CONDORCET JURY THEOREM IN A SPATIAL MODEL OF ELECTIONS*

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ABSTRACT. In this paper, we study conditions under which the Condorcet Jury Theorem extends to the spatial model of elections. In the model, individuals with ideal points distributed over a unidimensional policy space vote over two alternatives, the location of one of which is uncertain. By employing the techniques used in Bhattacharya (2013), we identify the entire set of symmetric equilibria for almost every voting rule. If there is uncertainty about whether the outcome induced by the policy alternative is to the right or to the left of the status quo (the certain alternative), then an election produces three outcomes, exactly one of which is full information equivalent. In the other two equilibria, the status quo always wins. This finding provides a novel explanation for status quo bias in referenda and incumbent advantage. The "bad equilibrium" is consistent with the ex-ante unlikely victory of the Brexit side in the UK referendum of 2016.

1. INTRODUCTION

The main lesson of Condorcet Jury Theorem (Condorcet, 1785, henceforth CJT) is that in large majoritarian elections, the commonly preferred alternative wins almost surely even if individuals are uncertain about which alternative is better. Most of the existing work on CJT assumes that individuals have the same preferences and receive independent, partially informative signals about which alternative is the better one. Under these assumptions, these models show that elections are *full information equivalent*, i.e., they produce outcomes that would occur if there were no individual uncertainty about the alternatives in competition. In this paper, we examine whether and to what extent this result holds up in the most commonly studied applied model of elections: the Downsian model. In the standard Downsian model (Downs 1957), there is a unidimensional space over which voter ideal points are distributed. Therefore, the main question here is whether the intuition derived with homogeneous preferences in a jury setup is robust to variation in voter preferences.[1] Our central result is that even with very precise signals, there is a class of situations under which an alternative that is both ex-ante and ex-post majority preferred may lose the election almost surely.[2]

In the standard Downsian model, there are two policies in competition, and each policy is identified with a location on the left-right ideological continuum $[-1, 1]$. Voter ideal points are distributed on the same space, and each voter prefers the policy closest to her ideal point. Our main innovation is to assume that while the location of one of the two policies (the "status quo", $\mathcal{Q}$) is known, there is uncertainty about the location of the other policy (the "policy alternative", $\mathcal{P}$). The interpretation is $\mathcal{P}$ induces one of two possible outcomes, but it is not known ex-ante which

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[1] McMurray (2017) analyzes Condorcet Jury Theorem in a setting where voter opinions have a natural order much like the Downsian model, but voters have common preferences.

[2] See Ali, Mihm and Siga (2017) for a model of aggregation failure when information is very scanty.

outcome it induces. Each outcome is associated with one *known* location on the policy space. Thus, there are two states of the world—in state L, $\mathcal{P}$ is located at $L \in [-1, 1]$ and in state R, $\mathcal{P}$ is located at $L \in [-1, 1]$. Voters may be mistaken in their assessment of which state has actually occurred, i.e., whether the location of $\mathcal{P}$ is at L or at R. To capture the possibility of such mistakes, we assume that each voter receives an independent but imperfectly informative signal about the state. Our main insight is that while the precise location of the alternatives is irrelevant for the outcome, what matters is whether there is uncertainty about the *order* in the location of the alternatives. In particular, if there is even a small uncertainty over which direction the policy alternative shifts the status quo, then the CJT fails to go through.

To see how such an uncertainty over the order of location of alternatives may arise, consider the (very stylized) example of a vote over trade liberalization. Suppose that a country facing a referendum over whether to allow free trade by joining an economic union with other countries (This example should be reminiscent of the failed Swiss referendum in 1992 on joining the EU). Because of its isolation, it has developed both an industrial sector and an agricultural sector to suit its own consumption needs. If the country allows free trade, the sector in which it has comparative advantage will grow and the other will shrink. If there is an ex-ante uncertainty over which sector the comparative advantage lies, i.e., whether the proposed trade reform will make those voters engaged in industry better off at the cost of those in agriculture or the other way round, we have a situation of unordered alternatives, leading to the reform being blocked. The possibility of trade reforms being blocked by voters due to uncertainty over final payoffs has been identified by Fernandez and Rodrik (1991). However, their model explains such outcomes through the perverse effects of the price mechanism while this paper shows that the failure is driven by the rational voting calculus.

Without loss of generality, assume that $L < R$ and $\mathcal{Q}$ is located at 0. We say that the alternatives are *ordered* if in each state, the policy alternative is located on the same side of the status quo, i.e., $L < R < 0$ or $0 < L < R$. On the other hand, we say that the alternatives are *unordered* if the policy lies to the left of the status quo in one state and to the right in the other, i.e., $L < 0 < R$. In this case, the uncertainty is really about the order in the location of the alternatives. The important feature of the unordered alternatives case is the fact that there are two groups of voters who have opposed interests in each state of the world. In particular, when the policy is to the left of the status quo (state L), the leftist voters prefer the policy over status quo while the rightist voters prefer the status quo to the policy. The ranking over alternatives is reversed for both these groups in state R. It is this state-contingent conflict that leads to the co-ordination problems among voter groups.

It must be noted that while we denote the alternative with no locational uncertainty as the "status quo", our result applies equally well to a setting where the pre-reform policy is the one corresponding to open trade and the proposal is about whether to contintute with such a policy, with the alternative being autarky. As long as there is current uncertainty about whether the policy will move to the left or the right, this situation would correspond to the now-famous Brexit referendum in Britain in 2016.

The exercise in the current paper involves finding the full set of symmetric equilibria for any plurality rule in large electorates. We show that if the alternatives are ordered, then the uncertainty does not matter—the election outcome is as if the state is known. On the other hand, if the alternatives are unordered, behavior of each voter depends on what she expects others to do. Using a methodology developed in Bhattacharya (2013), we find the full set of symmetric equilibria in the unordered alternatives case. We show that in this case, there are exactly three possible equilibrium outcomes. In one of these outcomes, the correct alternative wins almost surely in each state. But there are two other equilibria in which the status quo is always elected. In one of the two "bad" equilibria, each voter believes that almost everyone else is voting uninformatively, dampening his own incentive to use information. Independent of information received, the larger group votes for the status quo and almost everyone in the smaller group votes for the alternative $\mathcal{P}$. Voter behavior in this equilibrium is akin to what we know as block voting. In another "bad" equilibrium, only the extremists at either end of the ideological spectrum are responsive to information—but aggregation fails because most of the other voters vote for the status quo in either state.

An important application of our result is in direct democracies. When voters vote in a referendum, they face the choice over whether to adopt a new policy over a status quo. Any proposed reform creates winners and losers (compared to the status quo), but there is often ex-ante uncertainty about the identity of the winners and losers. There is a large literature in both economics and political science that equates policymaking with experimentation, making the point that choosing or electing a policy rarely the same as ascertaining or implementing an outcome.[3] Thus, voting for a policy alternative may often lead to substantial uncertainty over whether the outcome would lie to the left or to the right of the status quo. We show that in these situations, the voting outcome may simply favor the status quo even when it is not the correct choice. It is important to note here that it is not risk aversion but co-ordination problems between voter groups that leads to elections failing to aggregate information.

In fact, some proponents of direct democracy invoke the CJT in order to suggest that referenda aggregate information efficiently even if voters may be mistaken about the policy consequences (Matsusaka 2005, Lupia 2001).[4].This paper points out that such a position may be unwarranted: While there does exist one equilibrium that does indeed aggregate information, there may be others in which the alternative proposal always fails to pass.

Status quo bias in referenda have been well documented in empirical work. In general, the details of the referendum process and the rules for passage might vary, making comparison across countries or aggregation over instances difficult. In Australia, all amendments to the constitution are required to be passed via referenda in which voting is compulsory for everyone on the electoral roll, which makes it the closest approximation to the model we want to study. As of date, of the 44 proposals put forth for referendum in Australia, only 8 have passed. In Switzerland, the "gold standard" for direct democracy, only 36% of all optional referenda have passed in the period from 1991 till 2006, although the proportion was higher earlier (see footnote 7 in Kirchgassner (2007)). In fact, the status quo bias has been well-documented and studied in the case of Switzerland, where authors have held direct democracy responsible for its slow growth during the nineties, delays in reforms and so on (Kirchgassner 2008). In the United States, among 2360 statewide initiatives to appear on the ballot since the first such initiative in Oregon in 1904 till 2010, only 962 have passed.[5] The success rate of 40% in these initiatives seems a rather low figure, keeping in mind that initiatives appearing on the ballot are already those which are seen by their sponsors to be at least somewhat likely to pass.

Referenda, by definition, involve a single issue. To the extent we can think of a political race between two candidates or parties being reducible to a single, possibly ideological dimension, the current paper applies to elections too. In case of high profile national elections, we often know which candidate is to the right and which one is to the left simply from their party identities. However, in may other situations voters are faced with an uncertainty over the order of the alternatives. In primaries where both candidates are from the same party, the left-right order of candidates may not be clear. In local or municipal elections, candidates often run on the plank of efficiency or

[3]See Lindblom (1959) for an early enunciation of the idea of policymakers "muddling through" policies in search of good outcomes. A more recent example is Callander (2011).

[4]For example, Matsusaka (2005, p. 193) claims that "Direct democracy can be effective even when voters have no more or even worse information than legislators....aggregating the opinions of a million voters can be highly accurate by the Law of Large numbers even if each person's chance of being right is small (this is a version of Condorcet Jury Theorem..)"

[5]Source: Historical database maintained by the Initiative and Referendum Institute at the University of Southern California (http://www.iandrinstitute.org/data.htm)

local issues, making it difficult for voters to use party affiliation as an informational shortcut for candidate positions.

Even when the candidates do take clearly defined issue positions, there is substantial evidence to the effect that voters often fail to learn the positions or, worse still, fail to even identify the order of the candidates according to their positions. Lenz (2012, table 5.1, page 117-118) presents a survey where he studies several salient issues in US and European national elections (social security in 2000 US elections, EU integration in the British 1997 elections, public works jobs in the 1976 US elections, defense spending in the 1980 US elections, ideology in the 1992 US elections and Chernobyl in the 1986 Dutch elections) and shows that, in each case, less than half of the respondents could start out identifying the order of candidates correctly. These facts suggest that even in electoral races between candidates, there may be uncertainty in voters' minds about the order of candidates.

In case of electoral competitions, our results provide a new explanation for the phenomenon of incumbency advantage. It is well documented that incumbents enjoy a strong and growing advantage in US electorates - both in legislative and executive offices (Ansolabehere, Snyder and Stewart 2000, Ansolabehere and Snyder 2002). We hold that if there is incomplete information regarding wether the challenger lies to the left or right of the incumbent, then incumbency advantage may arise due to a co-ordination failure among voters. This explanation is in addition to those relying on issues relating to political structure (e.g. decline of the party (Cover 1977), campaign contribution and interest group activities (Jacobson 1980)) which can only explain the phenomenon in legislative offices. Our explanation applies to executive offices as well. In fact, our theory is more suited to lower offices where information regarding the challenger is harder to come by and party identification plays a smaller role.

Moreover, as Ansolabehere, Snyder and Stewart (2000) puts it, "measured in terms of vote share, the incumbency advantage grew from a modest 1-3 percentage point edge in the 1940s and early 1950s to a 7-10 percentage point edge in the 1980s and early 1990s." This growth has been linked with the increase in television coverage (Eriskson 1995, Ansolabehere, Snowberg and Snyder 2006): The idea being that since the incumbent gathers a larger share of television time than the challenger both in terms of news coverage and campaign advertisements. Our explanation is broadly in line with this position: the incumbent advantage stems from the voters being more informed about the incumbent than about the challenger.

In the main body of the paper, we compare the equilibrium properties of the ordered alternatives case ($L < R < 0$) with those of the unordered alternatives case ($L < 0 < R$). This not only allows us to show when elections may fail to aggregate information, it also sheds light on *why* aggregation fails. We show that in equilibrium only a subset of voters are *responsive* in the sense that changes in information received changes their voting decision. In the ordered alternatives case, for *all* responsive voters, the same information induces the same voting decision. But under unordered alternatives, the same information may induce opposite voting decisions among different groups of responsive voters. What is surprising is that the very possibility of different voters interpreting the same information in different ways leads to outcomes that are "wrong" with a very high probability. The fact that this breakdown does not depend on the size of conflicting groups, the accuracy of signals or on the exact distribution of voter preferences indicates that the source of informational inefficiency in voting is just the existence of two groups of voters that never agree with each other.

The paper is organized as follows. Section 2 discusses the relationship to the literature on Condorcet Jury Theorem in detail. Section 3 discusses the basic model and defines equilibrium of the game We study the information aggregation properties of the ordered alternatives case in section 4 and that of the unordered alternatives case in section 5. A final section concludes. Most proofs are relegated to the appendix.

2. Relationship to the literature on CJT

Most of the previous game-theoretic work on CJT (e.g. Austen-Smith and Banks (1996), Myerson (1998a, 1998b, 2000), Wit(1998), Meirowitz (2002)) assume that all voters have the same preference.[6] Feddersen and Pesendorfer (1997), henceforth FP, was the first to provide a proof of CJT allowing a limited heterogeneity of voter preferences. In FP, while not all voters have the same ranking over alternatives, any given increase in the state (e.g. extent of guilt) always increases the utility from voting for a specific alternative (e.g. conviction). As such, in FP, any change in state induces switches in favor of the same alternative in FP. In the ordered alternatives case in our model, a switch in state from L (the more extreme state) to R (the more moderate state) induces switches in ranking from $\mathcal{P}$ to $\mathcal{Q}$ but not the other way round. and this is the reason why, much like FP, information is always aggregated if the alternatives are ordered.[7]

Bhattacharya (2013) develops a methodology to analyze the limit of equilibria of a two-state, two-alternative election game with general voter preferences. In particular, Bhattacharya (2013) shows that if some voters prefer $\mathcal{P}$ in state L and $\mathcal{Q}$ in state R while others prefer $\mathcal{Q}$ in state L and P in state R, then *there exists* an equilibrium sequence where the full information equivalent outcome fails to obtain in the limit. The imposition of spatial structure allows us to use the same methodology to make clear predictions about the entire set of limit equilibrium outcomes. We show that while the equilibrium with the "wrong" outcome does exist as predicted by Bhattacharya (2013), there also exists one equilibrium in which the correct outcome obtains almost surely. Thus, the message here is that whether majoritarian elections lead to the efficient outcome or not depends entirely on voter co-ordination. If the achievement of informational efficiency is the objective of the government (or more generally, the election designer), then our work suggests that policies should be targeted towards co-ordinating on the right equilibrium. Characterization of the entire equilibrium set allows the designer to know precisely which outcomes to avoid while designing such a targeting mechanism.

