Asymmetric auctions with risk averse preferences

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Abstract

This paper studies asymmetric auctions for a generalized class of bidders' utility functions. We characterize all Bayesian equilibria of a first-price auction game under weaker assumptions. The necessary conditions of an equilibrium are strict monotonicity, continuity and pure strategy. Next, we establish a revenue ranking for the first-price and second-price auction. Finally, the bidders' preferences of the two auction mechanisms are compared for different types of absolute risk aversion.

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1 Introduction

We study asymmetric auctions without resale for a generalized class of bidders' utility functions. Two asymmetric bidders play an auction game with incomplete information for an indivisible object. The probability distributions of their types are independently distributed and are common knowledge among bidders and the seller. The bidders' von-Neumann-Morgenstern utility function is continuous, strictly increasing and concave on the real line. The bidder with the highest bid wins the object and pays the amount according to the underlying auction mechanism.

In auction literature, the key assumptions are (a) symmetric bidders, (b) independent and private types, (c) risk neutrality, (d) no collusion, and (e) the bidder with highest bid wins the object. Vickrey [16], Riley and

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Samuelson [15] and Myerson [13] study auction by considering assumptions (a)-(e). Holt [4], Riley and Samuelson [15] and Maskin and Riley [11] relax assumption (c). While Plum [14], Lebrun [6] and Maskin and Riley [10] relax assumption (a). We relax assumptions (a) and (c) simultaneously and study auctions with weaker assumptions.

Consider a risk neutral seller who conducts a first-price auction. Lebrun [6] characterizes all Bayesian equilibria of a first-price auction with asymmetric bidders having risk neutral preferences. He shows that no equilibrium exists in non-increasing, discountinuous and mixed strategies. Maskin and Riley [10] show that if one bidder is "strong" and the other is "weak", the weak bidder bids more aggressively and produces a "weaker bid distribution" than the strong bidder. Lebrun [7] shows that if the distribution function of a bidder changes "stochastically", then the bidder will produce a "stronger bid distribution". We extend their results for a generalized class of bidders' utility functions.

In Theorem 1 of this paper we characterize all Bayesian equilibria of a first-price auction for a generalized class of bidders' utility functions. The presumption of a continuous and a monotone pure strategy equilibrium has not been made while characterizing the equilibria. A priori, we just assume that a mixed strategy equilibrium exists which may be discontinuous and non-increasing. We show that the necessary conditions of an equilibrium are continuity, monotonicity and pure strategy.

In section 3 and henceforth, we assume that one bidder is "strong" and the other is "weak". In Proposition 1 and 2, we show that the weak bidder bids more aggressively and produces a weaker bid distribution than the strong bidder. In Proposition 3, we show that the strong (resp. weak) bidder bids more (resp. less) aggressively when playing against a strong (resp. weak) bidder than against a (resp. strong) weak bidder. Moreover, the strong (resp. weak) bidder produces a stronger (resp. weaker) bid distribution than the weak (resp. strong) bidder when both are playing against a strong (resp. weak) bidder. We also show a comparative result in Proposition 4.

An auction mechanism is "efficient" if the winner of the auction is the bidder with highest type. It is well known in the literature that, with symmetric bidders and risk neutral or risk averse preferences, any auction mechanism in which the bidder with the highest bid wins the auction is efficient. On the contrary, this is not generally true with asymmetric bidders. In asymmetric auctions, the winner of the object may not be the bidder with the highest type. More often, the sole motive of an auction is to maximize the expected revenue of the seller. Moreover, there is a trade-off between efficiency and revenue. Myerson [13] and Riley and Samuelson [15] independently show that revenue equivalence theorem holds if the assumptions (a)-(e) (as stated above) are satisfied. If any of the above condition is not satisfied, the revenue equivalence theorem may fail to hold. It is worthwhile to note that the revenue equivalence theorem does not rely on the payment structure. This means if all the conditions are satisfied, then the theorem holds for all types of auction mechanism such as first-price auction, second-price auction, third-price auction, all-pay auction, etc. Holt [4], Riley and Samuelson [15] and Maskin and Riley [11] show that, if the assumption of risk neutrality is not satisfied, the revenue equivalence theorem fails to hold. More specifically, they show that with risk averse bidders, the expected revenue generated from a first-price auction is more than from a second-price auction. Maskin and Riley [10] study revenue ranking by relaxing the assumption of symmetric bidders. They show that, in general, the rankings of the revenues cannot be established. Moreover, with the assumption of conditional stochastic dominance and "shifted" or "stretched" distribution functions, the revenue from a first-price auction exceeds that of a second-price auction. We extend the result by Maskin and Riley [10] for a generalized class of bidders' utility functions.

In Proposition 5, we show that the revenue generated from a first-price auction is more than from a second-price auction.

For symmetric bidders and different types of absolute risk aversion, the bidders' preferences for the two auction mechanisms has been compared by Matthews [12]. He shows, with increasing (resp. decreasing) absolute risk aversion, bidders prefer the first-price (resp. second-price) auction mechanism, and with constant absolute risk aversion, the bidders are indifferent between the two auction mechanisms. We compare the bidders' preferences over the two mechanisms when they are asymmetric. In Proposition 6, we show, with increasing (resp. decreasing) absolute risk aversion, the weak (resp. strong) bidder prefers first-price (resp. second-price) auction mechanism and, with constant absolute risk aversion, the weak (resp. strong) bidder prefers first-price (resp. second-price) auction mechanism.

The outline of the paper is as follows. In section 2, we formalize the model and characterize all Bayesian equilibria. In section 3, we study the bidding behavior and other properties of equilibrium. In section 4, we compute the general expression for seller's expected revenue and show a revenue ranking result. In section 5, we compare the bidders' preferences for different types of absolute risk aversion. We conclude the paper in section 6.

2 The model and characterization of equilibria

Consider an indivisible object for sale. There are two asymmetric bidders with risk neutral or risk averse preferences. Let $N = \{1, 2\}$ denote the set of bidders. Let $T_i = [0, a_i] \subset \Re$ be the type space of bidder i and $t_i \in T_i$ be his type. Assume $a_i \neq a_j$.[1] Let $T = T_1 \times T_2$ be the product type space. Nature draws a type profile $t \in T$ and privately informs t_i to bidder i, i.e., bidder i knows t_i and not t_j for every $j \neq i$. Let $B_i \subseteq \Re_+$ be the action (or bidding) space of bidder i and $b_i \in B_i$ be his bid. Let $B = B_1 \times B_2$ be the product action space. The von-Neumann-Morgenstern utility function for both the bidders is $u : \Re_+ \to \Re$ with $u(0) = 0$, $u' > 0$ and $u'' \leq 0$.[2]

The structure of the game is as follows. The seller of the object conducts a first-price auction. All the bidders submit their bids simultaneously in a sealed envelope. The bidder with the highest bid is the winner of the auction and pays his own bid. Whereas the loser of the auction does not pay anything. In case of a tie, the seller chooses the winner by a fair lottery. All the bidders are utility maximizers. Assume that the seller is risk neutral, i.e., he maximizes the expected profits. For simplicity, the reservation utility of the seller is assumed to be zero. The payoff function $\pi_i : T_i \times B \to \Re$ for bidder i is defined as

$$\pi_i(t_i, b_i, b_j) = \begin{cases} \frac{u(t_i - b_i)}{|Z|} & ; i \in Z \\ u(0) & ; i \notin Z \end{cases}$$

where $Z = \{i | \max_i b_i\}$ and $|Z|$ is the cardinality of Z.

Formally, a Bayesian game is defined as

$$\Gamma = \{N, ((T_i, \mathcal{T}_i, \mu_i), (B_i, \mathcal{B}_i), \pi_i)_{i \in N}\} \tag{1}$$

N is the set of bidders, the triplet $(T_i, \mathcal{T}_i, \mu_i)$ is a measure space, the pair $(B_i, \mathcal{B}_i)$ is a measurable space and π_i is the payoff function. Assume that the measure μ_i is absolutely continuous on $[0, a_i]$ for every $i \in N$. The probability measures on types are assumed to be independently distributed are common knowledge among the bidders and the seller. Let F_i be the

[1] In auction literature, the distribution functions on type spaces like T_i are known as "stretched" distributions.

[2] Alternatively, in a more general framework, $u_i = u(b, t_i)$ such that u_i is monotonic and weakly supermodular. In our framework, the assumption of strictly increasing and concave is equivalent to the assumption of monotonicity and weak supermodularity in a more general framework.

distribution function of the type such that $F_i(0) = 0$ and $F_i(a_i) = 1$. Assume that F_i is twice continuously differentiable and strictly concave on $(0, a_i]$. We also assume that the distribution function has no mass point and the density function $\mathrm{D}F_i \equiv f_i$ is locally bounded away from zero, i.e., $f_i > 0$.

We define the bidding strategy for the bidders. Let $\psi_i : T_i \times \mathcal{B}_i \to \Re$ be a transition function (or regular conditional distribution). Then, by definition $\psi_i(., A_i) : T_i \to \Re$ is $\mathcal{T}_i$-measurable function for every $A_i \in \mathcal{B}_i$; and $\psi_i(t_i, .) : \mathcal{T}_i \to \Re$ is a probability measure for every $t_i \in T_i$. The transition function $\psi_i(t_i, A_i)$ is the probability that $b_i \in A_i$ given his type is $t_i \in T_i$. Moreover, given a type t_i, $\psi_i(t_i, A_i)$ is the interim probability measure of the bid or the behavioral strategy of bidder i.

The strategy ψ_i is said to be **pure** if for every $t_i \in T_i$, $\psi_i(t_i, \{b_i\}) = 1$ for some $b_i \in T_i$ and $\psi_i(t_i, \{b_i'\}) = 0$ for every $b_i' \in T_i$ such that $b_i' \neq b_i$. With abuse of notation, if $\psi_i(t_i, \{b_i\}) = 1$, we write $\psi_i(t_i) = b_i$.

Let the space of transition functions ψ_i be $\Delta(B_i)$. Consider $P_i \in \mathcal{T}_i$ and $Q_i \in \mathcal{B}_i$. Then, $P_i \times Q_i$ is a measurable rectangle. Let the space of all measurable rectangles be $\mathcal{E}_i$. $\mathcal{T}_i \otimes \mathcal{B}_i$ is a product σ-algebra and $(T_i \times B_i, \mathcal{T}_i \otimes \mathcal{B}_i)$ is a product measurable space. A product measure $\varphi_i : \mathcal{T}_i \otimes \mathcal{B}_i \to \Re$ is defined as

$$\varphi_i(P_i \times Q_i) := \int_{P_i} \mu_i(\mathrm{d}t_i)\psi_i(t_i, Q_i)$$

for every $P_i \times Q_i \in \mathcal{T}_i \otimes \mathcal{B}_i$. Note $P_i \subseteq T_i$ and $Q_i \subseteq B_i$. The product measure $\varphi_i(P_i \times Q_i)$ is the probability that $t_i \in P_i$ and $b_i \in Q_i$.

The measure $\beta_i : \mathcal{B}_i \to \Re$ defined as $\beta_i(Q_i) := \varphi_i(T_i, Q_i)$ for every $Q_i \in \mathcal{B}_i$ is the marginal of the measure φ_i on T_i. It is interpreted as the ex-ante probability measure of the bid or the mixed strategy of bidder i. So, the triplet $(B_i, \mathcal{B}_i, \beta_i)$ is a measure space. Whereas the marginal of φ_i on B_i is $\mu_i(P_i) = \varphi_i(P_i, B_i)$.

The interim expected payoff $U_i : \Delta(B_i) \times \Delta(B_j) \times T_i \to \Re$ for bidder $i \in N$ is

$$U_i(\psi_i, \psi_j, t_i) = \int_{B_i \times T_j \times B_j} \psi_i(t_i, \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)\pi_i(t_i, b_i, b_j)$$

for every $t_i \in T_i$.

