Equivalence of Local and Global Strategy-proofness on Multi-dimensional Domains with Lexicographic Preferences *

Ujjwal Kumar †1 and Souvik Roy ‡1

[1]Economic Research Unit, Indian Statistical Institute, Kolkata

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Abstract

We explore the equivalence of local strategy-proofness and global strategy-proofness on domains where alternatives have multiple dimensions and agents have lexicographic preferences over those. We show that if there exists exactly one admissible preference over components, then local and global strategy-proofness for the multidimensional domain are equivalent if and only if the same holds for every marginal domain. We further show that if the marginal domains are either unrestricted or single-peaked, then local and global strategy-proofness are equivalent for the multi-dimensional domain for every set of component preferences.

KEYWORDS: Local strategy-proofness, global strategy-proofness, multi-dimensional lexicographic domains

JEL CLASSIFICATION CODES: D71, D82.

1. Introduction

1.1 Background of the Problem

We consider the situation where a designer has to choose an outcome from a feasible set of outcomes based on the preferences of a group of individuals in a society. Such a procedure is

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†Contact: ujju489@gmail.com

‡Contact: souvik.2004@gmail.com

called a social choice function. A well-known desirable property of such functions is strategy-proofness. This property ensures that the dishonest individuals in the society cannot bend the outcome in their favor by strategically misreporting their preferences.

In the seminal papers of Gibbard (1973)-Satterthwaite (1975), it is shown that if there are at least three social outcomes and the preferences of the individuals are unrestricted, then every strategy-proof and unanimous social choice function will be dictatorial. A dictatorial social choice function is one that selects the most preferred alternative of one particular individual at every collection of reported preferences.

Domain restrictions turn out to be the most practical way to evade Gibbard-Satterthwaite (Gibbard (1973), Satterthwaite (1975)) impossibility result. Well-known domain restrictions that are studied in literature are single-peaked, single-dipped, single-crossing etc. Moulin (1980) characterize the strategy-proof and unanimous rule on single-peaked domains, Peremans and Storcken (1999) characterize those on single-dipped domains, and Saporiti (2014) on single-crossing domains.

1.2 Our Motivation

Although the strategy-proof rules are characterized on several well-known domains, a general characterization of those on arbitrary domains is assumed to be a hard problem. In view of this, researchers started looking at simpler (easy to check) versions of strategy-proofness. One such version is local strategy-proofness. Local strategy-proofness requires that an individual cannot manipulate by a 'slight' misreport of his preferences. More formally, it ensures that an individual cannot manipulate by swapping two consecutive alternatives in his preference. This raises an interesting question as to when, that is under what condition on a domain, such a simple version of strategy-proofness becomes equivalent to strategy-proofness. This is the main question we deal with in this paper.

In Sato (2013), it is shown that if a domain satisfies 'no restoration' property, then every locally strategy-proof rule is strategy-proof. However, to the best of our knowledge, no necessary and sufficient condition is known till date for the equivalence of the two notions of strategy-proofness.

In many practical scenarios, a decision maker has to take decision on multiple issues simultaneously. Examples of such situations include deciding the optimum level of budget allocations over different sectors such as health, education, defense etc. Such situations are modeled as multi-dimensional decision problem and the usefulness of such models is well-established in literature.

It is shown in Breton and Sen (1999) that if a multi-dimensional separable domain satisfies richness property, then every strategy-proof rule on it is decomposable. However, the structure of such rules on arbitrary multi-dimensional domains is still open. This motivates us to establish the equivalence of local and global strategy-proofness on such domains.

1.3 Our Contribution

We consider multi-dimensional lexicographic domain and explore the equivalence of local and global strategy-proofness on such domains. In particular, we investigate how the said equivalence for one dimension gets translated to multiple dimensions.

We show that if there is exactly one admissible preference over components, then local and global strategy-proofness are equivalent for the multidimensional domain if and only if they are equivalent for every marginal domain. We further prove that if the marginal domains are single-peaked or unrestricted, then local and global strategy-proofness are equivalent for the multidimensional domain for every collection of component orderings. We also show by means of example that this result does not hold for arbitrary marginal domains. We leave the problem of finding the necessary and sufficient condition on marginal domains such that local and global strategy-proofness are equivalent on the multidimensional domain (for any collection of component orderings) for future research.

1.4 Remainder

The paper is organized as follows. In Section 2, we introduce the model and in Section 3, we present our results. Finally, we conclude the paper in Section 4. All the proofs are collected in the Appendix.

2. Model

We consider a social choice problem with finite set of agents $N = \{1, \ldots, n\}$ with $n \geq 2$. The set of alternatives is defined as $A = A^1 \times \ldots \times A^m$, where for all $k \in M = \{1, \ldots, m\}$, A^k is the finite set of alternatives available in the k-th component. For notational convenience, whenever it is clear from the context, we denote M by A^0. For $X \subseteq A^0$, we define $A^X = \prod_{l \in X} A^l$.

