On the Equivalence of Ordinal Bayesian Incentive Compatibility and Dominant Strategy Incentive Compatibility for Random Rules

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Abstract

We study random voting mechanisms and establish an equivalence relation between ordinal Bayesian incentive compatible (OBIC) random mechanisms and dominant strategy incentive compatible (DSIC) random mechanisms. We show that if a random social choice function on a domain is lower contour monotonic and locally OBIC with respect to strict generic priors, then it is also locally dominant strategy incentive compatible. Strict generic priors are strict subsets of generic priors as defined in Mishra (2016). The Lebesgue measure of the set of such priors is 1, in other words, every prior is almost surely strictly generic. We further show that under OBIC with strict generic priors, unanimity implies lower contour monotonicity on unrestricted domains. It follows from our results that *almost* every (with probability 1) OBIC and elementary monotonic (as defined Mishra (2016)) random rule on unrestricted, single-peaked, single-crossing, single-dipped domain is DSIC. Further, if one defines appropriate notion of locality for multi-dimensional separable domains, the same holds for such domains when the marginals are unrestricted, single-peaked, single-crossing, single-dipped. Thus, our result generalizes the result in Mishra (2016) in two ways: (i) by considering random rules, and (ii) by allowing domains that are not connected through adjacent locality.

1. Introduction

In this paper, we consider two notions of strategic manipulation, namely, ordinal Bayesian incentive compatibility (OBIC) and dominant strategy incentive compatibility (DSIC) and study their equivalence in case of random rules. We say that a voting mechanism is OBIC if for every agent, his expected outcome probability vector from truth-telling first-order stochastically-dominates any expected outcome probability vector obtained by misreporting. Similarly, a voting mechanism satisfies DSIC if for every agent, outcome probability vector from truth-telling first-order stochastically-dominates any outcome probability vector by deviating for every given collection of preferences of other agents. d'Aspremont and Peleg (1988) introduce the notion of OBIC which is weaker than that of DSIC as they claim that the latter notion is "very restrictive". This is demonstrated by the classical Gibbard-Satterthwaite theorem (Gibbard (1973) and Satterthwaite (1975)) which states that dictatorship is the only DSIC and unanimous voting mechanism in the unrestricted domain. Majumdar and Sen (2004) shows that if the domain of preferences under consideration is unrestricted, then for priors that are generic in the set of independent beliefs, a social choice function is OBIC only if it is DSIC. In case of deterministic voting mechanisms, the equivalence of OBIC with generic and independent beliefs and DSIC for a large class of preference domains (including the unrestricted domain, the single-peaked domain, the single-dipped domain, and some single-crossing domains) is established in Mishra (2016). He shows that if deterministic voting mechanisms are OBIC for generic and independent priors and also satisfy an additional condition called elementary monotonicity, then they are also DSIC. If we consider two preferences where only a pair of adjacent alternatives have switched their positions, then elementary monotonicity says that the outcome probability of the alternative which after the swap is at a relatively better position must not fall.

In this paper we establish an equivalence relationship between OBIC and DSIC in case of random social choice functions (RSCFs). In case of random voting mechanisms, genericity of priors and elementary monotonicity do not guarantee the equivalence between OBIC and DSIC. We show that an RSCF, φ that satisfies elementary monotonicity and is OBIC with generic for φ priors is also DSIC for a large class of domains. We consider agents can manipulate only via adjacent preferences and consider rules to prevent such manipulations. As in the existing literature, we use the terms locally OBIC (LOBIC) and locally DSIC (LDSIC) to refer to this weakened notion of OBIC and DSIC respectively. Gabriel (2012) and Sato (2013) identify domains where local incentive compatibility is equivalent to incentive compatibility. Such domains include the unrestricted domain, single-peaked, single-dipped and many such domains. Thus, for all these domains we establish an equivalence relationship between OBIC and DSIC by drawing an equivalence relationship between LOBIC and LDSIC.