It is important to mention the formal relationship between conditions on information aggregation in the spatial model (i.e., ordered vs. unordered alternatives) and those in the more general setting in Bhattacharya (2013). According to the Strong Preference Monotonicity (SPM) condition in Bhattacharya (2013), if the distribution of preferences is such that a *randomly chosen voter* is more likely to the same alternative over the other for *each* belief over states, then information is aggregated in all equilibria. Conversely, if SPM is not satisfied by the distribution of preferences, then there exist signal probabilities for which a "wrong" outcome obtains in at least one equilibrium. Bhattacharya (2013) also identifies a joint condition on signal probabilities and preference distribution called Weak Preference Monotonicity (WPM) that has the same flavor.[8] In the spatial model, if the alternatives are ordered, both SPM and WPM are satisfied. Hence, it follows directly that information is aggregated efficiently in every equilibrium. On the other hand, when the alternatives are unordered, both SPM and WPM are violated. Therefore, according to the main theorem in Bhattacharya (2013), there exist equilibria that fail to aggregate information for consequential rules.[9] Moreover, in the current paper, the spatial structure allows us to predict *the entire set of symmetric equilibria for all non-unanimous voting rules,* and not just consequential rules. Another interesting difference with Bhattacharya (2013) is in terms of exposition. Bhattacharya (2013) solves the electoral game in terms of beliefs held in equilibrium, while the current paper

[6]There is also an older literature on statistical proofs of CJT (see Nitzan and Paroush (1985), Ladha (1992) and Berg (1993) for example).

[7]Gersbach (1995) has also shown in an abstract setting with mechanistic voters that uncertainty over payoff distributions may lead to problems with aggregation.

[8]WPM is said to be satisfied if a change in the signal from a to b makes a randomly chosen voter more likely to prefer the same alternative for *each* belief over states. For a given distribution of preference, SPM holds if and only if WPM holds for every possible distribution over signals satisfying MLRP.

[9]Consequential rules are those voting rules that induce different outcomes in different states under full information.

derives the strategies held in equilibrium by the voters. This allows us to track the behavior of the swing (i.e., responsive) voters and provide conditions on what has to be true of such behavior for the election to achieve the correct outcomes in equilibrium. These conditions throw light on the reasons for why information may or may not be aggregated in different equilibria.

3. The Model

Suppose there is an electorate composed of a finite number $(n+1)$ of people who are voting for or against a policy proposal $\mathcal{P}$. If the proposed policy gets more than a proportion θ of the votes, then $\mathcal{P}$ wins; otherwise the status quo $\mathcal{Q}$ wins.[10] Assume that the policy space is $[-1,1]$, and that each policy leads to an outcome that is modelled as a location on the policy space. We shall stick to the conventional interpretation of the policy space as the left-right ideological continuum, and think of locations expressed as higher numbers as more rightist outcomes. We assume that the location of the status quo $\mathcal{Q}$ is 0. On the other hand, there is uncertainty about the location of the proposal $\mathcal{P}$. With equal probability, $\mathcal{P}$ likely to be located at L or R. The event that $\mathcal{P}$ is located at S, where $S \in \{L,R\}$ is referred to as state S. Additionally, we assume that the policy proposal never coincides with the status quo, i.e., both L and R are non-zero and that $L<R$.

At this stage, we introduce an important classification of the possible configurations of L, R and Q. Notice that with the assumptions made above, there are three possible configurations: $L<R<0$, $0<L<R$ and $L<0<R$. In the first (second) case, the proposed policy lies to the left (right) of the status quo irrespective of its actual location. In these two cases, we say that we have *ordered alternatives* in the sense that it is ex-ante known whether the proposed policy is to the right or left of the status quo. On the other hand, if we have $L<0<R$, then we say that we have *unordered alternatives*. In this case, the ordering between the proposed policy $\mathcal{P}$ and the status quo $\mathcal{Q}$ depends on the realized state: in state L, $\mathcal{P}$ is to the left of $\mathcal{Q}$ and in state R, $\mathcal{P}$ is to the right of $\mathcal{Q}$. The unordered alternatives case captures situations where there is uncertainty regarding whether the proposal shifts the status quo to the right or left. We study these two cases separately. In the ordered alternatives case, we assume that $-1<L<R<0$ without loss of generality, since the other case $(0<L<R<1)$ would be symmetric. In the unordered alternatives case, we assume that $L=-b$ and $R=b$ for some $b \in (0,1)$, i.e., the two alternatives are located equidistant from the status quo.

Voters have single peaked preferences defined on the policy space. Every individual has a privately known bliss point x that is drawn independently from a commonly known non-atomic distribution $F(\cdot)$ with support $[-1,1]$ and a positive, bounded and continuous density $f(\cdot)$ on the entire support. For a voter with ideal point x, the utility from a policy with location a is assumed to be $-(x-a)^2$. What matters for her voting decision is the difference in utility between $\mathcal{P}$ and $\mathcal{Q}$. Denote by $v(x,S)$ as the difference in utility between $\mathcal{P}$ and $\mathcal{Q}$ when the proposed policy is located at $S \in \{L,R\}$:

$$v(x,S) = (x-0)^2 - (x-S)^2 = S(2x-S) \tag{1}$$

If the state S is known, a voter votes for $\mathcal{P}$ if $v(x,S)>0$ and only if $v(x,S) \geq 0$.[11] The types x for which $v(x,S)$ is weakly positive for $S \in \{L,R\}$ vote for $\mathcal{P}$ irrespective of the signal they receive, and are called partisans for $\mathcal{P}$. Similarly, the types x for which $v(x,S)$ is weakly negative for $S \in \{L,R\}$ are called partisans for $\mathcal{Q}$. It is only those types for which $v(x,L)v(x,R)<0$ whose behavior is sensitive to information: they are called independent voters. Such a voter calculates the expected value of this function using the relevant probability distribution over states conditioning

[10] To simplify the analysis, assume the tie breaking rule that if the policy receives exactly θ proportion of votes, the status quo wins.

[11] Since $v(x,S)=0$ occurs with zero probability, it does not matter for the outcome what a type x does when $v(x,S)=0$.

on his vote being pivotal in determining the outcome (given his signal and the behavior of other voters) and votes $\mathcal{P}$ if the expectation is non-negative.

Each voter receives a private signal $\sigma \in \{l, r\}$ about the state. Signals are independent and identically distributed conditional on the state, with the distribution being:

$$\Pr(l|L) = \Pr(r|R) = q$$

The parameter q measures signal precision, and we assume that the signals are sufficiently precise, in particular, $q > \frac{1}{2} + \frac{b}{4}$. We denote this informativeness assumption by I. While we assume I throughout the paper, it has bite only when the alternatives are unordered.

At this stage, a few comments about the assumptions are due. First, symmetric location of alternatives is not too strong an assumption as we have very few restrictions on the distribution of voter ideal points except non-atomicity and full support. Second, assumption I ensures that it is never the case that all types of voters vote for the status quo irrespective of the signal. Therefore, the role of this assumption is to ensure that for every voter, there is always a positive probability of being pivotal.

The utility from the alternative A, when it is located at a, is given by:

$$U(x, A) = -(x - a)^2, \ A \in \{\mathcal{Q}, \mathcal{P}\}$$

Given a draw of x and S, we define $v(x, S)$ as the difference in utility between the policy alternative and the status quo:

We denote a voting environment by the tuple (F, q, L, R), and a voting game by an environment together with voting rule $\theta \in (0, 1)$ and a finite number $n + 1$ of voters. Now, we turn to the equilibrium of a voting game.

3.1. **Strategies and equilibrium.** The equilibrium concept we employ is symmetric Bayesian Nash equilibrium in undominated strategies. By symmetric, we mean that individuals with the same private information (bliss point $x \in [-1, 1]$ and signal $\sigma \in \{r, l\}$) vote the same way. Since F is nonatomic, we can concentrate on pure strategies. The strategy $\pi(x, \sigma)$ takes a value 0 or 1 for each (x, σ), and this value is a probability of voting for $\mathcal{P}$.

It is easy to see that any equilibrium strategy π^* of this game will be a "cut-off" strategy. To see that, for a voter with signal σ, denote the equilibrium belief of the state being L conditioning on a tie as $p^*_\sigma = \Pr(L|piv, \pi^*, \sigma)$. Note that this belief must be the same for all $x \in [-1, 1]$. The net expected utility of a voter with preference type x of voting for $\mathcal{P}$ is

$$\begin{aligned} E(v(x,S)|piv,\pi^*,\sigma) &= p^*_\sigma v(x,L) + (1-p^*_\sigma)v(x,R) \\ &= 2x(p^*_\sigma L + (1-p^*_\sigma)R) - (p^*_\sigma L^2 + (1-p^*_\sigma)R^2), \end{aligned} \tag{2}$$

which is linear in x. Therefore, in equilibrium, we will have a cut-off x_σ defined by $E(v(x_\sigma, S)|piv, \pi^*, \sigma) = 0$, such that all $x < x_\sigma$ vote for one alternative and all $x > x_\sigma$ vote for the other alternative. While a voter with $x = x_\sigma$ is indifferent and can potentially mix, since such voters occur with zero probability, their action does not matter for equilibrium.

The linearity of $E(v(x, S)|piv, \pi^*, \sigma)$ allows us a different way of characterizing the equilibrium of the voting game (F, q, L, R, θ, n). We have already seen that the equilibrium strategies can be characterized by a pair of cut-offs x^*_l and x^*_r satisfying $E(v(x_l, s)|piv, \pi^*, l) = 0$ and $E(v(x_r, s)|piv, \pi^*, r) = 0$. Equation (2) implies that finding a pair (x^*_l, x^*_r) is equivalent to finding a pair of beliefs (p^*_l, p^*_r), one for each signal. Now, denote the likelihood of a tie induced by any strategy profile π *without* conditioning on a signal as $\beta_L = \Pr(L|piv, \pi)$, and notice that we must have the posterior likelihoods (p_l, p_r) related by Bayes Rule to β_L in the following way.

$$\left.\begin{aligned} p_l(\beta_L) &\equiv \beta(L|piv,\pi,l) = \tfrac{q\beta_L}{q\beta_L + (1-q)(1-\beta_L)} \\ p_r(\beta_L) &\equiv \beta(L|piv,\pi,r) = \tfrac{(1-q)\beta_L}{(1-q)\beta_L + q(1-\beta_L)} \end{aligned}\right\} \tag{3}$$

It is easy to see that (i) both p_l and p_r are strictly increasing functions of β_L, (ii) $p_l(\beta_L) = p_r(\beta_L) = \beta_L$ at $\beta_L \in \{0,1\}$, and (iii) $p_l(\beta_L) > p_r(\beta_L)$ for $\beta_L \in (0,1)$. Therefore, given q, finding a pair of posterior beliefs (p_l, p_r) is equivalent to finding a value of β_L. Thus, there is a one-to-one correspondence between the cut-off strategy (x_l, x_r) and the β_L, the likelihood of state L conditional on a tie. We call $\beta_L \Pr(L|piv,\pi)$ the *induced prior* as it acts as a prior likelihood from which posteriors p_σ are determined, but it is induced in equilibrium by the strategy profile.

3.2. **Existence.** Given a strategy profile π, the likelihood that a randomly chosen voter votes for $\mathcal{P}$ in state S is

$$t(S,\pi) = \int_{-1}^{1} \Pr(l|S)\pi(x,l)dF(x) + \int_{-1}^{1} \Pr(r|S)\pi(x,r)dF(x),\ S = L, R \tag{4}$$

Expanding (4) we can write

$$\begin{aligned} t(L,\pi) &= q\int_{-1}^{1}\pi(x,l)dF(x) + (1-q)\int_{-1}^{1}\pi(x,r)dF(x) \\ t(R,\pi) &= (1-q)\int_{-1}^{1}\pi(x,l)dF(x) + q\int_{-1}^{1}\pi(x,r)dF(x) \end{aligned}$$

Under a rule θ a voter is pivotal if $n\theta$ votes are cast for the policy $\mathcal{P}$ from among the remaining n voters. So, the probability of being pivotal under state S is given by[12]:

$$\Pr(piv|\pi, S) = \binom{n}{n\theta} (t(S,\pi))^{n\theta} (1 - t(S,\pi))^{n-n\theta},\ S = L, R \tag{5}$$

Note that equation (5) actually denotes a pair of equations, one for each state. Call these the *pivot equations.* Note that if $t(S,\pi) \in (0,1)$ then $\Pr(piv|\pi,S) > 0$. We show later that in any equilibrium of the model, we must have $t(S,\pi) \in (0,1)$ which ensures that pivot probabilities in equations (5) are well defined. The belief on the state S conditional on being pivotal is given by:

$$\beta(S|piv,\pi) = \frac{\Pr(piv|\pi,S)}{\Pr(piv|\pi,L) + \Pr(piv|\pi,R)},\ S = L, R \tag{6}$$

We can rewrite this as

$$\frac{\beta_L}{1-\beta_L} = \frac{\beta(L|piv,\pi)}{\beta(R|piv,\pi)} = \frac{\Pr(piv|\pi,L)}{\Pr(piv|\pi,R)},$$

which, using the pivot equations, gives us

$$\frac{\beta_L}{1-\beta_L} = \left[\frac{(t(L,\pi))^{\theta}(1-t(L,\pi))^{1-\theta}}{(t(R,\pi))^{\theta}(1-t(R,\pi))^{1-\theta}}\right]^n \tag{7}$$

Equation (7) is called the equilibrium condition. Consider any strategy π that is comprised of cut-offs $x_l(\beta_L)$ and $x_r(\beta_L)$ arising from beliefs β_L. Clearly, it is an equilibrium if and only if it induces the same belief β_L, and therefore, leads to the same cut-offs $x_l(\beta_L)$ and $x_r(\beta_L)$ as best response. In other words, such a belief is characterized by a solution to the equilibrium condition (7).