Definition 1. *A profile of functions (ψ_i, ψ_j) is a Bayesian equilibrium if for every $i \in N$, $t_i \in T_i$ and $\psi_i' \in \Delta(B_i)$,*

$$U_i(\psi_i, \psi_j, t_i) \geq U_i(\psi_i', \psi_j, t_i).$$

Suppose (ψ_i, ψ_j) is a Bayesian equilibrium. Let a correspondence $\Sigma_i : T_i \rightrightarrows B_i$ be defined as

$$\Sigma_i(t_i) = \{b_i \in B_i | U_i(\psi_i, \psi_j, t_i) = U_i(b_i, \psi_j, t_i)\}$$

Let the functions $\Lambda_i : T_i \to B_i$ and $\Omega_i : T_i \to B_i$ be defined as

$$\Lambda_i(t_i) = \inf \Sigma_i(t_i), \text{ and}$$
$$\Omega_i(t_i) = \sup \Sigma_i(t_i)$$

for every $i \in N$. By the definition of a Bayesian equilibrium, $U_i(\psi_i, \psi_j, t_i)$ is the maximum expected payoff generated by bidder i given that bidder j follows ψ_j. Given a type t_i, $\Sigma_i(t_i)$ is the set of all those bids which bidder i bids with positive probability. Furthermore, $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are the infimum and supremum of all those bids respectively. Thus, by the definition of a mixed equilibrium, the sets $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are non-empty for every $t_i \in T_i$ and $i \in N$.

Consider a measure space $(B_i, \mathcal{B}_i, \beta_i)$. Let $g_i : B_i \to \Re$ be a $\mathcal{B}_i$-measurable function. Then, β_i is the ex-ante probability measure of the function g_i. The function g_i is the bid made by bidder i. The co-domain of the function g_i represents the set of "effective" bids when i follows a mixed strategy β_i.

Let $\{(B_i, \mathcal{B}_i, \psi_i(t_i, .))|t_i \in T_i\}$ be a family of measure spaces. Consider a type t_i and a measure space $(B_i, \mathcal{B}_i, \psi_i(t_i, .))$. Let $h_i(t_i) : B_i \to \Re$ be a $\mathcal{B}_i$-measurable function. Let $\{h_i(t_i)|i \in N\}$ be the family of $\mathcal{B}_i$-measurable functions. Then, $\psi_i(t_i, .)$ is the interim probability measure of the function h_i. The function $h_i(t_i)$ is the bid made by bidder i. The co-domain of the function $g_i(t_i)$ represents the set of bids that bidder i bids with positive probability when the behavioral strategy $\psi_i(t_i, .)$ is implemented.

We state the characterization of all Bayesian equilibria.

Theorem 1 (Characterization of equilibria). *The profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium if and only if for every $i \in N$, ψ_i is pure, strictly increasing, continuous and solves the following pair of differential equations*

$$\frac{F_i \circ \psi_i^{-1}(b)}{\mathrm{D}F_i \circ \psi_i^{-1}(b)} = \frac{u(\psi_j^{-1}(b) - b)}{u'(\psi_j^{-1}(b) - b)} \tag{2}$$

with boundary conditions $\psi_i(0) = 0$ and $\psi_j(a_i) = \bar{b}$ such that $\bar{b} \in \Re_{++}$.

Proof. Appendix A ■

The above theorem states that the necessary conditions of an equilibrium are continuity, strict monotonicity and pure strategy. This means we presume that an equilibrium may exists in mixed, discontinuous and non-decreasing strategy. We then show that if an equilibrium exists, it has to be in pure, continuous and strictly increasing strategy.

The proof of the above theorem is long and involved (The complete proof is given in Appendix A). The structure of the "if" part of the theorem is as follows. The idea is to show that the correspondence Σ_i is a function. This means that the functions Λ_i and Ω_i are identical. It then follows that Λ_i or Ω_i is indeed a pure strategy equilibrium.

To show Σ_i is a function requires some work. In Lemma A.1, we show that $U_i(\psi_i, \psi_j, t_i) > 0$ for every $t_i > 0$ and $i \in N$. In Lemma A.2, we show that $\Lambda_i(0) = \Omega_i(0) = 0$ for every i. In Lemma A.3, we show that $\Pr(i \text{ wins}|t_i)$ is non-decreasing in t_i. In Lemma A.4, we show that $\Lambda_i(t'_i) \geq \Omega_i(t_i)$ such that $t'_i > t_i$. This means that the infimum of $\Sigma_i(t'_i)$ for a higher type t'_i is at least as large as the supremum of $\Sigma_i(t_i)$ for a lower type t_i (Recall that for a given type t_i, $\Sigma_i(t_i)$ is the set of all those bids that bidder i bids with positive probability). We then show that $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are non-decreasing in t_i. In Lemma A.5, we show that $\Lambda_i(t_i) > 0$ and $\Omega_i(t_i) > 0$ for every $t_i \in T_i - \{0\}$ and $i \in N$. In Lemma A.7, we show that Λ_i is left continuous and Ω_i is right continuous. As Λ_i is left continuous, it follows if Λ_i is continuous, then $\Lambda_i(t_i) = \Omega_i(t_i)$ for every $t_i \in T_i$ and $i \in N$. Similarly, as Ω_i is right continuous, it follows if Ω_i is continuous, then $\Omega_i(t_i) = \Lambda_i(t_i)$ for every $t_i \in T_i$ and $i \in N$. Therefore, to show Σ_i is a function, we need to show Λ_i and Ω_i are indeed continuous. In Lemma A.8, we show that $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are strictly increasing in t_i. In Lemma A.9, we show that Λ_i and Ω_i are continuous. Hence, we conclude that Σ_i is a function.

We need to show that ψ_i is pure, strictly increasing and continuous. It suffices to show that Λ_i and Ω_i are identical, strictly increasing and continuous. As Σ_i is a function, it follows that ψ_i is pure. As Λ_i and Ω_i are strictly increasing and continuous, it follows that ψ_i is strictly increasing and continuous. Rest of the proof is routine.

The "only if" part of the proof requires to show that it is not profitable for any bidder to deviate.

3 Properties of an equilibrium

In this section and henceforth, we assume that bidder 1 is a "weak" bidder (w) and bidder 2 is a "strong" bidder (s). If the distribution function of bidder i "conditional stochastically dominates" that of bidder j, then we say i is strong and j is weak. So, the set of bidders can be re-written as $N = \{s, w\}$. We assume $a_s > a_w$ and the distribution function is twice continuously differentiable and strictly concave on $(0, a_i]$. We also assume that the distribution function has no mass point and the density function $\mathrm{D}F_i \equiv f_i$ is locally bounded away from zero, i.e., $f_i > 0$. F_i is said to be first-order stochastically dominant to F_j if $F_i(x) < F_j(x)$ for every $x \in$

$\Re$. Whereas F_i is said to be conditional stochastically dominant[3] to F_j if $F_i(x)/F_j(x) < F_i(y)/F_j(y)$ for every $x, y \in \Re$ such that $x < y$. It is easy to check that conditional stochastic dominance implies first-order stochastic dominance (in general, the converse is not true). We make the following assumption on the distribution functions of the bidders:

Assumption 1. *F_s is conditional stochastically dominant to F_w.*

The above assumption states that, given any two arbitrary types t_1 and t_2 such that $t_1 < t_2$, the relative probability of the strong bidder of getting a type higher than t_1 is more than the relative probability of the strong bidder of getting a type higher than t_2. In other words, the relative distribution function of the strong bidder is increasing in types.

In subsection 3.1, we study the bidding behavior when bidders are asymmetric. In subsection 3.2, we compare the bidding behavior of symmetric and asymmetric bidders. In subsection 3.3, we study comparative statics.

3.1 Bidding behavior of asymmetric bidders

Suppose (ψ_i, ψ_j) is a Bayesian equilibrium. By Theorem 1, ψ_i is pure, strictly increasing and continuous; $\psi_i(0) = 0$; and there exists $\bar{b} \in \Re_{++}$ such that $\psi_i(a_i) = \bar{b}$ for every $i \in N$. Let $\bar{B} = [0, \bar{b}]$. Note that T_i and $\bar{B}$ are compact and connected sets. Also, $B_i = \bar{B}$ for every $i \in N$. Since ψ_i is pure, from now on we shall refer to the equilibrium bidding strategy as a measurable function $\psi_i : T_i \to \bar{B}$. As ψ_i is strictly increasing and continuous, it follows that the inverse exists and it is also strictly increasing and continuous. We shall denote the equilibrium inverse bidding strategy by a measurable function $\theta_i : \bar{B} \to T_i$. It is noteworthy that the measurable functions ψ_i and θ_i are bijective.

Consider a bid $b \in \bar{B}$. Suppose bidder i with type t_i bids b and bidder j follows his equilibrium bidding strategy ψ_j with equilibrium inverse bidding strategy θ_j. It is more convenient to work with inverse bidding strategies rather than the bidding strategies. Bidder i wins if and only if $t_j < \theta_j(b)$. So, i wins the auction with probability $F_j \circ \theta_j(b)$ and thereby incurs a payoff of $u(t_i - b)$. While he loses the auction with probability $1 - F_j \circ \theta_j(b)$ and payoff $u(0) = 0$. Hence, the expected payoff of bidder i is

$$U_i(b, \psi_j, t_i) = F_j \circ \theta_j(b) u(t_i - b) \tag{3}$$

[3] In auction literature, the assumption of conditional stochastic dominance was first used by Maskin and Riley [10]. In statistics literature, conditional stochastic dominance is known as monotone probability ratio.

So, the maximization problem of bidder i is $\max_b U_i(b, \psi_j, t_i)$. Differentiating w.r.t. b, we get

$$\mathrm{D}_b U_i(b, \psi_j, t_i) = \mathrm{D}F_j \circ \theta_j(b) u(t_i - b) - F_j \circ \theta_j(b) u'(t_i - b)$$

As bidder i follows ψ_i in equilibrium, the first-order condition[4] can be written as

$$\frac{F_j \circ \theta_j(b)}{\mathrm{D}F_j \circ \theta_j(b)} = \frac{u(\theta_i(b) - b)}{u'(\theta_i(b) - b)} \tag{4}$$

The system of differential equations given by (4) along with the boundary conditions

$$\theta_i(0) = 0 \text{ and } \theta_i(\bar{b}) = a_i \tag{5}$$

characterize the equilibrium inverse bidding strategies. Later in this subsection, we shall characterize the equilibrium bidding strategies.

(4) can be re-written as

$$\mathrm{D}\theta_j(b) = \frac{F_j \circ \theta_j(b)}{f_j \circ \theta_j(b)} \frac{u'(\theta_i(b) - b)}{u(\theta_i(b) - b)} \tag{6}$$

and

$$\mathrm{D} \log F_j \circ \theta_j(b) = \frac{u'(\theta_i(b) - b)}{u(\theta_i(b) - b)} \tag{7}$$

(6) and (7) shall be used later. We compare the equilibrium bidding strategy of the two bidders.

Proposition 1. *Suppose the profile of measurable functions (ψ_i, ψ_j) is a Bayesian equilibrium and Assumption 1 is satisfied. Then, $\psi_w(t) > \psi_s(t)$ holds for every $t \in T_w - \{0\}$. (Equivalently, $\theta_w(b) < \theta_s(b)$ holds for every $b \in \bar{B} - \{0\}$.)*

Proof. Appendix B ■

The above result states that the weak bidder bids more aggressively than the strong bidder, i.e., for a given type the weak bidder bids more than the strong bidder.

In the next result, we compare the equilibrium bid distributions.

Proposition 2. *Suppose the profile of measurable functions (ψ_i, ψ_j) is a Bayesian equilibrium and Assumption 1 is satisfied. Then,*

$$F_s \circ \theta_s(b) < F_w \circ \theta_w(b)$$

holds for every $b \in \bar{B} - \{0, \bar{b}\}$.

[4] As F_j and u both are concave, the first-order condition is both necessary and sufficient.

Proof. Appendix B ■

It can be easily verified that at $b \in \{0, \bar{b}\}$, $F_s \circ \theta_s(b) = F_w \circ \theta_w(b)$. The above result states that, in equilibrium, weak bidder produces a more aggressive bid distribution than the strong bidder. In other words, the equilibrium probability of winning for the strong bidder is more than that for the weak bidder.

We now characterize the equilibrium bidding strategy of the bidders. Suppose (ψ_i, ψ_j) is a Bayesian equilibrium. Let $\zeta_j : T_i \to T_j$ be defined as

$$\zeta_j(t_i) = \theta_j \circ \psi_i(t_i) \tag{8}$$

for every $j \in N$. $\zeta_j(t_i)$ is interpreted as the type required by bidder j to bid the same as bidder i bids when his type is t_i. It is noteworthy that $\zeta_j(0) = 0$ and $\zeta_j(a_i) = a_j$. Moreover, it follows $\zeta_s(t_w) > t_w$ and $F_s \circ \zeta_s(t_w) < F_w(t_w)$ from Proposition 1 and 2 respectively. Differentiating (8) w.r.t. t_i and using (4), we have

$$\mathrm{D}\zeta_j(t_i) = \frac{F_j \circ \zeta_j(t_i)}{f_j \circ \zeta_j(t_i)} \frac{f_i(t_i)}{F_i(t_i)} \frac{u(\zeta_j(t_i) - \psi_i(t_i))}{u'(\zeta_j(t_i) - \psi_i(t_i))} \frac{u'(t_i - \psi_i(t_i))}{u(t_i - \psi_i(t_i))} \tag{9}$$

Similarly,

$$\mathrm{D}\psi_i(t_i) = \frac{f_i(t_i)}{F_i(t_i)} \frac{u(\zeta_j(t_i) - \psi_i(t_i))}{u'(\zeta_j(t_i) - \psi_i(t_i))} \tag{10}$$

for $t_i \in T_i - \{0, a_i\}$ and for every $i \in N$.

