rule. Finally, we present a characterization of the unanimous (tops-only) and group strategy-proof random rules on these domains.

JEL Classification: D71, D82.

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1. Introduction

1.1 Background of the Problem

We analyze the classical social choice problem of choosing an alternative from a set of feasible alternatives based on the preferences of the individuals in a society. Such a procedure is known as a *deterministic social choice function* (DSCF). Arrow, Gibbard, and Satterthwaite have identified some desirable properties of such a DSCF such as *unanimity* and *strategy-proofness*. A DSCF is strategy-proof if a strategic individual cannot change its outcome in her favor by misreporting her preferences, and it is unanimous if, whenever all the individuals have the same most preferred alternative, that alternative is chosen. The classic Gibbard (1973)-Satterthwaite (1975) impossibility theorem states that if there are at least three alternatives and the preferences of the individuals are *unrestricted*, then the only DSCFs that are unanimous and strategy-proof are *dictatorial*. This assures the presence of an individual, called the *dictator*, who is such that the DSCF always chooses her most preferred alternative.

Although unanimity and strategy-proofness are desirable properties of a DSCF, the assumption of an unrestricted domain made in the Gibbard-Satterthwaite Theorem is quite strong. There are many political and economic scenarios where the preferences of an individual satisfy natural restrictions such as single-peakedness, single-dippedness, single-crossingness etc. Moreover, the conclusion of Gibbard-Satterthwaite Theorem does not apply to such restricted domains. Consequently, domain restrictions turn out to be an obvious and useful way of evading the dictatorship result in social choice theory.

Single-peaked property is commonly used in public good location problem. Such domain restriction occurs in an environment where strictly quasi-concave utility functions are maximized over a linear budget set. The study of single-peaked domains can be traced back to Black (1948) where he shows that a Condorcet winner exists on such domains. Later, Moulin (1980) and Weymark (2011) show that a DSCF on a single-peaked domain is unanimous and strategy-proof if and only if it is a *min-max* rule. Single-dipped property is commonly used in public bad location problem. Peremans and Storcken (1999) show that a DSCF on such a domain is unanimous and strategy-proof if and only if it is a monotone rule between the left-most and the right-most alternatives. Single-crossing domains are well-known for their flexibility to accommodate the nonconvexities that appear in case of majority voting.[1] Single-crossing domains frequently appear in models of income taxation and redistribution (Roberts (1977), Meltzer and Richard (1981)), local public goods and stratification (Westhoff (1977), Epple and Platt (1998), Epple et al. (2001)), and coalition formation (Demange (1994), Kung (2006)).[2] Saporiti (2014) shows that a DSCF on a single-crossing domain is unanimous and strategy-proof if and only if it is an *augmented representative voter scheme*. Augmented representative voter schemes are min-max rules where all the parameters are chosen from the *top-set* of the domain. Top-set of a domain consists of those alternatives that appear as a top alternative in some preference in the domain.

The boundaries of social choice theory have been expanded by the notion of *random social choice functions* (RSCF). An RSCF assigns a probability distribution over the alternatives at every preference profile. Thus, RSCFs are generalization of DSCFs. The importance of RSCFs over DSCFs has been well-established in the literature (see, for example, Ehlers et al. (2002), Peters et al. (2014)).

The study of RSCFs dates back to Gibbard (1977) where he shows that an RSCF on the unrestricted domain is unanimous and strategy-proof if and only if it is a *random dictatorial* rule. A random dictatorial rule is a convex combination of dictatorial rules. Ehlers et al. (2002) characterize the unanimous and strategy-proof random rules on single-peaked domains, and Peters et al. (2014) show that such a rule is a convex combination of min-max rules. In a recent work, Peters et al. (2017) characterize the unanimous and strategy-proof RSCFs on single-dipped domains and show that they are convex combinations of unanimous and strategy-proof DSCFs on those domains. However, to the best of our knowledge, the unanimous and strategy-proof RSCFs on single-crossing domains are not characterized yet.

1.2 Motivation

It is worth noting that most restricted domains of practical importance, such as single-peaked, single-dipped, single-crossing, domains where preferences are derived using distances between alternatives etc., assume some prior ordering over the alternatives. This suggests that there might be some common structure of all these domains. However, there are separate characterizations of the unanimous and strategy-proof DSCFs on these domains. Moreover, as we have already

[1]See, for example, Romer (1975), p. 181, and Austen-Smith and Banks (2000), pp. 114-115.

[2]Moreover, models that study the selection of policies in the market for higher education (Epple et al. (2006)) and the choice of constitutional and voting rules (Barbera and Jackson (2004)), also use single-crossing domains. Saporiti (2009) has a detailed exposition on various applications, interpretations, and scopes of single-crossing domains.

mentioned, there is no characterization available in the literature of the unanimous and strategy-prof RSCFs on single-crossing and distance based domains. This makes it important to understand the general structure of such domains and to provide a unified characterization of the unanimous and strategy-proof RSCFs on all domains having similar structure. This comprises the main motivation of this paper.

The characterization of unanimous and strategy-proof DSCFs on single-peaked domain (Moulin (1980) and Weymark (2011)), single-crossing domain (Saporiti (2014)), and single-dipped domain (Peremans and Storcken (1999)) assume the domain to be *maximal*. Maximality requires that the domain contains *all* preferences satisfying the restriction. However, in many practical situations, even though these restrictions are natural, requirement of all such preferences is a strong prerequisite. On the other hand, *top-connectedness* is a property that any such domain should satisfy in most practical applications. Top-connectedness requires that for every two alternatives that are consecutive in the *top-set* [3] of the domain (in the prior order), there exists a preference that places one at the top and the other at the second-ranked position. Top-connectedness is well-studied (Barbera and Peleg (1990), Chatterji et al. (2013), Chatterji et al. (2016), Chatterji et al. (2014),Roy and Storcken (2016)) in the literature and is accepted as a mild and natural condition. This motivates us to characterize the unanimous and strategy-proof RSCFs on top-connected single-peaked or single-crossing domains.

Group strategy-proofness is a well-established refinement of strategy-proofness. Barberà et al. (2010) provide a sufficient condition for the equivalence of strategy-proofness and group strategy-proofness on a domain. Restricted domains like Single-peaked, single-dipped, single-crossing etc. satisfy his sufficient condition. However, to the best of our knowledge, the said equivalence is not explored in the literature for random rules. This motivates us to explore this equivalence for random rules on all the restricted domains we consider in this paper.

1.3 Our Contribution

We consider RSCFs on domains that are based on some prior ordering over the alternatives. In conformity with the existing literature, stochastic dominance is used to extend preferences over alternatives to preferences over probability distributions. We identify a condition called *top-richness* that is sufficient and almost necessary to ensure that every unanimous and strategy-proof

[3]Top-set of a domain consists of the alternatives that can appear as a top-ranked alternative in some preference in the domain.

RSCF is a convex combination of the tops-restricted min-max rules. We demonstrate by way of example the almost necessity of our condition. Moreover, we show that top-richness is necessary and sufficient under an additional natural assumption called *regularity*. A domain is regular if its top-set contains all alternatives. As an application of our result, we obtain a characterization of the unanimous and strategy-proof RSCFs on single-peaked, single-dipped, single-crossing, and some other domains of practical significance.

As we have discussed earlier, maximality is strong prerequisite for a restricted domain. In view of this, we characterize all regular domains $\mathcal{D}$ such that (i) every unanimous and strategy-proof RSCF on $\mathcal{D}^n$ is a random min-max rule, and (ii) every random min-max rule on $\mathcal{D}^n$ is strategy-proof. We call such domains *regular random min-max* domains. We show that a domain $\mathcal{D}$ is regular random min-max if and only if (i) each preference in $\mathcal{D}$ is single-peaked, and (ii) for every two adjacent (w.r.t. the prior ordering) alternatives, say a_j and a_{j+1}, there are two preferences of the form $P = a_j a_{j+1} \ldots$ and $P' = a_{j+1} a_j \ldots$ in $\mathcal{D}$.[4] Note that the number of preferences in a regular random min-max domain can range from $2m-2$ to 2^{m-1}, whereas that in the maximal single-peaked domain is 2^{m-1}. Thus, regular random min-max domains include a large class of restricted single-peaked domains.

We obtain as a bi-product of our result that unanimity and strategy-proofness guarantee tops-onlyness on top-rich domains. Chatterji and Zeng (2015) provide a sufficient condition for random tops-only domains, but top-rich domains do not satisfy their condition. Thus, our results are an exclusion of theirs on tops-onlyness.

Although unanimity implies tops-onlyness under strategy-proofness, tops-onlyness itself is well-accepted as a desirable property of a social choice function. Moulin (1980) characterizes the tops-only and strategy-proof DSCFs on single-peaked domains. In a similar attempt, we characterize the tops-only and strategy-proof RSCFs on top-rich domains. We demonstrate by way of examples that top-rich domains do not guarantee uncompromisingness for tops-only and strategy-proof rules, but it is guaranteed if an additional condition called top-connectedness is imposed on a top-rich domain. Top-connectedness is a commonly used property in social choice literature which says that for every two adjacent alternatives b_j and b_{j+1} in the top-set, there are preferences $P = b_j b_{j+1} \ldots$ and $P' = b_{j+1} b_j \ldots$ that differ only by their top two alternatives. We provide a complete characterization of the tops-only and strategy-proof RSCFs on top-rich domains satisfying top-connectedness. Our characterization shows that such an RSCF is a convex

[4] By $P = ab \ldots$, we mean a preference P where a is the top and b is the second-top alternatives.

combination of *tops-restricted generalized min-max* rules and *non-top constant* rules. A DSCF is a tops-restricted generalized min-max rule on a domain if it is a generalized min-max rule as defined in Weymark (2011) with parameters taking values in the top-set of the domain. A DSCF is a non-top constant rule on a domain if for all profiles, it selects a fixed alternative that is outside the top-set of the domain.

A domain is said to satisfy *deterministic extreme point* (DEP) property if every unanimous (tops-only) and strategy-proof RSCF on the domain is a convex combination of unanimous (tops-only) and strategy-proof DSCFs on it. The study of such domains is useful as it establishes a functional relationship between the DSCFs and the RSCFs satisfying unanimity (tops-onlyness) and strategy-proofness on those domains. Such a relationship can be utilized in finding the *optimum* RSCFs for a society. It follows from our result that top-rich domains satisfy DEP property for unanimous and strategy-proof RSCFs, and top-rich domains satisfying top-connectedness satisfy the same for tops-only and strategy-proof RSCFs. Gershkov et al. (2013) characterize the optimum DSCFs on single-crossing domains. It is worth noting that, by means of the DEP property of single-crossing domains, their result can be translated to RSCFs.

Barberà et al. (2010) show that strategy-proofness and group strategy-proofness are equivalent for the DSCFs on single-peaked domains and Saporiti (2009) shows the same for the DSCFs on single-crossing domains. In the same spirit, we show that the same holds for the RSCFs on all the domains we consider in this paper.

1.4 Remainder

The rest of the paper is organized as follows: Section 2 introduces the model and basic definitions. Section 3 presents our main results and Section 4 contains some application of those. Section 5 provides our results on group-strategy-proofness while Section 6 concludes the paper. Proofs of lemmas and theorems are gathered in the Appendix.

2. Preliminaries

Let $A = \{a_1, \ldots, a_m\}$ be a finite set of alternatives with a prior ordering $a_1 \prec \ldots \prec a_m$.[5] Let $N = \{1, \ldots, n\}$ be a finite set of agents. Except where otherwise mentioned, $n \geq 2$. For $a, b \in A$, we define $[a, b] = \{c \mid \text{ either } a \preceq c \preceq b \text{ or } b \preceq c \preceq a\}$, and for $B \subseteq A$, we define

[5]Whenever we write minimum or maximum of a subset of A, we mean it w.r.t. the ordering $\prec$ over A.

$[a, b]_B = [a, b] \cap B$.[6] A complete, antisymmetric, and transitive binary relation over A (also called a linear order) is called a preference. We denote by $\mathbb{L}(A)$ the set of all preferences over A. For $P \in \mathbb{L}(A)$ and $a, b \in A$, aPb is interpreted as "a is strictly preferred to b according to P", and $aP!b$ is interpreted as "a is contiguously ranked above b in P, i.e., aPb and there is there is no $c \in A$ such that aPc and cPb". For $P \in \mathbb{L}(A)$, by $r_k(P)$ we mean the k-th ranked alternative in P, i.e., $r_k(P) = a$ if and only if $|\{b \in A \mid bPa\}| = k - 1$. By P^a, we denote a preference with $r_1(P^a) = a$, and by $P^{a,b}$, we denote a preference with $r_1(P^{a,b}) = a$ and $r_2(P^{a,b}) = b$. For convenience, a preference $P^{a,b}$ is sometimes written as $ab\ldots$. We denote by $\mathcal{D} \subseteq \mathbb{L}(A)$ a set of admissible preferences. For $a \in A$, let $\mathcal{D}^a = \{P \in \mathcal{D} \mid r_1(P) = a\}$. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote sets $\{i\}$ by i. For a domain $\mathcal{D}$, the top-set of $\mathcal{D}$, denoted by $\tau(\mathcal{D})$, is defined as $\tau(\mathcal{D}) = \cup_{P \in \mathcal{D}} r_1(P)$. Whenever we write $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$, we assume without loss of generality that the indexation is such that $b_1 \prec \ldots \prec b_k$. A domain $\mathcal{D}$ is called *regular* if $\tau(\mathcal{D}) = A$. For $P \in \mathcal{D}$ and $a \in A$, the *upper contour set* of a at P, denoted by $U(a, P)$, is defined as the set of alternatives that are as good as a in P, i.e., $U(a, P) = \{b \in A \mid bPa\} \cup a$. A preference profile, denoted by $P_N = (P_1, \ldots, P_n)$, is an element of $\mathcal{D}^n = \mathcal{D} \times \ldots \times \mathcal{D}$.

A preference P is called *single-peaked* if for all $a, b \in A$, $[r_1(P) \preceq a \prec b$ or $b \prec a \preceq r_1(P)]$ implies aPb. A domain is called *single-peaked* if each preference in the domain is single-peaked and is called *maximal single-peaked* if it contains all single-peaked preferences.

For $P \in \mathbb{L}(A)$ and $B \subseteq A$, $P|_B \in \mathbb{L}(B)$ is defined as follows: for all $a, b \in B$, $aP|_B b$ if and only if aPb. For $\mathcal{D} \subseteq \mathbb{L}(A)$, $P_N \in \mathcal{D}^n$, and $B \subseteq A$, we define $\mathcal{D}|_B = \{P|_B \mid P \in \mathcal{D}\}$ and $P_N|_B = (P_1|_B, \ldots, P_n|_B)$. For $B \subseteq A$, a domain $\mathcal{D}$ of preferences is called a single-peaked domain restricted to B if $\mathcal{D}|_B$ is a single-peaked domain.

By $\triangle A$, we denote the set of probability distributions on A. For two sets X, Y, we define the symmetric difference of X and Y as $X \triangledown Y = (X \setminus Y) \cup (Y \setminus X)$.

A *Random Social Choice Function* (RSCF) is a function $\varphi : \mathcal{D}^n \to \triangle A$. For $B \subseteq A$ and $P_N \in \mathcal{D}^n$, we define by $\varphi_B(P_N) = \sum_{a \in B} \varphi_a(P_N)$, where $\varphi_a(P_N)$ is the probability of a at $\varphi(P_N)$.

REMARK 2.1. For all $L, L' \in \triangle A$ and all $P \in \mathbb{L}(A)$, if $L_{U(x,P)} \geq L'_{U(x,P)}$ and $L'_{U(x,P)} \geq L_{U(x,P)}$ for all $x \in A$, then $L = L'$.