We further show that in case of unrestricted domain an RSCF, φ that is unanimous and OBIC with generic for φ priors also satisfies elementary monotonicity. Hence, under unrestricted domain, an RSCF, φ that is unanimous and OBIC with generic for φ priors is DSIC. This result can be viewed as an extension of Majumdar and Sen (2004).

2. Preliminaries

Let A be the set of alternatives and let $\mathcal{D}$ be an arbitrary domain. Let $G = \langle \mathcal{D}, E \rangle$ be a graph over $\mathcal{D}$. Throughout this paper, we assume that G is arbitrary but fixed.

For $P, \bar{P} \in \mathcal{D}$ such that $(P, \bar{P}) \in E$, we define $\delta(P, \bar{P}) = \{(a,b) \mid aPb \text{ and } b\bar{P}a\}$ as the set of (ordered) pairs of alternatives that change their relative ordering from P to $\bar{P}$.

Definition 2.1. A random social choice function (RSCF) is a map $\varphi : \mathcal{D}_N \to \Delta A$.

Definition 2.2. For an agent i, a preference $P_i \in \mathcal{D}_i$, a prior μ_i of i, and an RSCF $\varphi : \mathcal{D}_N \to \Delta A$, define interim expected outcome of agent i, denote by $\varphi_\mu(P_i)$, as a probability distribution over A such that for all $a \in A$,

$$\varphi_\mu(P_i)(a) = \sum_{P_{-i} \in \mathcal{D}_{-i}} \mu(P_{-i}|P_i)\varphi_a(P_i, P_{-i}).$$

For $P_i \in \mathcal{D}_i$ and $a \in A$, we define $U(a, P_i) = \{b \in A \mid bR_i a\}$. For a preference P, by $P(k)$ we denote the k-th rank alternative, that is $P(k) = a$ if and only if $|\{b \in A \mid bPa\}| = k - 1$.

Definition 2.3. For given $P_i \in \mathcal{D}_i$ and $\nu, \nu' \in \Delta A$, ν is said to stochastically dominate ν', denoted by $\nu R_i^{sd} \nu'$, if $\nu(U(a, P_i)) \geq \nu'(U(a, P_i))$ for all $a \in A$.

Definition 2.4. An RSCF $\varphi : \mathcal{D}_N \to \Delta A$ is locally dominant strategy incentive compatible (LDSIC) if for every $i \in N$, every $P_i, P_i' \in \mathcal{D}_i$ such that $(P_i, P_i') \in E$, and every $P_{-i} \in \mathcal{D}_{-i}$,

$$\varphi(P_i, P_{-i}) R_i^{sd} \varphi(P_i', P_{-i}), \tag{1}$$

and is dominant strategy incentive compatible (DSIC) if (1) holds for all $P_i, P_i' \in \mathcal{D}_i$.

Definition 2.5. An RSCF $\varphi : \mathcal{D}_N \to \Delta A$ is locally ordinal Bayesian incentive compatible (LOBIC) with respect to a profile of priors $(\mu_i)_{i \in N}$ if for every $i \in N$, every $P_i, P_i' \in \mathcal{D}_i$ such that $(P_i, P_i') \in E$, we have

$$\varphi_\mu(P_i) R_i^{sd} \varphi_\mu(P_i'), \tag{2}$$

and is ordinal Bayesian incentive compatible (OBIC) with respect to a profile of priors $\{\mu_i\}_{i\in N}$ if (2) holds for all $P_i, P_i' \in \mathcal{D}_i$.

Definition 2.6. For an RSCF $\varphi : \mathcal{D}_N \to \Delta A$, a profile of priors $(\mu_i)_{i\in N}$ is called generic for φ if for all $i \in N$, all $P_i, P_i' \in \mathcal{D}_i$, and all $a \in A$,

$$\sum_{P_{-i}} \mu(P_{-i}|P_i)(\varphi_a(P_i, P_{-i}) - \varphi_a(P_i', P_{-i})) = 0$$

implies $\varphi_a(P_i, P_{-i}) - \varphi_a(P_i', P_{-i}) = 0$ for all P_{-i}.

