Revenue Management for Strategic Buyers with Time-inconsistency*

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Abstract

We consider the dynamic pricing problem of a seller who cannot cannot pre-commit to future price path. In the case when the buyers are *time-inconsistent*, we compare the revenue-maximizing price path of the seller with a benchmark case of *time-consistent* buyers. We show that under time-inconsistency, the seller has to lower the initial period prices and increase the more later period prices in order to induce the buyers to buy earlier. With large number of buyers, the seller's problem coincides with the time-consistent case.

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1 Introduction

Revenue management is the optimal pricing strategy practised in many industries like the airline pricing, pricing of tickets for packaged tours etc. There are two broad features of revenue management pricing that makes it different from other dynamic pricing problems. First, there is a *fixed quantity* of good that the seller can sell. Second, that there is a *fixed deadline* after which the good becomes obsolete. When an airline company prices tickets for passenger seats, it has to sell the tickets before the actual date of flight. The seller cannot bargain indefinitely with the buyers to sell the good.

Since the buyers' valuations of the good are private informations, the seller uses prices he sets in different periods to optimally price discriminate among different possible buyer types. The buyers make a strategic choice on whether to buy at the current price, or wait longer for the price to eventually decrease in future. If they reject, the seller updates his belief about the buyer types and sets price accordingly in the next period. This describes the dynamics of a *revenue management pricing problem*.

One common feature that can be observed in the buyers' purchase behaviors is *purchase delay*. There can be two broad reasons for purchase delay :

a) *Strategic* : Consumers delay purchases anticipating a lower price in future. Although most part of the revenue management literature assumes buyers to be myopic, Hörner and Samuelson (2011) introduced 'strategic buyers' into the revenue management literature.

b) *Behavioral* : This might be due to some consumer inertia. This behavioral issue of purchase delay is neglected in the revenue management literature. This paper models purchase delay by introducing 'time-inconsistent' buyers into the literature.

Standard revenue management literature assumes that the buyers are *time-consistent*. However, past empirical evidence suggests that economic agents are often *time-inconsistent*. Time-inconsistency is modelled in the literature with hyperbolic discounting where the relative discount factor between any two periods is different in different periods. This is in contrast to geometric discounting, where the discount factor between any two periods remains the same.

This paper introduces time-inconsistent buyers into the revenue-magement literature and shows that the pricing strategy for the revenue-management firm should change if it assumes that the buyers are time-inconsistent. Della Vigna and Malmendier (2004) shows that if the seller knows that the buyers are time-inconsistent, the seller may want to exploit the information in his pricing strategy.

Strotz(1956) and Phelps and Pollak(1968) introduced hyperbolic discounting into the literature. A variant of the model in the discrete time case, called the quasi-hyperbolic discounting, was introduced by Laibson(1997) and O'Donaghue and Rabin(1999, 2001). [1] Experimental studies like Thaler(1981), Benzion, Rapoport and Yagil(1989), Chapman (1996) and Chapman and Elstein(1995), support the existence of hyperbolic discounting.

Though there is enough empirical evidence of time-inconsistent behavior of economic agents, this aspect has been neglected in the revenue management literature. Our paper shows that when the buyers are time-inconsistent, and if they are not aware of that, then the optimal price path for the seller is *flatter* than the path predicted by the earlier literature. If the buyers wrongly assume that they will be more patient in future, they are less eager to accept the current price. This hurts the seller. So in order to make them buy earlier he has to lower the prices in the earlier periods a relative to that in the later periods. This makes the price path flatter than the time-consistent price path.

For other assumptions, we follow Hörner and Samuelson (2011). The standard revenue management literature assumes buyers as myopic (Talluri and van Ryzin, 2005). In reality, however, buyers are strategic because they often decide whether to buy now or in future. Hörner and Samuelson(2011) introduced strategic buyers into the revenue management literature. Our paper follows closely to that assumption.

Since a seller who does not have commitment power is tempted to lower prices in subsequent periods in order to make the buyers accept, this is similar to the durable-goods monopoly literature and Coase conjecture (Ausubel and Deneckere, 1989 and Gul, Sonnenschein and Wilson, 1986), but the durable goods literature differs from the revenue management literature in its infinite horizon setting, and also in the fact that in a durable-goods literature there is no scarcity of goods. In our model the deadline imposes a commitment on the seller. Thus the seller in our model, even if he is in Coasian dynamics, violates Coase conjecture, i.e. the price never equals the marginal cost even if we allow the time interval between price revisions to be close to zero.

