Utilitarianism without individual utilities

Thierry Marchant

Abstract We characterize anonymous utilitarianism in a multi-profile and purely ordinal framework, i.e. without assuming that utilities have been measured beforehand.

Keywords utilitarianism · scoring rule

JEL D60 · D71

1 Introduction

Many papers on cardinal Social Choice Theory begin with the assumption that individual utilities have been measured, somehow, and that we want to aggregate them into a collective preference represented by a Social Welfare Functional W. Some axioms like neutrality, anonymity and Pareto are then imposed. These are usually easy to interpret. But in order to arrive at a characterization more conditions are needed. That is why, in many papers, some invariance conditions are used [e.g. Sen, 1970, d'Aspremont and Gevers, 1977, Roberts, 1980, Bossert, 1991]. These conditions are supposed to represent the *informational basis* of the social preferences. Contrary to the other conditions, the invariance conditions are not easily interpretable and, often, misunderstood. The reason is that they fail to distinguish between a transformation of the well-being and a transformation of the numerical representation thereof. This problem has been discussed at length in Morreau and Weymark [2016]. Similar discussions, in different contexts, can be found in [Roemer, 1996, Sec. 2.5] and [Marchant, 2008].

To avoid the above-mentioned ambiguity, Morreau and Weymark [2016] introduce a new formalism making explicit reference to the measurement scale being used. This way, they can make a distinction between two kind of invariance conditions: those corresponding to a transformation of the well-being

Ghent University, Ghent 9000, Belgium. E-mail: thierry.marchant@ugent.be

(without any change of the measurement scale) and those corresponding to a change of the measurement scale (without any change of the well-being). Although this formalism is definitely precise and unambiguous, we think another approach can be enlightening. Instead of supposing that individual utilities have been measured before we consider their aggregation, we suggest to characterize the whole process of measuring individual utilities (by observing individual preferences) and aggregating them. The advantage of this approach is that the statement of characterization theorems are then exempt of any reference to individual utilities. Since individual utilities are not empirically observable, we deem it preferrable to avoid using them as primitives of our theories, so as to obtain theorems that are easier to interpret. In doing this, we follow a long tradition: for instance [Arrow, 1963, p.109, bottom] and [Dhillon and Mertens, 1999, p.473, top].

Characterizations of cardinal aggregation procedures without reference to individual utilities can be found in the literature. For instance, Harsanyi's Theorem [Harsanyi, 1955]. This single-profile result has often been criticized for not answering the question it aims to answer [e.g. Weymark, 1991]. Our approach will be multi-profile.

More recently, there have been a couple of papers about relative utilitarianism. Dhillon [1998][1] characterizes relative utilitarianism using inter alia a strong Pareto condition. The latter is very unusual in the sense that, for each bipartition of the set of voters, it assumes the existence of two social welfare functions satisfying some kind of Pareto condition. We will not assume the existence of some unobservable object. A remarkable paper by Dhillon and Mertens [1999] also characterizes relative utilitarianism. Our contribution differs from theirs in that we will use a variable-population framework and we will characterize not only relative utilitarianism, but also anonymous utilitarianism.

Another paper about relative utilitarianism is Börgers and Choo [2017b]. Their goal is to represent existing preferences of the individuals and of the social planner. Our contribution differs from theirs in that we do not assume we can observe sufficiently many choices of a social planner to infer the planner's preferences. Besides, we will also consider anonymous utilitarianism and not only relative utilitarianism.

Let us also mention a paper by Sprumont [2013] characterizing relative egalitarianism in a purely ordinal framework.

In the sequel, we will consider the following problem. A social planner wants to choose an alternative from some finite set X, taking preferences of voters into account. The social planner wants the choice to possibly depend on individual strengths of preferences or some kind of cardinal information. The voters therefore express their preferences by means of Von Neumann-Morgenstern preference relations defined on the set of all lotteries with prizes in X. The social planner then uses a social choice correspondence associating

[1] Börgers and Choo [2017a] have shown that the proof of the main result in Dhillon [1998] is incorrect. Yet, the main result might be correct.

a subset of X to each profile of VNM preference relations over lotteries. In Section 2, we will characterize the family of all anonymous utilitarian social choice correspondences, i.e. choice sets containing all alternatives for which the (unweighted) sum of individual VNM utilities is maximal. In section 3, we wil consider a special member of this family: relative utilitarianism. The fourth section is devoted to the logical independence of our axioms and the last section to some open problems or directions for future research.

2 Characterization of anonymous utilitarianism

2.1 Notation and definitions

Let $X = \{x, y, z, \ldots\}$ be the set (finite, with $\#X \geq 3$) of alternatives and $\Pi = \{p, q, r, \ldots\}$ be the set of all probability distributions on X. Each such probability distribution is called a lottery. Given the lottery p in Π, the probability that x obtains is denoted by p_x. The lottery such that x obtains with certainty is denoted by $\overline{x}$. It is called a safe lottery. The set of all binary relations on Π is $\mathcal{R} = 2^{\Pi \times \Pi}$. If $R \in \mathcal{R}$, then P and I respectively denote the asymmetric and the symmetric part thereof. A binary relation R on Π is a von Neumann-Morgenstern (VNM) relation [Jensen, 1967] if it satisfies

- weak order: it is transitive, reflexive and complete;
- independence: if $p \; P \; q$, then $\lambda p + (1-\lambda)r \; P \; \lambda q + (1-\lambda)r$ for all $\lambda \in \,]0, 1[$;
- continuity: if $p \; P \; q$ and $q \; P \; r$, then there are $\lambda, \lambda' \in \,]0, 1[$ such that $\lambda p + (1-\lambda)r \; P \; q$ and $q \; P \; \lambda' p + (1-\lambda')r$.

Most other definitions of VNM relations would work equally well. Let $V \subset \mathcal{R}$ be the set of all VNM relations on Π. We say a binary relation R has an expected utility representation if there exists a mapping $u : X \to \mathbb{R}$ such that

$$p \; R \; q \iff \sum_{x \in X} p_x u(x) \geq \sum_{x \in X} q_x u(x), \text{ for all } p, q \in \Pi. \tag{1}$$

A binary relation has an expected utility representation as in (1) if and only if it is a VNM relation [Jensen, 1967]. The utility function u in (1) is a VNM utility function; it is unique up to a positive affine transformation.