Now consider any given voting game (F, q, L, R, θ, n), and consider any strategy π that is comprised of cut-offs $x_l(\beta_L)$ and $x_r(\beta_L)$. First, note that $t(S,\pi)$ is continuous is β_L. We later show that $t(S,\pi) \in (0,1)$ for any cut-off strategy. So, the right hand side of equation (7) is continuous in β_L and bounded above and below. The left hand side, on the other hand, goes from 0 to ∞ continuously as β_L changes from 0 to 1. Hence, there exists a solution to the equation (7). This establishes the existence of cut-off equilibrium strategies in any given voting game.

Before identifying equilibrium strategies, we classify the voting rules and lay down the conditions that need to be satisfied for information to be aggregated in equilibrium given a voting rule.

[12]For technical convenience, we assume that $n\theta$ is an integer. The "integer problem" in dealt with in detail in Bhattacharya (2013). There, it is shown that the assumption that $n\theta$ is an integer does not affect the results in the limit as n becomes large.

3.3. **Conditions for Information Aggregation.** Before proceeding with the analysis of information aggregation properties, we set up some important definitions that will be useful throughout the paper. First, we introduce a classification of the voting rules according to the outcome induced in each state under full information in a large election. We shall call a voting rule *consequential* if under that rule, we get different outcomes under different states if the states were common knowledge. On the contrary, if the voting threshold is such that under full information, the same outcome obtains in each state, we call the it a *trivial* rule. Notice that this classification depends on the particular voting environment under consideration. We shall discuss this classification for each case (ordered vs unordered alternatives) in a later section.

Next, we define the standard for information aggregation in election.

Definition 1 (Full Information Equivalence)**.** *Suppose in a voting environment (F, q, L, R) under voting rule θ, alternative A wins in state L and A' wins in state R when the state is common knowledge. Now, consider a sequence π^n of equilibria in this environment, fixing voting rule θ and letting the number of voters increase unboundedly. The sequence of equilibria π^n is said to* aggregate information *for the voting rule θ if the probability of alternative A winning in state L and A' winning in state R converges to 1 along the sequence. An environment is said to be* full information equivalent *if every equilibrium sequence aggregates information for every voting rule $\theta \in (0, 1)$.*

We shall see that while all environments satisfying ordered alternatives will be full information equivalent, it will not be the case for environments with unordered alternatives. When alternatives are unordered, for each consequential rule, there will be three sequences of equilibria, one of which will aggregate information. In two others, the status quo $\mathcal{Q}$ will win almost surely in both states, and thus information aggregation will fail.

If in an equilibrium sequence, the vote of an individual with type x changes with the signal, i.e., if $\pi^n(x, l) \neq \pi^n(x, r)$ for all n large enough, then type x is said to be *responsive*. Given a voting rule, the characteristics of the responsive set of voters determines whether information will be successfully aggregated.

Definition 2 (Alignment)**.** *Suppose that given a consequential rule, under full information, $\mathcal{P}$ wins under state S and $\mathcal{Q}$ in the other state, i.e., the state $\{L, R\}\backslash S$. A type x is said to be* aligned *with the society if he prefers $\mathcal{P}$ in state S and $\mathcal{Q}$ in the other state. If on the other hand, a type x prefers $\mathcal{Q}$ in state S and $\mathcal{P}$ in the other state, then we call the type* mis-aligned*.*

Note that the responsive set of voters contains only independent types, and independents are either aligned or misaligned. If the majority of types in a set of voters is aligned, the set itself is said to be aligned. For voting with a consequential rule θ, we need the following conditions to be satisfied in equilibrium for the information to be aggregated in the limit.

(1) The responsive set should be *influential*, i.e., the overall voting outcome should change as the responsive types vote differently in the different states. In other words, the vote share for $\mathcal{P}$ should be higher than the threshold θ in one state and lower in the other.
(2) The responsive set should be *aligned* with the society, and thus contribute more votes for the "correct" alternative in each state.

Both conditions are satisfied under equilibrium in the common values situation, but in the non-common values situation, each can individually fail in the limit equilibrium.

On the other hand, for voting with trivial rules, we need the responsive types *not* to be influential for information to be aggregated.

4. Ordered Alternatives

First, we look at the benchmark case of ordered alternatives ($L < R < 0$). We show that the conditions for information aggregation above are satisfied for every voting rule.

We start by looking at the cut-offs x_l and x_r. From equation (2), for any p_σ, we obtain x_σ as the unique solution to $E(v(x,S)=0$.

$$x(p_\sigma) = \frac{1}{2}\left(\frac{(L)^2 p_\sigma + (R)^2(1-p_\sigma)}{Lp_\sigma + R(1-p_\sigma)}\right) \in \left[\frac{L}{2}, \frac{R}{2}\right]$$

The the cut-off strategies are given by equation (8):

$$\left\{ \begin{array}{c} \pi(x,l) = \left\{ \begin{array}{l} 1\ if\ x \le x_l \\ 0\ otherwise \end{array} \right. , \\ \pi(x,r) = \left\{ \begin{array}{l} 1\ if\ x \le x_r \\ 0\ otherwise \end{array} \right. \end{array} \right\} \text{ when } L < R < 0 \tag{8}$$

Since the cut-offs x_σ are functions of posteriors p_σ which are again functions of the induced prior β_L, we should understand the cut-offs x_σ as functions of β_L

Remark 1. *It is easy to check that (i) $x_\sigma(\beta_L)$ are decreasing functions of β_L,(ii)$x_r(\beta_L) > x_l(\beta_L)$ for $\beta_L \in (0,1)$, and (iii) For $\beta_L = 1,\ x_r = x_l = \frac{L}{2}$, and likewise for $\beta_L = 0,\ x_r = x_l = \frac{R}{2}$.*[13]

Thus, for any induced prior, the strategies in the benchmark case are characterized by cutpoints x_l and x_r, with $x_l \le x_r$. Thus, types left of x_l always vote for $\mathcal{P}$ and those right of x_r vote for $\mathcal{Q}$ while types in $[x_l, x_r]$ vote informatively (according to their signal). The set of types $[x_l, x_r]$ is the responsive set, while the other types vote according to their bias. Irrespective of the location of the cutoffs, the responsive set is always *aligned* with the society. This means that whenever the responsive set is influential, information will be aggregated. Thus, for consequential rules, all we need to show for information aggregation is that in any limit equilibrium, the responsive set is influential. For this, we need monotonicity of the vote shares under both states, which is again ensured by the ordered nature of the cut off strategies.

We define the probability of an individual voting for the alternative $\mathcal{P}$ given σ as z_σ, *i.e.*, $z_\sigma \equiv \int_{-1}^{1} \pi(x,\sigma)dF$. We have from equation (8),

$$z_\sigma = F(x_\sigma),\ \sigma = \{l, r\}$$

Therefore, using (4) we write:

$$\left. \begin{array}{l} t(L,\pi) = qz_l + (1-q)z_r = qF(x_l) + (1-q)F(x_r) \\ t(R,\pi) = (1-q)z_l + qz_r = (1-q)F(x_l) + qF(x_r) \end{array} \right\} \tag{9}$$

Note that since the cut-offs x_l and x_r are functions of the induced prior β_L, the vote shares $t(L,\pi)$ and $t(R,\pi)$ are also functions of β_L. The following lemma examines how the vote share in each state changes as a function of the induced prior.

Lemma 1. *The expected share of votes $t(S,\pi)$ in state S decreases strictly with the induced prior β_L. For all $\beta_L \in (0,1)$, the vote share $t(L,\pi)$ in state L is strictly less than the share $t(R,\pi)$ in state R. For $\beta_L = 0$, $t(L,\pi) = t(R,\pi) = F(\frac{R}{2})$ and for $\beta_L = 1$, $t(L,\pi) = t(R,\pi) = F(\frac{L}{2})$.*

Proof. By Remark 1, at $\beta_L = 0$, $z_l = z_r = F(\frac{R}{2}) \ \Rightarrow t(S,\pi) = F(\frac{R}{2})$ for $S \in \{L, R\}$. Similarly, at $\beta_L = 1$, $t(S,\pi) = F(\frac{L}{2})$ for $S \in \{L, R\}$. Also, since x_σ is decreasing in β_L, $t(S,\pi)$ is strictly decreasing in β_L. For the second part of the lemma, note that

$$t(L,\pi) - t(R,\pi) = (2q-1)\left(F(x_l) - F(x_r)\right)$$

By remark 1 again, for $\beta_L \in (0,1)$, $F(x_l) - F(x_r) < 0$, and since $q > \frac{1}{2}$, we have $t(L,\pi) < t(R,\pi)$. ■

Lemma 1 states that as the induced prior probability of the state being L (conditional on being pivotal) increases, the expected share of votes for the alternative policy decreases under either state

[13] The crucial observation driving this result is $\frac{dx(p)}{dp} < 0$.

since the state L is deemed to be more "extreme". Informative voting by the responsive set ensures that the policy receives more votes in the "moderate" state (R). Note also that at any induced prior, the difference in expected vote shares is increasing in the informativeness of the signal. The expected vote shares in the two states are plotted against the induced prior in figure 1.

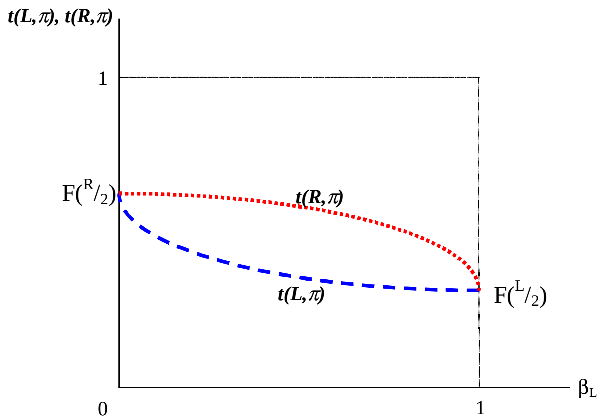

Figure 1: Vote shares in each state under ordered alternatives

Lemma 1 also ensures that since $t(S,\pi)$ lies strictly between 0 and 1, and $\beta(S|piv,\pi,\sigma)$ is always well-defined. Intuitively, since the types left of $\frac{L}{2}$ are $\mathcal{P}$-partisans and those to the right of $\frac{R}{2}$ are $\mathcal{Q}$-partisans, there is always a positive probability that any given type is pivotal..

4.1. **Limit Equilibria under ordered alternatives.** Now, we consider the properties of the voting equilibria as the electorate grows in size arbitrarily, keeping the environment and the voting rule constant. Therefore, every quantity is superscripted by the number of voters n. The superscript will be suppressed when there is no ambiguity Suppose, given an environment and voting rule for some n, the equilibrium is π^n, and the cutoffs are x_σ^n. Since x_σ^n lies in a compact space $\left[\frac{L}{2},\frac{R}{2}\right]$, existence of equilibrium for any n implies the existence of a convergent subsequence with an accumulation point as $n\to\infty$. If a limit of this sequence exists, we call it π^0. By continuity arguments, as $x_\sigma^n\to x_\sigma^0$, $t(S,\pi^n)$, β_L^n, p_l^n,and p_r^n all converge to finite limits $t(S,\pi^0)$, β_L^0, p_l^0, and p_r^0 respectively along the sequence.

Rewriting the equilibrium condition:

$$\text{(10)}\qquad \frac{\beta_L^n}{1-\beta_L^n}=\left[\frac{(t(L,\pi^n))^\theta(1-t(L,\pi^n))^{1-\theta}}{(t(R,\pi^n))^\theta(1-t(R,\pi^n))^{1-\theta}}\right]^n \quad for\ all\ n$$

We know that a solution to equation (10) exists for every n. From continuity, if a limit exists, we can also say that the above relation has to hold in the limit; call this the *limit equilibrium condition.*

$$\text{(11)}\qquad \frac{\beta_L^0}{1-\beta_L^0}=\lim_{n\to\infty}\left[\frac{\left(t(L,\pi^0)\right)^\theta\left(1-t(L,\pi^0)\right)^{1-\theta}}{(t(R,\pi^0))^\theta(1-t(R,\pi^0))^{1-\theta}}\right]^n$$

To avoid writing complicated expressions, we define:

$$\alpha_n=\frac{(t(L,\pi^n))^\theta(1-t(L,\pi^n))^{1-\theta}}{(t(R,\pi^n))^\theta(1-t(R,\pi^n))^{1-\theta}} \text{ and } \alpha_0=\frac{\left(t(L,\pi^0)\right)^\theta\left(1-t(L,\pi^0)\right)^{1-\theta}}{(t(R,\pi^0))^\theta(1-t(R,\pi^0))^{1-\theta}}$$

Note that the vote shares $t(S,\pi^n)$ are functions of β_L^n. Next, we look at the properties of the limit, assuming existence for the time being. We later show that in the ordered alternatives setting, for any voting rule, there is only one accumulation point of π^n which must be the limit.

Lemma 2. *If $\beta_L^0 \in (0,1)$, $\alpha_0 = \lim_{n\to\infty} \alpha_n = 1$*

Proof. See Appendix. Note that this lemma does not use the condition that $L < R < 0$, so it is true of unordered alternatives too. ■

Lemma 3. *If $\beta_L^0 = 1$, then $x_\sigma^n \to \frac{R}{2}$ from the left for $\sigma = l, r$. Similarly, if $\beta_L^0 = 0$, then $x_\sigma^n \to \frac{L}{2}$ from the right for $\sigma = l, r$*

Proof. Follows from continuity of x_σ^n in p_σ^n and of p_σ^n is β_L^n, along with Remark 1. ■

Note, as an aside to Lemma 3, that although under both signals the cutoffs converge to $\frac{R}{2}$ or $\frac{L}{2}$ as the induced prior converges to 1 or 0 respectively, by remark 2, we always have $x_l^n < x_r^n$. Thus, in the responsive set, the voters always vote for $\mathcal{Q}$ if they get moderate signal r and $\mathcal{P}$ if they get the extreme signal l. The responsive interval is vanishingly small as the induced prior distribution converges to state R , grows for intermediate values of the prior, and again shrinks to a vanishing size as the distribution converges to a degenerate distribution at state L. Thus, given q, a level of precision of the signals, the difference between expected shares in the two states is low for extreme values of the induced prior and high for intermediate values.