Note that if μ_i is generic, then μ_i is generic for every deterministic SCF f.

Definition 2.7. An RSCF $\varphi : \mathcal{D}_N \to \Delta A$ is called elementary monotonic if for all $i \in N$, all $P_i, P_i' \in \mathcal{D}_i$ with $(P_i, P_i') \in E$, and all $P_{-i} \in \mathcal{D}_{-i}$, $\varphi_a(P_N) \geq \varphi_a(P_i', P_{-i})$ for all $a \in A$ such that $(x, a) \in \delta(P_i, P_i')$ for some $x \in A$.

For $P, P' \in \mathbb{L}(A)$, we define $P \triangledown P'$ as the set of alternatives that change their relative ordering with some alternative from P to P', that is, $P \triangledown P' = \{x \mid \text{ there exists } y \in A \text{ such that } [xPy \text{ and } yP'x] \text{ or } [yPx \text{ and } xP'y]\}$.

Definition 2.8. An RSCF $\varphi : \mathcal{D}_N \to \Delta A$ satisfies lower contour monotonicity if for every $i \in N$, for every $P_i, P_i' \in \mathcal{D}$, we have for every $P_{-i} \in \mathcal{D}_{-i}$,

$$\varphi_L(P_N) \leq \varphi_L(P_i', P_{-i})$$

for all strict lower contour sets L of P restricted to $P \triangledown P'$.

3. Results

Theorem 3.1. *Let $\varphi : \mathcal{D}_N \to \Delta A$ satisfy lower contour monotonicity and let $(\mu_i)_{i\in N}$ be strictly generic for φ. Then, φ is LOBIC with respect to $(\mu_i)_{i\in N}$ implies φ is LDSIC.*

The proof of Theorem 3.1 is relegated to Appendix A.

Theorem 3.1 together with the fact that strictly generic priors have measure 1 yields the following corollary.

Corollary 3.1. *Almost every lower contour monotonic and LOBIC rule is LDSIC.*

Theorem 3.2. *Suppose $|A| \geq 3$ and $\varphi : \mathbb{L}(A)^n \to \Delta A$ is LOBIC with respect to $(\mu_i)_{i \in N}$ where $(\mu_i)_{i \in N}$ is strictly generic for φ. If φ satisfies unanimity, then it satisfies elementary monotonicity.*

The proof of Theorem 3.2 is relegated to Appendix B.

Corollary 3.2. *Almost every unanimous and OBIC random rule on the unrestricted domain is random dictatorial.*

4. Conclusion

In this paper we have shown that when priors are strictly generic for φ, then any RSCF which is OBIC and satisfies lower contour monotonicity is also LDSIC for a large class of domains including the unrestricted domain, single-peaked domain, single-crossing domain, single-dipped domain, etc. Also the requirement on priors to be strictly generic for the RSCF is rather weak as such priors occur with probability 1. Thus, our result establishes an almost sure equivalence between the OBIC and the DSIC random rules, and thereby they generalize the results in Mishra (2016) and Majumdar and Sen (2004).

A. Proof of Theorem 3.1

Definition A.1. An RSCF $\varphi : \mathcal{D}_N \to \Delta A$ satisfies block monotonicity if for every $i \in N$, for every $P_i, P_i' \in \mathcal{D}$, where $(P_i, P_i') \in E$, we have for every $P_{-i} \in \mathcal{D}_{-i}$,

$$\varphi_x(P_N) = \varphi_x(P_i', P_{-i})$$

for all $x \notin P \triangledown P'$

To prove Theorem 3.1, we first prove the following Lemma.