Given a set of agents $N \subset \mathbb{N}$, a profile $\succsim \, = (\succsim_i)_{i \in N}$ is an element of V^N indexed by the elements of N, where $\succsim_i$ is the preference relation of individual i. Let $\mathcal{P}_N$ be the set of all possible profiles given X and N and $\mathcal{P} = \bigcup_{N \subset \mathbb{N}} \mathcal{P}_N$. We define a VNM Social Choice Correspondence (SCC) as a mapping $f : \mathcal{P} \to 2^X \setminus \emptyset$, that is, a mapping from the set of all possible profiles to the set of all non-empty subsets of X. Notice that the choice set is a subset of X and not of Π. We want to choose alternatives, i.e. elements of X, even though the preferential information we use is defined on the richer set Π.

Let σ_X be a permutation on X and Σ the set of all such permutations. Then σ_Π is a permutation on Π defined by $(\sigma_\Pi(p))_x = p_{\sigma(x)}$ for all $x \in X$ and $p \in \Pi$. Similarly, $\sigma_{\mathcal{R}}$ is a permutation on $\mathcal{R}$ defined by $\sigma_\Pi(p) \; \sigma_{\mathcal{R}}(R) \; \sigma_\Pi(q)$ iff

$p\ R\ q$ for all $p, q \in \Pi$ and all $R \in \mathcal{R}$. And $\sigma_{\mathcal{P}}$ is a permutation on $\mathcal{P}$ defined by $\sigma_{\mathcal{P}}((\succsim_i)_{i\in N}) = (\sigma_{\mathcal{R}}(\succsim_i))_{i\in N}$ for all $(\succsim_i)_{i\in N} \in \mathcal{P}$. We will henceforth abuse notation and write σ without subscript for all these permutations.

The aim of this section is to characterize the anonymous utilitarian VNM Social Choice Correspondence, defined by

$$f((\succsim_i)_{i\in N}) = \operatorname*{argmax}_{x\in X} \sum_{i\in N} u(\succsim_i, x) \tag{2}$$

where $u : V \times X \to \mathbb{R}$ is such that

(i) $u(R, \cdot)$ is a VNM utility function, for any $R \in V$,
(ii) u is neutral, i.e., $u(\sigma(R), x) = u(R, \sigma(x))$, for any $R \in V$ and $x \in X$.

Note that u is not fixed. Yet, if we apply a transformation to u, we must respect (i) and (ii). We can distinguish two classes of transformation. First, we can apply a positive affine transformation to u, i.e., $u' = \alpha u + \beta$, with $\alpha > 0$. This results in exactly the same VNM SCC. Second, we can apply a positive affine transformation to $u\big(\sigma(R), \cdot\big)$ for some $R \in V$ and for all $\sigma \in \Sigma$, i.e., $u'\big(\sigma(R), \cdot\big) = \alpha u\big(\sigma(R), \cdot\big) + \beta$. This yields a different VNM SCC whenever $\alpha \neq 1$. Hence, (2) does not define a single VNM SCC but a family.

We can for instance choose u so that $u(R, \cdot)$ is normalized (except if R is the trivial preference relation[2]), that is, $u(R, x) = 1$ if $\overline{x}\ R\ \overline{z}$ for all $z \in X$ and $u(R, y) = 0$ if $\overline{z}\ R\ \overline{y}$ for all $z \in X$. The corresponding VNM SCC is then equivalent to relative utilitarianism [Dhillon and Mertens, 1999]. But we can also choose u so that the range of $u(R, \cdot)$ goes from zero to the number of equivalence classes of safe lottery in R. Many other choices are of course possible.

2.2 Standard axioms

In order to characterize the anonymous utilitarian VNM SCC, we will use a result by Pivato [2014] extending a result of Myerson [1995], which is itself an extension of a result of Young [1975][3]. Our axioms are therefore very similar to those of Young [1975]. We present them hereunder without much comment, because they have been extensively discussed elsewhere. The first condition says that all alternatives are treated equally.

A 1 *Neutrality. For each profile $\succsim \in \mathcal{P}$ and permutation σ on X,*

$$\sigma(f(\succsim)) = f(\sigma(\succsim)).$$

The second condition says that all agents are treated equally.

A 2 *Anonymity. For all profiles $\succsim, \succsim' \in \mathcal{P}$ and permutation γ on N such that $\succsim_i = \succsim'_{\gamma(i)}$, for all $i \in N$,*

$$f(\succsim) = f(\succsim').$$

[2] The preference relation R is trivial if $\overline{x}\ R\ \overline{y}$ for all $x, y \in X$.

[3] The latter is closely linked to [Smith, 1973]

Young [1975] groups these two conditions under the name 'Symmetry'.

We introduce a new piece of notation before next condition. Let $\succsim = (\succsim_i)_{i\in N}$ and $\succsim' = (\succsim_i)_{i\in M}$ be two profiles with $N \cap M = \emptyset$. Then $\succsim'' = \succsim \circ \succsim'$ is the profile in $\mathcal{P}_{N\cup M}$ defined by

$$\succsim_i'' = \begin{cases} \succsim_i \text{ if } i \in N \\ \\ \succsim_i' \text{ if } i \in M. \end{cases}$$

If $f(\succsim)$ is the choice set of an agent group N and $f(\succsim')$ is the choice set of another agent group M disjoint from N, and if $f(\succsim) \cap f(\succsim') \neq \emptyset$, then the group $N \cup M$ should choose precisely the alternatives in $f(\succsim) \cap f(\succsim')$. Formally,

A 3 *Separability. Let $\succsim = (\succsim_i)_{i\in N}$ and $\succsim' = (\succsim_i)_{i\in M}$ be two profiles with $N \cap M = \emptyset$. If $f(\succsim) \cap f(\succsim') \neq \emptyset$, then $f(\succsim \circ \succsim') = f(\succsim) \cap f(\succsim')$.*

This is what Young [1975] calls Consistency while Myerson [1995] calls it Reinforcement. We call it Separability, like Smith [1973].