Lemma 2 and Lemma 3 together imply that for any limit induced prior, given a voting rule under any equilibrium, the vote shares in each state must be related in a certain way. This is stated in Proposition 1 below. According to Lemma 2, if α_n is bounded away from 1, then β_L^0 must be either 0 or1. Under conditions of Lemma 3, if β_L^n is indeed 0 (or 1), then the voters are almost sure of the state in which they are pivotal and vote as if under full information. Every type except those in a vanishing set votes uninformatively, and the vote shares under either state are the same in the limit. Thus, in equilibrium, we have $\alpha_0 = 1$ for all values of the induced prior.

Proposition 1. *In all limit equilibria with ordered alternatives, we must have* $\alpha_0 = 1$, *i.e.,*

$$\left(t(L,\pi^0)\right)^\theta \left(1 - t(L,\pi^0)\right)^{1-\theta} = \left(t(R,\pi^0)\right)^\theta \left(1 - t(R,\pi^0)\right)^{1-\theta}, \textit{i.e., } \alpha_0 = 1$$

Proof. For any equilibrium with $\beta_L^0 \in (0,1)$, the proposition follows straightforwardly from Lemma 2. If $\beta_L^0 = 1$, the first part of Lemma 3 implies that $t(L,\pi^n)$ and $t(R,\pi^n)$ both converge to $F(\frac{R}{2})$. Therefore, $\alpha_0 = 1$, since $F(\frac{R}{2}) \in (0,1)$. If $\beta_L^0 = 0$, the proof follows in exactly the same way. ■

Note that Proposition 1 is based on a necessary condition that must be true for a β_L^0 to which induced belief converges in the limit equilibrium. It helps exclude certain voting rules that cannot support a given value of β_L in the limit. To say this formally, define $\Theta(\beta_L)$ as the set of voting rules that can support β_L as an induced belief in the *limit equilibrium condition* (equation 11) for *some* distribution of preferences in the cut-off equilibrium. To emphasize that $t(S,\pi)$ is a function of β_L, we write $t(S,\pi)$ as $t_S(\beta_L)$ for $S \in \{L, R\}$.

Lemma 4. *Under ordered alternatives, (i) If $\beta_L \in (0,1)$, then $\Theta(\beta_L)$ is a strictly increasing function $\theta^*(\beta_L)$, with $t_L(\beta_L) < \theta^*(\beta_L) < t_R(\beta_L)$. (ii) Otherwise, $\Theta(1) = \{\theta : \theta < F(\frac{L}{2})\}$, and $\Theta(0) = \{\theta : \theta > F(\frac{R}{2})\}$*

Proof. In Appendix. ■

The first part of the lemma is almost a corollary of Proposition 1. For each interior value β_L of the induced prior, it identifies a unique θ as the only possible voting rule to support β_L in the limit equilibrium. As long as the expected vote shares in the two states are different, the only voting rule that can satisfy Proposition 1 is one that lies strictly between the two shares. This has the implication that under one state the status quo wins, while in the other, the policy wins. If there are any equilibria with beliefs that place positive probability on both states, then the responsive set of types for these equilibria are always influential. The second part of the lemma says that the extreme beliefs can be supported only by extreme values of the voting rules. The main implication of the Lemma is that while the responsive set is influential for any possible equilibria with consequential rules, it is never influential for trivial rules.

Note that since $\theta^*(\beta_L)$ is strictly increasing, its inverse function $\beta_L^{-1}(\theta)$ exists for $\theta \in \left(F(\frac{L}{2}), F(\frac{R}{2})\right)$ and is strictly increasing. Thus, according to Lemma 5, for every θ, there is a unique β_L that can be supported as an equilibrium induced prior in the limit, *for any distribution of types*. Call it $\beta(\theta)$. We can write:

$$\beta(\theta) = \begin{cases} 1 \; if \; \theta < F(\frac{L}{2}) \\ \quad \beta_L^{-1}(\theta) \;\; if \; \theta \in \left(F(\frac{L}{2}), F(\frac{R}{2})\right) \\ 0 \; if \; \theta > F(\frac{R}{2}) \end{cases} \tag{12}$$

We plot the correspondence $\Theta(\beta_L)$ along with the expected vote shares in each state against the induced prior in Figure 2.

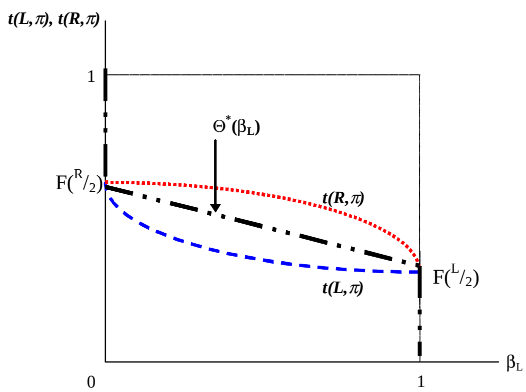

Figure 2: Correspondence $\Theta(\beta_L)$ under ordered alternatives

The next proposition gives a characterization of cut-off equilibria in large populations for different voting rules under ordered alternatives.

Proposition 2. *Assume that a voting environment (F, q, L, R) satisfies $L < R < 0$. Fix a voting rule $\theta \in (0,1)$. Then there is a unique limit equilibrium π^0 in cut-off strategies with the induced prior converging to β_L if and only if $\theta \in \Theta(\beta_L)$, or alternatively, if and only if $\beta_L = \beta(\theta)$.*

Proof. Since β_L lies in a compact set, for the sequence π^n, there is an accumulation point π^a, given θ. We show in the appendix that this π^a is the limit equilibrium π^0 given θ. Lemma 4 states that for any distribution of types, if a limit exists, there is a unique number $\beta(\theta)$ to which the induced prior converges in the limit along the sequence of equilibria under voting rule θ. ■

Note that once the limiting value of the induced prior β_L is established, the limit posterior distributions p_σ, the limit cut-offs x_σ etc. are all determined from β_L. Thus this proposition describes all relevant information about strategies, vote shares and statewise outcomes in equilibria with a voting rule when the population size becomes large. Also, by the Law of Large numbers, the actual vote shares are arbitrarily close to the expected vote shares[14].

[14]More specifically, given any $\epsilon > 0$ and $\delta > 0$, we can find some number N such that as long as the polupation size is larger than N, the actual vote share is within ϵ of the exoected share with a probability higher than $1 - \delta$.

4.2. **Outcomes and Information Aggregation.** In Section 2, we informally discussed a classification of voting rules according to the outcomes produced under full information. With full information, under state L, the policy would get $F(\frac{L}{2})$ share of votes; and similarly under state R, the policy would get $F(\frac{R}{2})$ share of votes. Therefore:

- Any voting rule $\theta < F(\frac{L}{2})$ is a $\mathcal{P}$-trivial rule, i.e., $\mathcal{P}$ wins under both states.
- Any voting rule $F(\frac{L}{2}) < \theta < F\left(\frac{R}{2}\right)$ is a consequential rule, i.e., $\mathcal{P}$ wins in state R and $\mathcal{Q}$ in state L.
- Any voting rule $\theta > F\left(\frac{R}{2}\right)$ is a $\mathcal{Q}$-trivial rule, i.e., outcome is status quo under both states.

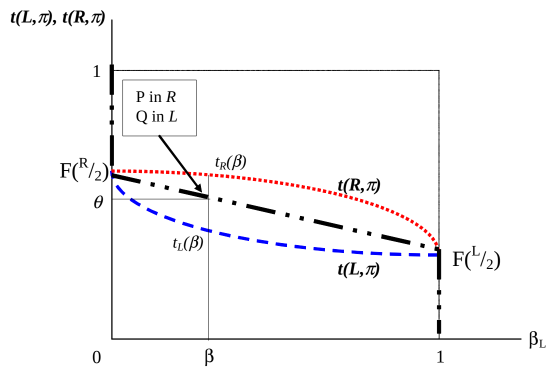

Figure 5(a): Outcome under a Consequential rule θ under Preference monotonicity *(L < R < 0)*

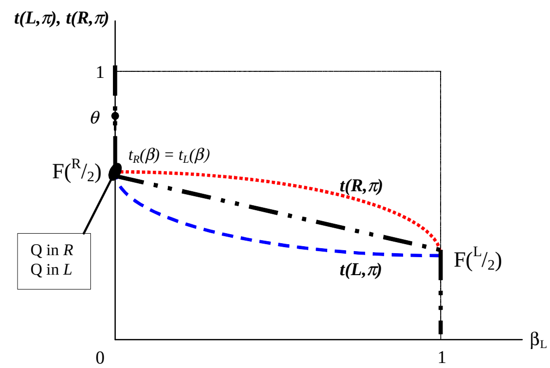

Figure 5(b): Outcome under a Q-trivial rule θ under preference monotonicity *(L < R < 0)*

Figure 3(a), 3(b)

Proposition 3. *A voting environment (F, q, L, R) satisfying $L < R < 0$ satisfies full information equivalence.*

Proof. In appendix. ■

According to the proposition, under ordered cut-offs, any voting rule aggregates information in all equilibrium sequences. Since the vote shares in each state is between $F(\frac{L}{2})$ and $F\left(\frac{R}{2}\right)$, any trivial rule aggregates information. Essentially, the responsive types lying between $\frac{L}{2}$ and $\frac{R}{2}$ can never be influential with trivial rules. With $\mathcal{P}$-trivial rules, everyone is virtually sure that conditional on being pivotal, the state is L. In other words, under such a rule, being pivotal at state L (when $\mathcal{P}$ receives least votes) is infinitely more probable than being pivotal at state R. Similarly, with any $\mathcal{Q}$-trivial rule, one has far higher chance of being pivotal in state R (when $\mathcal{P}$ receives most votes) than in state L. We depict the outcome in the limit equilibrium with a $\mathcal{Q}$-trivial rule in figure 3(a). On the other hand, for any consequential rule, the induced prior places positive probability on both states in the limit, and the responsive set is influential. Since the responsive types are aligned too, we have outcome $\mathcal{P}$ in state R and $\mathcal{Q}$ in state L almost surely, and hence we have information aggregation. The limit equilibrium outcome with a consequential rule is depicted in figure 3(b).

5. Unordered Alternatives

In this section, we study the case where $L < 0 < R$. We look at the strategies and equilibria in this situation and compare and contrast their properties with that of the benchmark model with ordered alternatives. Specifically, we show how voting can fail to aggregate information in the presence of heterogeneous groups with competing interests. Since the voters with $x \leq \frac{L}{2}$ support the alternative policy only in state L and those with $x \geq \frac{R}{2}$ do so only in state R, we will call these two groups of voters the L-group and the R-group respectively. Within the groups, the voters have the same ranking over alternatives, while across the groups, voters have exactly opposite ranking over alternatives in each state.

We shall simplify the model a bit and consider a slightly special case with $L = -b$ and $R = b > 0$. Note that this is not too strong an assumption as we consider all possible distributions of voter ideal points. An environment with unordered cut-offs is denoted by the tuple (F, q, b). In this section, we shall use the same methodology we used in the previous section to examine the unordered alternatives situation.

When $L = -b$ and $R = b$, the condition for type x voting for the policy $\mathcal{P}$ after having received σ is:

$$Ev(x, \sigma) \geq 0 \Rightarrow 2x(1 - 2p_\sigma) \geq b,$$

which gives us the following cut-offs

$$x_\sigma = \begin{cases} \min(1, \frac{b}{2(1-2p_\sigma)}), & 0 \leq p_\sigma < \frac{1}{2} \\ \max(-1, \frac{b}{2(1-2p_\sigma)}), & \frac{1}{2} \leq p_\sigma \leq 1 \end{cases} \tag{13}$$

and optimal strategies based on these cut-offs

$$\pi(x, \sigma) = \begin{cases} \left.\begin{array}{c} 1 \ for \ x \leq x_\sigma \\ 0 \ for \ x > x_\sigma \end{array}\right\} if \ \frac{1}{2} \leq p_\sigma \leq 1 \\ \left.\begin{array}{c} 1 \ for \ x \geq x_\sigma \\ 0 \ for \ x < x_\sigma \end{array}\right\} if \ \ 0 \leq p_\sigma < \frac{1}{2} \end{cases} \tag{14}$$

Or alternatively, combining equations (13) and (14), we define the strategies in terms of p_σ as follows:

$$\pi(x, \sigma) = \left\{ \begin{array}{c} \left.\begin{array}{c} 1 \ for \ x \leq \frac{b}{2(1-2p_\sigma)} \\ 0 \ for \ x > \frac{b}{2(1-2p_\sigma)} \end{array}\right\} \ if \ p_\sigma \geq \frac{1}{2} + \frac{b}{4} \\ 0 \ for \ all \ x \quad if \ p_\sigma \in \left(\frac{1}{2} - \frac{b}{4}, \frac{1}{2} + \frac{b}{4}\right) \\ \left.\begin{array}{c} 1 \ for \ x \geq \frac{b}{2(1-2p_\sigma)} \\ 0 \ for \ x < \frac{b}{2(1-2p_\sigma)} \end{array}\right\} \ if \ p_\sigma \leq \frac{1}{2} - \frac{b}{4} \end{array} \right\}$$

Any equilibria must have strategies of the above form. Note that $p_\sigma \in [0, 1] \Rightarrow -1 \leq 1 - 2p_\sigma \leq 1$ and so $x_\sigma \in [-1, -\frac{b}{2}] \cup [\frac{b}{2}, 1]$. Also, for all values of p_σ, $\pi(x, \sigma) = 0$ in the range $(-\frac{b}{2}, \frac{b}{2})$. Thus a voter with his bliss point in this range always votes for the status quo irrespective of the signal. Thus, although all equilibria must have cut-off strategies, the cut-offs are not nicely ordered as in the case with ordered alternatives.

One implication of the fact that all types in the range $(-\frac{b}{2}, \frac{b}{2})$ always vote for $\mathcal{Q}$ is that the vote share $t(S, \pi)$ for $\mathcal{P}$ is strictly less than 1. To ensure that $t(S, \pi) > 0$, we need to ensure that we can never have both p_l and p_r simultaneously in the range $\left(\frac{1}{2} - \frac{b}{4}, \frac{1}{2} + \frac{b}{4}\right)$, which is guaranteed by the informativeness assumption I.