Lemma A.1. *Let $\varphi : \mathcal{D}_N \to \Delta A$ be an RSCF and let $(\mu_i)_{i \in N}$ be generic for φ. If φ is LOBIC with respect to $(\mu_i)_{i \in N}$, then it must satisfy block monotonicity.*

Proof. Let $\varphi : \mathcal{D}_N \to \Delta A$ be an RSCF and let $(\mu_i)_{i \in N}$ be generic for φ. Further, let φ be LOBIC with respect to $(\mu_i)_{i \in N}$. We show that φ satisfies block monotonicity. Consider an agent $i \in N$ and two preferences P_i and P_i' be such that P_i' is an (a, b) swap of P_i. By definition $P_i(k) = a$, $P_i(k+1) = b$ and $P_i'(k) = b$, $P_i'(k+1) = a$ for some k and $P_i(l) = P_i'(l)$ for all $l \notin \{k, k+1\}$.

Take $x \in A \setminus \{a,b\}$ such that $P_i(k') = P'_i(k') = x$ where $k' < k$. We show that $\varphi_x(P_i, P_{-i}) = \varphi_x(P'_i, P_{-i})$ for all $P_{-i} \in \mathcal{D}_i$. If $k' = 1$, as $P_i(k'') = P'_i(k'')$ for all $k'' < k$, LOBIC implies that

$$\sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{P_i(1)}(P_i, P_{-i}) \geq \sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{P_i(1)}(P'_i, P_{-i}) \tag{3}$$

and

$$\sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{P_i(1)}(P_i, P_{-i}) \geq \sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{P_i(1)}(P'_i, P_{-i}) \tag{4}$$

Combining (3) and (4), we get for all $P_{-i} \in \mathcal{D}_{-i}$

$$\sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{P_i(1)}(P_i, P_{-i}) = \sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{P_i(1)}(P'_i, P_{-i}).$$

As $(\mu_i)_{i\in N}$ is generic for φ, this means $\varphi_{P_i(1)}(P_i, P_{-i}) = \varphi_{P_i(1)}(P'_i, P_{-i})$ for all $P_{-i} \in \mathcal{D}_{-i}$.

Now suppose the claim is true for all $k'' < k'$. Note that $U(x, P_i) = U(x, P'_i)$. Applying LOBIC to the top k' alternatives in P_i and P'_i we get

$$\sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{U(x,P_i)}(P_i, P_{-i}) \geq \sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{U(x,P_i)}(P'_i, P_{-i})$$

and

$$\sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{U(x,P'_i)}(P_i, P_{-i}) \geq \sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{U(x,P'_i)}(P'_i, P_{-i})$$

Combining the above inequalities, we have

$$\sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{U(x,P'_i)}(P_i, P_{-i}) = \sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{U(x,P'_i)}(P'_i, P_{-i}).$$

Using the induction hypothesis, we have for all $k'' < k'$, $\varphi_{P_i(k'')}(P_i, P_{-i}) = \varphi_{P_i(k'')}(P'_i, P_{-i})$ for all $P_{-i} \in \mathcal{D}_{-i}$.

Hence, we get

$$\varphi_x(P_i, P_{-i}) = \varphi_x(P'_i, P_{-i}) \text{ for all } P_{-i} \in \mathcal{D}_{-i}.$$

Next we show that $\varphi_{\{a,b\}}(P_i, P_{-i}) = \varphi_{\{a,b\}}(P'_i, P_{-i})$ for all $P_{-i} \in \mathcal{D}_{-i}$. Applying LOBIC, we get

$$\sum_{P_{-i}} \mu(P_{-i}|P_i)(\varphi_{U(b,P_i)}(P_i, P_{-i})) \geq \sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{U(b,P_i)}(P'_i, P_{-i}))$$

and

$$\sum_{P_{-i}} \mu(P_{-i}|P_i)(\varphi_{U(a,P_i')}(P_i, P_{-i})) \geq \sum_{P_{-i}} \mu(P_{-i}|P_i)\varphi_{U(a,P_i')}(P_i', P_{-i}))$$