For ease of presentation, we do not use braces for singleton sets. Also, whenever it is clear from the context, we use minus sign for setminus notation, that is, for two sets X and Y, we write $X - Y$ for $X \setminus Y$.

2.1 Domains and their properties

For any finite set X, a complete, transitive, and antisymmetric binary relation on X, denoted by P (also called a linear order) is called a strict preference over X. We denote by $\mathbb{L}(X)$ the set of all strict preferences over X. For $P \in \mathbb{L}(X)$ and $k \leq |X|$, we denote by $P(k)$ the k-th ranked alternative according to P, more formally, $P(k) = a$ if and only if $|\{b \mid bPa\}| = k - 1$.

Two alternatives are called adjacent in a preference if they are ranked consecutively in that preference. For $P, \bar{P} \in \mathbb{L}(X)$, we define

$$P \vartriangle \bar{P} = \{\{a,b\} \subseteq X \mid a \text{ and } b \text{ have different relative orderings in } P \text{ and } \bar{P}\}.$$

Two preferences P and $\bar{P}$ are called adjacent, denoted by $P \sim \bar{P}$, if and only if $|P \vartriangle \bar{P}| = 1$. In other words, $P \sim \bar{P}$ if and only if $\bar{P}$ is obtained by swapping exactly two adjacent alternatives in P.

Throughout this paper, we denote a generic element of A^k by a^k, and for $X \subseteq A^0$, a generic element of A^X by a^X. Also, we denote a generic preference in $\mathbb{L}(M)$ by P^0, and for $k \in M$, a generic preference in $\mathbb{L}(A^k)$ by P^k.

Definition 2.1. A preference P is called single-peaked with respect to an ordering $\prec$ over the set of alternatives if for all $x, y \in A$ with [either $x \prec y \prec r_1(P)$ or $r_1(P) \prec y \prec x$], we have yPx. A domain $\mathcal{S}$ is called single-peaked if it contains all single-peaked preferences with respect to some ordering over the alternatives.

Definition 2.2 (Lexicographic preference)**.** For $P^0 \in \mathbb{L}(M)$ and $P^k \in \mathbb{L}(A^k); k = 1, \ldots, m$, a preference $P \in \mathbb{L}(A)$ is called lexicographic with respect to $(P^0, P^1, \ldots, P^m)$ if for all $a, b \in A$, aPb if and only if there exist $j \in 1, \ldots, m$ such that $a^{P^0(l)} = b^{P^0(l)}$ for all $l < j$, and $a^{P^0(j)} P^{P^0(j)} b^{P^0(j)}$.

Whenever it is clear from the context, for a preference $P \in \mathcal{L}_i$, we denote by P^k the marginal preference of P over A^k. By $\mathcal{L}$, we denote a set of all lexicographic preferences over A. Each agent

$i \in N$ has a set of admissible lexicographic preferences $\mathcal{L}_i \subseteq \mathcal{L}$ over A. We denote $\mathcal{L}_N = \prod_i \mathcal{L}_i$. An element $P_N \in \mathcal{L}_N$ is called a preference profile.

For $k \in M$ and $\mathcal{L}_i \subseteq \mathcal{L}$, we define $\mathcal{L}_i^k = \{\hat{P}_i^k \in \mathbb{L}(A^k) \mid \text{ there is } P_i \in \mathcal{L}_i \text{ with } P_i^k = \hat{P}_i^k\}$ and $\mathcal{L}_N^k = \prod_i \mathcal{L}_i^k$. Similarly, we define $\mathcal{L}_i^0 = \{\hat{P}_i^0 \in \mathbb{L}(M) \mid \text{ there is } P_i \in \mathcal{L}_i \text{ with } P_i^0 = \hat{P}_i^0\}$.

Definition 2.3. The graph of a domain $\mathcal{L}_i$ is defined as the directed graph $G = (\mathcal{L}_i, E)$, where $P, \bar{P} \in E$ if and only if there exists unique $k \in \{0, 1, \ldots, m\}$ such that $P^k \sim \bar{P}^k$.

Definition 2.4 (Restoration)**.** For $a, b \in A; a \neq b$, a path $\pi(P, P') = (P_1, \ldots, P_k)$ in the graph of $\mathcal{L}_i$ is said to have (a, b) restoration if there exist $i, j \in \{1, \ldots, k\}$ such that $P_i \bigtriangleup P_{i+1} = P_j \bigtriangleup P_{j+1} = \{\{a, b\}\}$.

2.2 Social choice functions and their properties

Definition 2.5. A social choice function (SCF) on $\mathcal{L}_N$ is defined as a mapping $f : \mathcal{L}_N \to A$, and for all $k \in M$, an SCF on $\mathcal{L}_N^k$ is defined as a mapping $\hat{f}^k : \mathcal{L}_N^k \to A^k$.

For an SCF f and a component $k \in M$, we denote by f^k the k-th component of f, more formally, for all $P_N \in \mathcal{L}_N$, $f^k : \mathcal{L}_N \to A^k$ given by $f^k(P_N) = (f(P_N))^k$.