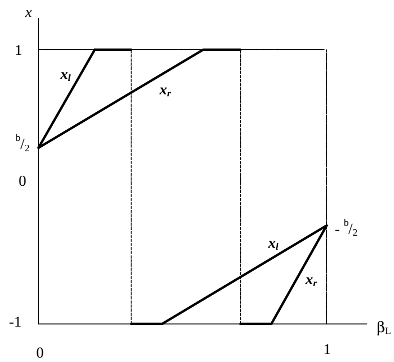

Figure 4: Cut-offs in an unordered alternatives setting as functions of induced prior

The cutoffs as functions of the induced prior are plotted in Figure 4. When a cut-off is in $[-1, -\frac{b}{2}]$ (the L-group), the types to the left of the cut-off vote for $\mathcal{P}$, and when the cut-off lies in $[\frac{b}{2}, 1]$ (the R-group), types to the right of the cut-off vote for $\mathcal{P}$. This has several implications. First, the responsive types lying in these two groups would vote in opposite ways based on the same information since one of the groups is aligned with the society and the other is not. Second, in each state, the vote share is a non-monotonic function of the induced belief. Note that the monotonicity in vote shares was crucial for information aggregation with consequential rules in the ordered alternatives case. Third, with unordered cut-offs, the existence of a well-defined induced prior is no longer trivial, and we need the informativeness assumption I on signals to guarantee that. Lastly, with a loss of the ordering property, uniqueness of the responsive set is no longer assured. This can give rise to equilibria with a certain feature that is not seen in the benchmark case of ordered alternatives, as we shall soon see in proposition.4.

Recall that the probability of an individual voting for the alternative $\mathcal{P}$ given σ is $z_\sigma, i.e.,\ z_\sigma \equiv \int_{-1}^{1} \pi(x,\sigma)dF$. In any equilibrium, we have:

$$z_\sigma = \begin{cases} F(x_\sigma)\ if\ x_\sigma \le -\frac{b}{2} \\ 1 - F(x_\sigma)\ if\ x_\sigma \ge \frac{b}{2} \\ 0\ otherwise \end{cases} \tag{15}$$

Although the definition of z_σ is different in the unordered alternatives case, the vote shares in the two states in terms of z_σ are still given by equation (9):

$$\begin{aligned} t(L,\pi) &= qz_l + (1-q)z_r \\ t(R,\pi) &= (1-q)z_l + qz_r \end{aligned}$$

We can immediately identify one particular equilibrium for the case with a distribution of types with density $f(\cdot)$ that is symmetric about 0.

Proposition 4. *Consider any environment with unordered alternatives* (F, q, b)*, for which the density* $f(\cdot)$ *is symmetric about* 0*. For any voting rule* $\theta \in (0,1)$ *and any finite number of voters* n*, there is an equilibrium with* $x_l^* = -\frac{b}{2(2q-1)}$ *and* $x_r^* = -x_l^*$*.*

Proof. Consider the situation where everyone else plays $x_\sigma = x_\sigma^*$, and $\sigma \in \{l, r\}$. Note that $x_l^* < -\frac{b}{2}$ and $x_r^* > \frac{b}{2}$. So, $z_l^* = F(x_l^*)$ and $z_r^* = 1 - F(x_r^*) = 1 - F(-x_l^*) = F(x_l^*) = z_l^*$, by symmetry of $f(\cdot)$. Therefore, $t(L,\pi) = t(R,\pi) = F(x_l^*)$ for each n, which implies that $\beta_L = \frac{1}{2}$ for every θ and n. Thus, the signals are fully informative, and we have $p_l = q$ and $p_r = 1-q$. These, coupled with the Assumption I, imply that the best response to x_σ^* is indeed x_σ^*, which establishes the claim. ∎

The proposition says that if the commonly held induced priors are uninformative, then sufficiently extreme types vote for the alternative $\mathcal{P}$ if and only if they get favorable signals, and everyone else votes uninformatively, disregarding their signal. There are a few things to be noted about the above equilibrium. First, this is the only "stable" equilibrium sequence in the sense that the strategies do not change with the number of players. Second, in this equilibrium, the expected vote share does not change with the state or the voting rule. If the required plurality for the policy to pass is higher than $F(x_\sigma^*)$, then the status quo always passes, and if the required share is lower than $F(x_\sigma^*)$, then the status quo always loses. If $\theta = F(x_\sigma^*)$, then we get either alternative (policy or status quo) with equal probability. We later show that even if the distribution of ideal points is not symmetric, there is always an equilibrium at some belief β_L^* (not necessarily equal to $\frac{1}{2}$) that has the same vote share for each state and is independent of the voting rule. As we shall establish later, this constitutes a failure of information aggregation.

Next, let us examine the vote share as a function of the induced prior in the unordered alternatives set-up.

Lemma 5. *In any environment with unordered alternatives* (F, q, b)*, there exists some number* β_L^* *satisfying* $0 < \beta_L^* < 1$ *such that* $\beta_L < \beta_L^*$*,* $t(R,\pi) > t(L,\pi)$*, for* $\beta_L > \beta_L^*$*,* $t(R,\pi) > t(L,\pi)$ *and for* $\beta_L = \beta_L^*$*,* $t(R,\pi) = t(L,\pi)$.

Proof. See Appendix. ■

This lemma says that if the commonly held induced prior probability that one is pivotal at state L falls below a critical value β_L^*, then the expected vote share in favour of the policy in state L is higher than that in state R. If, on the contrary, the belief is higher than β_L^*, then the alternative $\mathcal{P}$ is expected to get a higher vote share in state R. However, given a state, the expected share of the votes in favour of the policy alternative increases as one gets more and more extreme beliefs, i.e., as one is surer and surer of the state in which one is pivotal. As the voters get more unsure about the state, only the very extreme types vote for the policy. Note that at β_L^*, we have $F(x_l) + F(x_r) = 1$, and under a symmetric distribution of types, $\beta_L^* = \frac{1}{2}$, and we have an equilibrium at $\beta_L = \frac{1}{2}$ according to Proposition 4. The expected share of votes under the two states (when $L = -b$ and $R = b$) as functions of the induced prior are shown in figure 5. To illustrate how the shares are constructed according to (9), we also show the functions z_l and z_r (i.e., the probability of voting for $\mathcal{P}$ on getting the signal l and r respectively) in the figure.

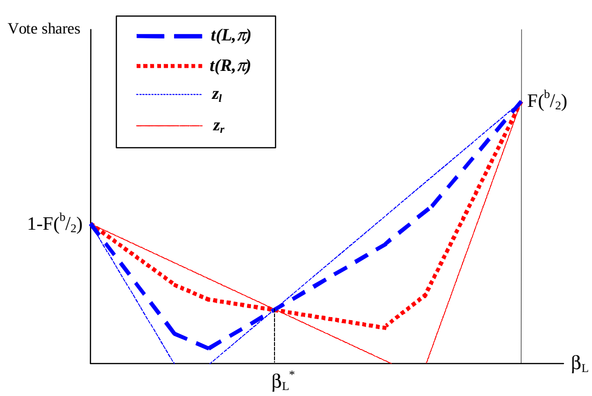

Figure 5: Construction of vote shares as functions of induced prior

5.1. **Limit equilibria with unordered alternatives.** We use the same notation as in Section 4.1. Since the cutoffs are bounded within a compact set, any sequence of x_σ^n will have a convergent subsequence. We look at such convergent subsequences x_σ^n as $n \to \infty$. We call an accumulation point of such a sequence of cutoffs as x^0_σ, and the resulting equilibrium as π^0. By the continuity arguments , as $x^n_\sigma \to x^0_\sigma$, $t(S, \pi^n)$, β^n_L, p^n_l,and p^n_r all converge to $t(S, \pi^0)$, β^0_L, p^0_l, and p^0_r respectively along the subsequence. In this section we examine which outcomes can be supported in the limit.

The *limit equilibrium condition* as identified in equation (11) remains exactly the same. Lemma 2 goes through without any change. Lemma 3 goes through too, with the slight modification that it is no longer true of all n, but it holds for large enough n. We state this in Lemma 6. For a sufficiently large electorate, if the induced prior converges to 0 (1), both cut-offs are in the L-group (R-group).

Lemma 6. *Consider any environment with unordered alternatives (F, q, b). If $\beta^0_L = 1$, (i) $\exists$ some m such that $x^n_l > x^n_r$ for all $n > m$; and (ii) $x^n_\sigma \to -\frac{b}{2}$ from the left for $\sigma = l, r$. Similarly, if $\beta^0_L = 0$, (i) $\exists$ some m_1 such that $x^n_l > x^n_r$ for all $n > m_1$; and (ii) $x^n_\sigma \to \frac{b}{2}$ from the right for $\sigma = l, r$*

Proof. See Appendix. ■

Next, we examine which voting rules can be supported by a given value of the induced prior in the limit, for which an equivalent of Lemma 4 is necessary. We define $\Theta(\beta_L)$ in the same way as before, i.e., the set of voting rules that may support an equilibrium sequence with induced prior converging to β_L.

Lemma 7. *Under unordered alternatives, (i)for $\beta_L \in (0, \beta^*_L) \cup (\beta^*_L, 1)$, $\Theta(\beta_L)$ is a continuous function $\theta^*(\beta_L)$, with $t_L(\beta_L) < \theta^*(\beta_L) < t_R(\beta_L)$ for $\beta_L < \beta^*_L$, and $t_L(\beta_L) > \theta^*(\beta_L) > t_R(\beta_L)$ for $\beta_L > \beta^*_L$, (ii)Otherwise, $\Theta(1) = \{\theta : \theta > F(-\frac{b}{2})\}$, $\Theta(0) = \{\theta : \theta > 1 - F(\frac{b}{2})\}$ and $\Theta(\beta^*_L) = \{\theta : \theta \in (0, 1)\}$.*

Proof. In Appendix. ■

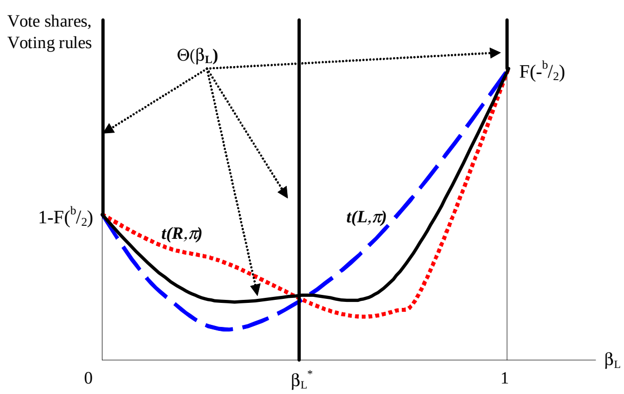

Figure 6: The correspondence $\Theta(\beta_L)$ under unordered alternatives

The correspondence $\Theta(\beta_L)$ for the unordered alternatives case, as inferred in Lemma 7, is depicted in figure 6. Note that in this case, if we invert the correspondence to get the supporting induced belief β_L for each voting rule θ, we no longer get a function $\beta(\theta)$ as defined in (12) in the ordered alternatives case, but rather a correspondence.

Denote $t(L, \beta^*_L) = t(R, \beta^*_L)$ by z. Because of the non-monotonic vote share functions, for any voting rule $\theta > z$, there can be three different limit equilibria. One equilibrium is an approximation to the symmetric equilibrium in Proposition 4. With the equilibrium belief at β^*_L, the vote shares are equal in both states and independent of the voting rule. For any consequential rule or a $\mathcal{P}$-trivial rule, information is not aggregated in this equilibrium. Of the two other limit equilibria, one has induced prior probability of state L less than β^*_L and has the responsive set of types entirely (or mostly) in the R-group. For voting rules less than $1 - F(\frac{b}{2})$, the responsive set in this equilibrium is influential, and $\mathcal{P}$ obtains in state R and $\mathcal{Q}$ in state L. For $\theta > 1 - F(\frac{b}{2})$, the responsive set in this equilibrium sequence cannot be influential, and the status quo obtains in both states. Similarly, there is another equilibrium with induced prior belief greater than β_L^*, where the responsive set is entirely or mostly in the L-group.

Note that so far we have only claimed that for a given voting rule there *can* be three limit equilibria. The next proposition states that all the equilibria discussed above exist for any distribution of ideal points. Given an induced prior β_L, any voting rule that is not ruled out by the necessary condition (11) can indeed support a limit equilibrium with beliefs converging to β_L.

Proposition 5. *For any voting environment (F, q, L, R) satisfying $L = -b$ and $R = b$, given a voting rule θ, there is a sequence of equilibria with cut-off strategies and with the equilibrium induced prior converging to β_L if $\theta \in \Theta(\beta_L)$*[15].

Proof. In appendix. ■

5.2. **Outcomes and Information Aggregation.** From Lemma 7 and Proposition 5, we can deduce possible outcomes for each value of the induced prior. All these outcomes occur almost surely, in the same way as in the ordered alternatives case.

- For $\beta_L = 0$, the only possible outcome is $\mathcal{Q}$ under both states. Here, the responsive set is in the R-group but is not influential.
- For $\beta_L \in (\beta_L^*, 1)$, the only possible outcome is $\mathcal{Q}$ under state L and $\mathcal{P}$ under state R. Here, the responsive set is in the R-group and is influential.
- For $\beta_L = \beta_L^*$, the vote share in each state is fixed at z and the outcome depends on whether the voting rule is greater or less than z.
- For $\beta_L \in (0, \beta_L^*)$, the only possible outcome is $\mathcal{P}$ under state L and $\mathcal{Q}$ under state R. Here, the responsive set is in the L-group and is influential.
- For $\beta_L = 0$, the only possible outcome is $\mathcal{Q}$ under both states. Here, the responsive set is in the L-group but is not influential.

From here onwards, we assume with a slight loss of generality that $F(-\frac{b}{2}) > 1 - F(\frac{b}{2})$. In other words, we assume that the L-group is the larger interest group, and hence the group that is aligned with the society. Therefore,

- Any voting rule $\theta < 1 - F\left(\frac{b}{2}\right)$ is $\mathcal{P}$-trivial
- Any voting rule $1 - F\left(-\frac{b}{2}\right) \leq \theta < F\left(\frac{b}{2}\right)$ is a consequential rule, i.e., the policy wins in state L and the status quo in state R.
- Any voting rule $\theta \geq F\left(-\frac{b}{2}\right)$ is a $\mathcal{Q}$-trivial rule.