As $U(b, P_i) = U(a, P_i')$, we get

$$\varphi_{\{a,b\}}(P_i, P_{-i}) = \varphi_{\{a,b\}}(P_i', P_{-i}) \text{ for all } P_{-i} \in \mathcal{D}_{-i}.$$

Finally, consider an alternative $x \in A \setminus \{a, b\}$ such that $P_i(k') = P_i'(k') = x$ where $k' > k + 1$. Using the same argument as in the case when $k' < k$, we obtain

$$\varphi_x(P_i, P_{-i}) = \varphi_x(P_i', P_{-i}) \text{ for all } P_{-i} \in \mathcal{D}_{-i}.$$

Thus, for every $P_{-i} \in \mathcal{D}_{-i}$, we have $\varphi_x(P_i, P_{-i}) = \varphi_x(P_i', P_{-i})$ for all $x \notin \{a, b\}$. ∎

Proof of Theorem 3.1. Let $\varphi : \mathcal{D}_N \to \Delta A$ be an RSCF satisfying local contour monotonicity and let $(\mu_i)_{i\in N}$ be generic for φ. Further, let φ be LOBIC with respect to $(\mu_i)_{i\in N}$. We show that φ is also LDSIC. By Lemma A.1, we know that φ satisfies block monotonicity. Take $i \in N$,$P_{-i} \in \mathcal{D}_{-i}$ and $P_i, P_i' \in \mathcal{D}_i$ such that P_i' is an (a, b)-swap of P_i, where $P_i(k) = a$, $P_i(k + 1) = b$ and $P_i'(k) = b$, $P_i'(k + 1) = a$ for some k. Let $P_i(k') = x$ for some $1 \leq k' \leq |A|$. If $k' < k$ or $k' > k + 1$ then $U(x, P_i) = U(x, P_i') = U(x)$. By block monotonicity we have $\varphi_{U(x)}(P_i, P_{-i}) = \varphi_{U(x)}(P_i', P_{-i})$. Thus, agent i cannot manipulate at (P_i, P_{-i}) via P_i' or at (P_i', P_{-i}) via P_i. If $k' = k$, then $x = a$ and by block monotonicity $\varphi_{U(P_i(k-1),P_i)}(P_i, P_{-i}) = \varphi_{U(P_i(k-1),P_i)}(P_i', P_{-i})$. Moreover, as φ satisfies local contour monotonicity, $\varphi_a(P_i, P_{-i}) \geq \varphi_a(P_i', P_{-i})$. This means $\varphi_{U(a,P_i)}(P_i, P_{-i}) \geq \varphi_{U(a,P_i)}(P_i', P_{-i})$. Thus, i cannot manipulate at (P_i, P_{-i}) via P_i'. Now, take $k' = k + 1$. In this case $x = b$. By block monotonicity $\varphi_{\{a,b\}}(P_i, P_{-i}) = \varphi_{\{a,b\}}(P_i', P_{-i})$. As P_i' is an $(a, b) - swap$ of P_i, we have $U(P_i(k - 1), P_i) = U(P_i'(k - 1), P_i')$ and $U(P_i(k'), P_i) = U(P_i'(k'), P_i') = U(P_i(k - 1), P_i) \cup \{a, b\} = U(P_i'(k - 1), P_i') \cup \{a, b\}$. Hence, $\varphi_{U(x,P_i)}(P_i, P_{-i}) = \varphi_{U(x,P_i)}(P_i', P_{-i})$. This means i cannot manipulate at (P_i, P_{-i}) via P_i'. This completes the proof of the theorem. ∎

B. PROOF OF THEOREM 3.2

First we show that an RSCF φ that is unanimous and LOBIC with respect to $(\mu_i)_{i\in N}$, then φ is Pareto efficient.