Let $\succsim = (\succsim_i)_{i\in N}$ and $\succsim' = (\succsim_i')_{i\in M}$ be two profiles. We say $\succsim$ and $\succsim'$ are isomorphic if there is a bijection $\mu : N \to M$ such that $\succsim_i = \succsim_{\mu(i)}'$ for all $i \in N$. If $\succsim$ and $\succsim'$ are isomorphic, we can consider $\succsim'$ as a copy of $\succsim$. If $f(\succsim^1)$ is the choice set of a certain group N^1, then given any second group M disjoint from N^1 and with preference profile $\succsim'$, we can replicate the first group (and its preference profile) a sufficient number of times so that it will overwhelm the second group in a combined profile and yield a subset of $f(\succsim^1)$ as choice set. This kind of continuity requirement is our Archimedean condition.

A 4 *Archimedeanness. Let $\{N^j\}_{j\in\mathbb{N}}$ be a collection of disjoint subsets of $\mathbb{N}$, all of size n. Suppose $\{\succsim^j\}_{j\in\mathbb{N}}$ is a collection of isomorphic profiles in $\mathcal{P}_{N^j}$ and $\succsim' \in \mathcal{P}_M$ with $(\bigcup_{j\in\mathbb{N}} N^j) \cap M = \emptyset$. Then there exists $h \in \mathbb{N}$ such that, for every $k > h$,*

$$f(\succsim^1 \circ \ldots \circ \succsim^k \circ \succsim') \subseteq f(\succsim^1).$$

This is exactly Myerson's (1995) Overwhelming Majority.

The next condition is a kind of monotonicity condition; it applies to profiles with one single agent. In such a case, the choice set does not contain his/her least preferred alternatives. Let $(R)_i$ denote a profile consisting of a single preference relation R corresponding to agent i.

A 5 *Weak Faithfulness. For all $R \in V$, if $\overline{x}\ P\ \overline{y}$ and $\overline{w}\ R\ \overline{y}$ $\forall w \in X$, then $\overline{y} \notin f((R)_i)$.*

This condition is weaker than Young's (1975) condition named Faithfulness. His condition requires that the choice set only contains the most preferred alternatives. In our first result, we will use a weakening of this condition:

A 6 *Non-Triviality. There exists $R \in V$ such that $f((R)_i) \neq X$.*

2.3 Preliminary result

Using the conditions of previous section, we state a preliminary result that can almost be considered as a corollary to a result by Pivato [2014].

Proposition 1 *Let $\#X \geq 3$. A VNM SCC f satisfies Neutrality, Anonymity, Separability, Archimedeanness and Non-Triviality iff there exists $u : V \times X \to \mathbb{R}$ such that*

$$f((\succsim_i)_{i \in N}) = \operatorname*{argmax}_{x \in X} \sum_{i \in N} u(\succsim_i, x). \tag{3}$$

with u neutral, i.e., $u(\sigma(R), x) = u(R, \sigma(x))$, for any $R \in V$, $\sigma \in \Sigma$ and $x \in X$.

Let us notice that, at this stage, the mapping u is not necessarily a VNM utility function and not even a utility function. It can be anything, provided it is not constant (so as to satisfy Non-Triviality) and neutral. It is probably possible to impose some monotonicity condition in order to guarantee that u is a utility function in the sense that $u(R, x) \geq u(R, y)$ iff $x \; R \; y$.

It is also important to notice that using a VNM SCC f defined by (3) already implies some interpersonal comparisons. Indeed, if two voters i, j have the same VNM preference relation R, then $u(R, x) - u(R, y)$ is the same number for both of them, although one could argue that the difference in well-being is not necessarily the same for i and j. Consequently, the choice set will be the same for the profile where voter i is replace by voter j. Moreover, if two voters have preferences R and R', such that $R' = \sigma(R)$ for some $\sigma \in \Sigma$, then the corresponding mappings $u(R, \cdot)$ and $u(R', \cdot)$ are identical up to a permutation. So, if we have a profile with only two voters, with preferences R and R' as above, and if $\sigma(x) = y$ and $\sigma(y) = x$, then both x and y belong to $f(R, R')$ or none of them does.

Proof of Proposition 1. Let us assume f satisfies all five axioms. Let $\delta : \mathcal{P} \times V \to \mathbb{N}$ be a mapping such that $\delta(\succsim, R)$ is the number of individuals having the preference R in the profile $\succsim$. Since f is anonymous, $f(\succsim)$ depends only on $(\delta(\succsim, R))_{R \in V}$. In other words, there exists $F : \mathbb{N}^V \to 2^X \setminus \emptyset$ such that $F((\delta(\succsim, R))_{R \in V}) = f(\succsim)$. Because f satisfies Neutrality, Non-Triviality, Separability and Archimedeanness, F satisfies Neutrality, Non-Triviality, Reinforcement and Overwhelming Majority as defined in Pivato [2014]. By Proposition A.1 in Pivato [2014], there exists $u : V \times X \to \mathbb{R}$ such that

$$F((\delta(\succsim, R))_{R \in V}) = \operatorname*{argmax}_{x \in X} \sum_{R \in V} \delta(\succsim, R) u(R, x) \tag{4}$$

with u neutral, i.e., $u(\sigma(R), x) = u(R, \sigma(x))$, for any $R \in V$, $\sigma \in \Sigma$ and $x \in X$. Clearly, Equation (4) can be rewritten in terms of f as (3). □

2.4 A new condition

All axioms presented so far are standard in the social choice literature. Usually, they are imposed on SCCs acting on profiles of preference relations on unstructured sets, but nothing prevents us from imposing them on a SCC acting on profiles of preference relations defined on a structured set (e.g., Π), as we just did. Yet, none of these axioms makes use of the structure of Π; none of them helps us to untap the potentially cardinal information contained in the VNM preference relations. Our last condition will precisely do this.