For all $\mathcal{Q}$-trivial rules, the beliefs that can be supported in equilibrium are $\beta = \{0, \beta_L^*, 1\}$. Since the maximum share of received by the alternative $\mathcal{P}$ in any state is $F\left(-\frac{b}{2}\right)$, $\mathcal{Q}$-trivial rules always aggregate information. Figure 7(a) depicts the limit equilibria for a $\mathcal{Q}$-trivial rule.

For information to be aggregated under consequential rules, we need the responsive set to be influential and in the L-group. For these rules however, there is always one equilibrium with $\beta_L = 0$ where the responsive set in the R-group and is not influential. Hence we get $\mathcal{Q}$ in both states. In another equilibrium for these rules, $\beta_L = \beta_L^*$, and here too, we get $\mathcal{Q}$ in both states with a very high probability. However, there is a third equilibrium with induced prior converging to some belief in $(0, \beta_L^*)$ with the responsive set entirely in the L-group and influential. This equilibrium aggregates information. Figure 7(b) depicts all the possible limit equilibria for a consequential rule.

For $\mathcal{P}$-trivial rules greater than z we have two equilibria with opposite outcomes in the different states: one with equilibrium induced prior in the set $(0, \beta_L^*)$ and the other in the set $(\beta_L^*, 1)$. The responsive sets are influential here when information aggregation requires that they not be so. So,

[15] This theorem requires an assumption that $\theta^*(\beta_L)$ is not constant over any range. We ignore that as a non-generic case.

for these voting rules we have no information-aggregating equilibrium. The third equilibrium has beliefs converging to β_L^*. Since at this belief, the vote share in both states is z, in this equilibrium we always get the status quo. Figure 7(c) shows the possible equilibria for one such rule. However, information is aggregated almost surely by the very low $\mathcal{P}$-trivial rules[16].

We summarize the inferences about information aggregation for different voting rules in an unordered alternatives setting in the next proposition. We use the definition of information aggregation as defined in Section 3.3. Fixing a voting environment and a voting rule, we say that an equilibrium sequence aggregates information if we obtain approximately the full information outcome in the limit. Notice that for the same environment and voting rule, we may have multiple equilibrium sequences, some of which do aggregate information while some do not.

Proposition 6. *Consider any voting environment (F, q, L, R) satisfying $L = -b$ and $R = b$. All (limit) voting equilibria with $\mathcal{Q}$-trivial voting rules aggregate information. For consequential rules, there is one equilibrium sequence that aggregate information and two that do not aggregate information. For $\mathcal{P}$-trivial rules that are sufficiently large, all equilibrium sequences are non-information aggregating. All $\mathcal{P}$-trivial rules below some threshold aggregate information.*

The above proposition establishes the bias in favour of the status quo. Unless the required vote share for the policy to win is very low, competition between two groups ensures that the status quo wins in at least one state. Note that the only voting rules for which information is aggregated in any equilibrium are all $\mathcal{Q}$-trivial rules and the very low $\mathcal{P}$-trivial rules.

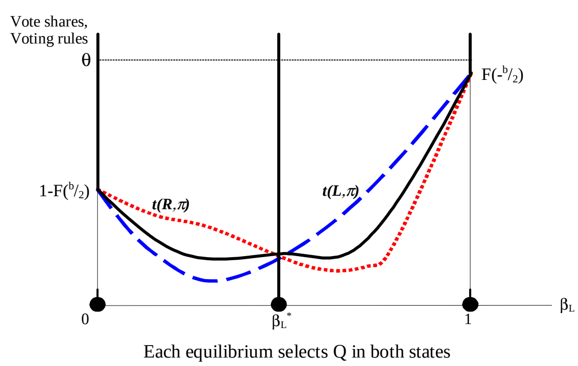

Figure 7(a): Equilibria under a $\mathcal{Q}$-trivial rule

[16]More speficically, the $\mathcal{P}$-trivial voting rules that aggregate information for sure for any distribution of preferences are those that are below the minimum share of votes received by $\mathcal{P}$ for any belief, i.e. those rules that satisfy $\theta < \min\{\min_{\beta_L} t(L,\pi), \min_{\beta_L} t(L,\pi)\}$. Equilibrium induced prior is β_L^* and equilibrium shares in both states are $z > \theta$ in the limit.

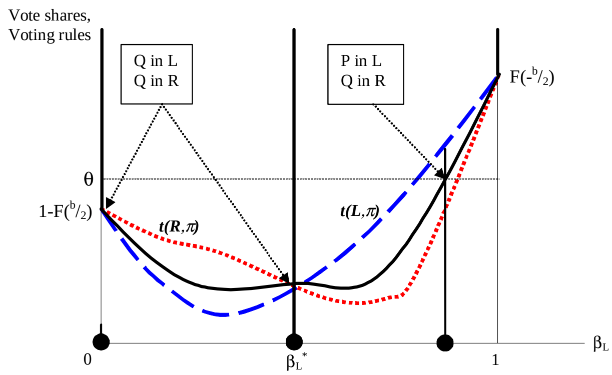

Figure 7(b): Equilibria under a consequential rule

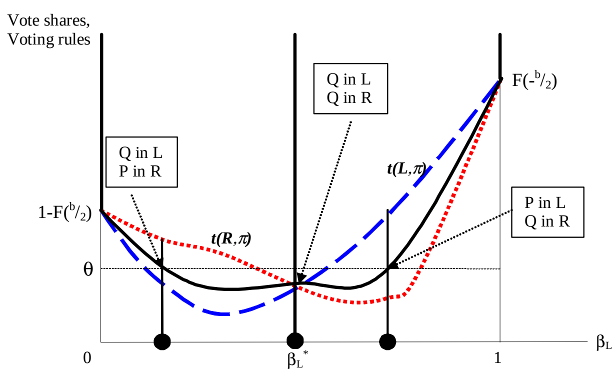

Figure 7(c): Equilibria under a large $\mathcal{P}$-trivial rule

6. Discussion

The chief idea of the paper is that the source of informational inefficiency in elections is the existence of groups of voters who always have opposed interests. While the method we have used allows us to pin down all (limit) equilibria for the unidimensional policy space and identify their aggregation properties, the analysis readily applies to any finite dimensional policy space. In some sense, it is easier to obtain groups with state-contingent conflicts when the policy space is multidimensional, i.e., when people care about many issues simultaneously. Therefore, our message is that the co-ordination problems is exacerbated when we move beyond the Downsian policy space.

The fact that voting under incomplete information may produce outcomes inferior to those under complete information is hardly a surprise. Several papers point out sources of aggregation failure using completely preference homogeneity or only limited heterogeneity in the form of common values: use of unanimity rules (Feddersen and Pesendorfer 1998), voters signaling their preferences through their votes (Razin 2003), information being costly (Persico 2004, Martinelli 2006), abstention (Oliveros 2005) and so on. Since the agenda of the current paper is to pinpoint that the fundamental source of aggregation failure is in competing interests among groups which is endemic to any democracy, we do not allow for these other possible causes of partial failure and simply consider the unidimensional policy space.

There are a few papers (Kim and Fey 2006, Meirowitz 2006, 2007a) that consider groups with opposed rankings in each state, but in these papers, the voters within the group have exactly same preferences. In the spatial model however, voters with the same *ranking* over the alternatives under full information need not have the same *intensity* of preference for them, leading to different behavior under uncertainty. Allowing any intra-group heterogeneity uncovers the deeper problem with inter-group conflict in preferences.

Another important contribution of our paper is that we show why the electoral system may fail to produce the outcomes desired by the majority. Since there are different equilibria, there are different reasons why elections can fail to aggregate information. The multiplicity of equilibria makes the role of beliefs in a political system crucial. The model endogenizes the process of formation of beliefs about which types are going to be responsive to information in equilibrium. Aggregation failure for consequential rules can simply be thought of a co-ordination failure because of "wrong" beliefs. For example, while a consequential rule needs the responsive set to be in the larger interest group, voters can believe that almost everyone is voting uninformatively. Independent of information received, the larger interest group votes for the status quo and almost everyone in the smaller interest group votes for the alternative. Voter behavior in this equilibrium is akin to what we know as block voting. In another "bad" equilibrium, only the extremists at either end of the ideological spectrum are responsive—but aggregation fails because most of the voters vote for the status quo in either state.

In each of these "bad" equilibria, whatever be the mode of failure of aggregation, the failure is of an extreme nature in the sense that the "wrong" outcomes occur with a very high probability in a large electorate. It is worth noting that these results do not depend on the relative size of the conflicting groups or on the extent of noise in the signals. Therefore, any improvement in the accuracy of information that individuals have will fail to produce superior outcomes in the limit.

One interpretation of Condorcet Jury Theorem is that communication among voters is not necessary in large elections for the information problem to be solved. This paper indicates that we are faced with the possibility of multiple equilibria, some or all of which produce informationally inferior outcomes. Thus, voting cannot perform the role of communication among voters. Can democratic deliberation improve election outcomes?[17] Note that since all members within each conflicting interest group have the same state-contingent rankings[18], members each group have an incentive to share information among themselves. However, this needs the voter preferences to be public information. We can think of each set of independent voters with similar rankings as belonging to a political party or a special interest group, and thus this paper highlights the role of political institutions like parties or interest groups as information aggregators in an electorate.

7. Bibliography

(1) ALI, N., M. MIHM AND L. SIGA (2017): "The Perverse Politics of Polarization", mimeo

(2) ANSOLABEHERE, S. AND J. SNYDER (2002): "The Incumbency Advantage in U.S. Elections: An Analysis of State and Federal Offices, 1942-2000", *Election Law Journal*.1(3)

(3) ANSOLABEHERE, S., J. SNYDER AND C. STEWART III. (2000): "Old Voters, New Voters and the Personal Vote: Using Redistricting to Measure the Incumbency Advantage", *American Journal of Political Science,* 44(1), 17-34

(4) ANSOLABEHERE, S., AND J. SNYDER (2006): "Television and the Incumbency Advantage in Elections", 31(4), *Legislative Studies Quarterly,* 469-90

(5) AUSTEN-SMITH, D., AND J. BANKS (1996): "Information Aggregation, Rationality and the Condorcet Jury Theorem", *American Political Science Review*, 90(1), 34-45

[17] See Coughlan(2000), Austen-Smith and Feddersen (2005, 2006), Gerardi and Yariv (2007), Meirowitz (2007a, 2007b, 2006) for models of deliberation before voting.

[18] In this case, the condition for full revelation of information between any two members of the same group is satisfied according to Baliga and Morris (2002)

(6) AUSTEN-SMITH, D., AND FEDDERSEN T. (2005): "Deliberation and Voting Rules", In Austen-Smith D. and J. Duggan (eds.) Social Choice and Strategic Decisions: Essays in Honor of Jeffrey S. Banks, Berlin: Springer

(7) AUSTEN-SMITH, D., AND FEDDERSEN T. (2006): "Deliberation, Preference Uncertainty and Voting Rules", *American Political Science Review*, 100, 209-218

(8) BALIGA, S. AND S. MORRIS (2002): "Co-ordination, Spillovers, and Cheap Talk", *Journal of Economic Theory*, 105, 450-468

(9) BATTAGLINI, M., R. MORTON AND T. PALFREY (2006): "The Swing Voters' Curse in the Laboratory", Mimeo, Princeton University

(10) BERG, S. (1993): "Condorcet's Jury Theorem, Dependency among Jurors", *Social Choice and Welfare*, 10, 87-96

(11) BHATTACHARYA, S (2013): "Preference Monotonicity and Information Aggregation in Elections", *Econometrica,* 81 (3), 1229-47

(12) CALLANDER, S. (2011): "Searching for Good Policies", *American Political Science Review*, 105(November), 643-62

(13) CONDORCET, MARQUIS DE. [1785] (1976): "Essay on the Application of Mathematics to the Theory of Decision-Making". Reprinted in *Condorcet: Selected Writings*, Keith Michael Baker, ed., 33-70. Indianapolis: Bobbs-Marrill Co.

(14) COUGHLAN, P.J. (2000): "In Defense of Unanimous Jury Verdicts: Mistrials, Communication and Strategic Voting", *American Political Science Review*, 94(June), 375-93

(15) COVER, A (1977): "One Good Term Deserves Another: The Advantage of Incumbency in Congressional Elections", *American Journal of Political Science,* 21: 523-41

(16) DOWNS, A. (1957): "An Economic Theory of Democracy", Chicago

(17) ERICKSON, S (1995): "The Entrenching of Incumbency: Reelections in the US House of Representatives", *Cato Journal* 14: 397-420

(18) FEDDERSEN, T., AND W. PESENDORFER (1996): "The Swing Voter's Curse", *American Economic Review*, 86(3), 408-424

(19) FEDDERSEN, T., AND W. PESENDORFER (1997): "Voting Behavior and Information Aggregation in Large Elections with Private Information", *Econometrica*, 65(5), 1029-1058

(20) FEDDERSEN, T., AND W. PESENDORFER (1998): "Convicting the Innocent: The Inferiority of Unanimous Jury Verdicts under Strategic Voting", *The American Political Science Review*, 92(1), 23-35

(21) FEDDERSEN, T., AND W. PESENDORFER (1999): "Abstention in Elections with Asymmetric Information and Diverse Preferences", *The American Political Science Review*, 93(2), 381-98

(22) FERNANDEZ, R. AND D. RODRIK (1991): "Resistance to Reform: Status Quo Bias in the Presence of Individual- Specific Uncertainty", *American Economic Review,* 81(5), 1146-55

(23) GERARDI, D. AND L. YARIV (2007): "Deliberative Voting", *Journal of Economic Theory*, 134(1), 317-338

(24) GERSBACH, H. (1995): "Informational Efficiency and Majority Decisions", *Social Choice and Welfare,* 12(4), 363-70

(25) JACOBSON, G (1980): "Money in Congressional Elections", New Haven, CT: Yale University Press

(26) KIM, J (2005): "A Model of Adversarial Committees", Mimeo, W. Allen Wallis Institue of Political Economy, University of Rochester

(27) KIM, J., AND M. FEY (2006): "The Swing Voter's Curse with Adversarial Preferences", *Journal of Economic Theory,* Forthcoming