For RSCFs φ^j, $j = 1, \ldots, k$ and nonnegative numbers λ^j, $j = 1, \ldots, k$, summing to 1, we define

[6] By $a \preceq b$, we mean $a = b$ or $a \prec b$.

the RSCF $\varphi = \sum_{j=1}^{k} \varphi^j$ by $\varphi_a(P_N) = \sum_{j=1}^{k} \lambda^j \varphi_a^j(P_N)$ for all $P_N \in \mathcal{D}^n$ and all $a \in A$. We call φ a *convex combination* of the RSCFs φ^j.

Definition 2.1. An RSCF $\varphi : \mathcal{D}^n \to \triangle A$ is called *unanimous* if for all $a \in A$ and all $P_N \in \mathcal{D}^n$,

$$[r_1(P_i) = a \text{ for all } i \in N] \Rightarrow [\varphi_a(P_N) = 1].$$

Definition 2.2. An RSCF $\varphi : \mathcal{D}^n \to \triangle A$ is called *strategy-proof* if for all $i \in N$, all $P_N \in \mathcal{D}^n$, all $P'_i \in \mathcal{D}$, and all $x \in A$,

$$\sum_{y \in U(x,P_i)} \varphi_y(P_i, P_{-i}) \geq \sum_{y \in U(x,P_i)} \varphi_y(P'_i, P_{-i}).$$

REMARK 2.2. An RSCF is called a DSCF if it selects a degenerate probability distribution at every preference profile. More formally, An RSCF $\varphi : \mathcal{D}^n \to \triangle A$ is called a DSCF if $\varphi_a(P_N) \in \{0, 1\}$ for all $a \in A$ and all $P_N \in \mathcal{D}^n$. The notions of unanimity and strategy-proofness for DSCFs are special cases of the corresponding definitions for RSCFs.

Definition 2.3. Two profiles $P_N, P'_N \in \mathcal{D}^n$ are called *tops-equivalent* if $r_1(P_i) = r_1(P'_i)$ for all $i \in N$.

Definition 2.4. An RSCF $\varphi : \mathcal{D}^n \to \triangle A$ is called *tops-only* if $\varphi(P_N) = \varphi(P'_N)$ for all tops-equivalent $P_N, P'_N \in \mathcal{D}^n$.

Definition 2.5. A DSCF f on $\mathcal{D}^n$ is called a *generalized tops-restricted min-max (GTM)* rule if for all $S \subseteq N$, there exists $\beta_S \in \tau(\mathcal{D})$ satisfying

$$\beta_T \preceq \beta_S \text{ for all } S \subseteq T$$

such that

$$f(P_N) = \min_{S \subseteq N} \left[\max_{i \in S} \{ r_1(P_i), \beta_S \} \right].$$

A GTM rule on $\mathcal{D}^n$ is called a *tops-restricted min-max (TM)* rule if $\beta_\emptyset = \max(\tau(\mathcal{D}))$ and $\beta_N = \min(\tau(\mathcal{D}))$. If $\tau(\mathcal{D}) = A$, then a GTM rule is called a *generalized min-max* rule and a TM rule is called a *min-max* rule.

Note that GTM rules are not unanimous, whereas TM rules are unanimous.

REMARK 2.3. Let $f : \mathcal{D}^n \to A$ be a GTM rule. Define $\hat{f} : (\mathcal{D}|_{\tau(\mathcal{D})})^n \to \tau(\mathcal{D})$ such that $\hat{f}(P_N|_{\tau(\mathcal{D})}) = f(P_N)$. This is well-defined since f is tops-only and $f(P_N) \in \tau(\mathcal{D})$ for all $P_N \in \mathcal{D}^n$. Then, f is strategy-proof if and only if $\hat{f}$ is strategy-proof.

Definition 2.6. An RSCF $\varphi : \mathcal{D}^n \to \triangle A$ is called a *tops-restricted random min-max (TRM)* rule if φ can be written as a convex combination of some TM rules. If $\tau(\mathcal{D}) = A$, then a TRM rule is called a *random min-max* rule.

Definition 2.7. An RSCF $\varphi : \mathcal{D}^n \to \triangle A$ is called *uncompromising* if $\varphi_B(P_N) = \varphi_B(P'_i, P_{-i})$ for all $i \in N$, all $P_N \in \mathcal{D}^n$, all $P'_i \in \mathcal{D}$, and all $B \subseteq A$ such that $B \cap [r_1(P_i), r_1(P'_i)] = \emptyset$.

REMARK 2.4. An uncompromising RSCF is tops-only by definition.

Definition 2.8. Let $\mathcal{D}$ be a domain with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$. Then, $\mathcal{D}$ is called *top-rich* if

(i) for all $b_j, b_{j+1} \in \tau(\mathcal{D})$, there exist $P = b_j b_{j+1} \ldots, P' = b_{j+1} b_j \ldots \in \mathcal{D}$ such that $U(b_l, P) \triangledown U(b_l, P') \subseteq \tau(\mathcal{D})$ for all $b_l \in \tau(\mathcal{D})$, and

(ii) for all $r < s < t$ and all $P^{b_r}, P^{b_t} \in \mathcal{D}$, $U(b_s, P^{b_r}) \cap U(b_s, P^{b_t}) = b_s$.

Condition (i) ensures some type of richness of the domain w.r.t. the top-set. It says the following. Take two alternatives, say b_j, b_{j+1}, that are consecutive in the top-set, that is, there is no other alternative in the top-set that lies in-between (w.r.t. the prior order $\prec$) those two alternatives. Then, there must be two preferences $P = b_j b_{j+1} \ldots, P' = b_{j+1} b_j \ldots$ such that for all $b \in \tau(\mathcal{D})$ and $c \notin \tau(\mathcal{D})$, bPc if and only if $bP'c$. Condition (ii) ensures that the domain respects the prior order $\prec$ over the alternatives in a suitable sense. For instance, as we show in Lemma A.1, one implication of this condition is that a top-rich domain restricted to its top-set is single-peaked. However, Condition (ii) is stronger than implying that the domain restricted to its top-set is single-peaked as it puts some restrictions on the ranking of the alternatives outside the top-set.

Below, we provide an example of a top-rich domain.

Example 2.1. Let the set of alternatives be $A = \{a_1, a_2, a_3, a_4, a_5\}$ with prior order $a_1 \prec \ldots \prec a_5$. Consider the domain $\mathcal{D} = \{P_1, \ldots, P_7\}$ given in Table 1.

P_1	P_2	P_3	P_4	P_5	P_6	P_7
a_1	a_1	a_3	a_3	a_3	a_5	a_5
a_2	a_3	a_1	a_1	a_5	a_3	a_4
a_3	a_2	a_2	a_4	a_1	a_1	a_3
a_4	a_4	a_4	a_2	a_2	a_2	a_2
a_5	a_5	a_5	a_5	a_4	a_4	a_1

Table 1

Note that $\tau(\mathcal{D}) = \{b_1, b_2, b_3\}$, where $b_1 = a_1$, $b_2 = a_3$, and $b_3 = a_5$. We show that $\mathcal{D}$ is a top-rich domain, in particular, we show that it satisfies Conditions (i) and (ii) in Definition 2.8.

For Condition (i), consider, for instance, b_1 and b_2. Take $P = P_2 = a_1a_3\ldots$ and $P' = P_4 = a_3a_1\ldots$. Then, $r_l(P) = r_l(P')$ for all $l \geq 3$, and hence $U(x, P) \triangledown U(x, P') = \emptyset$ for all $x \in \tau(\mathcal{D}) \setminus \{b_1, b_2\}$. So, Condition (i) is satisfied for b_1 and b_2. Similarly, Condition (i) can be verified for b_2 and b_3.

For Condition (ii), consider $r = 1, s = 2, t = 3$, and preferences $P^{b_1} = P_1$ and $P^{b_3} = P_5$. Then, $U(b_2, P^{b_1}) = \{a_1, a_2, a_3\}$ and $U(b_2, P^{b_3}) = \{a_3, a_4, a_5\}$. Therefore, $U(b_2, P^{b_1}) \cap U(b_2, P^{b_3}) = \{b_2\}$, as required by Condition (ii). Similarly, Condition (ii) can be verified for other cases.

3. Results

3.1 Strategy-proofness and Unanimity

In this subsection, we present our main result characterizing the unanimous and strategy-proof RSCFs on top-rich domains.

In the following theorem, we show that a unanimous and strategy-proof RSCF on a top-rich domain never assigns a positive probability to an alternative outside the top-set, and such an RSCF is uncompromising.

Theorem 3.1. *Let $\mathcal{D}$ be a top-rich domain and let $\varphi : \mathcal{D}^n \to \triangle A$ be a unanimous and strategy-proof RSCF. Then,*

(i) $\varphi_{\tau(\mathcal{D})}(P_N) = 1$ *for all* $P_N \in \mathcal{D}^n$*, and*

(ii) φ *is uncompromising.*

The proof of this theorem is relegated to Appendix A.

Our next theorem provides a characterization of the unanimous and strategy-proof RSCFs on top-rich domains.

Theorem 3.2. *Let $\mathcal{D}$ be a top-rich domain and let $\varphi : \mathcal{D}^n \to \triangle A$ be an RSCF. Then, φ is unanimous and strategy-proof if and only if it is a TRM rule.*

The proof of this theorem is relegated to Appendix B.

3.1.1 Top-connected TRM Domains

In this section, we provide a characterization of the domains on which (i) every TRM rule is strategy-proof, and (ii) every unanimous and strategy-proof rule is a TRM rule. For tractability and transparency, we restrict our attention to the *top-connected* domains. We call such domains *top-connected TRM* domains. Below, we provide a formal definition of such domains.

Definition 3.1. Two preferences $P, P' \in \mathcal{D}$ are called *top-connected*, denoted by $P \sim P'$, if $r_1(P) = r_2(P'), r_1(P') = r_2(P)$, and $r_l(P) = r_l(P')$ for all $l \geq 3$.

Definition 3.2. A domain $\mathcal{D}$ with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$ satisfies the top-connectedness property if for all $j = 1, \ldots, k-1$, there exist $P \in \mathcal{D}^{b_j}, P' \in \mathcal{D}^{b_{j+1}}$ such that $P \sim P'$.

Definition 3.3. A domain $\mathcal{D}$ is called a TRM domain if

(i) every TRM rule on $\mathcal{D}^n$ is strategy-proof, and

(ii) every unanimous and strategy-proof RSCF on $\mathcal{D}^n$ is a TRM rule.

Furthermore, a domain is called a top-connected TRM domain if it is a TRM domain and it satisfies top-connectedness.

In the following, we establish a crucial property of TRM domains.

Lemma 3.1. *Let $\mathcal{D}$ be a TRM domain. Then, $\mathcal{D}|_{\tau(\mathcal{D})}$ is a single-peaked domain.*

Proof. Let $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$. Assume for contradiction that there exists $Q \in \mathcal{D}$ such that $Q|_{\tau(\mathcal{D})}$ is not single-peaked. Without loss of generality, this means there exist b_r, b_s with $b_r \prec b_s \prec r_1(Q)$ such that $b_r Q b_s$. Consider the tops-restricted min-max rule f on $\mathcal{D}^n$ such that $\beta_S = b_r$ for all non-empty $S \subsetneq N$. Consider the profile $P_N \in \mathcal{D}^n$ such that $P_1 = Q$ and $r_1(P_i) = b_s$ for all $i \neq 1$. Then, by the definition of f, $f(P_N) = b_s$. Let $P_1' \in \mathcal{D}$ be such that $r_1(P_1') = b_r$. Again, by the definition of f, $f(P_1', P_{-1}) = b_r$. Because $b_r Q b_s$, this means agent 1 manipulates at P_N via P_1', which is a contradiction. ■

Our next theorem provides a characterization of top-connected TRM domains.

Theorem 3.3. *A domain $\mathcal{D}$ is a top-connected TRM domain if and only if it is a top-rich domain satisfying top-connectedness.*

The proof of this theorem is relegated to Appendix C.

3.2 Strategy-proofness and Tops-onlyness

In this subsection, we replace unanimity by tops-onlyness and characterize the tops-only and strategy-proof RSCFs on top-rich domains. First, we introduce the notion of a non-top constant rule that we use in our characterization.

Definition 3.4. A DSCF $f : \mathcal{D}^n \to A$ is called a *non-top constant rule* if there exists $c \in A \setminus \tau(\mathcal{D})$ such that $f(P_N) = c$ for all $P_N \in \mathcal{D}^n$.

The following example shows that tops-only and strategy-proof on top-rich domains is not necessarily uncompromising.

Example 3.1. Let the set of alternatives be $A = \{a_1, a_2, a_3, a_4, a_5\}$ with prior order $a_1 \prec \ldots \prec a_5$. Consider the domain $\mathcal{D} = \{a_1a_2a_3a_4a_5, a_2a_1a_4a_3a_5, a_2a_5a_1a_4a_3, a_5a_2a_1a_3a_4\}$. It is easy to see that $\mathcal{D}$ is a top-rich domain with $\tau(\mathcal{D}) = \{a_1, a_2, a_5\}$. Consider the RSCF, say φ, given in Table 2. It can be verified that φ is tops-only and strategy-proof . We show φ is not uncompromising. Consider the outcomes $\varphi(a_1a_2a_3a_4a_5, a_1a_2a_3a_4a_5) = (.3, .2, .2, .1, .2)$ and $\varphi(a_1a_2a_3a_4a_5, a_2a_1a_4a_3a_5) = (.2, .3, .1, .2, .2)$. Here, agent 2 changes his top from a_1 to a_2, but the probabilities of a_3 and a_4 are changed. Hence, φ is not uncompromising.

1 \ 2	$a_1a_2a_3a_4a_5$	$a_2a_1a_4a_3a_5$	$a_2a_5a_1a_4a_3$	$a_5a_2a_1a_3a_4$
$a_1a_2a_3a_4a_5$	$(.3, .2, .2, .1, .2)$	$(.2, .3, .1, .2, .2)$	$(.2, .3, .1, .2, .2)$	$(.2, .2, .2, .1, .3)$
$a_2a_1a_4a_3a_5$	$(.3, .2, .2, .1, .2)$	$(.2, .3, .1, .2, .2)$	$(.2, .3, .1, .2, .2)$	$(.2, .2, .1, .2, .3)$
$a_2a_5a_1a_4a_3$	$(.3, .2, .2, .1, .2)$	$(.2, .3, .1, .2, .2)$	$(.2, .3, .1, .2, .2)$	$(.2, .2, .1, .2, .3)$
$a_5a_2a_1a_3a_4$	$(.3, .2, .2, .1, .2)$	$(.2, .3, .1, .2, .2)$	$(.2, .3, .1, .2, .2)$	$(.2, .2, .1, .2, .3)$

Table 2

In view of Example 3.1, we look for an additional condition on top-rich domains to ensure uncompromisingness for the tops-only and strategy-proof RSCFs. We show in our next theorem that top-connectedness is one such condition. More formally, we show that if a top-rich domain satisfies top-connectedness, then every tops-only and strategy-proof RSCF defined on it is uncompromising. We further show that such a rule assigns a fixed probability to the top-set of the domain.

Theorem 3.4. *Let $\mathcal{D}$ be a top-rich domain satisfying top-connectedness and let $\varphi : \mathcal{D}^n \to \triangle A$ be a tops-only and strategy-proof RSCF. Then,* *(i)* φ *is uncompromising,*

(ii) there exists $0 \leq \alpha \leq 1$ *such that* $\varphi_{\tau(\mathcal{D})}(P_N) = \alpha$ *for all* $P_N \in \mathcal{D}^n$*, and*

(iii) $\varphi_c(P_N) = \varphi_c(P'_N)$ *for all* $P_N, P'_N \in \mathcal{D}^n$ *and all* $c \in A \setminus \tau(\mathcal{D})$*.*

The proof of this theorem is relegated to Appendix D.

Now, we provide a characterization of the tops-only and strategy-proof RSCFs on top-rich domains satisfying top-connectedness.