Lemma B.1. *Let $\varphi : \mathbb{L}(A)^n \to \Delta A$ be an RSCF and let $(\mu_i)_{i\in N}$ be generic for φ. Then, if φ is unanimous and LOBIC with respect to $(\mu_i)_{i\in N}$, then φ is Pareto efficient.*

Proof. Let $\varphi : \mathbb{L}(A)^n \to \Delta A$ be unanimous and LOBIC with respect to $(\mu_i)_{i\in N}$ where $(\mu_i)_{i\in N}$ is generic for φ. We show that φ is Pareto efficient. Assume for contradiction that it is not Pareto efficient. For this, we consider a profile $P_N \in \mathbb{L}(A)^n$ such that $\varphi_b(P_N) > 0$ and there exists $a \in A$ such that aP_ib for all $i \in N$. Consider an agent $i \in N$ such that $P_i(k) = a$ and $k \neq 1$. Suppose $P_i(k-1) = x$. Consider $P'_i \in \mathbb{L}(A)$ such that P'_i is an (x, a)-swap of P_i. By Lemma A.1, we know that φ satisfies block monotonicity. This means $\varphi_b(P'_i, P_{-i}) > 0$. Similarly, by such repeated swaps we reach a preference P''_i for agent i such that $P''_i(1) = a$ and $\varphi_b(P''_i, P_{-i}) > 0$. Note that as aP_ib such swaps are independent of b. Now we can repeat this procedure for every agent j such that $P_j(k) = a$ and $k \neq 1$ and arrive at a profile P''_N such that $P''_i(1) = a$ for all $i \in N$ and $\varphi_a(P''_N) < 1$. This contradicts the fact the φ is unanimous. ■

Proof of Theorem 3.2. Let $\varphi : \mathbb{L}(A)^n \to \Delta A$ be unanimous and LOBIC with respect to $(\mu_i)_{i\in N}$ where $(\mu_i)_{i\in N}$ is generic for φ. By Lemma B.1, φ is Pareto efficient and by Lemma A.1 φ satisfies block monotonicity. We show that φ satisfies elementary monotonicity. Consider an agent $i \in N$, a preference profile $P_{-i} \in \mathcal{D}_{-i}$ of other agents, and $P_i, \bar{P}_i \in \mathcal{D}_i$ such that $\bar{P}_i$ is an (a, b)-swap of P_i. Notice that there are some agents in P_{-i} who prefer a to b and some who prefer b to a. We show that $\varphi_b(P_i, P_{-i}) = \varphi_b(\bar{P}_i, P_{-i})$.

By block monotonicity, $\varphi_{\{a,b\}}(P_i, P_{-i}) = \varphi_{\{a,b\}}(\bar{P}_i, P_{-i})$. Assume for contradiction that $\varphi_b(P_i, P_{-i}) > \varphi_b(\bar{P}_i, P_{-i})$. We carry out the proof in the following steps.

P^1_N			
...	...	...	...
...	...	...	...
a	b	a	x
...	...	...	b
...	...	...	...
b	a	x	...
x	x	b	a
...	...	...	...
...	...	...	...

Table 1

Step 1: We modify the profile (P_i, P_{-i}) to bring one of the alternatives not in $\{a, b\}$ just below

$\{a, b\}$ for all agents. Let $x \notin \{a, b\}$ be some alternative. If aP_jx and bP_jx for some $j \in N$, then we can do a series of swaps to lift x up such that it is just below b if aP_jb or just below a if bP_ja (note that none of these swaps will involve b). By block monotonicity, the probability that the outcome at the new profile is b is unchanged. Using a similar argument, if bP_jx and xP_ja for some $j \in M$, then we can come to a preference where x is just below a maintaining the probability of the outcome being b.