The system of differential equations given by (9) and (10) along with the boundary conditions

$$\zeta_j(0) = 0 \text{ and } \zeta_j(a_i) = a_j \tag{11}$$

characterize the equilibrium bidding strategies. It is wothwhile to note that the system of differential equations given by (4) along with the boundary conditions given by (5) is equivalent to the system of differential equations given by (9) and (10) along with the boundary conditions given by (11). The characterization of equilibrium bidding strategies will be useful when we derive the expression of the expected revenue for the seller in section 4.

3.2 Comparison of bidding behavior for symmetric and asymmetric bidders

We say the bidders are symmetric if both the bidders are either weak or both strong. The main goal of this subsection is to compare the bidding behavior of symmetric and asymmetric bidders.

Suppose both the bidders are either strong or both weak. Let the equilibrium bidding strategies be (ξ_k, ξ_k) with inverse bidding strategies (λ_k, λ_k) for $k \in N$. Then, by Theorem 1, the following holds: (a) ξ_k is pure, strictly increasing and continuous, (b) $\xi_k(0) = 0$ and there exists $\bar{b}_k \in \Re_{++}$ such that $\xi_k(\bar{b}_k) = a_k$, and (c) the following differential equation holds

$$\frac{F_k \circ \lambda_k(b)}{\mathrm{D}F_k \circ \lambda_k(b)} = \frac{u(\lambda_k(b) - b)}{u'(\lambda_k(b) - b)} \equiv w(\lambda_k(b) - b) \tag{12}$$

The above condition can be re-written as

$$\mathrm{D}\lambda_k(b) = \frac{F_k \circ \lambda_k(b)}{f_k \circ \lambda_k(b)} \frac{1}{w(\lambda_k(b) - b)} \tag{13}$$

The differential equation given by (12) along with the boundary conditions

$$\lambda_k(0) = 0 \text{ and } \lambda_k(\bar{b}_k) = a_k \tag{14}$$

characterize the equilibrium inverse bidding strategies.

We now characterize the equilibrium bidding strategies. Let $\bar{B}_k = [0, \bar{b}_k]$. Consider a representative bidder i with type t who bids according to $\xi_k(s)$. Suppose the other bidder j bids according to his equilibrium bidding strategy. Then, the expected payoff of bidder i is

$$V_k(\xi_k(s), \xi_k(t), t) = F_k(t)u(t - \xi_k(s))$$

Differentiating w.r.t. t, we get

$$\mathrm{D}V_k(\xi_k(s), \xi_k(t), t) = f_k(s)u(t - \xi_k(s) -)F_k(s)u'(t - \xi_k(s))\mathrm{D}\xi_k(s)$$

Since bidder i follows $\xi_k(t)$ in equilibrium, the first-order condition is

$$\mathrm{D}\xi_k(t) = \frac{f_k(t)}{F_k(t)} w(t - \xi_k(t)) \tag{15}$$

The differential equation given by (15) along with the boundary conditions

$$\xi_k(0) = 0 \text{ and } \xi_k(a_k) = \bar{b}_k \tag{16}$$

characterize the equilibrium bidding strategies. It is noteworthy that the differential equation given by (12) along with the boundary conditions given by (14) is equivalent to the differential equation given by (15) along with the boundary conditions given by (16).

We compare the equilibrium bidding strategy of weak symmetric bidders and strong symmetric bidders.

Corollary 1. *Suppose the profiles of measurable functions* (ξ_w, ξ_w) *and* (ξ_s, ξ_s) *are a Bayesian equilibrium when both the bidders are weak and strong respectively, and Assumption 1 is satisfied. Then,*
(A) $\xi_w(t) < \xi_s(t)$ *for every* $t \in T_w - \{0\}$ *(Equivalently,* $\lambda_w(b) > \lambda_s(b)$ *for every* $b \in \bar{B}_w - \{0, \bar{b}_w\}$*),*
(B) $F_s \circ \lambda_s(b) < F_w \circ \lambda_s(b)$ *for every* $b \in \bar{B}_w - \{0, \bar{b}_w\}$*.*

Proof. Appendix B ■

The above result states that bidders bid aggressively and produce a weaker bid distribution when they are strong as compared to when they are weak.

In the next result we compare the bidding behavior of symmetric and asymmetric bidders.

Proposition 3. *Suppose the profile of measurable functions* (ψ_w, ψ_s) *is a Bayesian equilibrium when one bidder is weak and the other is strong. Suppose the profiles of measurable functions* (ξ_w, ξ_w) *and* (ξ_s, ξ_s) *are a Bayesian equilibrium when both the bidders are weak and strong respectively. If Assumption 1 is satisfied, then*
(A) $\psi_s(t) < \xi_s(t)$ *for every* $t \in T_s - \{0\}$ *(Equivalently,* $\theta_s(b) > \lambda_s(b)$ *for every* $b \in \bar{B}_s - \{0, \bar{b}_s\}$*),*
(B) $\psi_w(t) > \xi_w(t)$ *for every* $t \in T_w - \{0\}$ *(Equivalently,* $\theta_w(b) < \lambda_w(b)$ *for every* $b \in \bar{B} - \{0, \bar{b}\}$*),*
(C) $F_w \circ \theta_w(b) > F_s \circ \lambda_s(b)$ *for every* $b \in \bar{B}_s - \{0, \bar{b}_s\}$*,*
(D) $F_s \circ \theta_s(b) < F_w \circ \lambda_w(b)$ *for every* $b \in \bar{B} - \{0, \bar{b}\}$*.*

Proof. Appendix B ■

The above result states that the strong (resp. weak) bidder bids more (resp. less) aggressively when playing against a strong (resp. weak) bidder than a (resp. strong) weak bidder. Moreover, the strong (resp. weak) bidder produces a weaker (resp. stronger) bid distribution than the weak (resp. strong) bidder when both are playing against a strong (resp. weak) bidder.

3.3 Comparative statics

Consider a bidder j. Suppose the distribution function of j changes to $\hat{F}_j$ such that $\hat{F}_j$ is conditional stochastically dominant to F_j. Furthermore, $\hat{F}_j$ satisfies all the assumptions satisfied by F_j. As before, when distribution functions are (F_i, F_j), the equilibrium bidding strategies and equilibrium inverse bidding strategies are denoted by (ψ_i, ψ_j) and (θ_i, θ_j) respectively. When the distribution functions are $(F_i, \hat{F}_j)$, let the equilibrium bidding strategies and the equilibrium inverse bidding strategies be denoted by $(\hat{\psi}_j, \hat{\psi}_j)$ and $(\hat{\theta}_i, \hat{\theta}_j)$ respectively. From Theorem 1, $\hat{\psi}_i(0) = \hat{\psi}_j(0) = 0$ and there exists $\hat{b}$ such that $\hat{\psi}_i(a_i) = \hat{\psi}_j(0) = \hat{b}$.

When distribution functions are (F_i, F_j), the expected payoff of i is given by (3) with first-order condition given by (4). The expected payoff of j and the first-order condition is analogous to (3) and (4) respectively.

When distribution functions are $(F_i, \hat{F}_j)$, the expected payoff of bidder i is

$$\hat{U}_i(b, \hat{\psi}_j, t_i) = \hat{F}_j \circ \hat{\theta}_j(b) u(t_i - b)$$

and the expected payoff of bidder j is

$$\hat{U}_j(b, \hat{\psi}_i, t_j) = F_i \circ \hat{\theta}_i(b) u(t_j - b)$$

The first-order conditions for i and j are

$$\frac{\hat{F}_j \circ \hat{\theta}_j(b)}{\mathrm{D}\hat{F}_j \circ \hat{\theta}_j(b)} = \frac{u(\hat{\theta}_i(b) - b)}{u'(\hat{\theta}_i(b) - b)} \tag{17}$$

and

$$\frac{F_i \circ \hat{\theta}_i(b)}{\mathrm{D}F_i \circ \hat{\theta}_i(b)} = \frac{u(\hat{\theta}_j(b) - b)}{u'(\hat{\theta}_j(b) - b)} \tag{18}$$

respectively.

We compare the bidding behavior due to a change in the distribution function of one bidder.

Proposition 4. *Suppose the profiles of measurable functions (ψ_i, ψ_j) and $(\hat{\psi}_i, \hat{\psi}_j)$ are a Bayesian equilibrium when the distribution functions are (F_i, F_j) and $(F_i, \hat{F}_j)$ respectively such that $\hat{F}_j$ conditional stochastically dominates F_j. Then,*
(A) $\hat{\psi}_i(t_i) > \psi_i(t_i)$ for every $t_i \in T_i - \{0\}$,
(B) $\hat{F}_j \circ \hat{\theta}_j(b) < F_j \circ \theta_j(b)$ for every $b \in \bar{B} - \{0\}$.

Proof. Appendix B ∎

The above result states that due to a change in the distribution function of bidder j, (a) bidder i bids more aggressively than before, and (b) bidder j produces a weaker bid distribution than before.

When the distribution functions are (F_i, F_j), the interim expected payoff functions of i and j are

$$U_i(t_i, \psi_i, \psi_j) = \max_b F_j \circ \theta_j(b) u(t_i - b)$$

and

$$U_j(t_i, \psi_i, \psi_j) = \max_b F_i \circ \theta_i(b) u(t_j - b)$$

respectively. When the distribution functions are $(F_i, \hat{F}_j)$, the interim expected payoff functions of i and j are

$$\hat{U}_i(t_i, \hat{\psi}_i, \hat{\psi}_j) = \max_b \hat{F}_j \circ \hat{\theta}_j(b) u(t_i - b)$$

and

$$\hat{U}_j(t_i, \hat{\psi}_i, \hat{\psi}_j) = \max_b F_i \circ \hat{\theta}_i(b) u(t_j - b)$$

respectively.

In the next result, we compare the expected payoff of the bidders due to a change in the distribution function of one bidder.

Corollary 2. *Suppose the profiles of measurable functions (ψ_i, ψ_j) and $(\hat{\psi}_i, \hat{\psi}_j)$ are a Bayesian equilibrium when the distribution functions are (F_i, F_j) and $(F_i, \hat{F}_j)$ respectively such that $\hat{F}_j$ conditional stochastically dominates F_j. Then, $\hat{U}_i(t_i, \hat{\psi}_i, \hat{\psi}_j) < U_i(t_i, \psi_i, \psi_j)$ and $\hat{U}_j(t_j, \hat{\psi}_i, \hat{\psi}_j) < U_j(t_j, \psi_i, \psi_j)$ for every $t_i \in T_i - \{0\}$ and $t_j \in T_j - \{0\}$.*

Proof. Appendix B ■

The above corollary states that due to a change in the distribution function of one bidder, both the bidders are worse-off .