Theorem 3.5. *Let* $\mathcal{D}$ *be a top-rich domain satisfying top-connectedness and let* $\varphi : \mathcal{D}^n \to \triangle A$ *be an RSCF. Then,* φ *is tops-only and strategy-proof if and only if it is a convex combination of GTM rules and non-top constant rules.*

The proof of this theorem is relegated to Appendix E.

4. APPLICATION

In this section, we demonstrate the applicability of our results in characterizing the unanimous and strategy-proof random rules on some well-known domains.

4.1 SINGLE-PEAKED DOMAINS

As discussed earlier, many domains of practical importance satisfy single-peakedness property. Ehlers et al. (2002) characterize the unanimous and strategy-proof RSCFs on maximal single-peaked domains as fixed-probabilistic-ballots rules, and Peters et al. (2014) show that such an RSCF is a convex combination of min-max rules. In what follows, we characterize all single-peaked domains where every unanimous and strategy-proof RSCF is a convex combination of min-max rules. We call such domains regular random min-max domains.

4.1.1 REGULAR RANDOM MIN-MAX DOMAINS

We begin with the formal definition of regular random min-max domains.

Definition 4.1. A domain $\mathcal{D}$ is called a *regular random min-max domain* if

(i) $\mathcal{D}$ is regular,

(ii) every random min-max rule on $\mathcal{D}^n$ is strategy-proof, and

(iii) every unanimous and strategy-proof RSCF on $\mathcal{D}^n$ is a random min-max rule.

Now, we introduce the notion of a weakly top-connected single peaked domain. Two preferences $P, P' \in \mathcal{D}$ are called *weakly top-connected* if $P = ab\ldots$ and $P' = ba\ldots$ for some $a, b \in A$. Note that P and P' might have different orderings over the alternatives other than their top two alternatives.

Definition 4.2. A domain $\mathcal{D}$ is called a *weakly top-connected single peaked* domain if $\mathcal{D}$ is a single-peaked domain and for all $1 \leq j < m$, there exist $P \in \mathcal{D}^{a_j}$, $P' \in \mathcal{D}^{a_{j+1}}$ such that P and P' are weakly top-connected.

REMARK 4.1. A weakly top-connected single-peaked domain trivially satisfies Condition (i) in Definition 2.8.

Our next theorem provides a characterization of regular random min-max domains.

Theorem 4.1. *A domain is a regular random min-max domain if and only if it is a weakly top-connected single-peaked domain.*

Proof. (If Part) Let $\mathcal{D}$ be a weakly top-connected single-peaked domain. We show that $\mathcal{D}$ is a regular random min-max domain. By Remark 4.1, $\mathcal{D}$ satisfies Condition (i) in Definition 2.8. Moreover, since $\mathcal{D}$ is single-peaked, Condition (ii) in Definition 2.8 is also satisfied. Therefore, $\mathcal{D}$ is a top-rich domain with $\tau(\mathcal{D}) = A$. So, by Theorem 3.2, $\mathcal{D}$ is a regular random min-max domain.

(Only-if Part) Let $\mathcal{D}$ be a regular random min-max domain. Then, by definition, $\mathcal{D}$ is a TRM domain with $\tau(\mathcal{D}) = A$. Therefore, by Lemma 3.1, $\mathcal{D}$ is single-peaked. We show $\mathcal{D}$ is weakly top-connected. Assume for contradiction that for some $j < m$, there do not exist $P, P' \in \mathcal{D}$ such that $P = a_j a_{j+1}\ldots$ and $P' = a_{j+1} a_j \ldots$. Without loss of generality, assume that $r_2(P) \neq a_{j+1}$ for all $P \in \mathcal{D}^{a_j}$. Since $\mathcal{D}$ is single-peaked, this means $j \neq 1$ and $r_2(P) = a_{j-1}$ for all $P \in \mathcal{D}^{a_j}$. Consider the DSCF f on $\mathcal{D}^n$ as follows:[7]

$$f(P_N) = \begin{cases} r_1(P_1) \text{ if } r_1(P_1) \neq a_j, \\ a_j \text{ if } r_1(P_1) = a_j \text{ and } a_j P_2 a_{j-1}, \\ a_{j-1} \text{ otherwise.} \end{cases}$$

[7]Here $\mathcal{D}$ satisfies the *unique seconds* property defined in Aswal et al. (2003) and the SCF f considered here is similar to the one used in the proof of Theorem 5.1 in Aswal et al. (2003).

It can be verified that f is unanimous and strategy-proof. We show that f is not a min-max rule. In particular, we show that f is not uncompromising. This is sufficient as every min-max rule is uncompromising. Let $P_N \in \mathcal{D}^n$ be such that $r_1(P_2) = a_1$. Then, by the definition of f, $f(P_N) = a_{j-1}$ when $r_1(P_1) = a_j$, and $f(P'_1, P_{-1}) = a_{j+1}$ when $r_1(P'_1) = a_{j+1}$. This clearly violates uncompromisingness for agent 1. This completes the proof of the only-if part. ■

4.1.2 Strategy-proof and Tops-only Random Rules

In this subsection, we provide a characterization of the tops-only and strategy-proof RSCFs on top-connected single-peaked domains. First, we show by means of an example that weakly top-connected single-peaked domains do not guarantee uncompromisingness for the tops-only and strategy-proof RSCFs on those domains.

Example 4.1. Let the set of alternatives be $A = \{a_1, a_2, a_3, a_4\}$ with prior order $a_1 \prec \ldots \prec a_4$. Consider the domain $\mathcal{D} = \{a_1a_2a_3a_4, a_2a_1a_3a_4, a_2a_3a_1a_4, a_3a_2a_4a_1, a_3a_4a_2a_1, a_4a_3a_2a_1\}$. It can be easily verified that $\mathcal{D}$ is a weakly top-connected single-peaked domain. Consider the RSCF, say φ, given in Table 3. It is left to the reader to verify that φ is tops-only and strategy-proof . We show that φ is not uncompromising. Consider the outcomes $\varphi(a_1a_2a_3a_4, a_2a_3a_1a_4) = (.3, .3, .2, .2)$ and $\varphi(a_1a_2a_3a_4, a_3a_2a_4a_1) = (.2, .2, .3, .3)$. Here, agent 2 changes his top from a_2 to a_3, but the probabilities of a_1 and a_4 are changed. Hence, φ is not uncompromising.

1 \ 2	$a_1a_2a_3a_4$	$a_2a_1a_3a_4$	$a_2a_3a_1a_4$	$a_3a_2a_4a_1$	$a_3a_4a_2a_1$	$a_4a_3a_2a_1$
$a_1a_2a_3a_4$	$(.3, .3, .2, .2)$	$(.3, .3, .2, .2)$	$(.3, .3, .2, .2)$	$(.2, .2, .3, .3)$	$(.2, .2, .3, .3)$	$(.2, .2, .3, .3)$
$a_2a_1a_3a_4$	$(.3, .3, .2, .2)$	$(.3, .3, .2, .2)$	$(.3, .3, .2, .2)$	$(.2, .2, .3, .3)$	$(.2, .2, .3, .3)$	$(.2, .2, .3, .3)$
$a_2a_3a_1a_4$	$(.3, .3, .2, .2)$	$(.3, .3, .2, .2)$	$(.3, .3, .2, .2)$	$(.2, .2, .3, .3)$	$(.2, .2, .3, .3)$	$(.2, .2, .3, .3)$
$a_3a_2a_4a_1$	$(.3, .3, .2, .2)$	$(.3, .3, .2, .2)$	$(.3, .3, .2, .2)$	$(.2, .2, .3, .3)$	$(.2, .2, .3, .3)$	$(.2, .2, .3, .3)$
$a_3a_4a_2a_1$	$(.3, .3, .2, .2)$	$(.3, .3, .2, .2)$	$(.3, .3, .2, .2)$	$(.2, .2, .3, .3)$	$(.2, .2, .3, .3)$	$(.2, .2, .3, .3)$
$a_4a_3a_2a_1$	$(.3, .3, .2, .2)$	$(.3, .3, .2, .2)$	$(.3, .3, .2, .2)$	$(.2, .2, .3, .3)$	$(.2, .2, .3, .3)$	$(.2, .2, .3, .3)$

Table 3

In view of Example 4.1, we strengthen weak top-connectedness by requiring top-connectedness. Clearly, a top-connected single-peaked domain is a top-rich domain satisfying top-connectedness. Moreover, such a domain is regular by definition. Therefore, there does not exist any non-top constant rule on such domains. This yields the following corollary.

Corollary 4.1. *Let $\mathcal{D}$ be a top-connected single-peaked domain and let $\varphi : \mathcal{D}^n \to \triangle A$ be an RSCF. Then, φ is tops-only and strategy-proof if and only if it is a convex combination of generalized min-max rules.*

4.2 Single-crossing Domains

In this subsection, we introduce the notion of single-crossing domains and provide a characterization of the unanimous (tops-only) and strategy-proof RSCFs on these domains.

Definition 4.3. A domain $\mathcal{D}$ is called a *single-crossing domain w.r.t. an ordering $<$ over $\mathcal{D}$* if for all $a, b \in A$ and all $P, P' \in \mathcal{D}$,

$$[a \prec b, P < P', \text{ and } bPa] \implies bP'a.$$

A domain is called *single-crossing* if it is single-crossing w.r.t. some ordering over the domain.

Definition 4.4. A single-crossing domain $\bar{\mathcal{D}}$ is called *maximal* if there does not exist a single-crossing domain $\mathcal{D}$ such that $\bar{\mathcal{D}} \subsetneq \mathcal{D}$.

Remark 4.2. A maximal single-crossing domain with m alternatives contains $m(m-1)/2+1$ preferences.[8]

Definition 4.5. A domain $\mathcal{D}$ is called a *successive single-crossing domain* if there is a maximal single-crossing domain $\bar{\mathcal{D}}$ w.r.t. some ordering $<$, and $P', P'' \in \bar{\mathcal{D}}$ with $P' \leq P''$ such that $\mathcal{D} = \{P \in \bar{\mathcal{D}} \mid P' \leq P \leq P''\}$.

The notion of a successive single-crossing domain is introduced in Carroll (2012). In the following example, we present a maximal single-crossing domain and a successive single-crossing domain with 5 alternatives.

Example 4.2. Let the set of alternatives be $A = \{a_1, a_2, a_3, a_4, a_5\}$ with prior order $a_1 \prec \ldots \prec a_5$. Consider the domain $\bar{\mathcal{D}} = \{a_1a_2a_3a_4a_5,\ a_2a_1a_3a_4a_5,\ a_2a_3a_1a_4a_5,\ a_2a_3a_4a_1a_5,\ a_2a_4a_3a_1a_5,\ a_4a_2a_3a_1a_5,\ a_4a_2a_3a_5a_1,\ a_4a_3a_2a_5a_1, a_4a_3a_5a_2a_1,\ a_4a_5a_3a_2a_1,\ a_5a_4a_3a_2a_1\}$. Then, $\bar{\mathcal{D}}$ is a maximal single-crossing domain w.r.t. the ordering $<$ over $\bar{\mathcal{D}}$ given by $a_1a_2a_3a_4a_5 < a_2a_1a_3a_4a_5 < a_2a_3a_1a_4a_5 < a_2a_3a_4a_1a_5 < a_2a_4a_3a_1a_5 < a_4a_2a_3a_1a_5 < a_4a_2a_3a_5a_1 < a_4a_3a_2a_5a_1 < a_4a_3a_5a_2a_1 < a_4a_5a_3a_2a_1 < a_5a_4a_3a_2a_1$. Note that the cardinality of A is 5 and that of $\bar{\mathcal{D}}$ is $5(5-1)/2+1 = 11$. Now, consider the domain $\mathcal{D} = \{a_1a_2a_3a_4a_5,\ a_2a_1a_3a_4a_5,\ a_2a_3a_1a_4a_5,\ a_2a_3a_4a_1a_5\}$. Then, $\mathcal{D}$ is a successive single-crossing domain.

[8] For details see Saporiti (2009).

Our next lemma shows that every successive single-crossing domain is a top-rich domain satisfying top-connectedness.

Lemma 4.1. *Let $\mathcal{D}$ be a successive single-crossing domain. Then, $\mathcal{D}$ is a top-rich domain satisfying top-connectedness.*

Proof. Let $\mathcal{D}$ be a successive single-crossing domain. Then, by the definition of successive single-crossing domain, there is a a maximal single-crossing domain $\bar{\mathcal{D}}$ w.r.t. some ordering $<$ such that $\mathcal{D} = \{P \in \bar{\mathcal{D}} \mid \tilde{P} \leq P \leq \tilde{\tilde{P}}\}$ for some $\tilde{P}, \tilde{\tilde{P}} \in \bar{\mathcal{D}}$ with $\tilde{P} \leq \tilde{\tilde{P}}$. Suppose $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$. We show that for all $j = 1, 2, \ldots, k-1$, there are $P \in \mathcal{D}^{b_j}$ and $P' \in \mathcal{D}^{b_{j+1}}$ such that $P \sim P'$. Take $b_j, b_{j+1} \in \tau(\mathcal{D})$ and take $\bar{P} \in \mathcal{D}^{b_j}$ and $\hat{P} \in \mathcal{D}^{b_{j+1}}$. Since $b_j \bar{P} b_{j+1}$, $b_{j+1} \hat{P} b_j$, and $b_j \prec b_{j+1}$, it follows from the definition of single-crossing domain that $\bar{P} < \hat{P}$. Using a similar argument, $P^{b_l} < \bar{P}$ for all $l < j$, and $P^{b_l} > \hat{P}$ for all $l > j+1$. Therefore, there must be $P \in \mathcal{D}^{b_j}$ and $P' \in \mathcal{D}^{b_{j+1}}$ that are consecutive in the ordering $<$ meaning that there is no $P'' \in \mathcal{D}$ with $P < P'' < P'$. We show $P \sim P'$. Suppose not. Let a be the alternative such that $aP!b_{j+1}$. Consider the preference P'' that is obtained by switching the alternatives a and b_{j+1} in P, i.e., $P'' \triangledown P = \{(a, b_{j+1}), (b_{j+1}, a)\}$. We show $P'' \notin \bar{\mathcal{D}}$. In particular, we show that $P'' \not< P$ and $P' \not< P''$. This is sufficient since P and P' are consecutive in the ordering $<$. Suppose $P'' < P$. Since aPb_{j+1}, $P < P'$, and $b_{j+1}P'a$, by the single-crossing property of $\bar{\mathcal{D}}$, it must be that $a \prec b_{j+1}$. However, because $b_{j+1}P''a$ and aPb_{j+1}, this contradicts $P'' < P$. Now, suppose $P' < P''$. Since $P < P'$, there must be a pair of alternatives c, d with $c \prec d$ such that cPd and $dP'c$. Moreover, because P and P' are not top-connected, it must be that $\{c, d\} \neq \{a, b_{j+l}\}$. Since $c \prec d$, $dP'c$, and $P' < P''$, by the single-crossing property of $\bar{\mathcal{D}}$, we have $dP''c$. However, by the construction of P'', we have $cP''d$, a contradiction. Thus, we have $P'' \notin \bar{\mathcal{D}}$. However, this means $\bar{\mathcal{D}} \cup P''$ is a single-crossing domain w.r.t. the ordering $<'$ over $\bar{\mathcal{D}} \cup P''$, where $<'$ is obtained by placing P'' in-between P and P' in the ordering $<$, i.e., $<'$ coincides with $<$ over $\bar{\mathcal{D}}$ and $P <' P'' <' P'$. This contradicts that $\bar{\mathcal{D}}$ is a maximal single-crossing domain. Therefore, $P \sim P'$.