Now, consider $j \in N$, such that xP_jb. If x and b are not consecutive in P_j, then again we can do a series of swaps to come to a preference such that x is just above b (note again that none of these swaps will involve b). Let us denote this new profile by P^1_N. By block monotonicity, $\varphi_b(P^1_N) = \varphi_b(P_i, P_{-i})$. So, we have reached a profile P^1_N, where for every $j \in N$, either x is just above b in P^1_j or [x is just below b if aP^1_jb and x is just below a if bP^1_ja]. Table 1 shows the profile P^1.

Consider $P^2_j \in \mathcal{D}_j$ such that P^2_j is (x, b)-swap of P^1_j for every $j \in N$ such that x is just above b in P^1_j(Columns 3 and 4 in Table 1) and $P^2_j \in \mathcal{D}_j$ such that $P^2_j = P^1_j$ for others. By block monotonicity $\varphi_{\{x,b\}}(P^2_N) = \varphi_{\{x,b\}}(P^1_N)$. But b is preferred to x by all the agents, and hence, Pareto efficiency implies that $\varphi_b(P^2_N) = \varphi_{\{x,b\}}(P^1_N)$. For every j belonging to Column 4 in Table 1, we then do a sequence of swaps to get x just below a. Denote this new preference by P^3_j and this new profile by P^3_N. By block monotonicity, $\varphi_b(P^3_N) = \varphi_b(P^2_N)$.

Now, consider the (a, b)-swap of P^3_i and denote this preference as $\bar{P}^3_i$. By an analogous argument $\varphi_b(\bar{P}^3_i, P^3_{-i}) = \varphi_{\{x,b\}}(\bar{P}_i, P_{-i})$. This means, $\varphi_b(\bar{P}^3_i, P^3_{-i}) < \varphi_b(P^3_i, P^3_{-i})$ The two profiles $(\bar{P}^3_i, P^3_{-i})$ and (P^3_i, P^3_{-i}) are shown in Table 2.

P^3_i	P^3_{-i}		$\bar{P}^3_i$	P^3_{-i}	
.	$\dots$	$\dots$	.	$\dots$	$\dots$
.	$\dots$	$\dots$	.	$\dots$	$\dots$
a	$b\dots$	$a\dots$	b	$b\dots$	$a\dots$
b	$\dots$	$\dots$	a	$\dots$	$\dots$
x	$a\dots$	$b\dots$	x	$a\dots$	$b\dots$
.	$x\dots$	$x\dots$	.	$x\dots$	$x\dots$
.	$\dots$	$\dots$	.	$\dots$	$\dots$

Table 2

Step 2: In this step, we modify the profile $(\bar{P}^3_i, P^3_{-i})$ in a particular way. First, we look at an agent $j \in N$, such that $aP^3_j b$ and $bP^3_j x$. Consider the profile P^4_N such that P^4_j is (b, x)-swap of P^3_j for each of these agents and $P^4_j = P^3_j$ otherwise. The new profile is shown in Table 3. By block monotonicity, $\varphi_{\{b,x\}}(P^4_N) = \varphi_{\{b,x\}}(P^3_N)$. But since a is ranked higher than x for all the agents, Pareto efficiency implies $\varphi_b(P^4_N) = \varphi_{\{b,x\}}(P^3_N)$.

Agent i	Other	agents
.	$\dots$	$\dots$
.	$\dots$	$\dots$
a	$b\dots$	$a\dots$
x	$\dots$	$\dots$
b	$a\dots$	$x\dots$
.	$x\dots$	$b\dots$
.	$\dots$	$\dots$

Table 3: New profile in Step 2 (P^4_N)

Step 3: In this step, we modify the profile in Table 3 further. In particular, we lift x just above a. For agent i and for all $j \neq i$ such that x is just below a, this can be done by a (a, x)-swap. For all other agents, this requires a series of swaps which can be done by not involving b. The new profile denoted by P^5_N is shown in Table 4. Since none of the swaps involve b, block monotonicity implies that $\varphi_b(P^5_N) = \varphi_b(P^4_N)$.