4 Revenue ranking

In this section, we compare the seller's expected revenue in a first-price auction and a second-price auction. We make the following assumption on the density functions of the types:

Assumption 2. *$F_s = \rho F_w$ for $0 < \rho < 1$ and $f_w(t_w) \geq f_s(t_s)$ for every $t_w \in [0, a_w]$ and $t_s \in [a_w, a_s]$*

Suppose (ψ_i, ψ_j) is a Bayesian equilibrium of Γ. Consider $b \in \bar{B}$ and the weak bidder. Given b, the expected revenue of the seller is b times the winning probability of the weak bidder, i.e., $bF_s \circ \theta_s(b)$. Summing this expression over the bidding space $\bar{B}$, we get the ex-ante expected revenue of the seller as

$$P_w = \int_0^{\bar{b}} F_w \circ \theta_w(\mathrm{d}b) b F_s \circ \theta_s(b)$$

Integrating by-parts, we get

$$P_w = \int_0^{\bar{b}} \{(1 - F_w \circ \theta_w(b))\mathrm{D}(bF_s \circ \theta_s(b))\}\mathrm{d}b$$

Using (4), we get

$$\mathrm{D}(bF_s \circ \theta_s(b)) = \mathrm{D}\theta_s(b) f_s \circ \theta_s(b) \left(b + \frac{u(\theta_w(b) - b)}{u'(\theta_w(b) - b)} \right)$$

Using the above equation in the expression of P_w, we get

$$P_w = \int_0^{\bar{b}} \left\{ (1 - F_w \circ \theta_w(b)) f_s \circ \theta_s(b) \left(b + \frac{u(\theta_w(b) - b)}{u'(\theta_w(b) - b)} \right) \mathrm{D}\theta_s(b) \right\} \mathrm{d}b$$

Recall that $\zeta_j(t_i)$, given by (8), is the type required by bidder j to match bidder i's bid with type t_i. So, $\zeta_s \circ \theta_w(b) = \theta_s \circ \theta_w^{-1} \circ \theta_w(b) = \theta_s(b)$. Using this in the expression of P_w, we get

$$P_w = \int_0^{a_w} \left\{ (1 - F_w(t_w)) \left(\psi_w(t_w) + \frac{u(t_w - \psi_w(t_w))}{u'(t_w - \psi_w(t_w))} \right) \mathrm{D}F_s \circ \zeta_s(t_w) \right\} \mathrm{d}t_w$$

Integrating by-parts, we get the ex-ante expected revenue from the weak bidder

$$P_w = \int_0^{a_w} (1 - F_s \circ \zeta_s(t)) \mathrm{D} \left\{ (1 - F_w(t)) \left(\psi_w(t) + \frac{u(t - \psi_w(t))}{u'(t - \psi_w(t))} \right) \right\} \mathrm{d}t \quad (19)$$

We now compute the ex-ante expected revenue for the seller generated by the strong bidder. Applying symmetry to (19), we have

$$P_s = \int_0^{\bar{b}} \left\{ (1 - F_s \circ \theta(b)) \left(b + \frac{u(\theta_s(b) - b)}{u'(\theta_s(b) - b)} \mathrm{D}F_w \circ \theta_w(b) \right) \right\} \mathrm{d}b$$

We know $\zeta_s \circ \theta_w(b) = \theta_s(b)$. So, the expression for ex-ante expected revenue from the strong bidder is

$$P_s = \int_0^{a_w} \left\{ (1 - F_s \circ \zeta_s(t)) f_w(t) \left(\psi_w(t) + \frac{u(\zeta_s(t) - \psi_w(t))}{u'(\zeta_s(t) - \psi_w(t))} \right) \right\} \mathrm{d}t \quad (20)$$

The expressions given by (19) and (20) are the general expressions of ex-ante expected revenue of the seller in a first-price auction generated from the weak and strong bidder respectively. Since, these expressions will be quite useful, we state them in the following lemma.

Lemma 1. *Suppose the profile of measurable functions* (ψ_i, ψ_j) *is a Bayesian equilibrium of* Γ*. Then, the expressions for ex-ante expected revenue of the seller in a first-price auction generated by the weak and strong bidder are given by* (19) *and* (20) *respectively.*

The ex-ante expected revenue of the seller in a first-price auction is

$$\begin{aligned} P &= P_w + P_s \\ &= \int_0^{a_w} (1 - F_s \circ \zeta_s(t)) \Bigg\{ f_w(t) \Bigg(\frac{u(\zeta_s(t) - \psi_w(t))}{u'(\zeta_s(t) - \psi_w(t))} - \frac{u(t - \psi_w(t))}{u'(t - \psi_s(t))} \Bigg) \\ &+ (1 - F_w(t)) \mathrm{D} \Bigg(\psi_w(t) + \frac{u(t - \psi_w(t))}{u'(t - \psi_w(t))} \Bigg) \Bigg\} \mathrm{d}t \end{aligned} \tag{21}$$

We now compute the expression of ex-ante expected revenue for the seller in a second-price auction. Recall that, it is a weakly dominant strategy to bid its own type in a second-price auction. Consider the weak bidder with type t_w. Given t_w, the expected revenue of the seller is the probability that the type of strong bidder is less than t_w times t_s, i.e., $\int_0^{t_w} F_s(\mathrm{d}t_s) t_s$. Summing this expression over the type space of the weak bidder, we get the ex-ante expected revenue of the seller as

$$Q_w = \int_0^{a_w} \int_0^{t_w} F_s(\mathrm{d}t_s) F_w(\mathrm{d}t_w) t_s$$

Integrating by-parts, we have

$$Q_w = \int_0^{a_w} (1 - F_w(t_w)) t_w \mathrm{d}t_w$$

Integrating by-parts again, we have

$$Q_w = \int_0^{a_w} (1 - F_s(t_w)) \mathrm{D}(t_w (1 - F_w(t_w))) \mathrm{d}t_w \tag{22}$$

Applying symmetry, the ex-ante revenue of the seller generated from the strong bidder is

$$Q_s = \int_0^{a_w} (1 - F_s(t_w)) t_w \mathrm{d}t_w \tag{23}$$

The ex-ante expected revenue of the seller in a second-price auction is

$$\begin{aligned} Q &= Q_w + Q_s \\ &= \int_0^{a_w} (1 - F_w(t))(1 - F_s(t)) \mathrm{d}t \end{aligned} \tag{24}$$

Therefore, the general expression of the seller's expected revenue generated from a second-price auction is given by (24).[5]

The difference between the seller's ex-ante expected revenue generated from a first-price auction and a second-price auction is

$$\begin{aligned} P - Q = \int_0^{a_w} \{ &(1 - F_s \circ \zeta_s(t)) f_w(t) W(\zeta_s(t), \psi_w(t), t) + (1 - F_s \circ \zeta_s(t)) \\ &(1 - F_w(t)) \mathrm{D}_t Y(\psi_s(t), t) - (1 - F_w(t))(1 - F_s(t)) \} \mathrm{d}t \end{aligned} \tag{25}$$

where

$$W(\zeta_s(t), \psi_w(t), t) = \frac{u(\zeta_s(t) - \psi_w(t))}{u'(\zeta_s(t) - \psi_w(t))} - \frac{u(t - \psi_w(t))}{u'(t - \psi_w(t))}$$

and,

$$Y(\psi_w(t), t) = \psi_w(t) + \frac{u(t - \psi_w(t))}{u'(t - \psi_w(t))}$$

Note that as $u' > 0$ and $u'' \leq 0$, it follows $\mathrm{D}_t Y(\psi_w(t), t) \geq 1$. (25) can be re-written as

$$\begin{aligned} P - Q &= \int_0^{a_w} (1 - F_s \circ \zeta_s(t)) W(\zeta_s(t), \psi_w(t), t)(1 - F_w(t)) \left\{ \frac{f_w(t)}{1 - F_w(t)} \right. \\ &\quad \left. + \frac{\mathrm{D}_t Y(\psi_w(t), t)}{W(\zeta_s(t), \psi_w(t), t)} - \frac{1 - F_s(t)}{(1 - F_s \circ \zeta_s(t)) W(\zeta_s(t), \psi_w(t), t)} \right\} \mathrm{d}t \\ &\geq \int_0^{a_w} (1 - F_s \circ \zeta_s(t)) W(\zeta_s(t), \psi_w(t), t)(1 - F_w(t)) \left\{ \frac{f_w(t)}{1 - F_w(t)} \right. \\ &\quad \left. - \frac{F_s \circ \zeta_s(t) - F_s(t)}{(1 - F_s \circ \zeta_s(t)) W(\zeta_s(t), \psi_w(t), t)} \right\} \mathrm{d}t \\ &> \int_0^{a_w} W(\zeta_s(t), \psi_w(t), t)(1 - F_w(t)) \left\{ f_w(t) - \frac{F_s \circ \zeta_s(t) - F_s(t)}{W(\zeta_s(t), \psi_w(t), t)} \right\} \mathrm{d}t \end{aligned}$$

The last inequality follows from the fact that $\zeta_s(t) > t$ (Proposition 1) and $F_s \circ \zeta_s(t) < F_w(t)$ (Proposition 2).

We state the revenue ranking result.

Proposition 5. *Suppose (ψ_i, ψ_j) is a Bayesian equilibrium of Γ and Assumptions 1 and 2 are satisfied. Then, the ex-ante expected revenue of the seller generated from a first-price auction is more than from a second-price auction.*

[5]The expression for the expected revenue in a second-price auction is same as in the linear utilities framework as computed by Maskin and Riley [10]. This is because, in a second-price auction with generalized utility functions, it is still a dominant strategy to bid your own type.

Proof. Appendix B ■

The above result states that for the generalized class of utility functions, the expected revenue generated from a first-price auction is more than from a second-price auction.

5 Different types of risk aversion

We compare the bidders' preferences of first-price auction and second-price auction under different types of absolute risk aversion, i.e., increasing absolute risk aversion, constant absolute risk aversion and decreasing absolute risk aversion. In this section, we assume that the von-Neumann-Morgenstern utility is strictly increasing and strictly concave. Furthermore, we assume that the reserve price of $r \in \Re_{++}$ is exogeneously given.

The Arrow-Pratt measure of absolute risk aversion is defined as $R(x) = -u''(x)/u'(x)$ for every $x \in \Re$. The utility function u has increasing absolute risk aversion if $\mathrm{D}R(x) > 0$, has constant absolute risk aversion if $\mathrm{D}R(x) = 0$, and has decreasing absulute risk aversion if $\mathrm{D}R(x) < 0$.

Suppose (ψ_i, ψ_j) is a Bayesian equilibrium of a first-price auction. Recall that the equilibrium bidding strategy ψ_i with inverse bidding strategy θ_i are strictly increasing and continuous. Suppose bidder j follows his equilibrium bidding strategy. The maximization problem of bidder i is

$$V_i(t_i) = \max_b F_j \circ \theta_j(b) u(t_i - b)$$

Suppose b^* solves the above maximization problem. Then,

$$b^* \in \arg\max_b F_j \circ \theta_j(b) u(t_i - b)$$

So, the value function can be re-written as

$$V_i(t_i) = F_j \circ \theta_j(b^*) u(t_i - b^*) \tag{26}$$

Using Envelope theorem, we have

$$\mathrm{D}V_i(t_i) = F_j \circ \theta_j(b^*) u'(t_i - b^*) \tag{27}$$

Recall that, in a second-price auction, it is a dominant strategy to bid your own type. Let the equilibrium bidding strategy and equilibrium inverse bidding strategy be denoted by η_i and υ_i respectively. Note that $\eta_i(t_i) = t_i$ and $\upsilon_i(b) = b$. Suppose (η_i, η_j) be a Bayesian equilibrium of a second-price auction. Suppose bidder j follows his equilibrium bidding strategy. The maximization problem of bidder i is

$$K_i(t_i) = \max_b F_j \circ \upsilon_j(b) \int_0^b F_j(\mathrm{d}t_j) u(t_i - t_j)$$

As $\eta(t_i) = t_i$, we have

$$t_i \in \arg\max_b F_j \circ \upsilon_j(b) \int_0^b F_j(\mathrm{d}t_j) u(t_i - t_j)$$

So, the value function can be re-written as

$$K_i(t_i) = F_j(t_i) \int_0^{t_i} F_j(\mathrm{d}t_j) u(t_i - t_j) \tag{28}$$

Using Envelope theorem, we have

$$\mathrm{D}K_i(t_i) = F_j(t_i) \int_0^{t_i} F_j(\mathrm{d}t_j) u'(t_i - t_j) \tag{29}$$

Consider $x, y \in \Re_+$. It can be proved that with increasing (resp. decreasing) absolute risk aversion, $u(x-y) = \mathrm{E}_{\hat{y}}(u(x-\hat{y}))$ implies $u'(x-y) >$ (resp. $<$)$\mathrm{E}_{\hat{y}}(u'(x-\hat{y}))$ and with constant absolute risk aversion, $u(x-y) = \mathrm{E}_{\hat{y}}(u(x-\hat{y}))$ implies $u'(x-y) = \mathrm{E}_{\hat{y}}(u'(x-\hat{y}))$.[6]

We state the following result.

Proposition 6. *Suppose (ψ_i, ψ_j) is a Bayesian equilibrium of a first-price auction and Assumption 1 is satisfied. Suppose (η_i, η_j) is a Bayesian equilibrium of a second-price auction. Then,*
(A) With increasing (resp. decreasing) absolute risk aversion, the weak (resp. strong) bidder prefers a first-price (resp. second-price) auction over a second-price (resp. first-price) auction.
(B) With constant absolute risk aversion, the weak (resp. strong) bidders prefers first-price (resp. second-price) auction over (resp. first-price) second-price auction.