The main contribution of our paper is the introduction of time inconsistent buyers in a revenue management literature and examination of its effects on the pricing strategy of the seller. There has been some earlier work in the literature in applying time-inconsistency into different models on bargaining and Coasian dynamics (Sarafidis, 2006, 2005). Although Sarafidis (2005) studies inter-temporal pricing under time-inconsistent behavior, their model deals with only one buyer. In our model, the force of buyers' competition acts against that of time-inconsistency. We show that with large enough buyers, the model coincides with the time-consistent buyer model.

Our paper also contributes to the growing literature on dynamic mechanism design. Although most of the literature deals with situations where the mechanism designer can pre-

[1] Whenever we mention hyperbolic discounting in this paper, we shall more technically mean quasi-hyperbolic discounting, since our model is in the discrete time case.

commit to the entire path of mechanism, called dynamic mechanisms *with commitment*,[2] there is only a small body of literature on dynamic mechanisms *without commitment*. Without commitment literature assumes that the mechanism designer has limited or no commitment, and his strategy is sequentially rational. Skreta (2015), Hörner and Samuelson (2011), Banerjee (2017) are some examples. Our paper contributes to this branch of the literature. It shows the effects of time-inconsistency on dynamic price mechanisms.

2 The Model

We consider a general T-period dynamic game where the seller posts take-it-or-leave-it prices for the sale of one unit of an indivisible good to n buyers, where $n \geq 2$. The good is consumed at the end of the T periods after which it becomes valueless. So, the seller has to sell the good within these T periods. We denote time period t as the number of periods remaining in the game. In this $T-$period set-up, the first period is denoted by T and likewise, $t = T - 1$ denotes the next period while $t = 1$ is the last period.

The timeline for the game is as follows : In each period t, the seller announces a price $p_t \in R$, and the buyers simultaneously decide whether to accept or to reject the price. If only one buyer accepts the price, the game ends and the good is given to that buyer at price p_t. If more than one buyer accept, then the good is randomly allocated to one of the accepting buyers at the announced price. If no one accepts the good, the game moves to the next period $t - 1$.

Each buyer draws his private valuation v independently and identically from a known distribution F with support [0,1] . A buyer with valuation v who gets the good, derives a payoff $(v - p)$. The seller's valuation is the price p at which the good is sold.

The non-trivial history $h_t \in H_t$ is the history at period t where the game does not end effectively. A behavior strategy of the seller $\{\sigma_S^t\}_{t=1}^T$ is a sequence of prices p_t which maps from the history to a probability distribution of prices. A behavior strategy of a buyer i ,$\{\sigma_i^t\}_{t=1}^3$ is a map from his type, history of prices, and current price to a probability of acceptance, *i.e.* $\sigma_i^t : [0, 1] \times H_t \times R \to \{0, 1\}$.

The seller's optimization problem at each time period t is the following:

$$Max_{p_t}\pi_t(v_t) = Max_{p_t}[(1 - (\frac{F(v_t)}{F(v_{t+1})})^n)p_t + (\frac{F(v_t)}{F(v_{t+1})})^n\pi_{t-1}(v_{t-1})].$$

If the good gets sold at period t, he gets a payoff of p_t. This happens if any of the buyers accept the price p_t.In the event that the good gets unsold, i.e., when none of the buyers

[2] For example, see Board and Skrypacz, 2010, Gershkov and Moldovanu, 2010, Pai and Vohra, 2009,Pavan, Segal and Toikka, 2009 etc.

accept the price, the game moves on to the next period. In the next two subsections we introduce the buyers' games in both cases of *time consistency* and *time inconsistency.*

The solution we focus on in this paper is *perfect Bayesian equilibrium.*[3] We assume that the seller has no commitment power and each price is sequentially rational . These are symmetric perfect Bayesian equilibria where the buyers use symmetric strategies, $\sigma_i^t = \sigma_j^t$, *i.e*, the strategy depends only on the type of the buyer.

2.1 Time Consistent Buyers: the Benchmark Case

In the benchmark model of time-consistent buyers, we assume that the buyers have the same geometric discount factor δ between any two periods. In the T-period model, the discount factors are $\{1, \delta, \delta^2 ... \delta^{T-1}\}$. We shall compare this benchmark case with our main result in the case of time inconsistent buyers in the next subsection.

In the last period a buyer accepts a price if it is below or equal to his valuation. In the earlier periods each buyer faces a trade-off whether to accept at the posted price, or to wait till the next period . If he waits till the next period, he may get the good at a lower price, but the probability of getting the good decreases. If he accepts, he may get the good at a higher price, compared to waiting till next period, but the probability that he gets the good becomes higher.