Agent i	Other	agents
.	$\dots$	$\dots$
.	$\dots$	$\dots$
x	$b\dots$	$x\dots$
a	$\dots$	$\dots$
b	$x\dots$	$a\dots$
.	$a\dots$	$b\dots$
.	$\dots$	$\dots$

Table 4: New profile in Step 3 (P^5_N)

Step 4: In this step, we modify the profile in P^5_N by changing only agent is preference. We do this by doing an (a, b)-swap of the preference of agent i in the profile shown in Table 4. The new profile denoted by P^6_N is shown in Table 5. By block monotonicity, $\varphi_{\{a,b\}}(P^6_N) = \varphi_{\{a,b\}}(P^5_N)$. But x

is better than a for all agents, and hence, Pareto efficiency implies that $\varphi_b(P^6_N) = \varphi_{\{a,b\}}(P^5_N)$.

Agent i	Other	agents
.	...	...
.	...	...
x	$b\ldots$	$x\ldots$
b	...	...
a	$x\ldots$	$a\ldots$
.	$a\ldots$	$b\ldots$
.	...	...

Table 5: New profile in Step 4 (P^6_N)

Step 5: In this step, we modify the profile P^6_N by changing the preferences of those agents who prefer x to a and a to b(the third column of agents in Table 5). We perform a series of swaps to bring x just one position above b. The new profile P^7_N is shown in Table 6. By block monotonicity, $\varphi_b(P^7_N) = \varphi_b(P^6_N)$.

Agent i	Other	agents
.	...	...
.	...	...
x	$b\ldots$	$a\ldots$
b	...	...
a	$x\ldots$	$x\ldots$
.	$a\ldots$	$b\ldots$
.	...	...

Table 6: New profile in Step 5 (P^7_N)

Step 6: Now, we perform an (x,b)-swap of preferences of those agents who rank x just above b in the profile in Step 5 this will be agent i and agents in the third column in Table 6. The new profile P^8_N is shown in Table 7. By block monotonicity, $\varphi_{\{x,b\}}(P^8_N) = \varphi_{\{x,b\}}(P^7_N)$. But b is preferred to x for all the agents. Hence, Pareto efficiency implies $\varphi_b(P^8_N) = \varphi_{\{x,b\}}(P^7_N)$.

Step 7: Finally, we perform a (x,a)-swap for the preferences of all agents in the profile in Step 6 who rank x just above a this will include agent i and agents in the second column of Table 7. The new profile denoted by P^9_N is shown in Table 8. By block monotonicity, $\varphi_b(P^9_N) = \varphi_b(P^8_N)$. But

Agent i	Other	agents
.	...	...
.	...	...
b	$b\dots$	$a\dots$
x	...	...
a	$x\dots$	$b\dots$
.	$a\dots$	$x\dots$
.	...	...

Table 7: New profile in Step 6 (P^8_N)

the profile shown in Table 8 is exactly the profile $(\bar{P}^3_i, P^3_{-i})$(see Table 2) and we had assumed that $\varphi_b(\bar{P}^3_i, P^3_{-i}) < \varphi_b(P^3_i, P^3_{-i})$. So we have $\varphi_b(\bar{P}^3_i, P^3_{-i}) < \varphi_b(P^3_i, P^3_{-i}) \leq \varphi_b(P^9_N) = \varphi_b(\bar{P}^3_i, P^3_{-i})$. This is a contradiction. Hence, φ satisfies elementary monotonicity.

Agent i	Other	agents
.	...	...
.	...	...
b	$b\dots$	$a\dots$
a	...	...
x	$a\dots$	$b\dots$
.	$x\dots$	$x\dots$
.	...	...

Table 8: New profile in Step 7 (P^9_N)

Now, applying Theorem 3.1, we can say that φ LDSIC.

■

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