Definition 2.6 (Locally strategy-proof function)**.** An SCF $f : \mathcal{L}_N \to A$ is called locally manipulable if there exist $P_N \in \mathcal{L}_N$, $i \in N$, $P_i' \in \mathcal{L}_i$ with $P_i' \sim P_i$ such that

$$f(P_i', P_{N-i}) P_i f(P_N).$$

An SCF is locally strategy-proof if it is not locally manipulable.

Definition 2.7 (Globally strategy-proof function)**.** An SCF $f : \mathcal{L}_N \to A$ is called globally manipulable if there exist $P_N \in \mathcal{L}_N$, $i \in N$, $P_i' \in \mathcal{L}_i$ such that

$$f(P_i', P_{N-i}) P_i f(P_N).$$

An SCF is globally strategy-proof if it is not globally manipulable.

Definition 2.8 (Smooth domain)**.** A domain $\mathcal{L}_N$ is said to be smooth if each locally strategy-proof social choice function on $\mathcal{L}_N$ is globally strategy-proof.

Throughout this paper, by a *rough social choice function*, we mean an SCF that is locally strategy-proof but not globally. Also, by a *rough domain*, we mean a domain that is not smooth.

Definition 2.9. An SCF $f : \mathcal{L}_N \to A$ is called globally (locally) manipulable for component $k \in M$ if there exist $P_N^{M-k} \in \mathcal{L}_N^{M-k}$, $i \in N$, $P_i^k, \bar{P}_i^k \in \mathcal{L}_i^k$ (with $P_i^k \sim \bar{P}_i^k$), and $P_{N-i}^k \in \mathcal{L}_{N-i}^k$ such that

$$f((\bar{P}_i^k, P_{N-i}^k), P_N^{M-k}) P_i^k f((P_i^k, P_{N-i}^k), P_N^{M-k}).$$

An SCF is strategy-proof for component k if it is not manipulable for component k.

Definition 2.10 (Decomposable SCFs)**.** An SCF $f : \mathcal{L}_N \to A$ is called globally (locally) decomposable if f is globally (locally) strategy-proof for all $k \in A^0$ implies f is globally (locally) strategy-proof.

Definition 2.11 (Decomposable domains)**.** A domain $\mathcal{L}_N$ is called decomposable if $\mathcal{L}_i^k$ is smooth for all $k \in A^0$ implies $\mathcal{L}_N$ is smooth.

3. Results

Definition 3.1. For $a \in A$, $B \subseteq A$, and $\hat{P} \in \mathcal{P}$, a social choice function $f : \mathcal{P} \to A$ is called monotonic with respect to $(a, B, \hat{P})$ if $f(P) = a$ for all $P \in \mathcal{P}$ such that there is a path from $\hat{P}$ to P in which no element from B overtakes a, and $f(P) = \max_P(B)$ for all other preferences.

Lemma 3.1. *Let $a \in A$, $B \subseteq A$, and $\hat{P} \in \mathcal{P}$. Suppose $f : \mathcal{P} \to A$ is monotonic with respect to $(a, B, \hat{P})$. Then, f is locally strategy-proof.*

The proof of this lemma is relegated to Appendix A

Definition 3.2. A domain $\mathcal{P}$ is said to satisfy maximal restoration property if there exist $P, P' \in \mathcal{P}$ and $a \in A$ such that for every path $\pi(P, P')$ from P to P' has a (a, x) restoration for some $x \in A \setminus a$ with aPx.

Theorem 3.1. *If a domain $\mathcal{P}$ satisfies the maximal restoration property, then it is a rough domain.*

The proof of this theorem is relegated to Appendix B

Definition 3.3. A lexicographic domain $\mathcal{L}_i$ is called a product lexicographic domain if for all $k = 0, \ldots, m$ and all $P^k \in \mathcal{L}_i^k$, there exists $(P^0, P^1, \ldots, P^m) \in \mathcal{L}_i$.

Theorem 3.2. *Let $\mathcal{L}_i$ for all $i \in N$ be a product lexicographic domain such that $|\mathcal{L}_i^0| = 1$ and $\mathcal{L}_i^k$ is smooth for all $k \in A^0$. Then, $\mathcal{L}_N$ is a smooth domain.*

The proof of this theorem is relegated to Appendix C

Theorem 3.3. *Let $\mathcal{D} \subset \mathbb{L}(A)$ be such that for each $P, P' \in \mathcal{D}$ and each $a \in A$, there exist a path in $\mathcal{D}$ between P and P' which is without $\{a, z\}$ restoration for all $z \in A$ such that aPz. Then, $\mathcal{D}$ is a smooth domain.*

The proof of this theorem is relegated to Appendix D

Theorem 3.4. *Let $\mathcal{P}^0 \subseteq \mathbb{L}(M)$, and let $\mathcal{P}^k = \mathbb{L}(A^k)$ for all $k \in M$. Then, the product lexicographic domain defined by $\mathcal{D} = (\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2, \ldots, \mathcal{P}^m)$ is a smooth domain.*