Given our equilibrium concept as *perfect Bayesian equilibrium*, in any given period t, the buyers who accept the period t price are those whose valuations exceed a threshold valuation v_t. Our next lemma, taken from Horner and Samuelson (2011) illustrates in detail about the seller's and opponent buyers' posterior belief, given no sale has occurred till period $(t + 1)$.

Lemma 1. (Hörner and Samuelson (2011)) *Let $n \geq 2$. Fix an equilibrium, and suppose period t has been reached without a price having been accepted. Then the seller's posterior belief is that the buyers' valuations are identically and independently drawn from the distribution $F(v)/F(v_{t+1})$, with support $[0, v_{t+1}]$, for some $v_{t+1} \in (0, 1]$.*

Let us consider period t and a buyer i with valuation v. If he accepts the price, the expected payoff he receives is :

$$\sum_{j=0}^{n-1} \frac{1}{j+1}\left(1-\frac{F(v_t)}{F(v_{t+1})}\right)^j \left(\frac{F(v_t)}{F(v_{t+1})}\right)^{n-1-j}(v-p_t) = \frac{1-\left(\frac{F(v_t)}{F(v_{t+1})}\right)^n}{1-\frac{F(v_t)}{F(v_{t+1})}}\frac{(v-p_t)}{n}. \tag{1}$$

For buyer i , if he accepts the price, j is the number of buyers who also accept the same

[3]Existence of such an equilibrium in our setting is similar to that in Horner and Samuelson (2011), and follows from standard arguments (see Chen (2012)).

price in that period . In this case, the good is allocated to buyer i with probability $\frac{1}{j+1}$. $\frac{F(v_t)}{F(v_{t+1})}$ is the conditional probability that the opponent's valuation is less than v_t , where v_t is the threshold valuation above which the opponent buyer accepts p_t, given that his valuation is below v_{t+1}. $(1-\frac{F(v_t)}{F(v_{t+1})})$ is the corresponding conditional probability that his opponent's valuation is not less than his valuation.

On the other hand, if buyer i waits for one more period, his expected payoff is :

$$(\frac{F(v_t)}{F(v_{t+1})})^{n-1}\sum_{j=0}^{1}\frac{1}{j+1}(1-\frac{F(v_{t-1})}{F(v_t)})^j(\frac{F(v_{t-1})}{F(v_t)})^{1-j}(v-p_{t-1}) = \delta(\frac{F(v_t)}{F(v_{t+1})})^{n-1}\frac{1-(\frac{F(v_{t-1})}{F(v_t)})^n}{1-\frac{F(v_{t-1})}{F(v_t)}}\frac{(v-p_{t-1})}{n}. \tag{2}$$

The first term $(\frac{F(v_t)}{F(v_{t+1})})^{n-1}$ is the probability that the good is available for sale in the next period, i.e., none of his opponents has already bought the good. $\frac{F(v_{t-1})}{F(v_t)}$ is the probability that any of his rivals has valuation less than v_{t-1}, the new threshold in this period, given that his valuation was below v_t. All other terms are analogous to the previous expression. If this critical threshold v_t is interior, then v_t type is indifferent between accepting at price p_t in this period and waiting for the next period. So, the expressions (1) and (2) should be equal for a v_t type buyer.

$$\begin{aligned}
&\sum_{j=0}^{n-1}\frac{1}{j+1}(1-\frac{F(v_t)}{F(v_{t+1})})^j(\frac{F(v_t)}{F(v_{t+1})})^{n-1-j}(v_t-p_t) \\
= \quad &\delta(\frac{F(v_t)}{F(v_{t+1})})^{n-1}\sum_{j=0}^{1}\frac{1}{j+1}(1-\frac{F(v_{t-1})}{F(v_t)})^j(\frac{F(v_{t-1})}{F(v_t)})^{1-j}(v_t-p_{t-1})
\end{aligned} \tag{3}$$

$\Rightarrow$

$$\frac{1-(\frac{F(v_t)}{F(v_{t+1})})^n}{1-\frac{F(v_t)}{F(v_{t+1})}}(v_t-p_t) = \delta(\frac{F(v_t)}{F(v_{t+1})})^{n-1}\frac{1-(\frac{F(v_{t-1})}{F(v_t)})^n}{1-\frac{F(v_{t-1})}{F(v_t)}}(v_t-p_{t-1}). \tag{4}$$

The next subsection introduces the buyers problem in the time inconsistency case.