Proof. Appendix B ■

The above result holds for $t_i > r$ for every $i \in N$. It states that the weak bidder prefers first-price auction over second-price auction under increasing and constant absolute risk aversion. Whereas the strong bidder prefers second-price auction over first-price auction under constant and decreasing absolute risk aversion.

[6]The proof of the above result can be found in Lemma 1 of Maskin and Riley [9].

6 Conclusion

We have characterized all Bayesian equilibria in a first-price auction under weaker assumptions. The presumption of a continuous and a monotone pure strategy equilibrium has not been made while characterizing the equilibria. We have shown that the necessary conditions for an equilibrium are continuity, strict monotonicity and pure strategy.

By considering one weak and one strong bidder, we have shown that the weak bidder bids more aggressively and produces a weaker bid distribution than the strong bidder. Next, we have compared the bidding behavior of symmetric and asymmetric bidders and shown that the strong (resp. weak) bidder bids more (resp. less) aggressively when playing against a strong (resp. weak) bidder than against a (resp. strong) weak bidder. Moreover, the strong (resp. weak) bidder produces a weaker (resp. stronger) bid distribution than the weak (resp. strong) bidder when both are playing against a strong (resp. weak) bidder.

We have computed the general expression for seller's expected revenue and established a revenue ranking for the first-price and second-price auction. The revenue from a first-price auction is more than from a second-price auction when one bidder is weak and the other is strong.

Finally, we have shown that with increasing (resp. decreasing) absolute risk aversion, the weak (resp. strong) bidder prefers first-price (resp. second-price) auction and; with constant absolute risk aversion, the weak (resp. strong) bidder prefers first-price (resp. second-price) auction.

A Appendix: Proof of Theorem 1

Lemma A.1. *Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. Then, $U_i(\psi_i, \psi_j, t_i) > 0$ and $\Pr(i \text{ wins}|t_i) > 0$ for every $t_i \in T_i - \{0\}$ and $i \in N$.*

Proof. Consider any i with $t_i > 0$ such that $U_i(\psi_i, \psi_j, t_i) = 0$. Then, $\Pr(i \text{ wins}|t_i, b_i) = 0$. This implies $\Pr(b_j > t_i) = 1$. As $\Pr(t_j < t_i) > 0$, it follows $U_j(\psi_i, \psi_j, t_j) < 0$ which is a contradiction. Hence, $U_i(\psi_i, \psi_j, t_i) > 0$ for every $t_i \in T_i - \{0\}$. As $U_i(\psi_i, \psi_j, t_i) > 0$, it follows $\Pr(i \text{ wins}|t_i) > 0$ for every $t_i \in T_i - \{0\}$ and $i \in N$. ∎

Lemma A.2. *Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. Then, $\Lambda_i(0) = \Omega_i(0) = 0$ and there exists $\bar{b}$ such that $\Lambda_i(a_i) = \Omega_i(a_i) = \bar{b}$ for every $i \in N$.*

Proof. Let $\underline{g}_i = \inf\{g_i\}$ and $\underline{g} = \max\{\underline{g}_i, \underline{g}_j\}$. We show $\underline{g}_i = 0$ for every $i \in N$. We show by contradiction. Suppose $\underline{g} > 0$. Without loss of generality, assume $\underline{g} = \underline{g}_i$. Then, for $t_i \in (0, \underline{g})$, $\Pr(i \text{ wins}|t_i) > 0$ and $U_i(\psi_i, \psi_j, t_i) < 0$ which contradicts Lemma A.1. Hence, $\underline{g}_i = 0$ for every $i \in N$. As $\underline{g}_i = 0$, is follows $\Lambda_i(0) = \Omega_i(0) = 0$ for every $i \in N$.

Let $\bar{g}_i = \sup\{g_i\}$ and $\bar{g} = \max\{\bar{g}_i, \bar{g}_j\}$. We show there exists $\bar{b}$ such that $\bar{g}_i = \bar{b}$. We show by contradiction. Suppose $\bar{g}_i \neq \bar{g}_j$. Without loss of generality, suppose $\bar{g} = \bar{g}_i > \bar{b}$. Then, $\Pr(i \text{ wins}|t_i = a_i, b_i = \bar{g}) = 1$. This implies there exists $\epsilon > 0$ such that $U_i(\bar{g} - \epsilon, \psi_j, a_i) > U_i(\bar{g}, \psi_j, a_i)$, which is a cntradiction to Definition 1. Hence, $g_i = \bar{b}$ for every $i \in N$. ■

Lemma A.3. *Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. Then, $\Pr(i \text{ wins}|t_i)$ is non-decreasing in t_i.*

Proof. Consider $t_i, t_i' \in T_i$ such that $t_i' > t_i$. Then, by the definition of a Bayesian equilibrium, it follows

$$U_i(\psi_i(t_i, .), \psi_j, t_i) = \int \psi_i(t_i, \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)\pi_i(t_i, b_i, b_j) \geq$$
$$U_i(\psi_i(t_i', .), \psi_j, t_i) = \int \psi_i(t_i', \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)\pi_i(t_i, b_i, b_j)$$

As $u(.)$ is strictly increasing and concave, it follows $u(t_i - b_i) \geq u(t_i' - b_i) - u(t_i' - t_i)$. This implies $\pi_i(t_i, b_i, b_j) \geq \pi_i(t_i', b_i, b_j) - I_i u(t_i' - t_i)$ such that $I_i = 1/|Z|$ if $i \in Z$ and $I_i = 0$ if $i \notin Z$. As $U_i(\psi_i(t_i, .), \psi_j, t_i) \geq U_i(\psi_i(t_i', .), \psi_j, t_i)$ and $\pi_i(t_i, b_i, b_j) \geq \pi_i(t_i', b_i, b_j) - I_i u(t_i' - t_i)$, it follows that

$$\int \psi_i(t_i, \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)\pi_i(t_i, b_i, b_j) \geq \int \psi_i(t_i', \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)\pi_i(t_i', b_i, b_j) - u(t_i' - t_i) I_i \int \psi_i(t_i', \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)$$

This implies

$$U_i(\psi_i(t_i, .), \psi_j, t_i) \geq U_i(\psi_i(t_i', .), \psi_j, t_i') - u(t_i' - t_i) \Pr(i \text{ wins}|t_i')$$

Interchanging the roles of t_i and t_i', we get

$$U_i(\psi_i(t_i', .), \psi_j, t_i') \geq U_i(\psi_i(t_i, .), \psi_j, t_i) + u(t_i' - t_i) \Pr(i \text{ wins}|t_i)$$

The above two inequalities imply $\Pr(i \text{ wins}|t_i') \geq \Pr(i \text{ wins}|t_i)$. ■

Lemma A.4. *Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. Then, $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are non-decreasing in t_i for every $t_i \in T_i$ and $i \in N$.*

Proof. We show $\Lambda_i(t_i') \geq \Omega_i(t_i)$ such that $t_i' > t_i$. We show by contradiction. Suppose there exists some i such that $\Omega_i(t_i) > \Lambda_i(t_i')$. Consider two sequences $(b_n)_{n=1}^{\infty}$ and $(b_n')_{n=1}^{\infty}$ such that $b_n \downarrow \Omega_i(t_i)$ and $b_n' \uparrow \Lambda_i(t_i')$ for every $n \in \mathbb{N}$. By Definition 1, we have

$$U_i(b_n, \psi_j, t_i) = \Pr(i \text{ wins}|b_n)u(t_i - b_n) \geq \Pr(i \text{ wins}|b_n')u(t_i - b_n') = U_i(b_n', \psi_j, t_i)$$

and,

$$U_i(b_n', \psi_j, t_i') = \Pr(i \text{ wins}|t_i')u(t_i' - b_n') \geq \Pr(i \text{ wins}|t_i')u(t_i' - b_n) = U_i(b_n, \psi_j, t_i')$$

Adding the above two inequalities, we get

$$\Pr(i \text{ wins}|b_n')\{u(t_i' - b_n') - u(t_i - b_n')\} \geq \Pr(i \text{ wins}|b_n)\{u(t_i' - b_n) - u(t_i - b_n)\}$$

As $u(t_i' - b_n') - u(t_i - b_n') \leq u(t_i' - b_n) - u(t_i - b_n)$, it follows $\Pr(i \text{ wins}|b_n') \geq \Pr(\imath \text{ wins}|b_n)$. As $b_n > b_n'$, it follows from Lemma A.3 that $\Pr(i \text{ wins} b_n) \geq \Pr(i \text{ wins}|b_n')$. As $\Pr(i \text{ wins}|b_n') \geq \Pr(i \text{ wins}|b_n)$ and $\Pr(i \text{ wins}|b_n) \geq \Pr(i \text{ wins}|b_n')$, it follows $\Pr(i \text{ wins}|b_n) = \Pr(i \text{ wins}|b_n')$. As $U_i(b_n, \psi_j, t_i) \geq U_i(b_n', \psi_j, t_i')$, it follows $u(t_i - b_n) \geq u(t_i - b_n')$, which is a contradiction. Hence, $\Lambda_i(t_i') \geq \Omega_i(t_i)$.

We show that $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are non-decreasing in t_i. As $\Omega_i(t_i') \geq \Lambda_i(t_i)$, $\Lambda_i(t_i) \leq \Omega_i(t_i)$ and $\Lambda_i(t_i') \leq \Omega_i(t_i')$, it follows $\Lambda_i(t_i') \geq \Lambda_i(t_i)$ and $\Omega_i(t_i') \geq \Omega_i(t_i)$ for every $t_i, t_i' \in T_i$ such that $t_i' > t_i, i \in N$. ■

Lemma A.5. *Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. Then, $\Lambda_i(t_i) > 0$ and $\Omega_i(t_i) > 0$ for every $t_i \in T_i - \{0\}$ and $i \in N$.*

Proof. We show by contradiction. Suppose $\Lambda_i(t_i) = 0$ for some $i \in N$. Consider a sequence $(b_n)_{n=1}^{\infty}$ such that $b_n \uparrow \Lambda_i(t_i)$ for every $n \in \mathbb{N}$. Then, $U_i(b_n, \psi_j, t_i) = u(t_i - b_n)\Pr(i \text{ wins}|b_n)$ for every $n \in \mathbb{N}$. As $\lim_{n\to\infty} U_i(b_n, \psi_j, t_i) > 0$, it follows that $\lim_{n\to\infty} \Pr(i \text{ wins}|t_i, b_n) > 0$. As $\Lambda_i(t_i) = 0$, it follows from Lemma A.4 that $\Lambda_i(t_i') = 0$ for every $t_i' \in (0, t_i)$. As $\Omega_i(t_i') \leq \Lambda_i(t_i)$ (from Lemma A.4), it follows $\Omega_i(t_i) = 0$. As $\lim_{n\to\infty} \Pr(i \text{ wins}|t_i, b_n) > 0$, it follows that there is a strictly positive probability of a tie. This implies $U_i(\epsilon, \psi_j, t_i') > U_i(0, \psi_j, t_i')$ for a small enough $\epsilon > 0$ and for every $t_i' \in (0, t_i)$, which is a contradiction. Hence, $\Lambda_i(t_i) > 0$ and $\Omega_i(t_i) > 0$ for every $t_i \in T_i - \{0\}$ and $i \in N$. ■

Lemma A.6. *Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. Then, $\beta_i(\{b\}) = 0$ and $\Pr(i \text{ wins}|b_i)$ is continuous in every $b_i \in B_i$ and $i \in N$.*

Proof. Suppose $\varphi_i(T_i, \{b\}) > 0$ for some $i \in \{s, w\}$ and for every $b > 0$. We show that for every $\epsilon > 0$, $\varphi_j(T_j, (b-\epsilon, b]) > 0$. We show by contradiction. Suppose there exists a $\epsilon > 0$ such that $\varphi_j(T_j, (b-\epsilon, b]) = 0$. As $\varphi_i(T_i, \{b\}) > 0$, it follows $U_i(\psi_i, \psi_j | t_i, b) > 0$. As $\varphi_j(T_j, (b-\epsilon, b]) = 0$, it follows that there exists some ϵ' such that $U_i(b', \psi_j, t_i) > U_i(b, \psi_j, t_i)$ for every $b' \in (b-\epsilon+\epsilon', b)$, which is a contradiction. Hence, for every $\epsilon > 0$, $\varphi_j(T_j, (b-\epsilon, b]) > 0$.