Now, we show that $U(b_s, P^{b_r}) \cap U(b_s, P^{b_t}) = \{b_s\}$ for all $r < s < t$ and all P^{b_r}, $P^{b_t} \in \mathcal{D}$. Assume for contradiction that $a \in U(b_s, P^{b_r}) \cap U(b_s, P^{b_t})$ for some $a \neq b_s$. Take $P^{b_s} \in \mathcal{D}$. Using a similar argument as before, we have $P^{b_r} < P^{b_s} < P^{b_t}$. Because $a \in U(b_s, P^{b_r}) \cap U(b_s, P^{b_t})$, we have $aP^{b_r}b_s$, $b_sP^{b_s}a$, and $aP^{b_t}b_s$. Now, if $a \prec b_s$, then by the single-crossing property of $\mathcal{D}$, $b_sP^{b_s}a$ and $P^{b_s} < P^{b_t}$ together imply $b_sP^{b_t}a$, a contradiction. On the other hand, if $b_s \prec a$, then by the single-crossing property of $\mathcal{D}$, $aP^{b_r}b_s$ and $P^{b_r} < P^{b_s}$ together imply $aP^{b_s}b_s$, a contradiction. This completes the proof. ■

Our next two corollaries characterize the unanimous (tops-only) and strategy-proof RSCFs on successive single-crossing domains. Note that if a domain $\mathcal{D}$ satisfies Condition (ii) in Definition 2.8, then any subdomain (i.e., subset) of $\mathcal{D}$ also satisfies the same. Moreover, if a domain satisfies top-connectedness, then it satisfies Condition (i) in Definition 2.8.

Corollary 4.2. *Let $\mathcal{D}$ be a subset of a successive single-crossing domain satisfying Condition (i) in Definition 2.8 and let $\varphi : \mathcal{D}^n \to \triangle A$ be an RSCF. Then, φ is unanimous and strategy-proof if and only if it is a TRM rule.*

Corollary 4.3. *Let $\mathcal{D}$ be a subset of successive single-crossing domain satisfying top-connectedness and let $\varphi : \mathcal{D}^n \to \triangle A$ be an RSCF. Then, φ is tops-only and strategy-proof if and only if it is a convex combination of GTM and non-top constant rules.*

4.3 Single-dipped Domains

In this subsection, we introduce the notion of single-dipped domains and present a characterization of the unanimous (tops-only) and strategy-proof RSCFs on these domains. We begin with the formal definition of single-dipped domains.

Definition 4.6. A preference P is called *single-dipped* if P has a unique minimal element $d(P)$, the *dip* of P, such that for all $a, b \in A$, $[d(P) \preceq a \prec b$ or $b \prec a \preceq d(P)] \Rightarrow bPa$. A domain is called single-dipped if each preference in it is single-dipped.

A single-dipped domain vacuously satisfies Condition (ii) in Definition 2.8. Therefore, a single-dipped domain satisfying Condition (i) in Definition 2.8 is a top-rich domain. This yields the following two corollaries.

Corollary 4.4. *Let $\mathcal{D}$ be a single-dipped domain satisfying Condition (i) in Definition 2.8 and let φ : $\mathcal{D}^n \to \triangle A$ be an RSCF. Then, φ is unanimous and strategy-proof if and only if it is a TRM rule.*

Corollary 4.5. *Let $\mathcal{D}$ be a single-dipped domain satisfying top-connectedness and let $\varphi : \mathcal{D}^n \to \triangle A$ be an RSCF. Then, φ is tops-only and strategy-proof if and only if it is a convex combination of GTM and non-top constant rules.*

4.4 Binary-restricted Domains

The notion of binary-restricted domains is introduced in Peters et al. (2017). However, their notion is based on weak preferences, i.e., preferences with indifferences. For the sake of completeness, we present below the definition of binary-restricted domains for strict preferences.

Definition 4.7. A domain $\mathcal{D}$ is called a *stricly binary-restricted* domain if $|\tau(\mathcal{D})| = 2$.

The following corollaries are an immediate consequence of Theorem 3.2 and Theorem 3.5.

Corollary 4.6. *Let $\mathcal{D}$ be a strictly binary-restricted domain satisfying Condition (i) in Definition 2.8 and let $\varphi : \mathcal{D}^n \to \triangle A$ be an RSCF. Then, φ is unanimous and strategy-proof if and only if it is a TRM rule.*

Corollary 4.7. *Let $\mathcal{D}$ be a strictly binary-restricted domain satisfying top-connectedness and let $\varphi : \mathcal{D}^n \to \triangle A$ be an RSCF. Then, φ is tops-only and strategy-proof if and only if it is a convex combination of GTM and non-top constant rules.*

4.5 Other Domains: Distance based Domains

In this subsection, we present a class of top-rich domains that are neither single-peaked nor single-dipped nor single-crossing domains. Such domains are based on some type of (perceived) distance between the alternatives. We call these distance based domains.

In order to illustrate the practical significance of distance based domains, let us consider a public good location problem of the following type. Suppose that there are ten available locations $b_1, \ldots, b_5, c_1, \ldots, c_5$ to locate a public good amongst which only $b_1, \ldots, b_5$ are the residential areas. Assume that the agents base their preferences over the locations on some type of distances (geometric/traffic or so). Suppose further that the ties are broken on both sides. Naturally, the top-alternative in any such preference can only be one of the locations $b_1, \ldots, b_5$. Figure 1 presents the communication graph together with the distances between the locations, and Table 4 presents the domain that is based on the graph in Figure 1.

It is fairly straightforward to verify that the domain in Table 4 is a top-rich domain satisfying top-connectedness. This yields the following corollaries.

Corollary 4.8. *Let $\mathcal{D}$ be the domain in Table 4 and let $\varphi : \mathcal{D}^n \to \triangle A$ be an RSCF. Then, φ is unanimous and strategy-proof if and only if it is a TRM rule.*

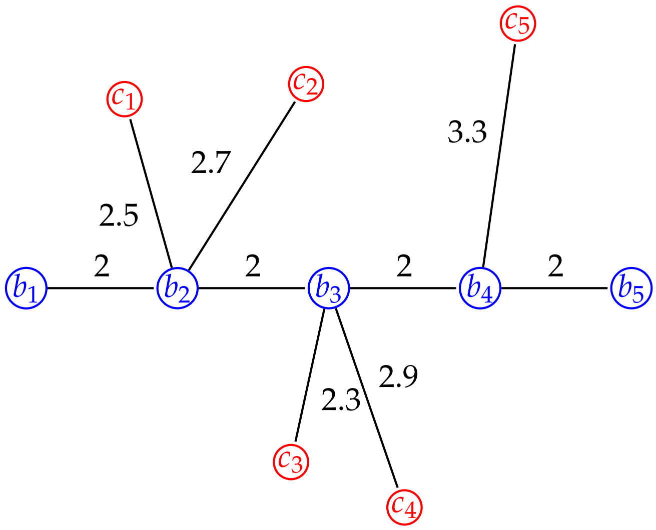

Figure 1

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
b_1	b_2	b_2	b_3	b_3	b_4	b_4	b_5
b_2	b_1	b_3	b_2	b_4	b_3	b_5	b_4
c_1	c_1	c_2	c_2	c_3	c_3	c_4	c_4
c_2	c_2	c_3	c_3	c_2	c_2	c_5	c_5
c_3	c_3	c_1	c_1	c_4	c_4	b_3	b_3
b_3	b_3	b_1	b_1	b_2	b_2	c_3	c_3
b_4	b_4	b_4	b_4	b_5	b_5	c_2	c_2
c_4	c_4	c_4	c_4	c_1	c_1	b_2	b_2
c_5	c_5	c_5	c_5	c_5	c_5	c_1	c_1
b_5	b_5	b_5	b_5	b_1	b_1	b_1	b_1

Table 4

Corollary 4.9. *Let $\mathcal{D}$ be the domain in Table 4 and let $\varphi : \mathcal{D}^n \to \triangle A$ be an RSCF. Then, φ is tops-only and strategy-proof if and only if it is a convex combination of GTM and non-top constant rules.*

4.6 Deterministic Extreme Point Property

In this subsection, we introduce the notion of deterministic extreme point property and show that top-rich domains satisfy this property.

Definition 4.8. A domain $\mathcal{D}$ is said to satisfy *deterministic extreme point* property for unanimous (tops-only) and strategy-proof RSCFs if every unanimous (tops-only) and strategy-proof RSCF on $\mathcal{D}^n$ can be written as a convex combination of unanimous (tops-only) and strategy-proof DSCFs on $\mathcal{D}^n$.

The following two corollaries are obtained from Theorem 3.2 and Theorem 3.4.

Corollary 4.10. *Top-rich domains satisfy deterministic extreme point property for unanimous and strategy-proof RSCFs.*

Corollary 4.11. *Top-rich domains satisfying top-connectedness satisfy deterministic extreme point property for tops-only and strategy-proof RSCFs.*

5. Group Strategy-proofness

In this section, we consider group strategy-proofness and provide a characterization of the unanimous (tops-only) and group strategy-proof RSCFs on top-rich domains. We begin with the formal definition of group strategy-proofness.

Definition 5.1. An RSCF $\varphi : \mathcal{D}^n \to \triangle A$ is *group strategy-proof* if for all non-empty coalitions $C \subseteq N$, all $P_N \in \mathcal{D}^n$, and all $P'_C \in \mathcal{D}^{|C|}$, there is $i \in C$ such that for all $x \in A$

$$\sum_{y \in U(x,P_i)} \varphi_y(P_N) \geq \sum_{y \in U(x,P_i)} \varphi_y(P'_C, P_{-C}).$$

Group strategy-proofness implies that, whenever a non-empty coalition of agents misreport their preferences, there is an agent in that coalition who does not strictly benefit from the misrepresentation.

Remark 5.1. A convex combination of some unanimous (tops-only) and group strategy-proof DSCFs is a unanimous (tops-only) and group strategy-proof RSCF.

Theorem 5.1. *Let $\mathcal{D}$ be a top-rich domain. Then, an RSCF $\varphi : \mathcal{D}^n \to \triangle A$ is unanimous and group strategy-proof if and only if it is a TRM rule.*

Proof. (If Part) A TRM rule is unanimous by definition. Therefore, in view of Remark 5.1, it is sufficient to show that each TM rule is group strategy-proof. Let f be a TM rule. Define $f' : (\mathcal{D}|_{\tau(\mathcal{D})})^n \to A$ such that $f'(P_N|_{\tau(\mathcal{D})}) = f(P_N)$. By Remark 2.3, f' is well-defined. Because $\mathcal{D}|_{\tau(\mathcal{D})}$ is single-peaked and f' is a min-max rule on $(\mathcal{D}|_{\tau(\mathcal{D})})^n$, it follows from Barberà et al. (2010) that f' is group strategy-proof. By the definition of f', this implies f is group strategy-proof. This completes the if part of the proof.
(Only-if Part) Since every group strategy-proof RSCF is strategy-proof, the proof of this part follows from Theorem 3.2. ■

Theorem 5.2. *Let $\mathcal{D}$ be a top-rich domain satisfying top-connectedness. Then, an RSCF $\varphi : \mathcal{D}^n \to \triangle A$ is tops-only and group strategy-proof if and only if it is a convex combination of GTM rules and non-top constant rules.*

Proof. (If Part) Since GTM rules and non-top constant rules are all tops-only, a convex combination of those is also tops-only. Moreover, non-top constant rules are clearly group strategy-proof. Therefore, in view of Remark 5.1, it is sufficient to show that each GTM rule is group strategy-proof. Let f be a GTM rule. Define $f' : (\mathcal{D}|_{\tau(\mathcal{D})})^n \to A$ as in the proof of Theorem 5.1. Since f is a GTM rule, f' is a generalized min-max rule. Moreover, because $\mathcal{D}|_{\tau(\mathcal{D})}$ is single-peaked and f' is a generalized min-max rule on $(\mathcal{D}|_{\tau(\mathcal{D})})^n$, it follows from Barberà et al. (2010) that f' is group strategy-proof.[9] By the definition of f', this implies f is group strategy-proof. This completes the if part of the proof.

(Only-if Part) Since every group strategy-proof RSCF is strategy-proof, the proof of this part follows from Theorem 3.5. ■

6. Conclusion

In this paper, we have shown that an RSCF on a top-rich domain is unanimous and strategy-proof if and only if it can be written as a convex combination of tops-restricted min-max rules. We have also provided a characterization of all domains where an RSCF is unanimous and strategy-proof if and only if it is a random min-max rule. Further, we have characterized the tops-only and strategy-proof RSCFs on top-rich domains satisfying the top-connectedness property. Finally, we establish the equivalence of strategy-proofness and group strategy-proofness on all the domains we consider in this paper.

A. Proof of Theorem 3.1

We present the proof of Theorem 3.1 in this section. First, we prove two lemmas that we use in our proof. The following lemma establishes a crucial property of the top-rich domains.

Lemma A.1. *Let $\mathcal{D}$ be a top-rich domain. Then, $\mathcal{D}|_{\tau(\mathcal{D})}$ is single-peaked.*

[9] Actually, Barberà et al. (2010) show that every min-max rule is group strategy-proof on the maximal single-peaked domain. However, employing the same set of arguments, one can show that generalized min-max rules are also group strategy-proof on the maximal single-peaked domains.

Proof. Let $\mathcal{D}$ be a top-rich domain with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$. We show that $\mathcal{D}|_{\tau(\mathcal{D})}$ is single-peaked. Without loss of generality, assume for contradiction that there exists $P \in \mathcal{D}$ such that $r_1(P) = b_j$ and $b_{l'} P b_l$ for some l, l' with $l' < l < j$. Take $P' \in \mathcal{D}^{b_{l'}}$. Then, $b_{l'} \subseteq U(b_l, P) \cap U(b_l, P')$, which is a contradiction to Condition (ii) in Definition 2.8. This completes the proof. ■

In the following, we prove a technical lemma that we use repeatedly in the proof of Theorem 3.1. We introduce a notation for the lemma. For $X, Y \subseteq A$ and a preference P, we write XPY to mean xPy for all $x \in X$ and $y \in Y$.

Lemma A.2. *Let $\mathcal{D}$ be a domain and let $\varphi : \mathcal{D}^n \to \triangle A$ be a strategy-proof RSCF. Let $P_N \in \mathcal{D}^n$, $P'_i \in \mathcal{D}$, and $B, C \subseteq A$ be such that BP_iC, BP'_iC, and $P_i|_C = P'_i|_C$. Suppose $\varphi_C(P_N) = \varphi_C(P'_i, P_{-i})$ and $\varphi_a(P_N) = \varphi_a(P'_i, P_{-i})$ for all $a \notin B \cup C$. Then, $\varphi_a(P_N) = \varphi_a(P'_i, P_{-i})$ for all $a \in C$.*

Proof. First note that since $\varphi_C(P_N) = \varphi_C(P'_i, P_{-i})$ and $\varphi_a(P_N) = \varphi_a(P'_i, P_{-i})$ for all $a \notin B \cup C$, $\varphi_B(P_N) = \varphi_B(P'_i, P_{-i})$. Suppose $b \in C$ is such that $\varphi_b(P_N) \neq \varphi_b(P'_i, P_{-i})$ and $\varphi_a(P_N) = \varphi_a(P'_i, P_{-i})$ for all $a \in C$ with aP_ib. In other words, b is the maximal element of C according to P_i that violates the assertion of the lemma. Without loss of generality, assume that $\varphi_b(P_N) < \varphi_b(P'_i, P_{-i})$. This, together with the facts that BP_iC, $\varphi_B(P_N) = \varphi_B(P'_i, P_{-i})$, and $\varphi_a(P_N) = \varphi_a(P'_i, P_{-i})$ for all $a \notin B$ with aP_ib, implies $\varphi_{U(b,P_i)}(P_N) < \varphi_{U(b,P_i)}(P'_i, P_{-i})$. This means agent i manipulates at P_N via P'_i, which is a contradiction. ■

Proof of Theorem 3.1

SKETCH OF THE PROOF. We prove the theorem by using the method of induction. We start with the base case $n = 1$. The theorem follows trivially for this case. Assuming that the theorem holds for all sets with $k < n$ agents, we proceed to prove it for n agents. First, we consider the set of profiles $\hat{\mathcal{D}}^n = \{P_N \in \mathcal{D}^n \mid P_1 = P_2\}$ where agents 1 and 2 have the same preference. Since the restriction of φ to $\hat{\mathcal{D}}^n$ induces a unanimous and strategy-proof RSCF on $\mathcal{D}^{n-1}$, by the induction hypothesis, we claim that the theorem holds for the restriction of φ to $\hat{\mathcal{D}}^n$, i.e., $\varphi_{\tau(\mathcal{D})}(P_N) = 1$ for all $P_N \in \hat{\mathcal{D}}^n$ and φ restricted to $\hat{\mathcal{D}}^n$ is uncompromising (in a suitable sense). Next, we show that the same holds for the profiles where agents 1 and 2 have the same top alternative. Finally, in order to prove the theorem for profiles where agents 1 and 2 have arbitrary top alternatives, we use another level of induction on the 'distance' between the top alternatives of agents 1 and 2. The distance between two alternatives $b_{j+l}, b_j \in \tau(\mathcal{D})$ is defined as l. Assuming that the theorem holds for profiles where the distance between the top alternatives of agents 1 and 2 is less than l, we prove it for the profiles where the said distance is l. By induction, this completes the proof of the theorem.