2.2 Time Inconsistent Buyers

This section introduces time inconsistency into our current framework and compares it with the benchmark case of time consistency. In the time consistent case, the discount factor is δ. In a T-period model, the buyers discount future periods as $\{1, \delta, \delta^2....\delta^T\}$. So, the discount factor between periods t and $t-1$, and that between $t-1$ and $t-2$ is the same, δ. So, looking from any given period t, the buyer discounts all the future periods in the same way. In this way he is time consistent. Instead, we model time inconsistency with a hyperbolic discount factor. Here we introduce two factors δ and α, such that in any given period t, the buyers discount future periods as $\{1, \delta, \delta\alpha ... \delta\alpha^{T-1}\}$, but in period $t-1$, they again discount as $\{1, \delta, \delta\alpha ... \delta\alpha^{T-2}\}$. Thus the discount factor between periods $t-1$ and $t-2$ is α, when the buyer is in period t. But once period $t-1$ comes, his actual discount factor between the current period and the next period is δ itself. Thus α is called the *perceived discount factor* for the buyers. His time preference changes with time, contrary to the time-consistent case where his discount factor is the same throughout.

We also assume that not only the buyers are time inconsistent, but they are *naive* time inconsistent, in the sense that they do not know that their time preference changes over time. It is assumed that the seller is perfectly rational and knows that the buyers are time inconsistent. The buyers are also naive to the fact that they do not know that the seller knows that they are naive. Also, the seller knows this fact, and so on. So, we can clearly see that the assumptions of rationality and common knowledge of rationality are violated in this case. All the players in the game are not perfectly rational, and even this irrationality is not common knowledge.

This hyperbolic discount factor may lead to time inconsistency by the buyers with their preferences being reversed over time. In this dynamic pricing problem, hyperbolic discounting leads to a reversal of the buyers' preferences on which period they want to accept the price. Also, since the seller is perfectly rational and is aware of the buyers' time inconsistency, we show that the optimal price mechanism in the case of time-consistent buyers is not optimal here.

To see this, first consider the time-consistent case. At any period t, v_t type buyer is the marginal buyer who is indifferent between buying in period t and buying in period $t-1$. And in period $t-1$, the marginal buyer with valuation v_{t-1} is indifferent between buying in period $t-1$ and in period $t-2$. By **Lemma 1,** any buyer with valuation strictly greater than v_{t-1} strictly prefers to buy in period $t-1$ than in period $t-2$. So, the marginal consumer in period t, having valuation $v_t > v_{t-1}$, strictly prefers to buy in period t than in period $t-2$.

At time period t, the incentives for a v_t type buyer is given by the indifference condition

$$\frac{1-(\frac{F(v_t)}{F(v_{t+1})})^n}{1-\frac{F(v_t)}{F(v_{t+1})}}(v_t - p_t) = \delta(\frac{F(v_t)}{F(v_{t+1})})^{n-1}\frac{1-(\frac{F(v_{t-1})}{F(v_t)})^n}{1-\frac{F(v_{t-1})}{F(v_t)}}(v_t - p_{t-1}). \tag{5}$$

Also, since he strictly prefers to buy in period t than in any later period $t-k$, $k>0$, we can write,

$$\frac{1-(\frac{F(v_t)}{F(v_{t+1})})^n}{1-\frac{F(v_t)}{F(v_{t+1})}}(v_t-p_t) > \delta^k(\frac{F(v_{t-k+1})}{F(v_{t+1})})^{n-1}\frac{1-(\frac{F(v_{t-k})}{F(v_t)})^n}{1-\frac{F(v_{t-k})}{F(v_t)}}(v_t-p_{t-k}). \tag{6}$$

The right hand side is the buyer's ex-ante payoff if he decides to buy in the $t-k^{th}$ period. Next we come to the case of hyperbolic discount factor. When the discount factors are hyperbolic, the expected payoff to the v_t type buyer standing in period t, is $\delta\alpha^{k-1}(\frac{F(v_{t-k+1})}{F(v_{t+1})})^{n-1}\frac{1-(\frac{F(v_{t-k})}{F(v_t)})^n}{1-\frac{F(v_{t-k})}{F(v_t)}}(v_t-p_{t-k})$. Now, when $\alpha > \delta$, the incentive to wait till the last period increases compared to the time consistent buyers. So, if the seller uses the same price path that is optimal in the case of time consistency, there is an increase in the ex-ante payoff for waiting. Now, with further increase in α, there must exist a threshold level of α, say $\overline{\alpha}$, such that when $\alpha > \overline{\alpha}$, the v_t type buyer strictly prefers to buy in period $t-k$ than buying in the current period. So, for all $\alpha > \overline{\alpha}$, the preference pattern reverses. We get,