Suppose a group of n^1 agents all have the same preferences: their most preferred alternatives is x and their least preferred one is z. All other alternatives are somewhere in-between. In particular, they are all indifferent between the safe lottery $\overline{y}$ and the mixture $\lambda\overline{x} + (1-\lambda)\overline{z}$, with $\lambda = 0.9$. We might consider this as relevant information about the relative standing of y vis-à-vis x and z. More specifically, we might consider that they support x more than y, but only slightly. Suppose another group of n^2 agents have preferences that result from permuting x and z in the preferences of the first group. Hence, all agents in the second group prefer z to x and are indifferent between the safe lottery $\overline{y}$ and the mixture $(1-\lambda)\overline{x} + \lambda\overline{z}$. Notice that λ is the same real number in both groups. We may again consider that the second group supports y much more that x. Suppose we consider a society consisting only of these two groups of agents. If we want to account for the strength or intensity of preferences, and if we accept the idea that many agents supporting x slightly more than y can be compensated by few agents supporting y much more that x, then we may be tempted to consider that this compensation occurs when $n^1/(n^1+n^2) = \lambda$. In that case, x is chosen if and only if y is chosen. The formal statement of this condition is more complex in two respects: (i) we consider a third group of n^3 agents that all have the same preferences and that are all indifferent between x and y; because of this indifference, they do not affect the ratio $n^1/(n^1+n^2)$ corresponding to an exact compensation; (ii) we do not impose this condition for all groups of agents, but only for all groups included in some infinite subset of $\mathbb{N}$ because we do not want to exclude the possibility that some agents have a different status and are treated differently. Let $(R)_{i\in N}$ denote a profile consisting of $\#N$ copies of the preference relation R.

A 7 *VNM-Comparability. There exists an infinite subset O of $\mathbb{N}$ such that, whenever*

- $R \in V$ *is such that* $\overline{x}\ R\ \overline{w}\ R\ \overline{z}$ *for all* $w \in X$, $\overline{x}\ P\ \overline{y}\ P\ \overline{z}$, $\overline{y}\ I\ \lambda\overline{x} + (1-\lambda)\overline{z}$,
- $R' \in V$ *is such that* $\overline{x}\ I'\ \overline{y}$,
- N^1, N^2 *and* N^3 *are disjoint subsets of* O,
- $\sigma \in \Sigma$ *is such that* $\sigma(x) = z, \sigma(z) = x$ *and* $\sigma(w) = w$ *for all* $w \neq x, z$, *and*
- $\#N^1/(\#N^1 + \#N^2) = \lambda$,

then

$$x \in f\big((R)_{i\in N^1} \circ (\sigma(R))_{i\in N^2} \circ (R')_{i\in N^3}\big) \iff y \in f\big((R)_{i\in N^1} \circ (\sigma(R))_{i\in N^2} \circ (R')_{i\in N^3}\big).$$

As far as we know, this condition has never been discussed in the literature. We do not claim that it is compelling or even appealing. Paraphrasing Sen [1976], p.254, it is not designed "to provide an axiomatic justification of" utilitarianism. Instead, we chose "a set of axioms with the focus on transparency rather than on immediate appeal" [Sen, 1976, p.259].[4]

The interest of our VNM-Comparability is that it will allow us to characterize the anonymous utilitarian VNM SCC exclusively in terms of empirically observable primitives, thereby escaping the ambiguities of the social welfare functionals approach, as discussed by Morreau and Weymark [2016].

2.5 Main result

We are now ready to state our main characterization theorem.

Theorem 1 *Let $\#X \geq 3$. A VNM SCC satisfies Neutrality, Anonymity, Separability, Archimedeanness, Weak Faithfulness and VNM-Comparability iff it is the anonymous utilitarian VNM SCC defined by (2).*

This result is not a justification of anonymous utilitarianism, because, as mentioned earlier, we do not consider our axioms as compelling. In some contexts, a social planner might consider them as appealing or reasonable and therefore decide to use an anonymous utilitarian SCC. In other contexts, another social planner might have strong arguments against our axioms and thus decide not to use an anonymous utilitarian SCC. In both cases, our axioms constitute unambiguous elements that can be used in a debate about the anonymous utilitarian VNM SCC.

Some kind of interpersonal comparability was already implied by Proposition 1. The addition of VNM Comparability in Theorem 1, for characterizing anonymous utilitarianism, makes clear how some kind of cardinal information, latent in the individual preference relations, is used to make interpersonal comparisons of differences of utilities. But, unlike invariance conditions such as, e.g. Cardinal Unit Comparability [Roberts, 1980], it does this without ever mentioning individual utilities.

Proof of Theorem 1. Let us assume f satisfies all six axioms. Weak Faithfulness implies Non-Triviality and, thanks to Proposition 1, f is defined by (3). We must now prove that $u(R, \cdot)$ is a VNM utility function representing R, for any $R \in V$. To this end, we consider three exhaustive cases.

1. R has only one equivalence class. Formally, $R \in V$ is such that $\overline{x}\ I\ \overline{y}$, for all $x, y \in X$. Since u is neutral, $u(R, x) = u(R, y)$ for all $x, y \in X$ and $u(R, \cdot)$ is therefore a VNM utility function representing R.
2. R has more than one equivalence class, but all safe lotteries are grouped in exactly two equivalence classes. Formally, $R \in V$ is such that $\overline{x}\ P\ \overline{y}$ and

[4] Our view of the axiomatic analysis is also close in spirit to that of Thomson [2001], in a different domain.

$[\overline{z}\ I\ \overline{x}$ or $\overline{z}\ I\ \overline{y}]$, for all $z \in X$. Define $A = \{z \in X : \overline{z}\ I\ \overline{x}\}$. By Weak Faithfulness, $f(R) \cap (X \setminus A) = \emptyset$. Since $f(R) \neq \emptyset$, there is $w \in A$ such that $w \in f(R)$. By Neutrality, $f(R) = A$ and $u(R,z) = u(R,x)$ for all $z \in A$. Since u is neutral, $u(R,z) = u(R,y)$ for all $z \notin A$. Since $x \in f(R)$ and $y \notin f(R)$, we have $u(R,x) > u(R,y)$. In conclusion, $u(R,\cdot)$ is a VNM utility function representing R.