Proof. We prove the theorem by using induction on the number of agents. Let $\mathcal{D}$ be a top-rich domain with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$.

Let $|N| = 1$ and let $\varphi : \mathcal{D} \to \triangle A$ be a unanimous and strategy-proof RSCF. Then, by unanimity, $\varphi_{\tau(\mathcal{D})}(P_N) = 1$ for all $P_N \in \mathcal{D}$, and hence φ satisfies uncompromisingness.

Assume that the theorem holds for all sets with $k < n$ agents. We prove it for n agents. Let $|N| = n$ and let $\varphi : \mathcal{D}^n \to \triangle A$ be a unanimous and strategy-proof RSCF. Suppose $N^* = N \setminus \{1\}$. Define the RSCF $g : \mathcal{D}^{n-1} \to \triangle A$ for the set of voters N^* as follows: for all $P_{N^*} = (P_2, P_3, \ldots, P_n) \in \mathcal{D}^{n-1}$,

$$g(P_2, P_3, \ldots, P_n) = \varphi(P_2, P_2, P_3, P_4, \ldots, P_n).$$

Evidently, g is a well-defined RSCF satisfying unanimity and strategy-proofness (See Lemma 3 in Sen (2011) for a detailed argument). Hence, by the induction hypothesis, $g_{\tau(\mathcal{D})}(P_{N^*}) = 1$ for all $P_{N^*} \in \mathcal{D}^{n-1}$ and g satisfies uncompromisingness. In terms of φ, this means $\varphi_{\tau(\mathcal{D})}(P_N) = 1$ for all $P_N \in \mathcal{D}^n$ with $P_1 = P_2$. In the next lemma, we show that $\varphi_{\tau(\mathcal{D})}(P_N) = 1$ and φ is tops-only over all profiles P_N where agents 1 and 2 have the same top alternative, i.e., $r_1(P_1) = r_1(P_2)$.

Lemma A.3. *Let $P_N, P'_N \in \mathcal{D}^n$ be two tops-equivalent profiles such that $P_1, P_2 \in \mathcal{D}^{b_j}$ for some $b_j \in \tau(\mathcal{D})$. Then, $\varphi_{\tau(\mathcal{D})}(P_N) = 1$ and $\varphi(P_N) = \varphi(P'_N)$.*

Proof. Note that since g is uncompromising, g satisfies tops-onlyness. Because g is tops-only and $P_1, P_2 \in \mathcal{D}^{b_j}$, we have $g(P_1, P_{-\{1,2\}}) = g(P_2, P_{-\{1,2\}})$, and hence $\varphi(P_1, P_1, P_{-\{1,2\}}) = \varphi(P_2, P_2, P_{-\{1,2\}})$. We show $\varphi(P_1, P_2, P_{-\{1,2\}}) = \varphi(P_1, P_1, P_{-\{1,2\}})$. Using strategy-proofness of φ for agent 2, we have $\varphi_{U(x,P_1)}(P_1, P_1, P_{-\{1,2\}}) \geq \varphi_{U(x,P_1)}(P_1, P_2, P_{-\{1,2\}})$ for all $x \in A$, and using that for agent 1, we have $\varphi_{U(x,P_1)}(P_1, P_2, P_{-\{1,2\}}) \geq \varphi_{U(x,P_1)}(P_2, P_2, P_{-\{1,2\}})$ for all $x \in A$. Since $\varphi(P_1, P_1, P_{-\{1,2\}}) = \varphi(P_2, P_2, P_{-\{1,2\}})$, it follows from Remark 2.1 that $\varphi(P_1, P_1, P_{-\{1,2\}}) = \varphi(P_1, P_2, P_{-\{1,2\}})$. Using a similar logic, we have $\varphi(P'_1, P'_1, P'_{-\{1,2\}}) = \varphi(P'_1, P'_2, P'_{-\{1,2\}})$. Because g is tops-only and P_N, P'_N are tops-equivalent, we have $g(P_1, P_{-\{1,2\}}) = g(P'_1, P'_{-\{1,2\}})$. This means $\varphi(P_1, P_1, P_{-\{1,2\}}) = \varphi(P'_1, P'_1, P'_{-\{1,2\}})$, and hence $\varphi(P_1, P_2, P_{-\{1,2\}}) = \varphi(P'_1, P'_2, P'_{-\{1,2\}})$. Moreover, as $\varphi_{\tau(\mathcal{D})}(P_1, P_1, P_{-\{1,2\}}) = 1$, it follows that $\varphi_{\tau(\mathcal{D})}(P_1, P_2, P_{-\{1,2\}}) = 1$. ∎

Lemma A.4. *Let $P_N, P'_N \in \mathcal{D}^n$ be such that $P_1, P_2 \in \mathcal{D}^{b_j}$ and $P'_1, P'_2 \in \mathcal{D}^{b_{j+l}}$ for some $1 \leq j \leq j+l \leq k$, and $r_1(P_i) = r_1(P'_i)$ for all $i \neq 1, 2$. Then, $\varphi_b(P_N) = \varphi_b(P'_N)$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}$.*

Proof. First note that because g is uncompromising and $g_{\tau(\mathcal{D})}(P_{N^*}) = 1$ for all $P_{N^*} \in \mathcal{D}^{n-1}$, $g_b(P_1, P_{-\{1,2\}}) = g_b(P'_1, P_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}$. Moreover, since g is tops-only and $r_1(P_i) = r_1(P'_i)$ for all $i \in \{3, 4, \ldots, n\}$, we have $g(P'_1, P_{-\{1,2\}}) = g(P'_1, P'_{-\{1,2\}})$. By the definition of g, $g(P_1, P_{-\{1,2\}}) = \varphi(P_1, P_1, P_{-\{1,2\}})$ and $g(P'_1, P_{-\{1,2\}}) = \varphi(P'_1, P'_1, P_{-\{1,2\}})$. As $r_1(P_1) = r_1(P_2)$ and $r_1(P'_1) = r_1(P'_2)$, Lemma A.3 implies $\varphi(P_1, P_2, P_{-\{1,2\}}) = \varphi(P_1, P_1, P_{-\{1,2\}})$ and $\varphi(P'_1, P'_2, P'_{-\{1,2\}}) = \varphi(P'_1, P'_1, P'_{-\{1,2\}})$. Combining all these, we have $\varphi_b(P_1, P_2, P_{-\{1,2\}}) = \varphi_b(P'_1, P'_2, P'_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}$. ■

Lemma A.5. *Let $P_N, P'_N \in \mathcal{D}^n$ be such that $P_1, P_2, P'_1 \in \mathcal{D}^{b_j}$ and $P'_2 \in \mathcal{D}^{b_{j+l}}$, and $r_1(P_i) = r_1(P'_i)$ for all $i \neq 1, 2$. Then, $\varphi_c(P_N) = \varphi_c(P'_N)$ for all $c \notin U(b_{j+l}, P'_1) \cap U(b_j, P'_2)$.*

Proof. By Lemma A.3, $\varphi(P_1, P_2, P_{-\{1,2\}}) = \varphi(P'_1, P'_1, P'_{-\{1,2\}})$. Hence, it suffices to show that $\varphi_c(P'_1, P'_1, P'_{-\{1,2\}}) = \varphi_c(P'_1, P'_2, P'_{-\{1,2\}})$ for $c \notin U(b_{j+l}, P'_1) \cap U(b_j, P'_2)$. We prove this for $c \notin U(b_{j+l}, P'_1)$, the proof of the same when $c \notin U(b_j, P'_2)$ follows from symmetric argument.

Take $c \notin U(b_{j+l}, P'_1)$. Note that by strategy-proofness of φ,

$$\varphi_{U(c,P'_1)}(P'_1, P'_1, P'_{-\{1,2\}}) \geq \varphi_{U(c,P'_1)}(P'_1, P'_2, P'_{-\{1,2\}}) \geq \varphi_{U(c,P'_1)}(P'_2, P'_2, P'_{-\{1,2\}}).$$

Moreover, by Lemma A.4, $\varphi_b(P'_1, P'_1, P'_{-\{1,2\}}) = \varphi_b(P'_2, P'_2, P'_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}$, and hence $\varphi_B(P'_1, P'_1, P'_{-\{1,2\}}) = \varphi_B(P'_2, P'_2, P'_{-\{1,2\}})$ for all $B \subseteq A$ such that $[b_j, b_{j+l}]_{\tau(\mathcal{D})} \subseteq B$. Since $c \notin U(b_{j+l}, P'_1)$ and $r_1(P'_1) = b_j$, by the definition of top-rich domain, we have $[b_j, b_{j+l}]_{\tau(\mathcal{D})} \subseteq U(c, P'_1)$, and hence $\varphi_{U(c,P'_1)}(P'_1, P'_1, P'_{-\{1,2\}}) = \varphi_{U(c,P'_1)}(P'_2, P'_2, P'_{-\{1,2\}})$. Thus, we have

$$\varphi_{U(c,P'_1)}(P'_1, P'_1, P'_{-\{1,2\}}) = \varphi_{U(c,P'_1)}(P'_1, P'_2, P'_{-\{1,2\}}). \tag{1}$$

Let d be the alternative such that $dP'_1!c$. Then, $[b_j, b_{j+l}]_{\tau(\mathcal{D})} \subseteq U(d, P'_1)$, and hence

$$\varphi_{U(d,P'_1)}(P'_1, P'_1, P'_{-\{1,2\}}) = \varphi_{U(d,P'_1)}(P'_1, P'_2, P'_{-\{1,2\}}). \tag{2}$$

Subtracting (2) from (1), we have $\varphi_c(P'_1, P'_1, P'_{-\{1,2\}}) = \varphi_c(P'_1, P'_2, P'_{-\{1,2\}})$, which completes the proof of the lemma. ■

Lemma A.6. *Let $P^{b_j,b_{j+1}}, P^{b_{j+1},b_j} \in \mathcal{D}$ be such that $U(b_l, P^{b_j,b_{j+1}}) \nabla U(b_l, P^{b_{j+1},b_j}) \subseteq \tau(\mathcal{D})$ for all $b_l \in \tau(\mathcal{D})$. Then, for all $i \in N$ and all $P_{-i} \in \mathcal{D}^{n-1}$,*

$$[\varphi_{\tau(\mathcal{D})}(P^{b_j,b_{j+1}}, P_{-i}) = 1] \implies [\varphi_{\tau(\mathcal{D})}(P^{b_{j+1},b_j}, P_{-i}) = 1].$$

Proof. Suppose not. Let $P^{b_j,b_{j+1}}$ and P^{b_{j+1},b_j} be as defined in the lemma. Let $c \in A \setminus \tau(\mathcal{D})$ be the highest ranked alternative in P^{b_{j+1},b_j} that receives positive probability at $\varphi(P^{b_{j+1},b_j}, P_{-i})$, i.e., $\varphi_c(P^{b_{j+1},b_j}, P_{-i}) > 0$ and $\varphi_d(P^{b_{j+1},b_j}, P_{-i}) = 0$ for all $d \in A \setminus \tau(\mathcal{D})$ such that $dP^{b_{j+1},b_j}c$.

First, we show $U(c, P^{b_j,b_{j+1}}) \cap \tau(\mathcal{D}) = U(c, P^{b_{j+1},b_j}) \cap \tau(\mathcal{D})$. Suppose not. Then, without loss of generality, there exists $b \in \tau(\mathcal{D})$ such that $b \in U(c, P^{b_j,b_{j+1}}) \setminus U(c, P^{b_{j+1},b_j})$. However, this means $c \in U(b, P^{b_{j+1},b_j})$ and $c \notin U(b, P^{b_j,b_{j+1}})$, and hence $c \in U(b, P^{b_j,b_{j+1}}) \nabla U(b, P^{b_{j+1},b_j})$. Since $c \notin \tau(\mathcal{D})$, this contradicts the assumption of lemma.

Next, we show $\varphi_{B'}(P^{b_j,b_{j+1}}, P_{-i}) = \varphi_{B'}(P^{b_{j+1},b_j}, P_{-i})$, where $B' = U(c, P^{b_j,b_{j+1}}) \cap \tau(\mathcal{D})$. Assume for contradiction that $\varphi_{B'}(P^{b_j,b_{j+1}}, P_{-i}) \neq \varphi_{B'}(P^{b_{j+1},b_j}, P_{-i})$. Suppose $\varphi_{B'}(P^{b_j,b_{j+1}}, P_{-i}) < \varphi_{B'}(P^{b_{j+1},b_j}, P_{-i})$. Since $\varphi_{\tau(\mathcal{D})}(P^{b_j,b_{j+1}}, P_{-i}) = 1$, we have $\varphi_{U(c,P^{b_j,b_{j+1}})}(P^{b_j,b_{j+1}}, P_{-i}) = \varphi_{B'}(P^{b_j,b_{j+1}}, P_{-i})$. Therefore, agent i manipulates φ at $(P^{b_j,b_{j+1}}, P_{-i})$ via P^{b_{j+1},b_j}. Now, suppose $\varphi_{B'}(P^{b_j,b_{j+1}}, P_{-i}) > \varphi_{B'}(P^{b_{j+1},b_j}, P_{-i})$. Let d be the alternative such that $dP^{b_{j+1},b_j}!c$. Then, by our assumption on c, $\varphi_{U(d,P^{b_{j+1},b_j})}(P^{b_{j+1},b_j}, P_{-i}) = \varphi_{B'}(P^{b_{j+1},b_j}, P_{-i})$. Hence, agent i manipulates at $(P^{b_{j+1},b_j}, P_{-i})$ via $P^{b_j,b_{j+1}}$.

Now, we complete the proof of the lemma. Since c is the highest ranked alternative in $A \setminus \tau(\mathcal{D})$ that receives positive probability at $\varphi(P^{b_{j+1},b_j}, P_{-i})$, it follows that $\varphi_{U(c,P^{b_j,b_{j+1}})}(P^{b_{j+1},b_j}, P_{-i}) = \varphi_c(P^{b_{j+1},b_j}, P_{-i}) + \varphi_{B'}(P^{b_{j+1},b_j}, P_{-i})$. Moreover, since $\varphi_{B'}(P^{b_j,b_{j+1}}, P_{-i}) = \varphi_{B'}(P^{b_{j+1},b_j}, P_{-i})$ and $\varphi_{\tau(\mathcal{D})}(P^{b_j,b_{j+1}}, P_{-i}) = 1$, we have $\varphi_{B'}(P^{b_j,b_{j+1}}, P_{-i}) = \varphi_{U(c,P^{b_j,b_{j+1}})}(P^{b_j,b_{j+1}}, P_{-i})$. Combining all these, we have $\varphi_{U(c,P^{b_j,b_{j+1}})}(P^{b_{j+1},b_j}, P_{-i}) > \varphi_{U(c,P^{b_j,b_{j+1}})}(P^{b_j,b_{j+1}}, P_{-i})$, which means i manipulates at $(P^{b_j,b_{j+1}}, P_{-i})$ via P^{b_{j+1},b_j}, a contradiction. This completes the proof of the lemma. ■

To simplify notation for the following lemma, for $j < l$, we define the distance from b_l to b_j, denoted by $b_l - b_j$, as $l - j$.