$$\frac{1-(\frac{F(v_t)}{F(v_{t+1})})^n}{1-\frac{F(v_t)}{F(v_{t+1})}}(v_t-p_t) < \delta\alpha^{k-1}(\frac{F(v_{t-k+1})}{F(v_{t+1})})^{n-1}\frac{1-(\frac{F(v_{t-k})}{F(v_t)})^n}{1-\frac{F(v_{t-k})}{F(v_t)}}(v_t-p_{t-k}). \tag{7}$$

For $\alpha > \overline{\alpha}$, the time-consistent optimal price path is no longer optimal for the seller. It gives incentives for the buyers to wait longer till the price becomes lower. The initial constraint for the time-consistent case is not binding in this case. As the value of α increases, the incentive to wait for longer period increases. Technically the right hand side of (7) increases with increase in α, which reaches its maximum at 1. The higher the value of α, the more later period constraint is the binding one. At $\alpha = 1$, an intermediate $(t-k)^{th}$ period constraint is the binding one. It is to be noted that for all values of α, k is bounded away from $t-1$, i.e. the last period constraint is never the binding one. The reason is that there are two *opposing forces* creating a trade-off for the buyers between buying in the current period and waiting longer. Higher is the value of α, the more patient the buyer thinks he will be in future, added with the belief of decreasing price path, gives the buyer an incentive to wait longer. On the other hand, the longer he waits, the lesser is the probability that he would actually get the good in future. The point in time up to which the first force dominates the second, *i.e.* the net marginal benefit of waiting is positive, the buyer wants to wait. If that point in time is after k periods, then we compare the payoffs for t^{th} period and $(t-k)^{th}$ period, and thus the $(t-k)^{th}$ period constraint is the binding one, contrary to the binding constraint in the time consistent case, where the binding constraint is always the one involving t and $t-1$. The value of k will depend on the value of α. Higher the value of α, higher is the value of k, *i.e.* more and more the later period constraints will be binding. So for all $\alpha > \overline{\alpha}$, it is not optimal for the seller to charge the same price mechanism as in the consistent case. This threshold $\overline{\alpha}$ is also an increasing function of δ. This can be formalized in the following proposition.

Lemma 2. *i) In the case of time inconsistency, if $t-k^{th}$ period constraint is the binding one, then the value of k is an increasing function of the perceived discount factor α.*

ii) If $\overline{\alpha}$ is the threshold level of perceived discount factor α, then $\overline{\alpha}$ is an increasing function of the true discount factor δ. Also, for all $\alpha > \overline{\alpha}$, the optimal price mechanism for the seller is different from that in the time consistent case, otherwise the mechanism is the same.

In the next subsection we describe a simple motivating example with two buyers and three periods.

2.2.1 Three-period example

In this section we assume that there are two buyers and three periods. The buyers' valuations are drawn independently from $U[0,1]$. First we consider the case of time consistency. The buyers have exponential discount factor $\{1, \delta, \delta^2\}$. Since this is a finite horizon problem, we solve it through backward induction. We start from the last period, *i.e.*, $t = 1$.

At $t = 1$, the seller maximizes his expected payoff :

$$Max_{v_1}\pi_1(v_1) = (1 - (\frac{v_1}{v_2})^2)p_1$$

$$s.t. p_1 \leq v_1.$$

A buyer accepts the price in the last period only if his valuation is at least as the price. $(1-(\frac{v_1}{v_2})^2)$ is the probability that at least one buyer has valuation greater than v_1, given that the good remained unsold in the earlier period. The first order condition gives $v_1 = p_1 = \frac{v_2}{\sqrt{3}}$ and $\pi_1(v_1) = \frac{2v_2}{3\sqrt{3}}$.

In the second-last period, i.e., $t = 2$, the incentive constraint for the buyers is :

$$\frac{1-(\frac{v_2}{v_3})^2}{1-\frac{v_2}{v_3}}(v_2 - p_2) = \delta.\frac{v_2}{v_3}.\frac{1-(\frac{v_1}{v_2})^2}{1-\frac{v_1}{v_2}}(v_2 - v_1). \tag{8}$$

From the previous subsection, we know that the left hand side is the expected pay-off of a v_2 type buyer in the second-last period, if he accepts the price. The right hand side is the expected payoff if he waits till the last period.