3. R has more than one equivalence class and the safe lotteries cannot be grouped in two equivalence classes. Formally, $R \in V$ is such that $\overline{x}\ P\ \overline{y}\ P\ \overline{z}$. Without loss of generality, we assume $\overline{x}\ R\ \overline{w}\ R\ \overline{z}$ for all $w \in X$. Let λ be such that $\overline{y}\ I\ \lambda\overline{x} + (1-\lambda)\overline{z}$ and assume for now $\lambda \in \mathbb{Q}$. Let $N^1, N^2 \subset O$ be such that
$$\frac{\#N^1}{\#N^1 + \#N^2} = \lambda.$$
This is possible because $\lambda \in \mathbb{Q}$. Let R' and σ be as in the statement of VNM-Comparability, with, in addition, $\overline{x}\ P\ \overline{w}$ for all $w \in X \setminus \{x,y\}$. Let $N^3 \subset O$ be as in the statement of VNM-Comparability, with N^3 large enough to guarantee that $x \in f\big((R)_{i\in N^1} \circ (\sigma(R))_{i\in N^2} \circ (R')_{i\in N^3}\big)$. Thanks to VNM-Comparability, y also belongs to $f\big((R)_{i\in N^1} \circ (\sigma(R))_{i\in N^2} \circ (R')_{i\in N^3}\big)$. By virtue of (3), this implies
$$\sum_{i\in N^1} u(R,x) + \sum_{i\in N^2} u(\sigma(R),x) + \sum_{i\in N^3} u(R',x)$$
$$= \sum_{i\in N^1} u(R,y) + \sum_{i\in N^2} u(\sigma(R),y) + \sum_{i\in N^3} u(R',y).$$
Since u is neutral and $x\ I'\ y$, we have $u(R',x) = u(R',y)$. Hence
$$\sum_{i\in N^1} u(R,x) + \sum_{i\in N^2} u(\sigma(R),x) = \sum_{i\in N^1} u(R,y) + \sum_{i\in N^2} u(\sigma(R),y).$$
Since u is neutral and $\sigma(x) = z$ and $\sigma(y) = y$,
$$\sum_{i\in N^1} u(R,x) + \sum_{i\in N^2} u(R,z) = \sum_{i\in N^1} u(R,y) + \sum_{i\in N^2} u(R,y).$$
Therefore
$$\#N^1 u(R,x) + \#N^2 u(R,z) = (\#N^1 + \#N^2)\ u(R,y).$$
Subtract $(\#N^1 + \#N^2)\ u(R,z)$ on both sides to obtain
$$\#N^1 u(R,x) - \#N^1 u(R,z) = (\#N^1 + \#N^2) u(R,y) - (\#N^1 + \#N^2)\ u(R,z)$$
or
$$\frac{u(R,y) - u(R,z)}{u(R,x) - u(R,z)} = \frac{\#N^1}{\#N^1 + \#N^2} = \lambda.$$

This proves that, when $\lambda \in \mathbb{Q}$, $u(R,\cdot)$ is a VNM utility function. If $\lambda \notin \mathbb{Q}$, we can "squeeze"

$$\frac{u(R,y)-u(R,z)}{u(R,x)-u(R,z)}$$

between two sequences of ratios $\#N^1/(\#N^1+\#N^2)$ both converging to λ, respectively from below and from above.

So, in any case, $u(R,\cdot)$ is a VNM utility function representing R. □

3 A characterization of relative utilitarianism

The aim of this section is to characterize the relative utilitarian VNM Social Choice Correspondence, defined by

$$f((\succsim_i)_{i\in N}) = \underset{x\in X}{\operatorname{argmax}} \sum_{i\in N} u(\succsim_i, x) \tag{5}$$

where $u : V \times X \to \mathbb{R}$ is such that

(a) $u(R,\cdot)$ is a VNM utility function, for any $R \in V$,
(b) $\overline{x}R\overline{y}$ (resp. $\overline{y}R\overline{x}$) for all $y \in X$, with at least one strict preference, implies $u(R,x) = 1$ (resp. 0).

Notice that this defines a unique VNM SCC. Besides, this VNM SCC is utilitarian, as defined by (2). Indeed, (a) and (b) together imply that u is neutral. So, the relative utilitarian VNM SCC satisfies all conditions of Theorem 1. The question is: which extra condition is satisfied by the relative utilitarian VNM SCC and only by this one. The answer turns out to be simple: when there are two agents, if a top ranked alternative of the first agent is a bottom ranked alternative of the second agent and vice versa, then both alternatives are chosen or none is chosen.

A 8 *Ordinal Comparability. There exists an infinite subset O of $\mathbb{N}$ such that, whenever*

- *$R \in V$ is not trivial and is such that $\overline{x}\ R\ \overline{w}\ R\ \overline{y}$ for all $w \in X$,*
- *$R' \in V$ is not trivial and is such that $\overline{y}\ R'\ \overline{w}\ R'\ \overline{x}$ for all $w \in X$,*
- *$R'' \in V$ is such that $\overline{x}\ I''\ \overline{y}$;*
- *$i, j \in O$, $i \neq j$, $N \subset O$ and $\{i,j\} \cap N = \emptyset$,*

then

$$x \in f\big((R)_i \circ (R')_j \circ (R'')_{k\in N}\big) \iff y \in f\big((R)_i \circ (R')_j \circ (R'')_{k\in N}\big).$$

The structure of this condition is similar to that of VNM-Comparability. We call it Ordinal Comparability because it does not exploit the structure of Π; it is possible to impose it on a SCC acting on profiles of preference relations defined on abstract unstructured sets.

Theorem 2 *A VNM SCC satisfies Neutrality, Anonymity, Separability, Archimedeanness, Weak Faithfulness, VNM-Comparability and Ordinal Comparability iff it is the relative utilitarian VNM SCC defined by (5).*

There are many differences between [Dhillon and Mertens, 1999] and our result. First, our set of voters is variable while theirs is fixed. Second, we characterize a procedure for aggregating preference relations into a choice set while they aggregate preference relations into a social preference relation. We consider these two differences as minor and almost technical. A more fundamental difference is that our choice set will never contain a lottery while their social preference relations is defined on the set of all lotteries. In our framework, lotteries play an instrumental role: we are not interested in lotteries, but we use them for obtaining cardinal information about the alternatives. Depending on what we are interested in (lotteries over alternatives of just alternatives), their result or ours may be more relevant.