The proof of this theorem is relegated to Appendix E

Theorem 3.5. *Let $\mathcal{P}^0 \subset \mathbb{L}(M)$, and let $\mathcal{P}^k \subset \mathbb{L}(A^k)$ be single peaked for all $k \in M$. Then, the product lexicographic domain defined by $\mathcal{D} = (\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2, \ldots, \mathcal{P}^m)$ is a smooth domain.*

The proof of this theorem is relegated to Appendix F

4. Conclusion

We have analyzed the equivalence of local and global strategy-proofness on multi-dimensional lexicographic domains. We have shown that the said equivalence holds if the same holds for every marginal domain and there exists exactly one component ordering. We have also proved that if the marginal domains are unrestricted or single-peaked, then local and global strategy-proofness are equivalent on the multi-dimensional domain for any collection of component orderings.

We leave the problem of finding the necessary and sufficient condition on the marginal domains and component orderings so that the local and global strategy-proofness are equivalent on multi-dimensional lexicographic domain for future research.

A. Proof of Lemma 3.1

Proof. Let $\mathcal{P}_a = \{P \in \mathcal{P} \mid f(P) = a\}$. Take two preferences $P, P' \in \mathcal{P}$ such that $(P, P') \in E$. Suppose $P, P' \in \mathcal{P}_a$. Then, $f(P) = f(P') = a$, and hence f is not manipulable at P via P'.

Suppose $P, P' \in \mathcal{P} \setminus \mathcal{P}_a$. Then, since $f(P'') = \max_{P''}(B)$ for all $P'' \in \mathcal{P} \setminus \mathcal{P}_a$, we have $f(P)Rf(P')$ and $f(P')R'f(P)$, which means f is not manipulable at P via P'. Now, take $P \in \mathcal{P}_a$ and $P' \in \mathcal{P} \setminus \mathcal{P}_a$. Since $(P, P') \in E$ and $P \in \mathcal{P}_a$, by the construction of f, there must exist some $b \in B$ that overtakes a along the edge (P, P'). By the definition of f, this means $f(P') = b$. Because aPb and $bP'a$, this means f is not manipulable at P via P' or at P' via P. ■

B. Proof of Theorem 3.1

Proof. Let $\mathcal{P}$ satisfy maximal restoration property. Then, there exist there exist $\hat{P}, \bar{P} \in \mathcal{P}$ and $a \in A$ such that for every path $\pi(\hat{P}, \bar{P})$ from $\hat{P}$ to $\bar{P}$ has a (a, x) restoration for some $x \in A \setminus a$ with $a\hat{P}x$.

Define $B = \{x \in A \mid a\hat{P}x$ and there exists a path $\pi(\hat{P}, \bar{P})$ having (a, x) restoration$\}$ and define $B_1 = \{x \in B \mid a\bar{P}x\}$. Without loss of generality we can assume that $B_1 \neq \emptyset$.

Suppose that each path from $\hat{P}$ to $\bar{P}$ has a (a, x) restoration for some $x \in B_1$. This in particular means that some element of B_1 overtakes a in each path from $\hat{P}$ to $\bar{P}$. Consider the SCF f that is monotonic with respect to $(a, B_1, \hat{P})$. By Lemma 3.1, f is locally strategy-proof. We show f is not globally strategy-proof. Since some element of B_1 overtakes a in each path from $\hat{P}$ to $\bar{P}$, by the definition of f, we have $f(\bar{P}) \max_{\bar{P}}(B_1)$. Because $f(\hat{P}) = a$ and $a\bar{P}x$ for all $x \in B_1$, this means f is globally manipulable at $\bar{P}$ via $\hat{P}$.

Now, suppose that there exists a path $\pi(\hat{P}, \bar{P})$ that has no (a, x) restoration for any $x \in B_1$. Then, by our assumption, there must $x \in B \setminus B_1$ such that $\pi(\hat{P}, \bar{P})$ has a (a, x) restoration. Let $\tilde{P} \in \pi(\hat{P}, \bar{P})$ be such that the path $\pi(\tilde{P}, \bar{P})$ does not have any (a, x) restoration for $x \in B \setminus B_1$, and there is a (a, b) flip for some $b \in B \setminus B_1$ such that $\pi(\hat{P}, \bar{P})$ has (a, b) restoration. We claim that every path from $\hat{P}$ to $\tilde{P}$ has (a, x) restoration for some $x \in B$. Assume for contradiction that there is a path $\hat{\pi}(\hat{P}, \tilde{P})$ from $\hat{P}$ to $\tilde{P}$ that has no (a, x) restoration for any $x \in B$. Consider the path $(\hat{\pi}(\hat{P}, \tilde{P}), \pi(\tilde{P}, \bar{P}))$ from $\hat{P}$ to $\bar{P}$. By our assumption, there exists $b \in B$ such that this path has a (a, b) restoration. Because both the paths $\hat{\pi}(\hat{P}, \tilde{P})$ and $\pi(\tilde{P}, \bar{P})$ do not have any (a, b) restoration, it must be that there is exactly one (a, b) flip in each of these paths. Suppose $b \in B_1$. Since $\tilde{P} \in \pi(\hat{P}, \bar{P})$ and there is no (a, x) restoration in $\pi(\hat{P}, \bar{P})$ for any $x \in B_1$, we must have $a\tilde{P}x$ for all $x \in B_1$. Because $a\hat{P}x$ for all $x \in B_1$, this means $\pi(\hat{P}, \tilde{P})$ cannot have exactly one (a, x) flip. This contradicts that $b \in B_1$. Now, suppose $b \in B \setminus B_1$. Note that a and b have different relative orderings in $\hat{P}$ and $\bar{P}$. Therefore, in any path from $\hat{P}$ to $\bar{P}$ there must be odd number of flips of a