Suppose $\varphi_i(T_i, (b-\epsilon, b]) > 0$ for some $i \in N, b > 0$ and for every $\epsilon > 0$. We show $\varphi_j(T_j, (b-\epsilon, b]) = 0$. We show by contradiction. Suppose $\varphi_j(T_j, (b-\epsilon, b]) > 0$. Consider a set $W_\epsilon \in \mathcal{B}(A_i)$ such that $\Pr(h_i(t_i) \in (b-\epsilon, b]) > 0$ for every $\epsilon > 0$ and $t_i \in W_\epsilon$. Then, $\epsilon' > \epsilon$ implies $W_\epsilon \subset W'_\epsilon$. We show there exists a $\hat{\epsilon} > 0$ such that $\hat{\epsilon} < b$ and $t_i > \hat{\epsilon} + b$ for every $t_i \in W_{\hat{\epsilon}}$. We show by contradiction. Suppose, for every W_n such that $n \in \mathbb{N}$, there exists some $t_{in} \in W_n$ such that $t_{in} \leq n + b$. It can be verified that $t_{in} > n - b$ for every $t_{in} \in W_n$ and $n \in \mathbb{N}$. As $U_i(\psi_i, \psi_j, t_i) = U_i(\psi_i, \psi_j, t_i | h_i(t_i) \in (b-\epsilon, b])$, it follows $U_i(\psi_i, \psi_j, t_{in}) \leq u(2n)$. As $n - b \leq t_{in} \leq n + b$, n approaches to zero implies t_{in} approaches b. Then, $\lim_{n\to 0} U_i(\psi_i, \psi_j | t_{in}) \leq \lim_{n \to 0} u(2n)$. This implies $U_i(\psi_i, \psi_j, b) = 0$, which is a contradiction, as $U_i(\psi_i, \psi_j, t_i) > 0$ for every $t_i > 0$. Hence, such a $\hat{\epsilon}$ exists. Consider $\epsilon > 0$ and $\delta > 0$ such that $\epsilon < \hat{\epsilon}$ and

$$\delta < \hat{\epsilon} - u^{-1}\left(u(\epsilon)\frac{\beta_j \circ (0, b) + 0.5\beta_j \circ \{b\}}{\beta_j \circ (0, b]}\right)$$

As $\epsilon \to 0$, it follows such a δ exists. Then,

$$\begin{aligned} U_i(\psi_i, \psi_j | t_i, g_i(t_i) \in (b-\epsilon, b]) &\leq u(t_i - (b-\epsilon))(\beta_j \circ (0, b) + 0.5\beta_j \circ \{b\}) \\ &< u(t_i - (b+\delta))\beta_j \circ (0, b] \\ &= U_i(\psi_i, \psi_j | t_i, b+\delta) \end{aligned}$$

which is a contradiction. Hence, $\varphi_j(T_j, (b-\epsilon, b]) = 0$.

We show $\beta_i(\{b\}) = 0$ for every $b > 0$ and $i \in N$. Suppose $\beta_i(\{b\}) > 0$ for some $i \in N$. Then, $\varphi_j(T_i, (b-\epsilon, b]) > 0$ for every $\epsilon > 0$. This implies $\varphi_i(T_i, \{b\}) = 0$, which is a contradiction. Hence, $\beta_i(\{b\}) = 0$ for every $b > 0$ and $i \in N$.

We show $\Pr(i \text{ wins} | b_i)$ is continuous in every $b_i \in B_i$. We know $\Pr(i \text{ wins} | b_i) = \beta_j \circ (0, b_i]$. As $\beta_j \circ \{b_i\} = 0$ (from Lemma A.2) for every b_i, it follows that $\beta_j \circ (0, b_i]$ is continuous in every b_i. Hence, $\Pr(i \text{ wins} | b_i)$ is continuous in every $b_i \in B_i$. ■

Lemma A.7. *Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. Then, $\Lambda_i(t_i)$ is left continuous and $\Omega_i(t_i)$ is right continuous.*

Proof. We show $\Lambda_i(t_i) = \lim_{t'_i \uparrow t_i} \Omega_i(t'_i)$ and $\Omega_i(t_i) = \lim_{t'_i \downarrow t_i} \Lambda_i(t'_i)$ for every $t'_i \in (0, a_i]$. As Ω_i is non-decreasing in t_i, it follows $\lim_{t'_i \uparrow t_i} \Omega_i(t'_i) =$

$\sup_{t_i'<t_i} \Omega_i(t_i')$. Then, $\lim_{t_i'\uparrow t_i} \Omega_i(t_i') \leq \Lambda_i(t_i)$. Consider a sequence $(t_{in})_{n=1}^{\infty}$ such that $t_{in} \uparrow t_i$ and $\lim_{n\to\infty} \Omega_i(t_{in}) = \sup_{t_i'<t_i} \Omega_i(t_i')$ for every $n \in \mathbb{N}$. Then, $U_i(\Omega_i(t_{in}), \psi_j, t_{in}) = u(t_{in} - \Omega_i(t_{in})) \Pr(i \text{ wins}|\Omega_i(t_{in}))$ for every $n \in \mathbb{N}$. As $n \to \infty$, we get $U_i(\psi_i, \psi_j, t_i) = u(t_i - \sup_{t_i'<t_i} \Omega_i(t_i')) \Pr(i \text{ wins}| \sup_{t_i'<t_i} \Omega_i(t_i'))$. Then, $\Lambda_i(t_i) = \inf \sup_{t_i'<t_i} \Omega_i(t_i')$. This implies $\Lambda_i(t_i) \leq \sup_{t_i'<t_i} \Omega_i(t_i')$. As $\Lambda_i(t_i) \geq \sup_{t_i'<t_i} \Omega_i(t_i')$ and $\Lambda_i(t_i) \leq \sup_{t_i'<t_i} \Omega_i(t_i')$, it follows $\Lambda_i(t_i) = \sup_{t_i'<t_i}$ $\Omega_i(t_i')$ for every $t_i' \in (0, a_i]$. Hence, $\Lambda_i(t_i) = \lim_{t_i'\uparrow t_i} \Omega_i(t_i')$. Similarly, it can be shown $\Omega_i(t_i) = \lim_{t_i'\downarrow t_i} \Lambda_i(t_i')$ for every $t_i' \in (0, a_i]$. As Λ_i is left continuous, it follows if Λ_i is continuous, then $\Lambda_i(t_i) = \Omega_i(t_i)$ for every $t_i \in T_i$. As Ω_i is right continuous, it follows if Ω_i is continuous, then $\Lambda_i(t_i) = \Omega_i(t_i)$ for every $t_i \in T_i$. ■

Lemma A.8. *Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. Then, $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are strictly increasing in t_i for every $i \in N$.*

Proof. We show by contradiction. Suppose Λ_i and Ω_i are not strictly increasing. From Lemma A.4, $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are non-decreasing in t_i. Suppose $\Omega_i(t_i) = c$ for all $t_i \in W_i \subset T_i$. Then, $\Lambda_i(t_i) = \Omega_i(t_i)$ for every $t_i \in W_i$. But, $\beta_i \circ \{\Lambda_i(t_i)\} = 0$ for some $t_i \in W_i$, which contradicts Lemma A.6. Therefore, $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are strictly increasing in t_i for every $t_i \in T_i$ and $i \in N$. ■

Lemma A.9. *Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. Then, $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are continuous for every $t_i \in T_i$ and $i \in N$.*

Proof. We show that Λ_i is right continuous. We show by contradiction. Suppose Λ_i is not right continuous at some $t_i \in T_i$. Then, for small enough $\epsilon > 0$, $\Pr(b_i \in (\Lambda_i(t_i), \Lambda_i(t_i + \epsilon)) = 0$. This implies $\Pr(b_j \in (\Lambda_i(t_i), \Lambda_i(t_i + \epsilon)) = 0$. As $\Pr(b_j \in (\Lambda_i(t_i), \Lambda_i(t_i + \epsilon)) = 0$, it follows $U_i(\Lambda_i(t_i), \psi_j, t_i) > U_i(\Omega_i(t_i), \psi_j, t_i)$, which is a contradiction. Hence, Λ_i is right continuous at t_i. Therefore, $\Lambda_i(t_i)$ is continuous for every $t_i \in T_i$ and $i \in N$. As $\Lambda_i(t_i) = \Omega_i(t_i)$ if Λ_i is continuous, it follows that Ω_i is continuous in every $t_i \in T_i$ and $i \in N$. ■

Proof of Theorem 1. Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. From Lemma A.9, it follows $\Lambda_i(t_i) = \Omega_i(t_i)$ for every $t_i \in T_i$ and $i \in N$. As Σ_i is a function (Lemma A.9) and, Λ_i and Ω_i are continuous, it follows that ψ_i is pure and continuous. As Λ_i and Ω_i are strictly increasing (Lemma A.8), it follows ψ_i is strictly increasing. From Lemmas A.2 and A.9, it follows $\psi_i(0) = 0$ and there exists $\bar{b}$ such that $\psi_i(a_i) = \bar{b}$ for every $i \in N$.

As ψ_i is pure, strictly increasing and continuous, it follows the inverse of ψ_i exists and is strictly increasing and continuous. Suppose bidder j follows his equilibrium strategy. Then, the maximization problem of bidder i is $\max_b F_j \circ \psi_j^{-1}(b) u(t_i - b)$. The first-order condition implies (3).

Conversely, suppose $\psi_i(0) = 0$, $\psi_i(a_i) = \bar{b}$ and the system of differential equations given by (3) are satisfied for every $i \in N$. Suppose bidder j follows ψ_j. We show that the best response for bidder i is ψ_i. Suppose bidder i with type t_i bids b. The expected utility of bidder i is $U_i(t_i, b) = F_j \circ \psi_i^{-1}(b) u(t_i - b)$. Using the monotonic transformation $V_i(t_i, b) = \log U_i(t_i, b)$, the expected utility of bidder i is $V_i(t_i, b) = \log F_j \circ$
$psi_j^{-1}(b) + \log u(t_i - b)$. The first-order derivative is

$$\mathrm{D}_b V_i(t_i, b) = \frac{\mathrm{D}F_j \circ \theta_j(b)}{F_j \circ \psi_j^{-1}(b)} - \frac{u'(t_i - b)}{u(t_i - b)}$$

The expected utility of bidder i gets maximized when $\mathrm{D}_b(t_i, b) = 0$, that is,

$$\frac{F_j \circ \psi_j^{-1}(b)}{\mathrm{D}F_j \circ \psi_j^{-1}(b)} = \frac{u(\psi_i^{-1}(b) - b)}{u'(\psi_i^{-1}(b) - b)}$$

Suppose bidder i over-bids by bidding b' such that $\psi_i^{-1}(b') > t_i$. As $\psi_i^{-1}(b') > t_i$, it follows $u'(t_i - b')/u(t_i - b') > u'(\psi_i^{-1}(b') - b')/u(\psi_i^{-1}(b') - b')$. Then,

$$\begin{aligned}
\mathrm{D}_{b'} V_i(t_i, b') &= \frac{\mathrm{D}F_j \circ \psi_j^{-1}(b')}{F_j \circ \psi_j^{-1}(b')} - \frac{u'(t_i - b')}{u(t_i - b')} \\
&< \frac{\mathrm{D}F_j \circ \psi_j^{-1}(b')}{F_j \circ \psi_j^{-1}(b')} - \frac{u'(\psi_i^{-1}(b') - b')}{u(\psi_i^{-1}(b') - b')} \\
&= 0
\end{aligned}$$

So, it is not profitable for bidder i to over-bid. Suppose, on the other hand, bidder i under-bids by bidding b' such that $\theta_i(b') < t_i$. Then,

$$\begin{aligned}
\mathrm{D}_{b'} V_i(t_i, b') &= \frac{\mathrm{D}F_j \circ \psi_j^{-1}(b')}{F_j \circ \psi_j^{-1}(b')} - \frac{u'(t_i - b')}{u(t_i - b')} \\
&> \frac{\mathrm{D}F_j \circ \psi_j^{-1}(b')}{F_j \circ \psi_j^{-1}(b')} - \frac{u'(\psi_i^{-1}(b') - b')}{u(\psi_i^{-1}(b') - b')} \\
&= 0
\end{aligned}$$

So, it is not profitable for bidder i to under-bid. Hence, it is optimal for bidder i to choose $\theta_i(b') = t_i$. Therefore, ψ is a Bayesian equilibrium. ■