Proof of Theorem 2. By Theorem 1, f is utilitarian. Let R and S be two non-trivial relations in V. We suppose without loss of generality that $\overline{x}\ R\ \overline{w}\ R\ \overline{y}$ for all $w \in X$. There is a relation $R' \in V$ and a permutation $\sigma \in \Sigma$ such that $R' = \sigma(S)$ and $\overline{y}\ R'\ \overline{w}\ R'\ \overline{x}$ for all $w \in X$. Let $R'' \in V$ be such that $\overline{x}\ I''\ \overline{y}$ and $\overline{x}\ P''\ \overline{w}$ for all $w \in X \setminus \{x, y\}$. Then R, R' and R'' are as in the statement of Ordinal Comparability. Let i, j and $N \subset O$ be as in the statement of Ordinal Comparability, with N large enough to guarantee that $x \in f\big((R)_i \circ (R')_j \circ (R'')_{k\in N}\big)$. Thanks to Ordinal Comparability, y also belongs to $f\big((R)_i \circ (R')_j \circ (R'')_{k\in N}\big)$. By virtue of (3), this implies

$$u(R,x) + u(R',x) + \sum_{i\in N} u(R'',x) = u(R,y) + u(R',y) + \sum_{i\in N} u(R'',y).$$

Thanks to Neutrality, we obtain $u(R,x) + u(R',x) = u(R,y) + u(R',y)$ or $u(R,x) - u(R,y) = u(R',y) - u(R',x)$. Since $R' = \sigma(S)$ and thanks to the neutrality of u, we can rewrite the last equation as $u(R,x) - u(R,y) = u(S,x') - u(S,y')$, with x', y' the maximal and minimal elements in S. So, the difference between the maximal and minimal utilities is the same for every preference relation. We can suppose without loss of generality that this difference is equal to 1 and that the minimal utility is zero. □

At this point, the reader may wonder what we obtain if we impose Ordinal Comparability without VNM-Comparability (on top of the standard axioms). It turns out that this is not really interesting. The first part of the proof of Theorem 1 is still valid, so that f must satisfy (3), with u neutral. Imposing Ordinal Comparability then implies that $u(R,\cdot)$ has values 0 and 1 for the minimal and maximal alternatives, but u has no other properties. In particular, $u(R,\cdot)$ can be larger than 1 for alternatives that are not maximal. If we replace Weak Faithfulness by a stronger monotonicity condition, then one obtains a criterion based on the sum of normalized utilities, but these utilities need not be VNM utilities.

4 Independence of the conditions in Theorems 1 and 2

We provide, for each of the seven conditions in Theorem 2, an example satisfying all conditions but one. This proves that the conditions in Theorem 2 are logically independent. The first six examples can be used for proving the logical independence of all conditions in Theorem 1.

Example 1 (Neutrality) *Let x, y be distinct elements of X and let V^* be a proper subset of V containing all relations $R \in V$ such that $\overline{x}\ R\ \overline{w}\ R\ \overline{y}$ or $\overline{y}\ R\ \overline{w}\ R\ \overline{x}$ for all $w \in X$. Define $g : V \times X \to \mathbb{R}$ so that, for all $R \in V$, $g(R, \cdot)$ is the normalized VNM utility function representing R (except for the trivial preference relation). Define $u : V \times X \to \mathbb{R}$ by*

$$u(R,\cdot) = \begin{cases} g(R,\cdot) & \text{if } R \in V^* \\ 2g(R,\cdot) & \text{otherwise} \end{cases}$$

and f by

$$f\big((\succsim_i)_{i\in N}\big) = \operatorname*{argmax}_{x\in X} \sum_{i\in N} u(\succsim_i, x).$$

This VNM SCC obviously violates Neutrality. The reason it satisfies VNM-Comparability is that the relations R and $\sigma(R)$ in the statement of VNM-Comparability both belong to V^* or both to $V \setminus V^*$. The reason it satisfies Ordinal Comparability is that the relations R and R' in the statement of Ordinal Comparability both belong to V^* or both to $V \setminus V^*$.

Example 2 (Anonymity) *Let $u : V \times X \to \mathbb{R}$ be such that, for all $R \in V$, $u(R, \cdot)$ is a VNM utility function representing R and u is neutral. Let O be any proper infinite subset of $\mathbb{N}$; for instance the set of all even natural numbers. Define*

$$f\big((\succsim_i)_{i\in N}\big) = \operatorname*{argmax}_{x\in X} \Big(\sum_{i\in N\setminus O} 2u(\succsim_i, x) + \sum_{i\in N\cap O} u(\succsim_i, x)\Big).$$

To understand why f satisfies VNM-Comparability and Ordinal Comparability, notice that both conditions only apply to agents in O. In that case, f can be rewritten as

$$f\big((\succsim_i)_{i\in N}\big) = \operatorname*{argmax}_{x\in X} \sum_{i\in N} u(\succsim_i, x),$$

which is the plain utilitarian VNM SCC.

Example 3 (Weak Faithfulness) *Let $u : V \times X \to \mathbb{R}$ be such that, for all $R \in V$, $u(R, \cdot)$ is a VNM utility function representing R and u is neutral. Define*

$$f\big((\succsim_i)_{i\in N}\big) = \operatorname*{argmin}_{x\in X} \sum_{i\in N} u(\succsim_i, x).$$

Example 4 (Separability) *Define*

$$f\big((\succsim_i)_{i\in N}\big)=\left\{x\in X:\sum_{i\in N}u(\succsim_i,x)>\frac{\#N-1}{\#N}\ \operatorname*{argmax}_{y\in X}\sum_{i\in N}u(\succsim_i,y)\right\}$$

with $u(R,\cdot)$ normalized for every $R\in V$ (except for the trivial preference relation).

In words, f selects not only the maximal alternatives according to the relative utilitarian criterion, but also those alternatives that are nearly maximal, i.e., exceeding the threshold $(\#N-1)/\#N$ times the maximal utility. This VNM SCC clearly satisfies Neutrality and Anonymity. When there is only one agent, we have $(\#N-1)/\#N=0$ and f selects all alternatives but the lowest ranked ones (with utility zero). It therefore satisfies Weak Faithfulness.