and b. However, this contradicts the fact that a and b flips exactly once in each paths $\pi(\hat{P},\tilde{P})$ and $\pi(\tilde{P},\bar{P})$ of a and b. This proves that every path from $\hat{P}$ to $\tilde{P}$ has a (a,x) restoration from some $x \in B$.

Let $B_2 = \{x \in B \mid a\tilde{P}x\}$. Since $b \in B$, (a,b) flip occurs exactly once in the path $\pi(\tilde{P},\bar{P})$, and $b\bar{P}a$, it must be that $a\tilde{P}b$. Therefore, it must be that $B_2 \supsetneq B_1$.

Now, if every path from $\hat{P}$ to $\tilde{P}$ has (a,x) restoration for some $x \in B_2$, then by using similar arguments as before f^{B_2} is a rough rule on $\mathcal{P}$. If not, then it must be that $B \setminus B_2 \neq \emptyset$ and there exists a path $\bar{\pi}(\hat{P},\tilde{P})$ from $\hat{P}$ to $\tilde{P}$ that has no (a,x) restoration for any $x \in B_2$. Then, using the assumption on the $\mathcal{P}$, $\pi(\hat{P},\tilde{P})$ must have (a,x) restoration for some $x \in B \setminus B_2$. Using similar argument as before, there must be some $\tilde{\tilde{P}}$ such that (i) every path from $\hat{P}$ to $\tilde{\tilde{P}}$ has (a,x) restoration for some $x \in B$, and (ii) $B_3 = \{x \in B \mid a\tilde{\tilde{P}}x\} \supsetneq B_2$.

Continuing in this manner, we get hold of P' such that (i) every path from $\hat{P}$ to P' has (a,x) restoration for some $x \in B$, and (ii) $aP'x$ for all $x \in B$. Then, the rule f^B is a rough rule on $\mathcal{P}$. This completes the proof of the theorem. ■

C. Proof of Theorem 3.2

Proof. Suppose not. Then, there exist a locally strategy-proof SCF $f : \mathcal{L}_N \to A$, $P_N \in \mathcal{L}_N$, $i \in N$, $\bar{P}_i \in \mathcal{L}_i$ such that f is globally manipulable at P_N via $\bar{P}_i$. We show that there exists a component $k \in A^0$ such that $\mathcal{L}^k_N$ is rough, which will contradict our supposition for contradiction. Since i and P_{N-i} are fixed, to minimize notation, we write P to mean P_i and write $f(P)$ to mean $f(P_i, P_{N-i})$.

Without loss of generality, assume that $\mathcal{L}^0 = \{P^0\}$ where $1P^0 2P^0 \ldots P^0 m$. Because f is globally manipulable at P_N via $\bar{P}_i$, we have (using our reduced notation) $f(\bar{P})Pf(P)$. Let $f(P) = x$ and $f(\bar{P}) = y$.

Step 1. In this step, we show $x^1 = y^1$. Because yPx, it must be that $y^1 R^1 x^1$. Suppose $y^1 P^1 x^1$.

Claim 1. $f^1(P^0, P^1, \hat{P}^{M-1}) = f^1(P^0, P^1, \tilde{P}^{M-1})$ for all $P^1 \in \mathcal{L}^1$ and all $\hat{P}^{M-1}, \tilde{P}^{M-1} \in \mathcal{L}^{M-1}$.

Suppose not. Then, there must exist $(P^0, P^1, \hat{P}^2, \ldots, \hat{P}^k, \tilde{P}^{k+1}, \ldots, \tilde{P}^m)$ and $(P^0, P^1, \hat{P}^2, \ldots, \hat{P}^{k-1}, \tilde{P}^k, \ldots, \tilde{P}^m)$ such that $f(P^0, P^1, \hat{P}^2, \ldots, \hat{P}^k, \tilde{P}^{k+1}, \ldots, \tilde{P}^m) \neq f(P^0, P^1, \hat{P}^2, \ldots, \hat{P}^{k-1}, \tilde{P}^k, \ldots, \tilde{P}^m)$. Let $P(1), \ldots, P(l)$ be such that $P(1) = \hat{P}^k$, $P(l) = \tilde{P}^k$, and $P(j) \sim P(j+1)$ for all $j = 1, \ldots, k-1$. Then, there must be $j \in \{1, \ldots, k-1\}$ such that $f^1(P^0, P^1, \hat{P}^2, \ldots, P(j), \tilde{P}^{k+1}, \ldots, \tilde{P}^m) \neq f^1(P^0, P^1, \hat{P}^2, \ldots, \hat{P}^{k-1}, P(j+$