(28) KIRCHGASSNER, G., (2007): “Status Quo Bias in Direct Democracy: Empirical Results for Switzerland, 1981-1999”, paper presented at Annual Conference of the International Institute of Public Finance, August 27-30, 2007, https://editorialexpress.com/cgi-bin/conference/download.cgi?db_name=IIPF63&paper_id=199
(29) KIRCHGASSNER, G., (2008): “Direct Democracy: Obstacle to Reform?”, *Constitutional Political Economy,* 19, 81-93
(30) LADHA, K., (1992): “The Condorcet Jury Theorem, Free Speech, and Correlated Votes, *American Journal of Political Science*, 36, 617-34
(31) LENZ, G. (2012): “Follow the Leader? How Voters Respond to Politicians’ Performance and Policies”, University of Chicago Press
(32) LI, H., ROSEN, S., AND W. SUEN (2001): “Conflicts and Common Interests in Committees”, *American Economic Review,* 91(5), 1478-97
(33) LINDBLOM, C., (1959): “The Science of ‘Muddling Through’”, *Public Administration Review,* 19, 75-88
(34) LUPIA, A. (2001): “Dumber than Chimps? An Assessment of Direct Democracy Voters”, in *Dangerous Democracy? The Battle over Ballot Initiatives in America.* Larry Sabato, Howard Ernst and Bruce Larson eds. Lanham, Md.: Rowman and Littlefield Publishers, 66-70
(35) MATSUSAKA, J. G., (2005): “Direct Democracy Works”, *Journal of Economic Perspectives,* 19 (2), 185-206
(36) MARTINELLI, C. (2005): “Would Rational Voters Acquire Costly Information?”, *Journal of Economic Theory,* 129, 225-51
(37) McLENNAN, A. (1998): “Consequences of the Condorcet Jury Theorem for Beneficial Information Aggregation by Rational Agents”, *The American Political Science Review*, 92(2), 413-18
(38) McMURRAY, J. (2017): “Ideology as Opinion: A Spatial Model of Common Value Elections”, *AEJ Microeconomics*, forthcoming
(39) MEIROWITZ, ADAM (2006): “Designing institutions to Aggregate Preferences and Information”, *Quarterly Journal of Political Science,* 2006 1(4), 373-92
(40) MEIROWITZ, ADAM (20007a):."In Defence of Exclusionary Deliberation: Communication and Voting with Private Beliefs and Values", *Journal of Theoretical Politics,* 19, 301-327
(41) MEIROWITZ, ADAM (2007b):."Communication and Bargaining in the Spatial Model", *International Journal of Game Theory,* 35, 251-66
(42) MEIROWITZ, ADAM (2002):“Informative Voting and Condorcet Jury Theorems with a Continuum of Types”, *Social Choice and Welfare*, 19, 219-236
(43) MYERSON, R.B. (1998): “Population Uncertainty and Poisson Games”, *International Journal of Game Thoery*, 27(3), 375-392
(44) MYERSON, R.B. (1998): “Extended Poisson Games and the Condorcet Jury Theorem”, *Games and Economic Behaviour*, 25, 111-131
(45) MYERSON, R.B. (2000): “Large Poisson Games”, *Journal of Economic Theory,* 94 (1), 7-45
(46) NITZAN, S., AND J. PAROUSH (1985): *Collective Decision Making*, Cambridge: Cambridge University Press
(47) OLIVEROS, S. (2013): “Abstention, Ideology and Information Acquisition”, *Journal of Economic Theory,* 148(3), 871-902
(48) PERSICO, N. (2004): “Committee Design with Endogenous Information”, *Review of Economic Studies*, 71(1), 165-91
(49) WIT, J. (1998): “Rational Choice and the Condorcet Jury Theorem”, *Games and Economic Behavior*, 22, 364-376

8. Appendix

8.1. **Proof of Lemma 2.** By hypothesis of the lemma, $\lim_{n\to\infty} \frac{\beta_L^n}{1-\beta_L^n} = \frac{\beta_L^0}{1-\beta_L^0}$ is a finite, positive number. Now suppose $\exists$ some $\varepsilon > 0$ such that $\alpha_n > 1+\varepsilon$ for all n. Then $\frac{\beta_L^n}{1-\beta_L^n} = (\alpha_n)^n > (1+\varepsilon)^n \to \infty$ *as* $n \to \infty$ which is a contradiction. On the other hand, suppose $\exists$ some $\varepsilon \in (0,1)$ such that $\alpha_n < 1-\varepsilon$ for all n. Then $\frac{\beta_L^n}{1-\beta_L^n} = (\alpha_n)^n < (1-\varepsilon)^n \to 0$ *as* $n \to \infty$, which is again a contradiction.

8.2. **Proof of Lemma 4.**

Proof. For part (i) of the lemma, since $\beta_L \in (0,1)$, Proposition 1 holds. Suppose $0 < y < x < 1$, and $f(z,\theta) = z^\theta(1-z)^{1-\theta}$, with both z and θ lying in $(0,1)$. Note that if we fix θ, the function $f(z,\theta)$ is continuous and single peaked in z with the peak lying at θ. From the properties of this function, we can show that for any $0 < y < x < 1$, there exists a unique θ^*such that $f(x,\theta^*) = f(y,\theta^*)$, and $x < \theta^* < y$. To be specific, $\theta^* = \frac{\log\frac{1-y}{1-x}}{\log\frac{x(1-y)}{y(1-x)}}$. Also, if both x and y increase, θ^* must increase. Since $0 < F(\frac{L}{2}) < t_L(\beta_L) < t_R(\beta_L) < F(\frac{R}{2}) < 1$, taking $t_R(\beta_L) = x$ and $t_L(\beta_L) = y$ and noting that $t_R(\beta_L)$ and $t_L(\beta_L)$ are strictly increasing functions of β_L, part (i) of the Lemma is established.

For part (ii), note that for any n, by Remark 1, we have $x_l^n < x_r^n$. Since $z_\sigma^n = F\left(x_\sigma^n\right)$, we have $z_r^n > z_l^n > 0$. Define, for any n, $h^n = z_r^n - z_l^n > 0$. Substituting, we have: $t(R,\pi^n) = z_l^n + qh^n$, and $t(L,\pi^n) = z_l^n + (1-q)h^n$. Therefore:

$$\frac{1-\beta_L^n}{\beta_L^n} = \left[\frac{(t(R,\pi^n))^\theta\,(1-t(R,\pi^n))^{1-\theta}}{(t(L,\pi^n))^\theta\,(1-t(L,\pi^n))^{1-\theta}}\right]^n = \left[\frac{(z_l^n+qh^n)^\theta\,(1-z_l^n-qh^n)^{1-\theta}}{\left(z_l^n+(1-q)h^n\right)^\theta\left(1-z_l^n-(1-q)h^n\right)^{1-\theta}}\right]^n$$

If $\beta_L^0 = 0$ (or 1), the left hand side of the above equation goes to infinity (or 0). This requires the term in the bracket large enough n to be greater (or less) than unity, or its logarithm to be positive (or negative). We can write,

$$\log\frac{(z_l^n+qh^n)^\theta\,(1-z_l^n-qh^n)^{1-\theta}}{\left(z_l^n+(1-q)h^n\right)^\theta\left(1-z_l^n-(1-q)h^n\right)^{1-\theta}} > 0 \Leftrightarrow \theta > \zeta(z_l^n,h^n)\ \forall n$$

where the function $\zeta(z_l^n,h^n)$ is defined as:

$$\zeta(z_l^n,h^n) \equiv \frac{-\log\left[\frac{1-z_l^n-qh^n}{1-z_l^n-(1-q)h^n}\right]}{\log\left[\frac{\left(z_l^n+qh^n\right)\left(1-z_l^n-(1-q)h^n\right)}{\left(z_l^n+(1-q)h^n\right)\left(1-z_l^n-qh^n\right)}\right]}$$

By Lemma 3, we know that for any sequence, with $\beta_L^0 \in \{0,1\}$, $h^n \to 0^+$. Hence,

$$\lim_{h^n\to0^+,\ z_l^n=t}\zeta(z_l^n,h^n) = \lim_{h^n\to0^+,\ z_l^n=t}\left(\frac{-\log\left[\frac{1-z_l^n-qh^n}{1-z_l^n-(1-q)h^n}\right]}{\log\left[\frac{\left(z_l^n+qh^n\right)\left(1-z_l^n-(1-q)h^n\right)}{\left(z_l^n+(1-q)h^n\right)\left(1-z_l^n-qh^n\right)}\right]}\right) = \lim_{z_l^n=t} z_\sigma^n = t$$

By Lemma 3, if $\beta_L^0 = 0$, $t = F(\frac{R}{2})$, and $\theta > \zeta(z_l^n,h^n)\ \forall n \Rightarrow \theta > \lim_{h^n\to0^+,\ z_l^n=t}\zeta(z_l^n,h^n) = F(\frac{R}{2})$. Similarly, if $\beta_L^0 = 1$, $t = F(\frac{L}{2})$, and $\theta < \zeta(z_l^n,h^n)\ \forall n \Rightarrow \theta < \lim_{h^n\to0^+,\ z_l^n=t}\zeta(z_l^n,h^n) = F(\frac{L}{2})$. ■

8.3. **Proof of Proposition 2.** Here we only show that the only accumulation point is also the limit. For this, it is enough to show that given $\theta \in \Theta\left(\beta_L^0\right)$, for any neighbourhood ϵ of β_L^0, there is some large enough N, such that β_L^n in the equilibrium sequence must lie within the nighbourhood for all values of $n > N$.

First consider $\beta_L^0 \in (0,1)$.Suppose the accumulation point is not the limit, and there is an infinite equilibrium subsequence β_L^m of the sequence β_L^n, such that for any $\epsilon > 0$, there is some M so that for all values of m larger than M, β_L^m lies outside $\left(\beta_L^0 - \epsilon, \beta_L^0 + \epsilon\right)$. Since even this subsequence must have an accumulation point, it must be either 0 or 1. But, by the second part of Lemma 5, since the limit equilibrium condition must hold for accumulation points too, there cannot be an accumulation point for θ in $\Theta\left(\beta_L^0\right)$ at 0 or 1. Hence there is no such infinite subsequence.

The proof for $\beta_L^0 \in \{0,1\}$ is similar.

8.4. **Proof of Proposition 3.**

Proof. Proposition 2 guarantees existence of limit equilibrium for all θ.

Consider $\theta < F\left(\frac{L}{2}\right)$. We know that $t(S,\pi^n) > F\left(\frac{L}{2}\right)$ $\forall n$ for $S = L, R$. Let $\delta = F\left(\frac{L}{2}\right) - \theta$. By Law of large numbers, given ϵ we can find N such that actual share of votes $\tau(S,\pi^n,\theta)$ under rule θ in any state S is greater than $F\left(\frac{L}{2}\right) - \delta > \theta$ for any $n > N$ with a probability larger than $1-\epsilon$. Thus, under both states, $\mathcal{P}$ wins with a probability larger than $1-\epsilon$.

Since $t(S,\pi^n) < F\left(\frac{R}{2}\right)$ $\forall n \forall S$, by the same logic as above, any $\mathcal{Q}$-trivial rule aggregates information too.

Consider a consequential rule θ, for which the only equilibrium induced prior in the limit is $\beta_L^{-1}(\theta)$. By Lemma 4, $t_L\left(\beta_L^{-1}(\theta)\right) < \theta < t_R\left(\beta_L^{-1}(\theta)\right)$.

Also, for any consequential rule θ, we can find a positive number η such that $F\left(\frac{L}{2}\right) + \eta < \theta < F\left(\frac{R}{2}\right) - \eta$. By Lemma 4, we can find a similar number $\kappa > 0$ such that $\kappa < \beta_L^{-1}(\theta) < 1-\kappa$. Also, we can find some $\lambda > 0$ such that $t_R\left(\beta_L^{-1}(\theta)\right) - t_L\left(\beta_L^{-1}(\theta)\right) > \lambda$. Now, from Proposition 1, we can derive θ from $t_R\left(\beta_L^{-1}(\theta)\right)$ and $t_L\left(\beta_L^{-1}(\theta)\right)$ and can find another number $\mu > 0$ such that $t_L\left(\beta_L^{-1}(\theta)\right) + \mu < \theta < t_R\left(\beta_L^{-1}(\theta)\right) - \mu$. Since t_R, t_L and θ^* are all continuous functions of β_L, we can find a number $\xi > 0$ such that for a range $\left(\beta_L^{-1}(\theta) - \xi, \beta_L^{-1}(\theta) + \xi\right)$ around $\beta_L^{-1}(\theta)$, $t_L - \frac{\mu}{2} < \theta < t_R + \frac{\mu}{2}$. Given ξ,we can find M_1 such that $\beta_L^n \in \left(\beta_L^{-1}(\theta) - \xi, \beta_L^{-1}(\theta) + \xi\right)$ in any π^n whenever $n > M_1$. Now consider $\delta = \min\left((t_R\left(\beta_L^{-1}(\theta) - \xi\right) + \frac{\mu}{2} - \theta,\ \theta - t_L(\beta_L^{-1}(\theta) + \xi) - \frac{\mu}{2}\right)$. By Law of large numbers, given ϵ we can find M_2 such that actual share of votes under rule θ under state R, $\tau(R,\pi^n,\theta)$ is less than $t_R\left(\beta_L^{-1}(\theta) - \xi\right) + \frac{\mu}{2} - \delta < \theta$ for any $n > M_2$ and the actual share under state L, $\tau(L,\pi^n,\theta)$ is greater than $t_L\left(\beta_L^{-1}(\theta) + \xi\right) + \frac{\mu}{2} - \delta > \theta$ for any $n > M_2$ with a probability larger than $1-\epsilon$. Set $N = \max(M_1, M_2)$ and we are done. ■

8.5. **Proof of Lemma 5.**

Proof. At $\beta_L = 0$, $x_l = x_r = \frac{b}{2} \Rightarrow z_l = z_r = 1 - F\left(\frac{b}{2}\right)$. Now, consider the interval of β_L such that p_l lies in $(0, \frac{1}{2} + \frac{b}{4}]$. In this interval, $x_l \in (\frac{b}{2}, 1] \cup \{-1\} \Rightarrow z_l = 1 - F(x_l)$. Also, in this interval of β_L, $p_r < \frac{1}{2} - \frac{b}{4} \Rightarrow x_r \in (\frac{b}{2}, 1) \Rightarrow z_r = 1 - F(x_r) > 0$, by assumptions F and I. For values of β_L such that $x_l \leq 1$, $x_r < x_l \Rightarrow z_l = 1 - F(x_l) < 1 - F(x_r) = z_r$, again by assumption F. For values of β_L such that $x_l = -1$, $z_l = 1 - F(-1) = 0 < z_r$. Thus, over this entire interval $z_r > z_l$. Note also that over this set of values of β_L, z_r is strictly decreasing, while z_l first strictly decreases and then stays at 0. For β_L such that $p_l = \frac{1}{2} + \frac{b}{4}$, $z_r = \overline{z_r}$, say. In the same way, consider the interval of β_L such that p_r lies in $[\frac{1}{2} - \frac{b}{4}, 1]$. Here, by the same token, $z_r < z_l$ except for $\beta_L = 1$ where $z_l = z_r = F\left(-\frac{b}{2}\right)$. z_l increases strictly from $\overline{z_l} > 0$ to $F\left(-\frac{b}{2}\right)$ over this interval, while z_r is initially 0 and then strictly increases.