Lemma A.7. *The RSCF φ is tops-only and $\varphi_{\tau(\mathcal{D})}(P_N) = 1$ for all $P_N \in \mathcal{D}^n$.*[10]

Proof. We prove this lemma by using induction on the distance between the top alternatives of

[10] Chatterji and Zeng (2015) provides a sufficient condition for a domain to be tops-only for RSCFs. However, top-rich domains do not satisfy their condition.

agents 1 and 2.

Take $0 \leq l \leq k-1$. Suppose $\varphi_{\tau(\mathcal{D})}(P_N) = 1$ and $\varphi(P_N) = \varphi(\tilde{P}_N)$ for all tops-equivalent profiles $P_N, \tilde{P}_N \in \mathcal{D}^n$ with $|r_1(P_2) - r_1(P_1)| \leq l$. We show $\varphi_{\tau(\mathcal{D})}(P'_N) = 1$ and $\varphi(P'_N) = \varphi(\tilde{P}'_N)$ for all tops-equivalent profiles $P'_N, \tilde{P}'_N \in \mathcal{D}^n$ with $|r_1(P'_2) - r_1(P'_1)| = l+1$.

Let P_N, P'_N be such that $P_1, P'_1 \in \mathcal{D}^{b_j}$, $P_2 \in \mathcal{D}^{b_{j+l}}$, $P'_2 \in \mathcal{D}^{b_{j+l+1}}$, and $r_1(P_i) = r_1(P'_i)$ for all $i \neq 1,2$. Further, let $\bar{P}_1 = P^{b_j, b_{j+1}}$, $\hat{P}_1 = P^{b_{j+1}, b_j}$, $\hat{P}_2 = P^{b_{j+l}, b_{j+l+1}}$, and $\bar{P}_2 = P^{b_{j+l+1}, b_{j+l}}$ be such that $U(b_l, \bar{P}_i) \nabla U(b_l, \hat{P}_i) \subseteq \tau(\mathcal{D})$ for all $i = 1,2$ and all $b_l \in \tau(\mathcal{D})$. Note that such preferences exist by the definition of top-rich domain. Then, by the induction hypothesis, $\varphi(P_N) = \varphi(P'_1, \hat{P}_2, P'_{-\{1,2\}})$. Now, we prove the following claims.

Claim 1. $\varphi_{\tau(\mathcal{D})}(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = 1$ and $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, P'_2, P'_{-\{1,2\}})$.

By the induction hypothesis, $\varphi_{\tau(\mathcal{D})}(P'_1, \hat{P}_2, P'_{-\{1,2\}}) = 1$ and $\varphi(P_N) = \varphi(\bar{P}_1, \hat{P}_2, P'_{-\{1,2\}}) = \varphi(P'_1, \hat{P}_2, P'_{-\{1,2\}})$. Let $P''_1 \in \{P'_1, \bar{P}_1\}$. By Lemma A.5,

$$\varphi_c(P''_1, P''_1, P'_{-\{1,2\}}) = \varphi_c(P''_1, \hat{P}_2, P'_{-\{1,2\}}) \quad \text{for all } c \notin U(b_{j+l}, P''_1) \cap U(b_j, \hat{P}_2), \tag{3}$$

and

$$\varphi_c(P''_1, P''_1, P'_{-\{1,2\}}) = \varphi_c(P''_1, \bar{P}_2, P'_{-\{1,2\}}) \quad \text{for all } c \notin U(b_{j+l+1}, P''_1) \cap U(b_j, \bar{P}_2). \tag{4}$$

As $r_1(\hat{P}_2) - r_1(P''_1) \leq l$, it follows from the induction hypothesis that $\varphi_{\tau(\mathcal{D})}(P''_1, P''_1, P'_{-\{1,2\}}) = \varphi_{\tau(\mathcal{D})}(P''_1, \hat{P}_2, P'_{-\{1,2\}}) = 1$. Since $U(b_{j+l}, P''_1) \cap U(b_j, \hat{P}_2) \cap \tau(\mathcal{D}) = [b_j, b_{j+l}]_{\tau(\mathcal{D})}$, (3) implies

$$\varphi_b(P''_1, P''_1, P'_{-\{1,2\}}) = \varphi_b(P''_1, \hat{P}_2, P'_{-\{1,2\}}) \quad \text{for all } b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}. \tag{5}$$

Moreover, since $\hat{P}_2 = P^{b_{j+l}, b_{j+l+1}}$, $\bar{P}_2 = P^{b_{j+l+1}, b_{j+l}}$, and $\varphi_{\tau(\mathcal{D})}(P''_1, \hat{P}_2, P'_{-\{1,2\}}) = 1$, by Lemma A.6, $\varphi_{\tau(\mathcal{D})}(P''_1, \bar{P}_2, P'_{-\{1,2\}}) = 1$. This, in particular, means $\varphi_{\tau(\mathcal{D})}(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = 1$. Because $U(b_{j+l+1}, P''_1) \cap U(b_j, \bar{P}_2) \cap \tau(\mathcal{D}) = [b_j, b_{j+l+1}]_{\tau(\mathcal{D})}$, (4) implies

$$\varphi_b(P''_1, P''_1, P'_{-\{1,2\}}) = \varphi_b(P''_1, \bar{P}_2, P'_{-\{1,2\}}) \quad \text{for all } b \notin [b_j, b_{j+l+1}]_{\tau(\mathcal{D})}. \tag{6}$$

Combining (5) and (6), $\varphi_b(P''_1, \hat{P}_2, P'_{-\{1,2\}}) = \varphi_b(P''_1, \bar{P}_2, P'_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l+1}]_{\tau(\mathcal{D})}$. Since $\hat{P}_2 = P^{b_{j+l}, b_{j+l+1}}$ and $\bar{P}_2 = P^{b_{j+l+1}, b_{j+l}}$, we have by strategy-proofness that $\varphi_{\{b_{j+l}, b_{j+l+1}\}}(P''_1, \hat{P}_2, P'_{-\{1,2\}}) = \varphi_{\{b_{j+l}, b_{j+l+1}\}}(P''_1, \bar{P}_2, P'_{-\{1,2\}})$. Let $B' = [b_j, b_{j+l+1}]_{\tau(\mathcal{D})} \setminus \{b_{j+l}, b_{j+l+1}\}$. Then, $\varphi_{B'}(P''_1, \hat{P}_2, P'_{-\{1,2\}}) = \varphi_{B'}(P''_1, \bar{P}_2, P'_{-\{1,2\}})$. Note that by Lemma A.1, $\hat{P}_2|_{B'} = \bar{P}_2|_{B'}$. Therefore, by applying

Lemma A.2 with $B = \{b_{j+l}, b_{j+l+1}\}$ and $C = B'$,

$$\varphi_b(P''_1, \hat{P}_2, P'_{-\{1,2\}}) = \varphi_b(P''_1, \bar{P}_2, P'_{-\{1,2\}}) \quad \text{for all } b \neq b_{j+l}, b_{j+l+1}. \tag{7}$$

By the induction hypothesis, $\varphi(\bar{P}_1, \hat{P}_2, P'_{-\{1,2\}}) = \varphi(P'_1, \hat{P}_2, P'_{-\{1,2\}})$. Again, by Lemma A.1, $b_{j+l}\bar{P}_1 b_{j+l+1}$ and $b_{j+l}P'_1 b_{j+l+1}$, which means $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(P'_1, \bar{P}_2, P'_{-\{1,2\}})$. Using a similar logic, $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, P'_2, P'_{-\{1,2\}})$. This completes the proof of Claim 1.

Claim 2. $\varphi_c(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_c(P'_N)$ for all $c \notin U(b_{j+l+1}, P'_1) \cap U(b_j, P'_2)$.

By (6), $\varphi_b(P'_1, P'_1, P'_{-\{1,2\}}) = \varphi_b(P'_1, \bar{P}_2, P'_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l+1}]_{\tau(\mathcal{D})}$. Since $[b_j, b_{j+l+1}]_{\tau(\mathcal{D})} \subseteq U(b_{j+l+1}, P'_1) \cap U(b_j, P'_2)$, we have $\varphi_c(P'_1, P'_1, P'_{-\{1,2\}}) = \varphi_c(P'_1, \bar{P}_2, P'_{-\{1,2\}})$ for all $c \notin U(b_{j+l+1}, P'_1) \cap U(b_j, P'_2)$. Moreover, by Lemma A.5, $\varphi_c(P'_1, P'_1, P'_{-\{1,2\}}) = \varphi_c(P'_N)$ for all $c \notin U(b_{j+l+1}, P'_1) \cap U(b_j, P'_2)$. Hence, $\varphi_c(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_c(P'_N)$ for all $c \notin U(b_{j+l+1}, P'_1) \cap U(b_j, P'_2)$. This completes the proof of Claim 2.

Claim 3. $\varphi_b(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(P'_N)$ for all $b \in [b_j, b_{j+l+1}]_{\tau(\mathcal{D})}$.

First, we show $\varphi_{b_j}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{b_j}(P'_N)$. By Claim 1, $\varphi(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, P'_2, P'_{-\{1,2\}})$. Moreover, as $r_1(\bar{P}_1) = r_1(P'_1) = b_j$, by strategy-proofness, $\varphi_{b_j}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) = \varphi_{b_j}(P'_N)$. Combining, we have $\varphi_{b_j}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{b_j}(P'_N)$.

Now, we complete the proof of Claim 3 by induction. Take $s < l+1$. Suppose $\varphi_{b_{j+r}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{b_{j+r}}(P'_N)$ for all $0 \leq r \leq s$. We show $\varphi_{b_{j+s+1}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{b_{j+s+1}}(P'_N)$. We show this in two steps. In Step 1, we show that if an alternative outside $\tau(\mathcal{D})$ appears above b_{j+s+1} in the preference P'_1, then it receives zero probability at $\varphi(P'_N)$. In Step 2, we use this fact to complete the proof of the claim.

STEP 1. Take $c \in A \setminus \tau(\mathcal{D})$ such that $cP'_1 b_{j+s+1}$. We show $\varphi_c(P'_N) = 0$. Assume for contradiction that $\varphi_c(P'_N) > 0$. Since $cP'_1 b_{j+s+1}$, by the definition of top-rich domain, we have $b_{j+s+1}P'_2 c$. Let $t \in \{2, \ldots, k-j-l\}$ be such that $U(b_{j+s+1}, P'_2) \cap \tau(\mathcal{D}) = [b_{j+s+1}, b_{j+l+1}]_{\tau(\mathcal{D})} \cup [b_{j+l+2}, b_{j+l+t}]_{\tau(\mathcal{D})}$. By Claim 1, $\varphi_{\tau(\mathcal{D})}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = 1$, and hence

$$\begin{aligned}\varphi_{U(b_{j+s+1}, P'_2)}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) &= \varphi_{[b_{j+s+1}, b_{j+l+1}]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) + \varphi_{[b_{j+l+2}, b_{j+l+t}]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) \\ &= 1 - \varphi_{[b_1, b_{j+s}]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) - \varphi_{[b_{j+l+t+1}, b_k]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}).\end{aligned} \tag{8}$$

By Claim 2, $\varphi_{b_i}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{b_i}(P'_N)$ for all $i \in [1, j-1] \cup [j+l+t+1, k]$, and by the assumption of Claim 3, $\varphi_{b_i}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{b_i}(P'_N)$ for all $i \in [j, j+s]$. Combining all these, we have $\varphi_{[b_1,b_{j+s}]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{[b_1,b_{j+s}]_{\tau(\mathcal{D})}}(P'_N)$ and $\varphi_{[b_{j+l+t+1},b_k]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{[b_{j+l+t+1},b_k]_{\tau(\mathcal{D})}}(P'_N)$. Note that the sets $[b_1, b_{j+s}]_{\tau(\mathcal{D})}$, $U(b_{j+s+1}, P'_2)$, $[b_{j+l+t+1}, b_k]_{\tau(\mathcal{D})}$, and $\{c\}$ are pairwise disjoint. Therefore, $\varphi_{[b_1,b_{j+s}]_{\tau(\mathcal{D})}}(P'_N) + \varphi_{U(b_{j+s+1},P'_2)}(P'_N) + \varphi_{[b_{j+l+t+1},b_k]_{\tau(\mathcal{D})}}(P'_N) + \varphi_c(P'_N) \leq 1$. Hence,

$$\begin{aligned}\varphi_{U(b_{j+s+1},P'_2)}(P'_N) &\leq 1 - \varphi_{[b_1,b_{j+s}]_{\tau(\mathcal{D})}}(P'_N) - \varphi_{[b_{j+l+t+1},b_k]_{\tau(\mathcal{D})}}(P'_N) - \varphi_c(P'_N) \\ &= 1 - \varphi_{[b_1,b_{j+s}]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) - \varphi_{[b_{j+l+t+1},b_k]_{\tau(\mathcal{D})}}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) - \varphi_c(P'_N).\end{aligned} \tag{9}$$

As $\varphi_c(P'_N) > 0$, (8) and (9) imply $\varphi_{U(b_{j+s+1},P'_2)}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) > \varphi_{U(b_{j+s+1},P'_2)}(P'_N)$, which means agent 2 manipulates at P'_N via $\bar{P}_2$, a contradiction. This completes Step 1.

STEP 2. In this step, we complete the proof of the claim. In view of Claim 1, it is sufficient to show that $\varphi_{b_{j+s+1}}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) = \varphi_{b_{j+s+1}}(P'_N)$.

Suppose $\varphi_{b_{j+s+1}}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) > \varphi_{b_{j+s+1}}(P'_N)$. If $d \in U(b_{j+s+1}, P'_1) \setminus \tau(\mathcal{D})$, then by Step 1, $\varphi_d(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_d(P'_N)$, and by Claim 1, $\varphi_d(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_d(\bar{P}_1, P'_2, P'_{-\{1,2\}})$. On the other hand, if $d \in U(b_{j+s+1}, P'_1) \cap \tau(\mathcal{D})$ and $d \neq b_{j+s+1}$, which in turn means $d = b_{j'}$ for some $j' \leq j+s$, then by Claim 2 and the assumption of Claim 3, $\varphi_d(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_d(P'_N)$. Also, by Claim 1, $\varphi(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, P'_2, P'_{-\{1,2\}})$. Combining all these, we have $\varphi_d(\bar{P}_1, P'_2, P'_{-\{1,2\}}) = \varphi_d(P'_N)$ for all $d \in U(b_{j+s+1}, P'_1) \setminus b_{j+s+1}$. Therefore, $\varphi_{b_{j+s+1}}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) > \varphi_{b_{j+s+1}}(P'_N)$ implies $\varphi_{U(b_{j+s+1},P'_1)}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) > \varphi_{U(b_{j+s+1},P'_1)}(P'_N)$, which means agent 1 manipulates at P'_N via $\bar{P}_1$.

Now, suppose $\varphi_{b_{j+s+1}}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) < \varphi_{b_{j+s+1}}(P'_N)$. By Claim 1, $\varphi_{\tau(\mathcal{D})}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) = 1$. Let $u \leq j$ be such that $U(b_{j+s+1}, \bar{P}_1) \cap \tau(\mathcal{D}) = [b_u, b_{j+s+1}]_{\tau(\mathcal{D})}$. Then, by the assumption of Claim 3, $\varphi_b(\bar{P}_1, P'_2, P'_{-\{1,2\}}) = \varphi_b(P'_N)$ for all $b \in [b_j, b_{j+s}]_{\tau(\mathcal{D})}$, and by Claim 2, $\varphi_b(\bar{P}_1, P'_2, P'_{-\{1,2\}}) = \varphi_b(P'_N)$ for all $b \in [b_u, b_{j-1}]_{\tau(\mathcal{D})}$. Therefore, $\varphi_{b_{j+s+1}}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) < \varphi_{b_{j+s+1}}(P'_N)$ implies $\varphi_{U(b_{j+s+1},\bar{P}_1)}(\bar{P}_1, P'_2, P'_{-\{1,2\}}) < \varphi_{U(b_{j+s+1},\bar{P}_1)}(P'_N)$, which means agent 1 manipulates at $(\bar{P}_1, P'_2, P'_{-\{1,2\}})$ via P'_1. This completes the proof of Claim 3.