$1), \dots, \tilde{P}^m)$. However, since in both the preferences $(P^0, P^1, \hat{P}^2, \dots, P(j), \tilde{P}^{k+1}, \dots, \tilde{P}^m)$ and $f^1(P^0, P^1, \hat{P}^2, \dots$ $1), \dots, \tilde{P}^m)$, the marginal preference over component 1 is the same and component 1 is lexicographic best, this means agent 1 will manipulate. This completes the proof of the claim.

Define the SCF $f^1 : \mathcal{L}^1 \to A^1$ as follows: for all $\hat{P}^1$, $f^1(\hat{P}^1) = (f(\hat{P}^1, P^{M-1}))^1$.

Claim 2. f^1 is locally strategy-proof.
Assume for contradiction that there are $\hat{P}^1$ and $\tilde{P}^1$ with $\hat{P}^1 \sim^1 \tilde{P}^1$ (i.e. when $\hat{P}^1$ and $\tilde{P}^1$ are viewed as preferences over A^1, they are adjacent.) such that $f^1(\tilde{P}^1)\hat{P}^1 f^1(\hat{P}^1)$. However, by the definition of f^1 and the fact that component 1 is lexicographic best, this means $f(\tilde{P}^1, P^{M-1})\hat{P} f(\hat{P}^1, P^{M-1})$. As $(\tilde{P}^1, P^{M-1}) \sim (\hat{P}^1, P^{M-1})$, this contradicts the local strategy-proofness of f, which completes the proof of the claim.

Consider $(\bar{P}^1, P^{M-1})$. Because $y^1 P^1 x^1$ and $f^1(P^1) = x^1$, it follows that f^1 is globally manipulable at P^1 via $\bar{P}^1$, which is a contradiction. This completes Step 1.

Step 2. In this step, we show $x = y$. Since yPx, this completes the proof of the theorem by contradiction.

Let $k \in A^0$ be such that $x^l = y^l$ for all $l < k$ and $x^k \neq y^k$. Because yPx, it must be that $y^k P^k x^k$. We distinguish the follows cases.

Case 1. Suppose that there exists a path from P to $\bar{P}$ such that for each P' in the path $f^l(P') = x^l$ for all $l \leq k - 1$.

Claim 3. $f^j(P^0, P^1, P^2, \dots, P^l, \hat{P}^{M-\{1,\dots,l\}}) = f^j(P^0, P^1, \dots, P^l, \tilde{P}^{M-\{1,\dots,l\}})$ for all $j \in 1, 2, \dots, l$, all $\hat{P}^{M-\{1,\dots,l\}}$, and all $\tilde{P}^{M-\{1,\dots,l\}}$.
The proof of this claim follows by using similar arguments as for the proof of Claim 1.

In what follows, we construct a rough SCF for the k-th component. Define $f^k(\tilde{P}^k) = f(P^1, ...P^{k-1}, \tilde{P}^k, P$ for all $\tilde{P}^k \in \mathcal{L}_i^k$. It follows from arguments similar to Claim2 that f^k is locally strategy-proof. However, note that since $f^l(P) = x^l$ for all $l \leq k - 1$ and $y^k P^k x^k$, it follows by using similar argument as for the proof of f^1 is globally manipulable in Step 1, that f^k is not globally strategy-proof. This contradicts our assumption that $\mathcal{L}_i^k$ is a smooth domain.

Case 2. Suppose Case 1 does not hold. Let $\bar{k} < k - 1$ be the maximum component such that there is a path $\pi(P, \bar{P})$ from P to $\bar{P}$ such that $f^l(P') = x^l$ for all $l \leq \bar{k}$ and all $P' \in \pi(P, \bar{P})$. Such a $\bar{k}$ must exist since $f^1(P') = x^1$ for all P' that lies in any path from P to $\bar{P}$.

Take $\tilde{P} \in \pi(P, \bar{P})$ such that $\tilde{P}^l = \bar{P}^l$ for all $l \leq \bar{k}$. By our assumption, $f^l(\tilde{P}) = x^l$ for all $l \leq \bar{k}$.