Next we come to the seller's problem. The seller maximizes :

$$Max_{v_2}\ \pi_2(v_2) = [(1-(\frac{v_2}{v_3})^2)p_2 + (\frac{v_2}{v_3})^2\pi_1(v_1)]$$

$$s.t.\ \frac{1-(\frac{v_2}{v_3})^2}{1-\frac{v_2}{v_3}}(v_2-p_2) = \delta.\frac{v_2}{v_3}.\frac{1-(\frac{v_1}{v_2})^2}{1-\frac{v_1}{v_2}}(v_2-v_1).$$

The seller chooses optimal v_2 to maximize his expected payoff. If any of the buyers accept the price with probability $(1-(\frac{v_2}{v_3})^2)$, he gets p_2, otherwise the game proceeds to the last period, in which case he gets $\pi_1(v_1)$.

Finally in the initial period, i.e., $t=3$, the buyers' incentive constraint is :

$$\frac{1-v_3^2}{1-v_3}(v_3-p_3) = \delta.v_3.\frac{1-(\frac{v_1}{v_2})^2}{1-\frac{v_1}{v_2}}(v_3-p_2). \tag{9}$$

The seller's problem in this period is :

$$Max_{v_2}\ \pi_3(v_3) = [(1-v_3^2)p_3 + v_3^2.\pi_2(v_2)]$$

$$s.t.\ \frac{1-v_3^2}{1-v_3}(v_3-p_3) = \delta.v_3.\frac{1-(\frac{v_1}{v_2})^2}{1-\frac{v_1}{v_2}}(v_3-p_2).$$

Solving the model explicitly, we can see that the prices form a decreasing path over time, and that as the discount factor increases, the first two periods' prices tend to decrease. As the buyers become more and more patient, the seller has to lower the initial prices in order to incentivize the buyers to buy in the initial periods, who otherwise would wait till the last period. This is formalized in the following proposition.

Proposition 1 *: Let δ be the discount factor in the dynamic pricing problem with time consistent buyers. Then the optimal pricing path for the seller is decreasing over time and the final price is strictly greater than the marginal cost of the seller. Also, we find that $\frac{\partial p_3}{\partial \delta} < 0$ and $\frac{\partial p_2}{\partial \delta} < 0$, i.e., as the buyers grow more and more patient the initial two prices tend to fall.*

Rearranging and solving the two constraints (3) and (4), we get

$$p_3 = \frac{2\delta v_2^2 + 3v_3 + 3v_3^2 - 3\delta v_3^2}{3(1+v_3)}, \text{and } p_2 = \frac{3v_2^2 - 2\delta v_2^2 + 3v_2v_3}{3(v_2+v_3)}.$$

Differentiating, we get, for given v_2 and v_3, and the equilibrium belief,

$$\frac{\partial p_3}{\partial \delta} < 0 \text{ and } \frac{\partial p_2}{\partial \delta} < 0.$$

Also, $p_1 = v_1 = \frac{v_2}{\sqrt{3}} > 0$ i.e. the final price is above the marginal cost.

In the case of time-inconsistency, in period 3 they discount as $\{1, \delta, \delta\alpha\}$, but once period 2 arrives, the discount rate between periods 2 and 1 is δ. If the threshold level of α is $\overline{\alpha}$, then for all $\alpha > \overline{\alpha}$, we have

$$\frac{1 - v_3^2}{1 - v_3}(v_3 - p_3) < \delta\alpha.v_2.\frac{1 - (\frac{v_1}{v_2})^2}{1 - \frac{v_1}{v_2}}(v_3 - v_1). \tag{10}$$

In order to calculate the value of $\overline{\alpha}$, one has to change the inequality of (10) to equality.

$$\frac{1 - v_3^2}{1 - v_3}(v_3 - p_3) = \delta\alpha.v_2.\frac{1 - (\frac{v_1}{v_2})^2}{1 - \frac{v_1}{v_2}}(v_3 - v_1). \tag{11}$$

Putting the optimal values of p_3, v_3, v_2 and v_1 derived from the time consistent case into (7), we can solve for the value of $\overline{\alpha}$ which is increasing in δ.

Thus there is a difference in price mechanism for the seller. The difference in the analysis in this case appears only at $t = 3$. For the last two periods, the analysis is the same as that in the case of time consistency. For the last two periods, the time inconsistency or the perceived discount factor does not have any effect to distort the result. It is only at $t = 3$ that the perceived discount factor comes into effect. This is because technically there should be at least a time difference of two periods in order to have time inconsistency. In a general T period version of the model, time inconsistency should have an effect in the behavior of the buyers in all the periods leaving only the last two periods.