We are now ready to complete the proof of Lemma A.7. First, we show $\varphi_{\tau(\mathcal{D})}(P'_N) = 1$. By Claim 3, $\varphi_b(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(P'_N)$ for all $b \in [b_j, b_{j+l+1}]_{\tau(\mathcal{D})}$. By Claim 2, $\varphi_b(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(P'_N)$ for all $b \in [b_1, b_{j-1}]_{\tau(\mathcal{D})} \cup [b_{j+l+2}, b_k]_{\tau(\mathcal{D})}$. Combining, we have $\varphi_{\tau(\mathcal{D})}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{\tau(\mathcal{D})}(P'_N)$. Moreover, by Claim 1, $\varphi_{\tau(\mathcal{D})}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = 1$, and hence $\varphi_{\tau(\mathcal{D})}(P'_N) = 1$.

Now, we show $\varphi(P'_N) = \varphi(\tilde{P}'_N)$ for all tops-equivalent profiles $P'_N, \tilde{P}'_N \in \mathcal{D}^n$. By Claim 1, 2, and 3, $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(P'_N)$. Moreover, as $\tilde{P}'_1 \in \mathcal{D}^{b_j}$ and $\tilde{P}'_2 \in \mathcal{D}^{b_{j+l+1}}$, applying Claim 1, 2, and 3 to $\tilde{P}'_N$, we have $\varphi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}}) = \varphi(\tilde{P}'_N)$. Hence, to show $\varphi(P'_N) = \varphi(\tilde{P}'_N)$, it is enough to show $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$. Recall that $\hat{P}_2 = P^{b_{j+l}, b_{j+l+1}}$. Since $r_1(\hat{P}_2) - r_1(P'_1) = l$ and $r_1(P'_i) = r_1(\tilde{P}'_i)$ for all $i \neq 1, 2$, by the assumption of Lemma A.7, we have $\varphi(\bar{P}_1, \hat{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, \hat{P}_2, \tilde{P}'_{-\{1,2\}})$. Also, by (7), $\varphi_b(\bar{P}_1, \hat{P}_2, P'_{-\{1,2\}}) = \varphi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}})$ for all $b \neq b_{j+l}, b_{j+l+1}$, which means $\varphi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for all $b \neq b_{j+l}, b_{j+l+1}$. Using a similar argument as for the proof of (7), it follows that $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\hat{P}_1, \bar{P}_2, P'_{-\{1,2\}})$ for all $b \neq b_j, b_{j+1}$, and hence $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for all $b \neq b_j, b_{j+1}$. Note that if $l \geq 1$, then $\varphi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for all $b \in A$. Therefore, we show $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for $r_1(\bar{P}_1) = b_j$ and $r_1(\bar{P}_2) = b_{j+1}$. Because $\varphi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for all $b \neq b_j, b_{j+1}$ and all tops-equivalent $P'_{-\{1,2\}}, \tilde{P}'_{-\{1,2\}} \in \mathcal{D}^{n-2}$, we have $\varphi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(\bar{P}_1, \bar{P}_2, \tilde{P}'_3, P'_{-\{1,2,3\}})$ for all $b \neq b_j, b_{j+1}$. As $r_1(P'_3) = r_1(\tilde{P}'_3)$, by Lemma A.1, $b_j P'_3 b_{j+1}$ if and only if $b_j \tilde{P}'_3 b_{j+1}$. Therefore, if $\varphi_{b_j}(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) \neq \varphi_{b_j}(\bar{P}_1, \bar{P}_2, \tilde{P}'_3, P'_{-\{1,2,3\}})$, then agent 3 manipulates either at $(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}})$ via $\tilde{P}'_3$ or at $(\bar{P}_1, \bar{P}_2, \tilde{P}'_3, P'_{-\{1,2,3\}})$ via P'_3. Hence, $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, \bar{P}_2, \tilde{P}'_3, P'_{-\{1,2,3\}})$. Continuing in this manner, we have $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$. Therefore, $\varphi(P'_N) = \varphi(\tilde{P}'_N)$ for all tops-equivalent profiles $P'_N, \tilde{P}'_N \in \mathcal{D}^n$. This completes the proof of the lemma. ■

Lemma A.8. *The RSCF φ satisfies uncompromisingness.*

Proof. We prove this in two steps. In Step 1, we provide a sufficient condition for uncompromisingness, and in Step 2, we use that to prove the lemma.

STEP 1. We show that if for all $j, j+1 \in \{1, \ldots, k\}$, all $P_i = P^{b_j, b_{j+1}} \in \mathcal{D}$, all $P'_i = P^{b_{j+1}, b_j} \in \mathcal{D}$, all P_{-i} and all $b \notin [r_1(P_i), r_1(P'_i)]$,

$$\varphi_b(P_i, P_{-i}) = \varphi_b(P'_i, P_{-i}), \tag{10}$$

then φ is uncompromising.

Suppose (10) holds. Since φ is tops-only, (10) implies for all $P_i \in \mathcal{D}^{b_j}$, all $P'_i \in \mathcal{D}^{b_{j+1}}$, all P_{-i}, and all $b \notin [r_1(P_i), r_1(P'_i)]$,

$$\varphi_b(P_i, P_{-i}) = \varphi_b(P'_i, P_{-i}). \tag{11}$$

Similarly, for all $\bar{P}_i \in \mathcal{D}^{b_{j+1}}$, all $\bar{P}'_i \in \mathcal{D}^{b_{j+2}}$, all P_{-i}, and all $b \notin [r_1(\bar{P}_i), r_1(\bar{P}'_i)]$,

$$\varphi_b(\bar{P}_i, P_{-i}) = \varphi_b(\bar{P}'_i, P_{-i}). \tag{12}$$

Combining (11) and (12), we have for all $P_i \in \mathcal{D}^{b_j}$, all $\bar{P}'_i \in \mathcal{D}^{b_{j+2}}$, all P_{-i}, and all $b \notin [r_1(P_i), r_1(\bar{P}'_i)]$, $\varphi_b(P_i, P_{-i}) = \varphi_b(\bar{P}'_i, P_{-i})$. Continuing in this manner, we have for all $P_i, P'_i \in \mathcal{D}$, all P_{-i}, and all $b \notin [r_1(P_i), r_1(P'_i)]$, $\varphi_b(P_i, P_{-i}) = \varphi_b(P'_i, P_{-i})$, which means φ is uncompromising.

STEP 2. In this step, we show that φ satisfies (10). First, we show (10) for agent 1. Without loss of generality, assume $r_1(P_2) = b_{j+l}$. Note that by Lemma A.7, $\varphi_{\tau(\mathcal{D})}(P_N) = 1$. Therefore, by Lemma A.5, $\varphi_b(P_1, P_2, P_{-\{1,2\}}) = \varphi_b(P_2, P_2, P_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}$ and $\varphi_b(P'_1, P_2, P_{-\{1,2\}}) = \varphi_b(P_2, P_2, P_{-\{1,2\}})$ for all $b \notin [b_{j+1}, b_{j+l}]_{\tau(\mathcal{D})}$. This means $\varphi_b(P_1, P_2, P_{-\{1,2\}}) = \varphi_b(P'_1, P_2, P_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathcal{D})}$. By strategy-proofness, $\varphi_{\{b_j, b_{j+1}\}}(P_1, P_2, P_{-\{1,2\}}) = \varphi_{\{b_j, b_{j+1}\}}(P'_1, P_2, P_{-\{1,2\}})$. Let $B' = [b_j, b_{j+l}]_{\tau(\mathcal{D})} \setminus \{b_j, b_{j+1}\}$. Since $P_1|_{B'} = P'_1|_{B'}$, by applying Lemma A.2 with $B = \{b_j, b_{j+1}\}$ and $C = B'$, we have $\varphi_b(P_1, P_2, P_{-\{1,2\}}) = \varphi_b(P'_1, P_2, P_{-\{1,2\}})$ for all $b \neq b_j, b_{j+l}$. This proves (10) for agent 1. Using a symmetric argument, (10) can be shown for agent 2. Therefore, by Step 1, we have

$$\varphi_b(P_i, P_{-i}) = \varphi_b(P'_i, P_{-i}) \quad \text{for all } P_{-i}, \text{ all } b \notin [r_1(P_i), r_1(P'_i)], \text{ and all } i = 1, 2. \tag{13}$$

Now, we show (10) for agents $i \in \{3, \ldots, n\}$. It is enough to show this for $i = 3$. If $P_1 = P_2$, then by the induction hypothesis, $\varphi_b(P_3, P_{-3}) = g_b(P_1, P_3, P_{-\{1,2,3\}}) = g_b(P_1, P'_3, P_{-\{1,2,3\}}) = \varphi_b(P'_3, P_{-3})$ for all $P_3, P'_3 \in \mathcal{D}$ and all $b \notin [r_1(P_3), r_1(P'_3)]$. Let $r_1(P_1) = b_p$ and $r_1(P_2) = b_q$. Since $\varphi_{\tau(\mathcal{D})}(P_N) = 1$ for all $P_N \in \mathcal{D}^n$, it follows from Lemma A.5 that $\varphi_b(P_1, P_1, P_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_2, P_3, P_{-\{1,2,3\}})$ for all $b \notin [b_p, b_q]_{\tau(\mathcal{D})}$ and $\varphi_b(P_1, P_1, P'_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}})$ for all $b \notin [b_p, b_q]_{\tau(\mathcal{D})}$. Combining all these, we have

$$\varphi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}}) \text{ for all } b \notin [b_p, b_q]_{\tau(\mathcal{D})} \cup [b_j, b_{j+1}]_{\tau(\mathcal{D})}. \tag{14}$$

Also, by strategy-proofness,

$$\varphi_{\{b_j, b_{j+1}\}}(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \varphi_{\{b_j, b_{j+1}\}}(P_1, P_2, P'_3, P_{-\{1,2,3\}}). \tag{15}$$

Now, we distinguish two cases.

Case 1. Suppose $[p, q \leq j+1]$ or $[p, q \geq j]$.

Let $B' = [b_p, b_q]_{\tau(\mathcal{D})} \setminus [b_j, b_{j+1}]_{\tau(\mathcal{D})}$. Then, by (14) and (15), $\varphi_{B'}(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \varphi_{B'}(P_1, P_2, P'_3, P_{-\{1,2,3\}})$. Since $P_3|_{B'} = P'_3|_{B'}$, by applying Lemma A.2 with $B = \{b_j, b_{j+1}\}$ and $C = B'$, $\varphi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}})$ for all $b \in B'$. Therefore,

$$\varphi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}}) \text{ for all } b \notin \{b_j, b_{j+1}\}. \tag{16}$$

This completes the proof of the lemma for Case 1.

Case 2. Suppose $p < j \leq j+1 < q$ or $q < j \leq j+1 < p$.

We prove the lemma for the case $p < j \leq j+1 < q$, the proof of the same for the case $q < j \leq j+1 < p$ follows from symmetric arguments. By (13), for all $b \notin [b_j, b_q]_{\tau(\mathcal{D})}$, $\varphi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_3, P_3, P_{-\{1,2,3\}})$ and $\varphi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_3, P'_3, P_{-\{1,2,3\}})$. Moreover, since $r_1(P_1) \leq b_{j+1}$, $r_1(P_3) = b_j$ and $r_1(P'_3) = b_{j+1}$, it follows from (16) that $\varphi_b(P_1, P_3, P_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_3, P'_3, P_{-\{1,2,3\}})$ for all $b \notin [b_j, b_{j+1}]_{\tau(\mathcal{D})}$. Combining all these, $\varphi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}})$ for all $b \notin [b_j, b_q]_{\tau(\mathcal{D})}$. By strategy-proofness, $\varphi_{\{b_j, b_{j+1}\}}(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \varphi_{\{b_j, b_{j+1}\}}(P_1, P_2, P'_3, P_{-\{1,2,3\}})$. Let $B' = [b_j, b_q]_{\tau(\mathcal{D})} \setminus \{b_j, b_{j+1}\}$. Since $P_3|_{B'} = P'_3|_{B'}$, by applying Lemma A.2 with $B = \{b_j, b_{j+1}\}$ and $C = B'$, we have $\varphi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}})$ for all $b \in B'$. Hence,

$$\varphi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}}) \text{ for all } b \notin \{b_j, b_{j+1}\},$$

which completes the proof of the lemma for Case 2.

Since Cases 1 and 2 are exhaustive, this completes the proof of Lemma A.8. ■

Theorem 3.1 now follows by applying Lemma A.7 and Lemma A.8. ■

B. Proof of Theorem 3.2

In this section, we present the proof of Theorem 3.2. Our proof uses the following theorem, which is taken from Peters et al. (2014).

Theorem B.1 (Theorem 3(a) in Peters et al. (2014))**.** *Let $\mathcal{D}$ be the maximal single-peaked domain. Then, every tops-only and strategy-proof RSCF $\varphi : \mathcal{D}^n \to \triangle A$ is a convex combination of tops-only and strategy-proof DSCFs $f : \mathcal{D}^n \to A$.*

The following lemma establishes the structure of an uncompromising RSCF on a regular single-peaked domain.

Lemma B.1. *Let $\mathcal{D}$ be a regular single-peaked domain and let $\varphi : \mathcal{D}^n \to \triangle A$ be uncompromising and strategy-proof. Then, φ is a convex combination of the generalized min-max rules on $\mathcal{D}^n$.*[11]

Proof. Note that since φ is uncompromising, by Remark 2.4, φ is tops-only. Let $\hat{\mathcal{D}}$ be the maximal single-peaked domain. Let $\hat{\varphi} : \hat{\mathcal{D}}^n \to \triangle A$ be the tops-only extension of φ on $\hat{\mathcal{D}}$. More formally, for all $\hat{P}_N \in \hat{\mathcal{D}}^n$, $\hat{\varphi}(\hat{P}_N) = \varphi(P_N)$, where $P_N \in \mathcal{D}^n$ is such that $P_N, \hat{P}_N$ are tops-equivalent. This is well-defined as φ is tops-only and $\mathcal{D}$ is regular. Since $\hat{\mathcal{D}}$ is single-peaked and φ is uncompromising and strategy-proof, $\hat{\varphi}$ is also strategy-proof. Hence, by Theorem B.1, $\hat{\varphi}$ is a convex combination of the generalized min-max rules on $\hat{\mathcal{D}}^n$. By the definition of $\hat{\varphi}$, this means φ is a convex combination of the generalized min-max rules on $\mathcal{D}^n$, which completes the proof. ■

Now, we proceed to prove Theorem 3.2.

Proof. (If Part) Let $\mathcal{D}$ be a top-rich domain with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$ and let $\varphi : \mathcal{D}^n \to \triangle A$ be a TRM rule. Since φ is a TRM rule, it is unanimous by definition. We show that φ is strategy-proof. Let $\varphi = \sum_{l=1}^{t} \lambda_l f_l$, where λ_ls are non-negative numbers summing to 1 and f_ls are TM rules. To show that φ is strategy-proof, it is enough to show that f_ls are strategy-proof. For all $l \in \{1, \ldots, t\}$, define $\hat{f}_l : (\mathcal{D}|_{\tau(\mathcal{D})})^n \to \tau(\mathcal{D})$ as $\hat{f}_l(P_N|_{\tau(\mathcal{D})}) = f_l(P_N)$. Note that by Lemma A.1, $\mathcal{D}|_{\tau(\mathcal{D})}$ is a single-peaked domain. Therefore, it follows from Moulin (1980) that $\hat{f}_l$ is strategy-proof for all l. By Remark 2.3, this means f_l is strategy-proof for all l. This completes the proof of the if part.