So by Claim 3 $f^l(\bar{P}^1, \ldots, \bar{P}^{\bar{k}}, P^{\bar{k}+1}, \ldots, P^m) = x^l$ for all $l \leq \bar{k}$. Suppose not. Consider the shortest path from $(\bar{P}^1, \ldots, \bar{P}^{\bar{k}}, P^{\bar{k}+1}, \ldots, P^m)$ to $(\bar{P}^1, \ldots, \bar{P}^{\bar{k}}, \tilde{P}^{\bar{k}+1}, \ldots, \tilde{P}^m)$. Because $\hat{P}^l = P^l$ for all $l \leq \bar{k}$ and all preference $\hat{P}$ in this path, we have by Claim 3 that $f^l(\hat{P}) = x^l$ for all $l \leq \bar{k}$. Since $f^{\bar{k}+1}(\bar{P}^1, \ldots, \bar{P}^{\bar{k}}, P^{\bar{k}+1}, \ldots, P^m) \neq f^{\bar{k}+1}(\bar{P}^1, \ldots, \bar{P}^{\bar{k}}, \tilde{P}^{\bar{k}+1}, \ldots, \tilde{P}^m)$, this means f is locally manipulable, which is a contradiction. This completes the proof of the claim.

Now, we complete the proof of Step 2. Consider all paths from $f(\bar{P}^1, \ldots, \bar{P}^{\bar{k}}, P^{\bar{k}+1}, \ldots, P^m) = x^l$ to $\bar{P}$. By our assumption on $\bar{k}$, in every such path $x^{\bar{k}+1}$ must have a restoration with some $z^{\bar{k}+1}$. If $z^{\bar{k}+1} P^{\bar{k}+1} x^{\bar{k}+1}$, then f is locally manipulable. Therefore, it must be that in every such path $x^{\bar{k}+1}$ has a restoration with some $z^{\bar{k}+1}$ such that $x^{\bar{k}+1} P^{\bar{k}+1} z^{\bar{k}+1}$. However, by Theorem 3.1, this means $\mathcal{L}_i^{\bar{k}+1}$ is not smooth, a contradiction. This completes Step 2, and hence the proof of the Theorem. ∎

D. Proof of Theorem 3.3

Proof. Suppose not. Then $\mathcal{D}$ is a rough domain.So we have a rule $f : \mathcal{D} \to A$ which is locally strategy proof but globally manipulable. So there exist P, P' such that $f(P')Pf(P)$.Let $f(P) = x$ and $f(P') = x'$. Also there exist a path between P and P' which has no $\{x', z\}$ restoration for all $z \in A$ such that $x'Pz$. Let this path be denoted by $(P^1 = P, P^2, \ldots, p^l = P')$ Now looking along this path, while going from P to P', the outcome has to change from x to x'.Let P^k, where $k \in \{1, 2, \ldots, l\}$ be the first preference in the path $(P^1 = P, P^2 \ldots, p^l = P')$, where the outcome is x'.Let $f(P^{k-1}) = y$. Then since f is locally strategy proof, we have $x'P^k y$ and $yP^{k-1}x'$

Claim: yPx'

Proof: Suppose not. Then $x'Py$ as $x \neq y$. Also we have $yP^{k-1}x'$ and $x'P^k y$. This means the path $(P^1 = P, P^2, \ldots, p^l = P')$ has $\{x', y\}$ restoration. Hence a contradiction. So yPx'.

Now there exist a path from P to P^k which has no $\{y, z\}$ restoration for all $z \in A$ such that yPz.Then following the same technique as before and doing it repeatedly untill we get $\bar{P}$ such that $f(\bar{P}) = r_1(P)$. Now according to the property of the domain, there exist a path from P to $\bar{P}$ which has no $\{r_1(P), z\}$ restoration for all $z \in A - r_1(P)$.Since f is locally strategy proof, so any any preference the outcome is same as that of its neighbour preference or the outcome overtakes the outcome at its neighbour preference. So Since $f(\bar{P}) = r_1(P)$, this a contradiction tothere exist a path from P to $\bar{P}$ which has no $\{r_1(P), z\}$ restoration for all $z \in A - r_1(P)$. This proves $\mathcal{D}$ is a smooth domain. ∎

E. PROOF OF THEOREM 3.4

Proof. We will show that for each $P, P' \in \mathcal{D}$ and each $a \in A$, there exist a path in $\mathcal{D}$ between P and P' which is without $\{a, z\}$ restoration for all $z \in A$ such that aPz and hence by Theorem 2.3, it follows that $\mathcal{D}$ is a smooth domain.

Let $(P^0, P^1, P^2, \ldots, P^m), (\bar{P}^0, \bar{P}^1, \bar{P}^2, \ldots, \bar{P}^m)$ be any preferences and a be any alternative in A.Let us denote $a = a_1 a_2 \ldots, a_m$ Without loss of generality we can assume that P^0 is such that $1P^0 2P^0 \ldots P^0 m$ holds.Now we look at P^1, we construct a path $\pi^1(P^1, \tilde{P}^1)$ from P^1 to $\tilde{P}^1$ such that starting from P^1, a_1 overtakes its adjacent alternative which lies above it one by one. In this process once a_1 reaches the top, then we arrange the remaining alternatives in that preference according to their relative positions amongst themselves in the preference $\bar{P}^1$. After this, we get a preference which we call $\tilde{P}^1$. So the path $\pi^1(P^1, \tilde{P}^1)$ consists of preferences that is adjacent to the ones on their right and left of the path. Notice that since according to P^0, 1 is lexicographically the best, so along the path from $(P^0, P^1, P^2, \ldots, P^m)$ to $(P^0, \tilde{P}^1, P^2, \ldots, P^m)$ a has no flips with any alternative which was below in the preference $(P^0, P^1, P^2, \ldots, P^m)$.