We analyze the case of $\alpha > \overline{\alpha}$. At $t = 3$, the seller's optimization problem is

$$Max_{v_3}\ \pi_3(v_3) = [(1 - v_3^2)p_3 + v_3^2\pi_2(v_2)]$$

$$s.t. \qquad \frac{1 - v_3^2}{1 - v_3}(v_3 - p_3) \geq \delta.v_3.\frac{1 - (\frac{v_1}{v_2})^2}{1 - \frac{v_1}{v_2}}(v_3 - p_2)$$

$$\frac{1 - v_3^2}{1 - v_3}(v_3 - p_3) \geq \delta\alpha.v_2.\frac{1 - (\frac{v_1}{v_2})^2}{1 - \frac{v_1}{v_2}}(v_3 - v_1).$$

The optimization problems at $t = 2$ and $t = 1$ remain the same. The two constraints imply that the v_3 type buyer should weakly prefer to buy in period 3 than in either period 2 or period 1. Now which constraint will be binding depends on the values of δ and α. The difference with the time consistent case is that in that case only the first constraint was binding. In this case, for $\alpha > \overline{\alpha}$, there is a preference reversal for the buyers. The buyer is still indifferent between buying in periods 3 & 2, but he may strictly prefer to buy in period 1. Then the second constraint becomes binding. The seller's problem becomes

$$Max_{v_3}\ \pi_3(v_3) = [(1-v_3^2)p_3 + v_3^2.\pi_2(v_2)]$$

$$s.t.\quad \frac{1-v_3^2}{1-v_3}(v_3-p_3) = \delta\alpha.v_2.\frac{1-(\frac{v_1}{v_2})^2}{1-\frac{v_1}{v_2}}(v_3-v_1).$$

Solving,

$$p_3 = \frac{-(\sqrt{3}+1)\delta\alpha v_2^2 - 3v_3 + (3+\sqrt{3})\delta\alpha v_2 v_3 - 3v_3^2}{-3(1+v_3)}$$

Lemma 3. $\frac{\partial p_3}{\partial \alpha} < 0$

Proof: $\frac{\partial p_3}{\partial \alpha} = \frac{-(\sqrt{3}+1)\delta v_2^2 + (\sqrt{3}+3)\delta v_2 v_3}{-3(1+v_3)}$

Now, since the denominator is negative, we need to show that $-(\sqrt{3}+1)\delta v_2^2 + (\sqrt{3}+3)\delta v_2 v_3 > 0$.

Suppose to the contrary let $-(\sqrt{3}+1)\delta v_2^2 + (\sqrt{3}+3)\delta v_2 v_3 < 0$.

$\Rightarrow v_2 > \frac{\sqrt{3}+3}{\sqrt{3}+1} v_3$

$\Rightarrow v_2 > v_3$, which is impossible.

Thus when the buyers are time-inconsistent, the seller has to decrease the initial prices to incentivize the buyers to accept earlier. The price path is flatter than the time-consistent price path. When $\alpha = \delta$, there is no time inconsistency and this is exactly the same as the time consistent case. For $\overline{\alpha} \geq \alpha \geq \delta$, there is time inconsistency, but the time inconsistency is not that much to affect the behavior of the buyers. The result is still the same as the time consistent case. For $\alpha > \overline{\alpha}$, time inconsistency comes into play, and affects the behavior of the buyers. The first period price p_3 falls, the last period price p_1 rises, and the price path becomes flatter than that in the time consistent case. We can show this explicitly using definite values of δ and derive price paths for both the case and show that it is flatter in the case of time inconsistency.

Let $\delta = 0.5$. Then putting the optimal values of v_3, v_2, p_3 and p_2 in (11), we compute the threshold α, $\overline{\alpha} = 0.88$. Figure 1 shows the optimal price paths of the seller when the discount factors are geometric vs when they are hyperbolic. It shows that the price path is flatter when the buyers have hyperbolic discount factors.