Suppose $X=\{x,y,z\}$ and the normalized VNM representation of R is $u(R,x)=1, u(R,y)=0.5$ and $u(R,z)=0$. Let $N^1=\{1\}, N^2=\{2\}, \succsim=(\succsim_i)_{i\in N^1}=(R)$ and $\succsim'=(\succsim_i)_{i\in N^2}=(R)$. Then $f(\succsim)=\{x,y\}=f(\succsim')$, but $f(\succsim\circ\succsim')=\{x\}$, thereby violating Separability.

We now prove f satisfies Archimedeanness. The set $f(\succsim^1)$ contains all maximal or nearly maximal alternatives according to the relative utilitarian criterion applied to $\succsim^1$. When k grows to infinity, the total utilities in the profile $\succsim^1\circ\ldots\circ\succsim^k\circ\succsim'$ converge to k times the utilities in $\succsim^1$. At the same time, the threshold $(\#N-1)/\#N$ converges to 1, from below. So, $f(\succsim^1\circ\ldots\circ\succsim^k\circ\succsim')$ contains only the maximal alternatives according to the relative utilitarian criterion applied to $\succsim^1$. So, $f(\succsim^1\circ\ldots\circ\succsim^k\circ\succsim')\subseteq f(\succsim^1)$.

Two alternatives x and y as in the statement of VNM-Comparability or Ordinal Comparability have the same total utility. So, depending on the threshold, both are selected or both are not selected. Hence f satisfies VNM-Comparability and Ordinal Comparability.

Example 5 (Archimedeanness) *Let $u:V\times X\to\mathbb{R}$ be such that, for all $R\in V$, $u(R,\cdot)$ is a normalized VNM utility function representing R and u is neutral. Define*

$$h\big((\succsim_i)_{i\in N}\big)=\operatorname*{argmax}_{x\in X}\sum_{i\in N}u(\succsim_i,x).$$

$$f\big((\succsim_i)_{i\in N}\big)=\operatorname*{argmax}_{x\in h\left((\succsim_i)_{i\in N}\right)}\#\{i\in N:\overline{x}\succsim_i\overline{z}\ \ \textit{for all } z\in X\}.$$

Put differently, this VNM SCC successively applies the argmax to two different criteria: first the utilitarian one and, then, a criterion based on the number of times an alternative i smaximal in individual preferences. This VNM SCC clearly satisfies Neutrality, Anonymity and Weak Faithfulness.

If an alternative x is selected in $f(\succsim)$, it is maximal in $\succsim$ according to the utilitarian criterion and according to the second criterion. If the same alternative x is selected in $f(\succsim')$, it is also maximal in $\succsim'$ according to both criteria. Since both criteria are additive, x is again maximal in $\succsim\circ\succsim'$ according to both criteria and, hence, Separability holds.

Suppose $X = \{x, y, z\}$, the normalized VNM representation of R is $u(R, x) = 1, u(R, y) = u(R, z) = 0$ and the normalized VNM representation of R' is $u(R, y) = 1, u(R, x) = 0.5$ and $u(R, z) = 0$. Let $N^1 = \{2, 3, 4\}, M = \{1\}, \succsim^1 = (\succsim_i^1)_{i \in N^1} = (R, R', R')$ and $\succsim' = (\succsim_i)_{i \in M} = (R)$. Then $h(\succsim^1) = \{x, y\}$ and $f(\succsim^1) = \{y\}$. For any $k > 0$, $h(\succsim^1 \circ \ldots \circ \succsim^k \circ \succsim') = \{x\}$ and $f(\succsim^1 \circ \ldots \circ \succsim^k \circ \succsim') = \{x\}$, thereby violating Archimedeanness.

To see that f satisfies VNM-Comparability and Ordinal Comparability, notice that, for all profiles as in the statement of these conditions, we have

$$\#\{i \in N : \overline{x} \succsim_i \overline{z} \text{ for all } z \in X\} = \#\{i \in N : \overline{y} \succsim_i \overline{z} \text{ for aal } z \in X\}.$$

Hence the second criterion does not play a role and f is the plain utilitarian VNM SCC and we have already shown that it satisfies both conditions.

Example 6 (VNM-Comparability) *Let $g : V \times X \to \mathbb{R}$ be such that, for all $R \in V$, $g(R, \cdot)$ is a VNM utility function representing R, with g neutral. Define $u : V \times X \to \mathbb{R}$ by $u(R, x) = \big(g(R, x)\big)^3$ for all $R \in V$ and $x \in X$. Define*

$$f\big((\succsim_i)_{i \in N}\big) = \operatorname*{argmax}_{x \in X} \sum_{i \in N} u(\succsim_i, x).$$

This VNM SCC violates VNM-Comparability because $u(R, \cdot)$ is the third power of a VNM utility function representing R. It is therefore not linearly related to a VNM utility function representing R. It clearly satisfies all other conditions.

Example 7 (Ordinal Comparability) *Let $u : V \times X \to \mathbb{R}$ be such that, for all $R \in V$, $u(R, \cdot)$ is a VNM utility function representing R and u is neutral but not normalized. Define*

$$f\big((\succsim_i)_{i \in N}\big) = \operatorname*{argmax}_{x \in X} \sum_{i \in N} u(\succsim_i, x).$$

This VNM SCC clearly violates Ordinal Comparability and satisfies all other conditions.

5 Discussion

The choice of u

Suppose a social planner buys all axioms of Theorem 1 and therefore wants to use an anonymous utilitarian VNM SCC. Yet, she does not adhere to Ordinal Comparability and, hence, she does not want to use the relative utilitarian VNM SCC. She then faces a choice: among the infinite family of anonymous utilitarian VNM SCCs, which one is she going to use? Is there a reasoned way to select a specific member of this family? Our results do not answer this question, but Theorem 2 shows a possible direction: instead of imposing Ordinal Comparability in terms of maximal and minimal elements, we could enrich our primitives with two particular alternatives with an identical meaning to all voters (called interpersonally significant norm by Blackorby and Donaldson [1982]), and restate Ordinal Comparability in terms of these two particular alternatives. The existence and meaningfulness of such interpersonally significant norms is another debate. Other directions are perhaps possible.