Now we repeat the process for 2nd component and then 3rd till mth component. So throughout this path, a goes to the top by overtaking alternatives that were above in the prefernece $(P^0, P^1, P^2, \ldots, P^m)$. Now along the path $(P^0, \tilde{P}^1, \tilde{P}^2, \ldots, \tilde{P}^m)$ to $(\bar{P}^0, \tilde{P}^1, \tilde{P}^2, \ldots, \tilde{P}^m)$ a remains on top throughout.Since a_i is the top of $\tilde{P}^i$ and $\tilde{P}^i$ is arranged such that apart from a_i other alternatives are arranged in the same relative positions according to $\bar{P}^i$. Consider a path from $\tilde{P}^i$ to $\bar{P}^i$ by bringing a_i down one by one.Now we look at $\bar{P}^0$, whichever is the lexicographically best according to $\bar{P}^0$, say j, then first we construct a path from $(\bar{P}^0, \tilde{P}^1, ..., \tilde{P}^j, ..\tilde{P}^m)$ to $(\bar{P}^0, \tilde{P}^1, ..., \bar{P}^j, ..\tilde{P}^m)$. So along this path, a is overtaken by other altenatives only once.Similarly for all components. So in the path from $(\bar{P}^0, \tilde{P}^1, \ldots, \tilde{P}^m)$ to $(\bar{P}^0, \bar{P}^1, \ldots, \bar{P}^m)$, a flips with some alternatives only once. Hence in the entire path from $(P^0, P^1, P^2, \ldots, P^m)$ to $(\bar{P}^0, \bar{P}^1, \bar{P}^2, \ldots, \bar{P}^m)$, a has no restoration with any alternative below. Hence by Theorem 3.3, $\mathcal{D}$ is a smooth domain. ■

F. PROOF OF THEOREM 3.5

Proof. We once again use Theorem 2.3 to show that $\mathcal{D}$ is smooth.

Let $(P^0, P^1, P^2, \ldots, P^m), (\bar{P}^0, \bar{P}^1, \bar{P}^2, \ldots, \bar{P}^m)$ be any preferences and a be any alternative in A.Let us denote $a = a_1 a_2 \ldots, a_m$ Without loss of generality we can assume that P^0 is such that

$1P^02P^0\dots P^0m$ holds.Let $\tilde{P}^i$ be the preference whose top is a_i for all $i \in \{1,2,\dots m\}$. Since $\mathcal{P}^i$'s are single peaked so there exist a path from P^i to $\tilde{P}^i$ without restoration.(Here we have used the fact that for any two preferneces in a single peaked domain, there exist a path between them without restoration.). Also there exist a path from $\tilde{P}^i$ to $\bar{P}^i$ without restoration.

Now we look at P^1, we have a path $\pi^1(P^1,\tilde{P}^1)$ without restoration from P^1 to $\tilde{P}^1$.Notice that since according to P^0, 1 is lexicographially the best, so along the path from $(P^0,P^1,P^2,\dots,P^m)$ to $(P^0,\tilde{P}^1,P^2,\dots,P^m)$ a has no flips with any alternative which was below in the preference $(P^0,P^1,P^2,\dots,P^m)$.Now we repeat the process for 2nd component and then 3rd till mth component. So throughout this path, a goes to the top by overtaking alternatives that were above in the prefernece $(P^0,P^1,P^2,\dots,P^m)$. Now along the path $(P^0,\tilde{P}^1,\tilde{P}^2,\dots,\tilde{P}^m)$ to $(\bar{P}^0,\tilde{P}^1,\tilde{P}^2,\dots,\tilde{P}^m)$, a remains on top throughout.

Now also there exist a path from $\tilde{P}^i$ to $\bar{P}^i$ without restoration for all i. Now we look at $\bar{P}^0$, whichever is the lexicographically best according to $\bar{P}^0$, say j, then first we construct a path from $(\bar{P}^0,\tilde{P}^1,...,\tilde{P}^j,..\tilde{P}^m)$ to $(\bar{P}^0,\tilde{P}^1,...,\bar{P}^j,..\tilde{P}^m)$. So along this path, a is overtaken by other altenatives only once.Similarly for all components. So in the path from $(\bar{P}^0,\tilde{P}^1,\dots,\tilde{P}^m)$ to $(\bar{P}^0,\bar{P}^1,\dots,\bar{P}^m)$, a flips with some alternatives only once. Hence in the entire path from $(P^0,P^1,P^2,\dots,P^m)$ to $(\bar{P}^0,\bar{P}^1,\bar{P}^2,\dots,\bar{P}^m)$, a has no restoration with any alternative below. Hence by Theorem 3.3, $\mathcal{D}$ is a smooth domain. ■

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