B Appendix: Proofs of Proposition 1-6 and Corollary 1-2

Proof of Proposition 1. We prove by contradiction. Suppose $\theta_w(b) \not< \theta_s(b)$ for every $b \in [0, \bar{b}]$. As $\theta_w(\bar{b}) < \theta_s(\bar{b})$ (by assumption), it follows that there exists some $\epsilon > 0$ such that $\theta_w(\bar{b} - \epsilon) < \theta_s(\bar{b} - \epsilon)$. As $\theta_w(b) \not< \theta_s(b)$ for every $b \in [0, \bar{b}]$ and $\theta_w(\bar{b}) < \theta_s(\bar{b})$, it follows that there exists some b^* such that $\theta_w(b^*) = \theta_s(b^*)$ and $\theta_w(b) < \theta_s(b)$ for every $b \in (b^*, \bar{b}]$. At $b = b^*$, from (4), we have

$$\frac{F_w \circ \theta_w(b)}{\mathrm{D}F_w \circ \theta_w(b)} = \frac{F_s \circ \theta_s(b)}{\mathrm{D}F_s \circ \theta_s(b)}$$

Equivalently,

$$\frac{F_s \circ \theta_s(b)}{f_s \circ \theta_s(b)} \mathrm{D}\theta_w(b) = \frac{F_w \circ \theta_w(b)}{f_w \circ \theta_w(b)} \mathrm{D}\theta_s(b)$$

From the assumption of conditional stochastic dominance, at $b = b^*$, it follows that $\mathrm{D}\theta_w(b) > \mathrm{D}\theta_s(b)$. Then, there exists a $\delta > 0$ such that $\theta_w(b^* + \delta) > \theta_s(b^* + \delta)$, which is a contradiction. Hence, no such b^* exists. Therefore, $\theta_w(b) < \theta_s(b)$ for every $b \in [0, \bar{b}]$. ■

Proof of Proposition 2. From Proposition 1, we know that $\theta_w(b) < \theta_s(b)$. As $u' > 0$ and $u'' < 0$, we have $u(\theta_w(b) - b) < u(\theta_s(b) - b)$ and $u'(\theta_w(b) - b) > u'(\theta_s(b) - b)$. From (1.2), it follows that

$$\frac{F_s \circ \theta_s(b)}{\mathrm{D}F_s \circ \theta_s(b)} < \frac{F_w \circ \theta_w(b)}{\mathrm{D}F_w \circ \theta_w(b)}$$

This implies

$$\mathrm{D}\left(\frac{F_s \circ \theta_s(b)}{F_w \circ \theta_w(b)}\right) > 0$$

As $F_s \circ \theta_s(\bar{b}) = F_w \circ \theta_w(\bar{b}) = 1$ and $\mathrm{D}(F_s \circ \theta_s(b)/F_w \circ \theta_w(b)) > 0$, it follows that $(F_s \circ \theta_s(b)/F_w \circ \theta_w(b)) < 1$. Therefore, $F_s \circ \theta_s(b) < F_w \circ \theta_w(b)$ for every $b \in (0, \bar{b})$. ■

Proof of Corollary 1. (A) Suppose $\xi_s(t) < \xi_w(t)$ for every $T_w - \{0\}$. Then, by (15), it follows $\mathrm{D}\xi_s(t) > \mathrm{D}\xi_w(t)$. This implies there exists $\epsilon > 0$ such that $\xi_s(t + \epsilon) > \xi_w(t + \epsilon)$, which is a contradiction. Hence, $\xi_w(t) < \xi_s(t)$ for every $t \in T_w - \{0\}$.
(B) As $\lambda_w(b) > \lambda_s(b)$ for every $b \in (0, \bar{b}_w)$, it follows from (12)

$$\frac{F_w \circ \lambda_w(b)}{\mathrm{D}F_w \circ \lambda_w(b)} > \frac{F_s \circ \lambda_s(b)}{\mathrm{D}F_s \circ \lambda_s(b)}$$

This implies

$$\mathrm{D}\frac{F_s \circ \lambda_s(b)}{F_w \circ \lambda_w(b)} > 0$$

As $F_s \circ \lambda_s(\bar{b}_s) = F_w \circ \lambda_w(\bar{b}_w) = 1$, it follows $F_s \circ \lambda_s(b) < F_w \circ \lambda_w(b)$. ■

Proof of Proposition 3. We prove (A) and (C). The proof of (B) and (D) are analogous to (A) and (C) respectively.
(A) As $\mathrm{D}^2\theta_s(0) > \mathrm{D}^2\lambda_s(0)$ (Lemma C.1), it follows that there exists $\epsilon > 0$ such that $\theta_s(b) > \lambda_s(b)$ for every $b \in (0, \epsilon)$. It suffices to show if $\theta_s(b) = \lambda_s(b)$ implies $\mathrm{D}\theta_s > \mathrm{D}\lambda_s(b)$, then $\theta_s(b) > \lambda_s(b)$ for every $b > 0$. Suppose $\theta_s(b) = \lambda_s(b)$. Then, from (6), (13) and Proposition 1, it follows $\mathrm{D}\theta_s(b) > \mathrm{D}\lambda_s(b)$. Hence, $\theta_s(b) > \lambda_s(b)$.
(C) As $\theta_s(b) > \lambda_s(b)$, from (4) and (12), it follows

$$\frac{F_w \circ \theta_w(b)}{\mathrm{D}F_w \circ \theta_w(b)} > \frac{F_s \circ \lambda_s(b)}{\mathrm{D}F_s \circ \lambda_s(b)}$$

This implies

$$\mathrm{D}\frac{F_s \circ \lambda_s(b)}{F_w \circ \theta_w(b)} > 0$$

As $F_w \circ \theta_w(\bar{b}) = F_s \circ \lambda_s(\bar{b}_s) = 1$, it follows $F_w \circ \theta_w(b) > F_s \circ \lambda_s(b)$. ■

Proof of Proposition 4. Suppose $\theta_i(c) \leq \hat{\theta}_i(c)$ and $\theta_j(c) \leq \hat{\theta}_j(c)$ for every $c \in (0, \min\{\bar{b}, \hat{b}\})$. We show that there exists $\epsilon > 0$ such that $\theta_i(b) < \hat{\theta}_i(b)$ and

$$\frac{F_j \circ \theta_j(b)}{\hat{F}_j \circ \hat{\theta}_j(b)} < \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} \equiv \pi(c) \tag{30}$$

for every $b \in [c - \epsilon, c)$. As $\theta_j(c) \leq \hat{\theta}_j(c)$, it follows $F_j \circ \theta_j(b) < F_j \circ \hat{\theta}_j(c)$ for every $b < c$. Dividing both sides by $\hat{F}_j \circ \hat{\theta}_j(b)$, we get

$$\frac{F_j \circ \theta_j(b)}{\hat{F}_j \circ \hat{\theta}_j(b)} < \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(b)}$$

Case 1: $\theta_i(c) < \hat{\theta}_i(c)$ and $\theta_j(c) < \hat{\theta}_j(c)$
It is straightforward to see that there exists $\epsilon > 0$ such that $\theta_i(b) < \hat{\theta}_i(b)$ and

$$\frac{F_j \circ \theta_j(b)}{\hat{F}_j \circ \hat{\theta}_j(b)} < \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} \equiv \pi(c)$$

for every $b \in [c-\epsilon, c)$.

Case 2: $\theta_i(c) = \hat{\theta}_i(c)$ and $\theta_j(c) < \hat{\theta}_j(c)$
From (4), $\mathrm{D}\theta_i(c)$ is strictly decreasing in $\theta_j(c)$. This implies $\mathrm{D}\theta_i(c) > \mathrm{D}\hat{\theta}_i(c)$. Then, there exists $\epsilon > 0$ such that $\theta_i(b) < \hat{\theta}_i(b)$ and

$$\frac{F_j \circ \theta_j(b)}{\hat{F}_j \circ \hat{\theta}_j(b)} < \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} \equiv \pi(c)$$

for every $b \in [c-\epsilon, c)$.

Case 3: $\theta_i(c) < \hat{\theta}_i(c)$ and $\theta_j(c) = \hat{\theta}_j(c)$
It is straightforward to see that there exists $\epsilon > 0$ such that $\theta_i(b) < \hat{\theta}_i(b)$ and

$$\frac{F_j \circ \theta_j(b)}{\hat{F}_j \circ \hat{\theta}_j(b)} < \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} \equiv \pi(c)$$

for every $b \in [c-\epsilon, c)$.

Case 4: $\theta_i(c) = \hat{\theta}_i(c)$ and $\theta_j(c) = \hat{\theta}_j(c)$
From (6), $\mathrm{D}\theta_i(c) = \mathrm{D}\hat{\theta}_i(c)$ and the assumption of conditional stochastic dominance, it follows $\mathrm{D}\theta_j(c) > \mathrm{D}\hat{\theta}_j(c)$. From (7), $\mathrm{D}^2 \log F_i \circ \theta_i(c)$ is strictly decreasing in $\mathrm{D}\theta_j(c)$. This implies $\mathrm{D}^2 \log F_i \circ \theta_i(c) < \mathrm{D}^2 \log F_i \circ \hat{\theta}_i(c)$. So, there exists $\epsilon > 0$ such that $\theta_i(b) < \hat{\theta}_i(b)$ and

$$\frac{F_j \circ \theta_j(b)}{\hat{F}_j \circ \hat{\theta}_j(b)} < \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} \equiv \pi(c)$$

for every $b \in [c-\epsilon, c)$.

Define $\vartheta = \inf\{g \in [0, c-\epsilon] \mid \theta_i(b) < \hat{\theta}_i(b)$ and $F_j \circ \hat{\theta}_j(b) < \pi(c) F_j \circ \hat{\theta}_j(b)$ for every $b \in (g, c)\}$. We show $\vartheta = 0$. We show by contradiction. Suppose $\vartheta > 0$. Then, either $\theta_i(\vartheta) = \hat{\theta}_i(\vartheta)$ or $F_j \circ \hat{\theta}_j(\vartheta) = \pi(c) F_j \circ \hat{\theta}_j(c)$. As $\theta_i(b) < \hat{\theta}_i(b)$ for every $b \in [\vartheta, c)$, it follows from (4) and (17) that

$$\frac{F_j \circ \theta_j(b)}{DF_j \circ \theta_j(b)} < \frac{\hat{F}_j \circ \hat{\theta}_j(b)}{D\hat{F}_j \circ \hat{\theta}_j(b)}$$

This implies $\mathrm{D} \log F_j \circ \theta_j(b) > \mathrm{D} \log(\pi(c) \hat{F}_j \circ \hat{\theta}_j(b))$. Then, as $\vartheta < c - \epsilon$, it follows $\log(F_j \circ \theta_j(c-\epsilon)) - \log F_j \circ \theta_j(\vartheta) > \log(\pi(c)\hat{F}_j \circ \hat{\theta}_j(c-\epsilon)) - \log(\pi(c)\hat{F}_j \circ \hat{\theta}_j(\vartheta))$. Rearranging the terms, we get $\log(\pi(c)\hat{F}_j \circ \hat{\theta}_j(\vartheta)) - \log F_j \circ \theta_j(\vartheta) >$

$\log(\pi(c)\hat{F}_j \circ \hat{\theta}_j(c-\epsilon)) - \log(F_j \circ \theta_j(c-\epsilon))$. From the definition of ϑ, it follows that $\log(\pi(c)\hat{F}_j \circ \hat{\theta}_j(c-\epsilon)) - \log(F_j \circ \theta_j(c-\epsilon)) > 0$. This implies $\log(\pi(c)\hat{F}_j \circ \hat{\theta}_j(\vartheta)) - \log F_j \circ \theta_j(\vartheta) > 0$. Taking exponential both sides, we get $\pi(c)\hat{F}_j \circ \hat{\theta}_j(\vartheta) > F_j \circ \theta_j(\vartheta)$. So, $\theta_i(\vartheta) = \hat{\theta}_i(\vartheta)$ must be the case. As $F_j \circ \theta_j(\vartheta) < \pi(c)\hat{F}_j \circ \hat{\theta}_j(\vartheta)$ and $\vartheta < c$, it follows from the assumption of conditional stochastic dominance that

$$\frac{F_j \circ \theta_j(\vartheta)}{\hat{F}_j \circ \hat{\theta}_j(\vartheta)} < \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} < \frac{F_j \circ \hat{\theta}_j(\vartheta)}{\hat{F}_j \circ \hat{\theta}_j(\vartheta)}$$

This implies $F_j \circ \theta_j(\vartheta) < F_j \circ \hat{\theta}_j(\vartheta)$, or equivalently $\theta_j(\vartheta) < \hat{\theta}_j(\vartheta)$. As $\theta_i(\vartheta) = \hat{\theta}_i(\vartheta)$ and $\theta_j(\vartheta) < \hat{\theta}_j(\vartheta)$, it follows from (6) that $\mathrm{D}\theta_i(\vartheta) > \mathrm{D}\hat{\theta}_i(\vartheta)$. This implies that there exists $\delta > 0$ such that $\theta_i(\vartheta + \delta) > \hat{\theta}_i(\vartheta + \delta)$, which is a contradiction. Hence, $\vartheta = 0$. Therefore, $\theta_i(b) < \hat{\theta}_i(b)$ and $F_j \circ \theta_i(b) < \pi(c)\hat{F}_j \circ \hat{\theta}_j(b)$ for every $b \in (0, c)$.