SCC vs SWF

Our results are stated in terms of a social choice correspondence while most of the literature about cardinal social choice is stated in terms of social welfare function (SWF). In particular, the characterization of relative utilitarianism by Dhillon and Mertens [1999] is in terms of a SWF. This difference is without much consequences because it is probably easy to reformulate Proposition A1 in [Pivato, 2014] in terms of a SWF.[5] It would then be easy to reformulate our results in terms of a SWF, after a slight adaptation of our conditions.

Other measurement techniques

In this paper, in order to obtain cardinal preferential information about the finite set of alternatives, we embed them in a rich set (the set of all lotteries) and we observe the preferences of the voters on this rich set. Provided the preferences satisfy some properties, it is possible to infer some cardinal preferential information about the alternatives.

Notice that there are other ways to obtain cardinal preferential information about a finite set of alternatives. Suppose the alternatives are elements of a Cartesian product. For instance $X = X_1 \times X_2$ where $X_1 = \{100, 110, 120\}$ and $X_2 = \{30, 40, 50\}$ are amounts to be invested in two different projects. We can embed the set X in the richer set $[100, 120] \times [30, 50]$ and observe the preferences of the voters over this richer set. Provided their preferences satisfy some conditions, using techniques of conjoint measurement, it is possible to represent the preferences by means of two utility functions unique up to positive affine transformations [Debreu, 1960, Krantz et al., 1971]. Restating Proposition 1 for such preference relations is immediate. It is then probably not too difficult to devise a new comparability condition, akin to VNM Comparability, in order to characterize anonymous utilitarianism in this context. Techniques of algebraic difference measurement or extensive measurement can also be used [Krantz et al., 1971]. In the latter case, the utilities are unique up to a positive linear transformation (ratio scale) and this may lead to a subset of the anonymous utilitarian family.

An interesting consequence of our approach (without individual utilities) is that the exact form of the comparability condition would depend on the measurement technique (decision under risk, conjoint measurement, etc.) while, with the classical approach (social welfare functional acting on profiles of individual utilities), the same invariance condition is used irrespective of what utilities mean and of the way they have been measured.

[5] This has already been done in Marchant [1996] for the characterization of scoring rules by Myerson [1995].

Acknowledgements I am grateful to Antoinette Baujard, Denis Bouyssou, Marc Pirlot and John Weymark for comments and discussions.

References

K. J. Arrow. *Social choice and individual values.* Wiley, New York, 2nd edition, 1963.

C. Blackorby and D. Donaldson. Ratio-scale and translation-scale full interpersonal comparability without domain restrictions: Admissible social-evaluation functions. *International Economic Review*, 23(2):249–268, 1982.

T. Börgers and Y. M. Choo. A counterexample to dhillon (1998). *Social Choice and Welfare*, In press, 2017a.

T. Börgers and Y. M. Choo. Revealed relative utilitarianism. Unpublished, 2017b.

W. Bossert. On intra- and interpersonal utility comparisons. *Social Choice and Welfare*, 8:207–219, 1991.

C. d'Aspremont and L. Gevers. Equity and the informational basis of collective choice. *Review of Economic Studies*, 44:199–209, 1977.

G. Debreu. Topological methods in cardinal utility theory. In K. J. Arrow, S. Karlin, and P. Suppes, editors, *Mathematical methods in the Social Sciences*, pages 16–26. Stanford University Press, Stanford, 1960.

A. Dhillon. Extended pareto rules and relative utilitarianism. *Social Choice and Welfare*, 15(4):521–542, 1998.

A. Dhillon and J. F. Mertens. Relative utilitarianism. *Econometrica*, 67(3): 471–498, 1999.

J. C. Harsanyi. Cardinal welfare, individual ethics, and interpersonal comparisons of utility. *Journal of Political Economy*, 63(4):309–321, 1955.

N. E. Jensen. An introduction to Bernouillian utility theory I: utility functions. *Swedish Journal of economics*, 69:163–183, 1967.

D. H. Krantz, R. D. Luce, P. Suppes, and A. Tversky. *Foundations of measurement: Additive and polynomial representations.* Academic Press, New York, 1971.

T. Marchant. *Agrégation de relations valuées par la méthode de Borda, en vue d'un rangement: considérations axiomatiques.* PhD thesis, Université Libre de Bruxelles, 1996.

T. Marchant. Scale invariance and similar invariance conditions for bankruptcy problems. *Social Choice and Welfare*, 31:693–707, 2008.

M. Morreau and J. A. Weymark. Measurement scales and welfarist social choice. *Journal of Mathematical Psychology*, 75:127–136, 2016.

R. B. Myerson. Axiomatic derivation of scoring rules without the ordering assumption. *Social choice and welfare*, 12:59–74, 1995.

M. Pivato. Formal utilitarianism and range voting. *Mathematical Socials Sciences*, 67:50–56, 2014.

K. W. S. Roberts. Interpersonal comparability and social choice theory. *Review of Economic Studies*, 47:421–439, 1980.

J. E. Roemer. *Theories of distributive justice*. Harvard University Press, 1996.

A. K. Sen. *Collective choice and social Welfare*. Holden-Day, San Francisco, 1970.

A. K. Sen. Welfare inequalities and Rawlsian axiomatics. *Theory and decision*, 7:243–262, 1976.

J. H. Smith. Aggregation of preferences with variable electorate. *Econometrica*, 41:1027–1040, 1973.

Y. Sprumont. On relative egalitarianism. *Social Choice and Welfare*, 40(4): 1015–1032, 2013.

W. Thomson. On the axiomatic method and its recent applications to game theory and resource allocation. *Social Choice and Welfare*, 18:327–386, 2001.

J. A. Weymark. A reconsideration of the Harsanyi–Sen debate on utilitarianism. In J. Elster and J. E. Roemer, editors, *Interpersonal Comparisons of Well-Being*. Cambridge University Press, Cambridge, 1991.

H. P. Young. Social choice scoring functions. *SIAM Journal on Applied Mathematics*, 28:824–838, 1975.