(Only-if Part) Let $\mathcal{D}$ be a top-rich domain with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$ and let $\varphi : \mathcal{D}^n \to \triangle A$ be a unanimous and strategy-proof RSCF. Define $\hat{\varphi} : (\mathcal{D}|_{\tau(\mathcal{D})})^n \to \triangle\tau(\mathcal{D})$ as $\hat{\varphi}_b(P_N|_{\tau(\mathcal{D})}) = \varphi_b(P_N)$ for all $b \in \tau(\mathcal{D})$. This is well-defined as by Theorem 3.1 and Remark 2.4, $\varphi_{\tau(\mathcal{D})}(P_N) = 1$ for all $P_N \in \mathcal{D}^n$ and φ is tops-only. Because φ satisfies uncompromisingness, $\hat{\varphi}$ also satisfies uncompromisingness. Hence, by Lemma B.1, $\hat{\varphi}$ is convex combination of generalized min-max rules on $(\mathcal{D}|_{\tau(\mathcal{D})})^n$. Moreover, since φ is unanimous, $\hat{\varphi}$ is a also unanimous. This means $\hat{\varphi}$ is a convex combination of min-max rules on $(\mathcal{D}|_{\tau(\mathcal{D})})^n$. By the definition of $\hat{\varphi}$, this means φ is a TRM rule. This completes the proof of the only-if part. ■

[11] If the set of alternatives is an interval of real numbers, then uncompromisingness implies strategy-proofness for the RSCFs on the maximal single-peaked domain (see Lemma 3.2 in Ehlers et al. (2002)). However, the same does not hold for the case of finitely many alternatives.

C. Proof of Theorem 3.3

Proof. If part of the theorem follows from Theorem 3.2. We prove the only-if part. Let $\mathcal{D}$ be a top-connected TRM domain. We show that $\mathcal{D}$ is a top-rich domain satisfying top-connectedness. In particular, we show that $\mathcal{D}$ satisfies Condition (i) and (ii) in Definition 2.8.

Lemma C.1. *The domain $\mathcal{D}$ satisfies Condition (i) in Definition 2.8.*

Proof. Let $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$. Take $j < k$. Since $\mathcal{D}$ is top-connected, there are $P, P' \in \mathcal{D}$ such that $r_1(P) = b_j$, $r_1(P') = b_{j+1}$, and $P \sim P'$. Therefore, $U(b_j, P) \triangledown U(b_j, P') = b_{j+1}$ and $U(b_{j+1}, P) \triangledown U(b_{j+1}, P') = b_j$. Moreover, by top-connectedness, for all $l \notin \{j, j+1\}$, $U(b_l, P) \triangledown U(b_l, P') = \emptyset$. This shows $\mathcal{D}$ satisfies Condition (i) in Definition 2.8. ■

Lemma C.2. *The domain $\mathcal{D}$ satisfies Condition (ii) in Definition 2.8.*

Proof. Let $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$. By Lemma 3.1, for all $r < s < t$ and all $P^{b_r}, P^{b_t} \in \mathcal{D}$, we have $U(b_s, P^{b_r}) \cap U(b_s, P^{b_t}) \cap \tau(\mathcal{D}) = b_s$. We show that $U(b_s, P^{b_r}) \cap U(b_s, P^{b_t}) = b_s$. Assume for contradiction that $U(b_s, P^{b_r}) \cap U(b_s, P^{b_t}) \supsetneq b_s$ for some r, t with $r < s < t$. Without loss of generality, assume that b_s is the maximum alternative for which $U(b_s, P^{b_r}) \cap U(b_s, P^{b_t}) \supsetneq b_s$ for some $r < s < t$, that is, for all $v > s$ and all u, w with $u < v < w$, $U(b_v, P^{b_u}) \cap U(b_v, P^{b_w}) = b_v$. Since $\mathcal{D}|_{\tau(\mathcal{D})}$ is single-peaked by Lemma 3.1, there cannot be any $b \in \tau(\mathcal{D}) \setminus b_s$ such that $b \in U(b_s, P^{b_r}) \cap U(b_s, P^{b_t})$. Therefore, suppose $x \notin \tau(\mathcal{D})$ is such that $x \in U(b_s, P^{b_r}) \cap U(b_s, P^{b_t})$. In the following, we construct a unanimous and strategy-proof DSCF on $\mathcal{D}^n$ that is not a TRM rule. Consider the following DSCF:

$$f(P_N) = \begin{cases} r_1(P_2) \text{ if } r_1(P_2) \preceq r_1(P_1), \\ x \text{ if } r_1(P_1) \prec b_s \prec r_1(P_2) \text{ and } x P_i b_s \text{ for all } i \in \{1, 2\}, \\ \text{med}\{b_s, r_1(P_1), r_1(P_2)\} \text{ otherwise,} \end{cases}$$

where med represents the median w.r.t. $\prec$.

Clearly, f is not a TRM rule since $f(P_N) = x$ for some $P_N \in \mathcal{D}$ and $x \notin \tau(\mathcal{D})$. Moreover, f is unanimous by definition. We show that f is strategy-proof. Note that agents $i \in N \setminus \{1, 2\}$ cannot manipulate as f does not depend on their preferences. Let $\mathcal{P} = \{P_N \in \mathcal{D}^n \mid f(P_N) \neq x\}$. Then, by the definition of f, f restricted to $\mathcal{P}$ is a tops-restricted min-max rule. This, together with the facts that $\mathcal{D}|_{\tau(\mathcal{D})}$ is single-peaked and $\mathcal{P}|_{\tau(\mathcal{D})} \subseteq \mathcal{D}^n|_{\tau(\mathcal{D})}$, means f cannot be manipulated by some agent $i \in N$ at a profile $P_N \in \mathcal{P}$ via some P'_i such that $(P'_i, P_{-i}) \in \mathcal{P}$. Now, we distinguish the following cases.

CASE 1. Suppose $P_N \in \mathcal{D}^n \setminus \mathcal{P}$.

By the definition of $\mathcal{P}$, this means $r_1(P_1) \prec b_s \prec r_1(P_2)$, xP_ib_s for all $i \in \{1,2\}$, and $f(P_N) = x$. We show that agent 1 cannot manipulate f at P_N, the proof of the same for agent 2 is symmetric. Let $P'_1 \in \mathcal{D}$ be such that $r_1(P'_1) \prec b_s$. Then, by the definition of f, $f(P'_1, P_{-1}) \in \{x, b_s\}$. Since xP_1b_s, agent 1 cannot manipulate at P_N via P'_1. Let $P'_1 \in \mathcal{D}$ be such that $r_1(P'_1) \succeq b_s$. Suppose $f(P'_1, P_{-1}) = b$. Then, by the definition of f, $b \in \tau(\mathcal{D})$ and $b \succeq b_s$. Becuase $P_1|_{\tau(\mathcal{D})}$ is single-peaked and xP_1b_s, we have xP_1b. Therefore, agent 1 cannot manipulate f at P_N via P'_1.

CASE 2. Suppose $P_N \in \mathcal{P}$. We distinguish the following subcases.

CASE 2.a. Suppose $r_1(P_1) \prec r_1(P_2)$. Suppose further that $r_1(P_1) \prec r_1(P_2) \prec b_s$. Then, by the definition of f, $f(P'_1, P_{-1}) \neq x$ for all $P'_1 \in \mathcal{D}$, and hence agent 1 cannot manipulate f at P_N. Note that since $f(P_N) = r_1(P_2)$, agent 2 does not have any incentive to manipulate at P_N. Now, suppose $r_1(P_1) \prec b_s \prec r_1(P_2)$. Then, $f(P_N) = b_s$. Moreover, since $P_N \in \mathcal{P}$, there exists $i \in \{1,2\}$ such that b_sP_ix. So, by the definition of f, the other agent $j \in \{1,2\} \setminus i$ cannot change the outcome to x, and hence cannot manipulate. Also, as $f(P_N) = b_s$ and b_sP_ix, agent i does not have any incentive to manipulate. Finally, suppose $b_s \prec r_1(P_1) \prec r_1(P_2)$. Then, by the definition of f, $f(P'_2, P_{-2}) \neq x$ for all $P'_2 \in \mathcal{D}$, and hence agent 2 cannot manipulate f at P_N. Moreover, since $f(P_N) = r_1(P_1)$, agent 1 does not have any incentive to manipulate at P_N.

CASE 2.b. Suppose $r_1(P_2) \prec r_1(P_1)$. Then, by the definition of f, $f(P_N) = r_1(P_2)$. Therefore, agent 2 does not have any incentive to manipulate at P_N. If $r_1(P_2) \preceq b_s$, then $f(P'_1, P_{-1}) \neq x$ for all $P'_1 \in \mathcal{D}$, and hence agent 1 cannot manipulate at P_N via P'_1. Now, suppose $b_s \prec r_1(P_2)$. We show $r_1(P_2)P_1x$. Recall that $x \notin \tau(\mathcal{D})$, which in particular means $r_1(P_2) \neq x$. Therefore, assume for contradiction that $xP_1r_1(P_2)$. Since $x \in U(b_s, P^{b_r}) \cap U(b_s, P^{b_t})$, we have $xP^{b_r}b_s$. Also, since $\mathcal{D}|_{\tau(\mathcal{D})}$ is single-peaked and $b_s \prec r_1(P_2)$, we have $b_sP^{b_r}r_1(P_2)$. Combining all these, we have $xP^{b_r}r_1(P_2)$. This means $b_r \prec r_1(P_2) \prec r_1(P_1)$ and $x \in U(r_1(P_2), P^{b_r}) \cap U(r_1(P_2), P_1)$. Since $b_s \prec r_1(P_2)$, this is a contradiction to the maximality of b_s. So, agent 1 cannot manipulate f at P_N. This shows that f is strategy-proof. ■

Now, the proof of Theorem 3.3 follows from Lemma C.1 and Lemma C.2. ■

D. Proof of Theorem 3.4

Proof. Let $\mathcal{D}$ be a top-rich domain satisfying top-connectedness with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$ and let $\varphi : \mathcal{D}^n \to \triangle A$ be a tops-only and strategy-proof RSCF. Take $P_N \in \mathcal{D}^n$ and $i \in N$. Suppose $r_1(P_i) = b_j$. Let $\varphi_{\tau(\mathcal{D})}(P_N) = \alpha$ for some $0 \leq \alpha \leq 1$. Take $P^{b_j, b_{j+1}} \in \mathcal{D}$. By tops-onlyness, $\varphi(P^{b_j,b_{j+1}}, P_{-i}) = \varphi(P_N)$. Now, take $P^{b_{j+1},b_j} \in \mathcal{D}$ such that $P^{b_j,b_{j+1}} \sim P^{b_{j+1},b_j}$. By strategy-proofness, $\varphi_d(P^{b_j,b_{j+1}}, P_{-i}) = \varphi_d(P^{b_{j+1},b_j}, P_{-i})$ for all $d \in A \setminus \{b_j, b_{j+1}\}$. Again, take $P_i'' \in \mathcal{D}$ with $r_1(P_i'') = b_{j+1}$. By tops-onlyness, $\varphi(P^{b_{j+1},b_j}, P_{-i}) = \varphi(P_i'', P_{-i})$. Therefore, $\varphi_{\tau(\mathcal{D})}(P_i'', P_{-i}) = \alpha$ and $\varphi_B(P_i, P_{-i}) = \varphi_B(P_i'', P_{-i})$ for all $B \subseteq A \setminus \{b_j, b_{j+1}\}$. Continuing in this manner, it follows that $\varphi_{\tau(\mathcal{D})}(P_N) = \varphi_{\tau(\mathcal{D})}(\hat{P}_i, P_{-i}) = \alpha$ and $\varphi_B(P_N) = \varphi_B(\hat{P}_i, P_{-i})$ for all $B \subseteq A \setminus [r_1(P_i), r_1(\hat{P}_i)]|_{\tau(\mathcal{D})}$ and all $\hat{P}_i \in \mathcal{D}$. Since i is arbitrary, this proves $\varphi_{\tau(\mathcal{D})}(P_N) = \alpha$ for all $P_N \in \mathcal{D}^n$, and φ is uncompromising and $\varphi_c(P_N) = \varphi_c(P_N')$ for all $P_N, P_N' \in \mathcal{D}^n$ and all $c \in A \setminus \tau(\mathcal{D})$. ■

E. Proof of Theorem 3.5

Proof. (If Part) Let $\mathcal{D}$ be a top-rich domain satisfying top-connectedness with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$ and let $\varphi : \mathcal{D}^n \to \triangle A$ be a convex combination of GTM rules and non-top constant rules. Then, φ is tops-only by definition. We show that φ is strategy-proof. Note that non-top constant rules are trivially strategy-proof. Therefore, since φ is a convex combination of GTM rules and non-top constant rules, it is enough to show that every GTM rule is strategy-proof. Let $f : \mathcal{D}^n \to A$ be a GTM rule. Define $\hat{f} : (\mathcal{D}|_{\tau(\mathcal{D})})^n \to \tau(\mathcal{D})$ as $\hat{f}(P_N|_{\tau(\mathcal{D})}) = f(P_N)$. By Lemma A.1, $\mathcal{D}|_{\tau(\mathcal{D})}$ is single-peaked domain. Therefore, it follows from Moulin (1980) that $\hat{f}_l$ is strategy-proof for all l. Hence, by Remark 2.3, f_l is strategy-proof.

(Only-if Part) Let $\mathcal{D}$ be a top-rich domain satisfying top-connectedness with $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$ and let $\varphi : \mathcal{D}^n \to \triangle A$ be a tops-only and strategy-proof RSCF. In view of Theorem 3.4, let $0 \leq \alpha \leq 1$ be such that $\varphi_{\tau(\mathcal{D})}(P_N) = \alpha$ for all $P_N \in \mathcal{D}^n$. If $\alpha = 0$, then $\varphi_b(P_N) = 0$ for all $b \in \tau(D)$ and all $P_N \in \mathcal{D}^n$. Therefore, by Theorem 3.4, φ is a convex combination of non-top constant rules. Suppose $\alpha > 0$. Define $\hat{\varphi} : (\mathcal{D}|_{\tau(\mathcal{D})})^n \to \triangle\tau(\mathcal{D})$ as $\hat{\varphi}_b(P_N|_{\tau(\mathcal{D})}) = \frac{1}{\alpha}\varphi_b(P_N)$ for all $b \in \tau(\mathcal{D})$ and all $P_N|_{\tau(\mathcal{D})} \in (\mathcal{D}|_{\tau(\mathcal{D})})^n$. Then, the RSCF $\hat{\varphi}$ is well-defined by Theorem 3.4. Clearly, $\hat{\varphi}$ satisfies uncompromisingness. Hence, by Lemma B.1, $\hat{\varphi}$ is a convex combination of generalized min-max rules. Suppose $\hat{\varphi} = \sum_{l=1}^{L} \lambda_l f_l$, where for $l = 1, \ldots, L$, λ_ls are non-negative numbers summing to 1, and f_l is a generalized min-max rule for all l. By Theorem 3.4, we have $\varphi_c(P_N) = \varphi_c(P_N')$ for all

$c \in A \setminus \tau(\mathcal{D})$ and all $P_N, P'_N \in \mathcal{D}^n$. Let $\lambda_c = \varphi_c(P_N)$ for all $c \in A \setminus \tau(\mathcal{D})$ and all $P_N \in \mathcal{D}^n$. Then, $\varphi = \sum_{l=1}^{L} \alpha\lambda_l f_l + \sum_{c \in A \setminus \tau(\mathcal{D})} \lambda_c f_c$, where f_c is the non-top constant rule given by $f_c(P_N) = c$ for all $P_N \in \mathcal{D}^n$. Since $\sum_{c \in A \setminus \tau(\mathcal{D})} \varphi_c(P_N) = 1 - \alpha$, this means φ is a convex combination of generalized min-max and non-top constant rules. ■

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