We show that $\theta_i(c) \leq \hat{\theta}_i(c)$ and $\theta_j(c) \leq \hat{\theta}_j(c)$ cannot hold simultaneously. We show by contradiction. Suppose there exists $c \in (0, \bar{b})$ such that $\theta_i(c) \leq \hat{\theta}_i(c)$ and $\theta_j(c) \leq \hat{\theta}_j(c)$. As $\theta_i(c) \leq \hat{\theta}_i(c)$ and $\theta_j(c) \leq \hat{\theta}_j(c)$, it follows that $\theta_i(b) < \hat{\theta}_i(b)$ and $F_j \circ \theta_j(b) < \pi(c)\hat{F}_j \circ \hat{\theta}_j(b)$ for every $b \in (0, c)$. Taking the limits $b \downarrow 0$, we have $F_j(0) < \pi(c)\hat{F}_j(0)$. As $\hat{F}_j/F_j$ is strictly increasing, it follows $F_j(0) > \pi(c)\hat{F}_j(0)$, which is a contradiction. Therefore, $\theta_i(c) \leq \hat{\theta}_i(c)$ and $\theta_j(c) \leq \hat{\theta}_j(c)$ cannot hold simultaneously.

We show that $\hat{b} > \bar{b}$. We show by contradiction. Suppose $\hat{b} \leq \bar{b}$. Then, $\theta_i(\hat{b}) \leq \hat{\theta}_i(\hat{b})$ and $\theta_j(\hat{b}) \leq \hat{\theta}_j(\hat{b})$, a contradiction. Hence, $\hat{b} > \bar{b}$.

We show $\theta_i(b) > \hat{\theta}_i(b)$ for every $b \in [0, \bar{b}]$. We show by contradiction. Suppose $\theta_i(b) \not> \hat{\theta}_i(b)$ for every $b \in [0, \bar{b}]$. As $\theta_i(\bar{b}) > \hat{\theta}_i(\bar{b})$, it follows that there exists $\epsilon > 0$ such that $\theta_i(\bar{b} - \epsilon) > \hat{\theta}_i(\bar{b} - \epsilon)$. As $\theta_i(b) \not> \hat{\theta}_i(b)$ for every $b \in [0, \bar{b}]$ and $\theta_i(\bar{b}) > \hat{\theta}_i(\bar{b})$, it follows that there b^* such that $\theta_i(b^*) = \hat{\theta}_i(b^*)$ and $\theta_i(b) > \hat{\theta}_i(b)$ for every $b \in (b^*, \bar{b}]$. As $\theta_i(b^*) = \hat{\theta}_i(b^*)$ implies $\theta_j(b^*) > \hat{\theta}_j(b^*)$, at $b = b^*$, it follows from (4) and (17) that

$$\frac{\hat{F}_j \circ \hat{\theta}_j(b)}{\mathrm{D}\hat{F}_j \circ \hat{\theta}_j(b)} = \frac{F_j \circ \theta_j(b)}{\mathrm{D}F_j \circ \theta_j(b)}$$

Equivalently, at $b = b^*$,

$$\frac{\hat{F}_j \circ \hat{\theta}_j(b)}{\mathrm{D}\hat{F}_j \circ \hat{\theta}_j(b)}\mathrm{D}\theta_i(b) = \frac{F_j \circ \theta_j(b)}{\mathrm{D}F_j \circ \theta_j(b)}\mathrm{D}\hat{\theta}_i(b)$$

From the assumption of conditional stochastic dominance, at $b = b^*$, $\mathrm{D}\hat{\theta}_j(b) > \mathrm{D}\theta_j(b)$. Then, there exists $\delta > 0$ such that $\hat{\theta}_j(b^* + \delta) > \theta_j(b^* + \delta)$, which is a contradiction. Hence, $\theta_i(b) > \hat{\theta}_i(b)$ for every $b \in [0, \bar{b}]$.

We show $\hat{F}_j \circ \hat{\theta}_j(b) < F_j \circ \theta_j(b)$ for every $b \in (0, \bar{b})$. We will use the fact that $\theta_i(b) > \hat{\theta}_i(b)$ for every $b \in [0, \bar{b}]$. As $u' > 0$ and $u'' < 0$, we have $u(\theta_i(b) - b) > u(\hat{\theta}_i(b) - b)$ and $u'(\theta_i(b) - b) < u'(\hat{\theta}_i(b) - b)$. As $\theta_i(b) > \hat{\theta}_i(b)$, it follows from (4) and (17) that

$$\frac{F_j \circ \theta_j(b)}{\mathrm{D}F_j \circ \theta_j(b)} > \frac{\hat{F}_j \circ \hat{\theta}_j(b)}{\mathrm{D}\hat{F}_j \circ \hat{\theta}_j(b)}$$

This implies

$$\mathrm{D}\left(\frac{\hat{F}_j \circ \hat{\theta}_j(b)}{F_j \circ \theta_j(b)}\right) > 0$$

As $F_j \circ \theta_j(\bar{b}) = 1 = \hat{F}_j \circ \hat{\theta}_j(\hat{b})$, it follows $\hat{F}_j \circ \hat{\theta}_j(b) < F_j \circ \theta_j(b)$. ■

Proof of Corollary 2. As $\hat{F}_j \circ \hat{\theta}_j(b) < F_j \circ \theta_j(b)$ for every $b \in (0, \bar{b})$, it follows that $\hat{U}_j(t_j, \hat{\psi}_i, \hat{\psi}_j) < U_j(t_j, \psi_i, \psi_j)$. As $\hat{\theta}_i(b) < \theta_i(b)$ for every $b \in (0, \bar{b})$, it follows that $\hat{U}_i(t_i, \hat{\psi}_i, \hat{\psi}_j) < U_i(t_i, \psi_i, \psi_j)$. ■

Proof of Propostion 5. We show that

$$f_w(t) - \frac{F_s \circ \zeta_s(t) - F_s(t)}{W(\zeta_s(t), \psi_w(t), t)} \geq 0$$

As $\zeta_s(t) > t$, $u' > 0$ and $u'' \leq 0$, it follows $W(\zeta_s(t), \psi_w(t), t) \geq \zeta_s(t) - t$. It suffices to show

$$f_w(t) - \frac{F_s \circ \zeta_s(t) - F_s(t)}{\zeta_s(t) - t} \geq 0$$

A $\zeta_s(t) > t$, it follows from the mean value theorem that there exists a $\hat{t} \in (t, \zeta_s(t)]$ such that

$$\frac{F_s \circ \zeta_s(\hat{t}) - F_s(\hat{t})}{\zeta_s(\hat{t}) - \hat{t}} = f_s(\hat{t})$$

It suffices to show $f_w(t) \geq f_s(\hat{t})$. Suppose $\hat{t} > a_w$. Then, $f_w(t) \geq f_s(\hat{t})$ follows from Assumption 2. Suppose $\hat{t} < a_w$. Then,

$$\frac{f_s(\hat{t})}{F_s(\hat{t})} = \frac{f_w(\hat{t})}{F_w(\hat{t})} < \frac{f_w(t)}{F_w(t)}$$

The first inequality follows from Assumption 1. The second inequality follows from the fact that F is concave. As $\hat{t} \leq \zeta_s(t)$, it follows from Proposition 2 $F_s(\hat{t}) \leq F_s \circ \zeta_s(t) < F_s(t)$. Hence $f_w(t) > f_s(\hat{t})$. Therefore $P > Q$. ■

Proof of Proposition 6. We prove for increasing absolute risk aversion. Consider the weak bidder. Remark that $V_w(r) = 0 = K_w(r)$. We show $V_w(t_w) > K_w(t_w)$ for every $t_w > r$. It suffices to show if $V_w(t_w) = K_w(t_w)$, then $\mathrm{D}V_w(t_w) > \mathrm{D}K_w(t_w)$ for every $t_w > r$.

Suppose $V_w(t_w) = K_w(t_w)$. Then, from (27) and (29), $\mathrm{D}V_w(t_w) > \mathrm{D}K_w(t_w)$. Hence, the weak bidder prefers first-price auction over second-price auction. A similar proof holds for constant and decreasing absolute risk aversion. ■

C Appendix: Some results

Lemma C.1. *Suppose the profile of measurable functions (ψ_w, ψ_s) is a Bayesian equilibrium when one bidder is weak and the other is strong. Suppose the profiles of measurable functions (ξ_w, ξ_w) and (ξ_s, ξ_s) are a Bayesian equilibrium when both the bidders are weak and strong respectively. If Assumption 1 is satisfied, then $\mathrm{D}^2\theta_s(0) > \mathrm{D}^2\lambda_s(0)$ and $\mathrm{D}^2\theta_w(0) < \mathrm{D}^2\lambda_w(0)$.*

Proof. Let

$$y_i(t) = t\frac{f_i(t)}{F_i(t)}$$

Then,

$$y_i(0) = 1, y_i'(0) = \frac{f_i'(0)}{2f_i(0)} \text{ and } \frac{f_s'(0)}{f_s(0)} > \frac{f_w'(0)}{f_w(0)} \tag{31}$$

And,

$$y_i \circ \theta_i(b) = \theta_i(b)\frac{f_i \circ \theta_i(b)}{F_i \circ \theta_i(b)}$$

Using (4), we have

$$\mathrm{D}\theta_i(b) y_i \circ \theta_i(b) = \frac{\theta_i(b)}{w(\theta_j(b) - b)} \tag{32}$$

Similarly,

$$\mathrm{D}\lambda_i(b) y_i \circ \lambda_i(b) = \frac{\lambda_i(b)}{w(\lambda_i(b) - b)} \tag{33}$$

Taking the limit of b at 0, we have

$$\mathrm{D}\theta_w(0) = \mathrm{D}\theta_s(0) = \mathrm{D}\lambda_w(0) = \lambda_w(0) = 1 + \frac{1}{w'(0)} \tag{34}$$

Taking logarithms both sides of (32) and differentiating w.r.t. b, we have

$$\frac{\mathrm{D}^2\theta_i(b)}{\mathrm{D}\theta_i(b)} + \frac{y_i' \circ \theta_i(b)}{y_i \circ \theta_i(b)}\mathrm{D}\theta_i(b) = \frac{\mathrm{D}\theta_i(b)}{\theta_i(b)} - \frac{w'(\theta_j(b) - b)}{w(\theta_j(b) - b)}(\mathrm{D}\theta_j(b) - 1)$$

Using (32) and solving, we have

$$\frac{w'(0)}{w(0)+1}\mathrm{D}^2\theta_i(0) + w'(0)\mathrm{D}^2\theta_j(0) = -\frac{w'(0)+1}{w'(0)}\frac{f_i'(0)}{f_i(0)} - \frac{w''(0)}{(w'(0))^2}$$

Using analogy, we have

$$\frac{w'(0)}{w(0)+1}\mathrm{D}^2\theta_j(0) + w'(0)\mathrm{D}^2\theta_i(0) = -\frac{w'(0)+1}{w'(0)}\frac{f_j'(0)}{f_j(0)} - \frac{w''(0)}{(w'(0))^2}$$

Solving the above two equations, we get

$$\mathrm{D}^2\theta_i(0) = \frac{(w'(0)+1)^2}{(w'(0))^3}\left\{\frac{f_i'(0)}{f_i(0)} - (w'(0)+1)\frac{f_j'(0)}{f_j(0)}\right\} \tag{35}$$

and

$$\mathrm{D}^2\theta_j(0) = \frac{(w'(0)+1)^2}{(w'(0))^3}\left\{\frac{f_j'(0)}{f_j(0)} - (w'(0)+1)\frac{f_i'(0)}{f_i(0)}\right\} \tag{36}$$

Similarly,

$$\mathrm{D}^2\lambda_i(0) = \frac{(w'(0)+1)^2}{(w'(0))^2}\frac{f_i'(0)}{f_i(0)} \tag{37}$$

Comparing the above equations, we get the desired results. ■

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