Now, consider the remaining interval of β_L which is $\left(p_l^{-1}(\frac{1}{2} + \frac{b}{4}), p_r^{-1}(\frac{1}{2} - \frac{b}{4})\right)$. That this is a valid nonempty interval is guaranteed by assumption I. In this interval, $x_r \in (\frac{b}{2}, 1]$, and x_r increases with

β_L. Thus, $z_r = 1 - F(x_r)$ is a strictly falling continuous function, going from $\overline{z_r} > 0$ to 0 over this interval. Similarly, z_l strictly and continuously increases from 0 to $\overline{z_l} > 0$. Therefore, there exists a unique β_L^* in this interval where $z_l = z_r$. This implies that at β_L^*, $t(L,\pi) = t(R,\pi)$. For all $\beta_L < \beta_L^*$, $z_l < z_r \Rightarrow t(L,\pi) = qz_l + (1-q)z_r < qz_r + (1-q)z_l = t(R,\pi)$. Similarly, for $\beta_L > \beta_L^*$, where $z_l > z_r$, we have $t(L,\pi) > t(R,\pi)$. ■

8.6. **Proof of Lemma** 6.

Proof. We prove the result for the case $\beta_L^0 = 1$, the other one follows symmetrically. First we look at how $\frac{p_l}{p_r}$ changes with β_L.

$$\frac{p_l}{p_r} = \left(\frac{q}{1-q}\right)\left(\frac{q\beta_R + (1-q)\beta_L}{q\beta_L + (1-q)\beta_R}\right) = \left(\frac{q}{1-q}\right)\left(\frac{q + (1-q)\alpha}{q\alpha + (1-q)}\right),$$

where $\alpha = \frac{\beta_L}{\beta_R}$. Therefore, we have:

$$\frac{d}{d\beta_L}\left(\frac{p_l}{p_r}\right) = \frac{d\alpha}{d\beta_L}\cdot\frac{d}{d\alpha}\left(\frac{p_l}{p_r}\right) = \frac{1}{(1-\beta_L)^2}\left(\frac{q}{1-q}\right)\frac{(1-q)^2 - q^2}{(q\alpha + (1-q))^2} < 0$$

At $\beta_L = 1$, we have $p_l = p_r = 1$. Thus, for $\beta_L \in [0,1)$, we always have $p_l > p_r$ by the above strictly monotonic relationship. Since $\beta_L^0 = 1 \Rightarrow p_r^n \to 1$, by continuity we can find some m large enough such that for all $n > m$, we have $p_r^n > \frac{1}{2} + \frac{b}{4}$. Since $p_l^n > p_r^n$, for all $n > m$, $p_l^n > \frac{1}{2} + \frac{b}{4}$ too. Since we always have $\beta_L^n < 1$, $p_\sigma^n < 1$. Therefore, for all $n > m$, both x_l^n and x_r^n lie in the open interval $(-1, -\frac{b}{2})$. Also, $p_l^n > p_r^n \Rightarrow x_l^n > x_r^n$ for all $n > m$. This proves part (i). Part (ii) follows trivially from $p_\sigma^n \to 1$. ■

8.7. **Proof of Lemma** 7.

Proof. Part (i) follows from Lemma 4 and 5.

For part (ii), we first consider the case with $\beta_L^0 = 1$. By Lemma 6, we know that for any such sequence, $x_\sigma^n \to \left(-\frac{b}{2}\right)^-$ for $\sigma = \{l,r\}$, and $x_l^n > x_r^n$ for all large enough n. For large enough n, $p_\sigma^n > \frac{1}{2} + \frac{b}{4} \Rightarrow z_\sigma^n = F(x_\sigma^n) \Rightarrow z_l^n > z_r^n > 0$ and $z_\sigma^n \to F(-\frac{b}{2})$. Define $h^n = z_l^n - z_r^n \to 0^+$. Substituting, we have: $t(L,\pi^n) = z_r^n + qh^n$, and $t(R,\pi^n) = z_r^n + (1-q)h^n$. Therefore:

$$\frac{\beta_L^n}{1-\beta_L^n} = \left[\frac{(t(L,\pi^n))^\theta (1-t(L,\pi^n))^{1-\theta}}{(t(R,\pi^n))^\theta (1-t(R,\pi^n))^{1-\theta}}\right]^n = \left[\frac{(z_r^n + qh^n)^\theta (1 - z_r^n - qh^n)^{1-\theta}}{(z_r^n + (1-q)h^n)^\theta (1 - z_r^n - (1-q)h^n)^{1-\theta}}\right]^n$$

If $\beta_L^0 = 1$, the left hand side of the above equation goes to infinity. This requires the term in the bracket large enough n to be greater than unity, or its logarithm to be positive .

For the case with $\beta_L^0 = 0$, we again use Lemma 6 which tells us that $x_\sigma^n \to \left(\frac{b}{2}\right)^+$ for $\sigma = \{l,r\}$, and $x_l^n > x_r^n$ for all large enough n.We also know that for large enough n, $p_\sigma^n > \frac{1}{2} - \frac{b}{4} \Rightarrow z_\sigma^n = 1 - F(x_\sigma^n) \Rightarrow z_r^n > z_l^n > 0$ and $z_\sigma^n \to 1 - F(-\frac{b}{2})$. Define $h^n = z_r^n - z_l^n \to 0^+$. Substituting, we have: $t(R,\pi^n) = z_r^n + qh^n$, and $t(L,\pi^n) = z_r^n + (1-q)h^n$. Therefore:

$$\frac{\beta_L^n}{1-\beta_L^n} = \left[\frac{(t(L,\pi^n))^\theta (1-t(L,\pi^n))^{1-\theta}}{(t(R,\pi^n))^\theta (1-t(R,\pi^n))^{1-\theta}}\right]^n = \left[\frac{(z_r^n + qh^n)^\theta (1 - z_r^n - qh^n)^{1-\theta}}{(z_r^n + (1-q)h^n)^\theta (1 - z_r^n - (1-q)h^n)^{1-\theta}}\right]^{-n}$$

Since the LHS goes to 0 in the limit, the term within the bracket in the RHS has to be greater than 1. Thus we have the exact same situation as in the proof of Lemma 5, and therefore, we need.

$$\log\frac{(z_r^n + qh^n)^\theta (1 - z_r^n - qh^n)^{1-\theta}}{(z_r^n + (1-q)h^n)^\theta (1 - z_r^n - (1-q)h^n)^{1-\theta}} > 0 \Leftrightarrow \theta > \zeta(z_r^n, h^n)\ \forall n$$

where the function $\zeta(z_l^n, h^n)$ is defined as in the proof of lemma 4.

By Lemma 4, if $\beta_L^0 = 0$, $t = 1 - F(\frac{b}{2})$, and $\theta > \zeta(z_l^n, h^n)\ \forall n \Rightarrow \theta > \lim_{h^n \to 0^+,\ z_l^n = t} \zeta(z_l^n, h^n) = 1 - F(\frac{b}{2})$. Similarly, if $\beta_L^0 = 1$, $t = F(-\frac{b}{2})$, and $\theta > F(-\frac{b}{2})$.

For $\beta_L^0 = \beta_L^*$, from Proposition 4, no value of θ can be ruled out. ■

8.8. **Proof of Proposition 5.** This is a proof by construction. Consider any unordered alternatives environment $(F(\cdot), q, b)$. We show that every $\beta \in [0,1]$ can be supported by any $\theta \in \Theta(\beta)$ for any $F(\cdot)$ satisfying full support.

Define the function

$$f_n(\beta, \theta) = \frac{1}{1 + \left[\frac{t_R(\beta)^\theta (1-t_R(\beta))^{1-\theta}}{t_L(\beta)^\theta (1-t_L(\beta))^{1-\theta}}\right]^n}$$

If given (n, θ) we can show that there is some fixed point β_n of the function $f_n(\beta, \theta)$, then that β_n is the solution to the equilibrium condition (10), proving that π^n exists for that θ. We prove proposition 5 by showing that for any $\theta \in \Theta(\beta^0)$, there is a sequence of fixed points of beliefs β_n such that $\beta_n \to \beta^0$ as $n \to \infty$. We prove this separately for different values ranges of β^0.

Case 1: $\beta^0 \in (0, \beta_L^*) \cup (\beta_L^*, 1)$

Proof. First, consider some β^0 the range of beliefs $(0, \beta_L^*) \cup (\beta_L^*, 1)$. By Lemma 7, in this range, $\Theta(\beta_L)$ is a continuous function $\theta^*(\beta_L)$. Since F admits a pdf f, $\theta^*(\beta_L)$ is differentiable too. Thus, there exists a neighbourhood $(\beta^0 - \epsilon, \beta^0 + \epsilon)$ where $\theta^*(\beta_L)$ is either only increasing, only decreasing or constant.

Suppose first that $\theta^*(\beta_L)$ is decreasing in $(\beta^0 - \epsilon, \beta^0 + \epsilon)$. Now, for $\beta \in (\beta^0, \beta^0 + \epsilon)$, we must have $f_n(\beta, \theta^*(\beta^0)) \to 0$ as $n \to \infty$. On the other hand, for $\beta \in (\beta^0 - \epsilon, \beta^0)$, we must have $f_n(\beta, \theta^*(\beta^0)) \to 1$ as $n \to \infty$. Thus, for δ small enough, there must exist some m such that $f_n(\beta + \epsilon, \theta^*(\beta^0)) < \delta$ and $f_n(\beta - \epsilon, \theta^*(\beta^0)) > 1 - \delta$ for all $n > m$. In particular, choose $\delta < \epsilon$. Then, for all $n > m$, if $f_n(\beta, \theta^*(\beta^0))$ is plotted against β, it intersects the 45^0 line for some $\beta \in (\beta^0 - \epsilon, \beta^0 + \epsilon)$, which is the fixed point of the function. Call it β_n. To be specific, β_n is the solution of $f_n(\beta, \theta^*(\beta^0)) = \beta$, and for all $n > m$, $\beta_n \in (\beta^0 - \epsilon, \beta^0 + \epsilon)$. Thus, there exists a sequence β_n such that for any $\epsilon > 0$ small enough, there is some m such that for all $n > m$, $f_n(\beta_n, \theta^*(\beta^0)) = \beta_n$ and $\left|\beta_n - \beta^0\right| < \epsilon$.

If $\theta^*(\beta_L)$ is increasing in $(\beta^0 - \epsilon, \beta^0 + \epsilon)$, then we can prove the proposition in an analogous way.

However, if $\theta^*(\beta_L)$ is constant in the range $(\beta^0 - \epsilon, \beta^0 + \epsilon)$, the theorem may not hold. To be clear, this case requires that $t_L(\beta_L)$ increases (decreases) and $t_L(\beta_L)$ decreases (increases) so as to keep $\frac{t_R(\beta)^\theta (1-t_R(\beta))^{1-\theta}}{t_L(\beta)^\theta (1-t_L(\beta))^{1-\theta}}$ constant over the range. We ignore this case as non-generic. ■

Case 2: $\beta^0 \in \{0, \beta_L^*, 1\}$

Proof. Consider the case $\beta^0 = 0$. Note that $\Theta(0) = \{\theta : \theta > 1 - F(\frac{R}{2})\}$.

Select $\epsilon > 0$ small enough such that $t_R(\beta_L) > t_L(\beta_L)$ in the range $\beta_L \in (0, 2\epsilon)$. Choose $\delta < \epsilon$. By Case 1, for voting rule $\theta^*(\epsilon)$ there exists a sequence of equilibria β_n such that $f_n(\beta_n, \theta^*(\epsilon)) = \beta_n$ and $\beta_n \in (\epsilon - \delta, \epsilon + \delta)$ for n large enough. This implies $\beta_n < 2\epsilon$ for all n large enough.

Now consider the sequence β_n such that $f_n(\beta_n, \theta^*(\epsilon)) = \beta_n$. For $\theta > 1 - F(\frac{R}{2}) > \theta^*(\epsilon)$,we must have $f_n(\beta_n, \theta) < \beta_n$. Now, consider the function $f_n(\beta, \theta) - \beta$. At $\beta = \beta_n$, the function is negative while at $\beta = 0$, the function is positive due to the boundedness of the shares. Since $f_n(\beta, \theta)$ is continuous, there is some $0 < \beta_n' < \beta_n < 2\epsilon$ such that $f_n(\beta_n', \theta) = \beta_n'$.

Thus, given $\theta \in \Theta(0)$, for any ϵ small enough, there exists a sequence β_n'.such that $f_n(\beta_n', \theta) = \beta_n'$ and $\left|\beta_n' - 0\right| < 2\epsilon$ for all n large enough.

In the same way, we can prove the theorem for $\beta^0 = 1$.

Next, consider the case with $\beta^0 = \beta_L^*$. Note that $\Theta(\beta_L^*) = (0, 1)$. To show existence of a limit equilibrium for $\theta < t_L(\beta_L^*)$, use the neighbourhood $(\beta_L^* - \epsilon, \beta_L^*)$ to the left of β_L^*, and to show existence of a limit equilibrium for $\theta > t_L(\beta_L^*)$, use the neighbourhood $(\beta_L^*, \beta_L^* + \epsilon)$ to the right of β_L^* and apply the same method. ■

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