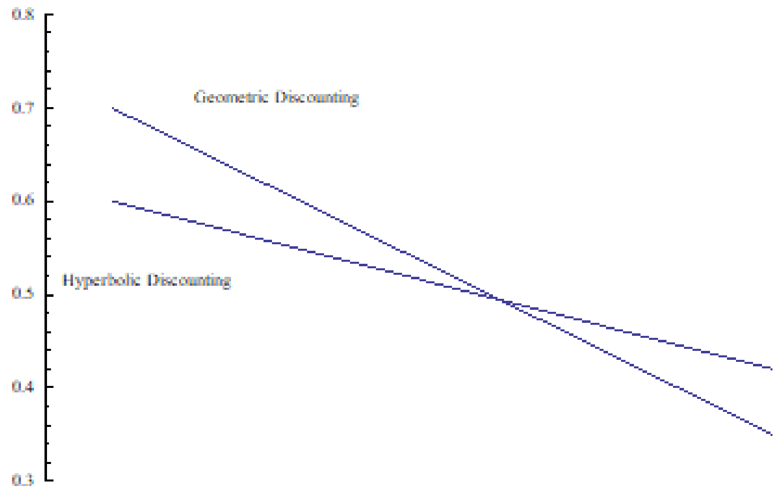

Figure 1: Price Paths for Geometric and Hyperbolic Discounting

2.3 T period Characterization of Equilibrium

In this subsection, our assumption of Uniform distribution still holds. The buyers' game is a game of strategic complementarity. The marginal gain from waiting one extra period increases for a buyer, the more he believes that his opponent also waits. In general, a game of strategic complementarity has multiple equilibria. To avoid this issue, we take the specific case of uniform distribution, in which case there is an unique solution to the problem.

In the time consistent case, we know from the previous sections that the buyers' indifference condition can be written as :

$$\frac{1-(\frac{v_t}{v_{t+1}})^n}{1-\frac{v_t}{v_{t+1}}}(v_t - p_t) = \delta(\frac{v_t}{v_{t+1}})^{n-1}\frac{1-(\frac{v_{t-1}}{v_t})^n}{1-\frac{v_{t-1}}{v_t}}(v_t - p_{t-1}). \tag{12}$$

Recursively substituting, equation (12) can be rewritten as:

$$\frac{1-{\gamma_t}^n}{1-\gamma_t}(1-\frac{p_t}{v_t}) = \sum_{\tau=1}^{t-1}\delta^{t-\tau}(1-\gamma_\tau^n)(\Pi_{l=\tau+1}^t \gamma_l^{n-1})\frac{v_{\tau+1}}{v_t}, \tag{13}$$

where $\gamma_t = \frac{v_t}{v_{t+1}}$. Substituting $\frac{p_t}{v_t} = \frac{p_t}{v_{t+1}}\frac{1}{\gamma_t}$, the equation gives full characterization of p_t in terms of $\gamma_{1,}\gamma_2, ... \gamma_t$ and v_{t+1}. $\gamma_{1,}\gamma_2, ... \gamma_t$ are given by the optimality conditions and so p_t is a linear function of v_{t+1}. In the limit as $t \to T$, *i.e.*, in the initial period, $v_{t+1} = 1$.

Equation (13) helps us to find the limiting price at the starting period as,

$$p_T = v_T - \delta \frac{1 - v_T}{1 - v_T^n} v_T^n [1 - (1 - \delta)(v_{T-1}^n + \delta v_{T-1}^n v_{T-2}^n + ...)]. \tag{14}$$

This would help us to compare the initial starting price between the time consistent case and the time inconsistent case. We would show that in the time inconsistent case the seller has to lower the starting price to incentivize the buyers to buy earlier.

The seller thus sets price in each period according to the threshold cut-off rules such that the cut-off type is indifferent between accepting the price and waiting for the next period. The buyers on the other hand follow the strategy in any period to accept the price if his valuation(or type) is strictly greater than the cutoff valuation in that period, otherwise he waits for the next period. This gives the unique perfect Bayesian equilibrium of the continuation game, which is stated in the following proposition.

Proposition 2: *In the time consistent case, at any period t, if the seller's posterior belief is* $[0, v_{t+1}]$, *then in the unique perfect Bayesian equilibrium, the* t^{th} *period price is given by*

$$p_t = 1 - \delta \frac{1 - \gamma_t}{1 - \gamma_t^n} \gamma_t^{n-1} [1 - (1 - \delta)(\gamma_t^n + \delta \gamma_{t-1}^n \gamma_{t-2}^n + ...)], \tag{15}$$

and given a price $\widetilde{p}_t$, *all buyers with valuations* $v > v_t(\widetilde{p}_t, v_{t+1})$ *, the threshold type at time period* t, *accept the price, and buyers with valuations* $v < v_t(\widetilde{p}_t, v_{t+1})$ *reject the price, where* $v_t(\widetilde{p}_t, v_{t+1})$ *is given by the equation,*

$$(1 - \frac{p_t}{v_t}) = \delta \frac{v_{t+1} - v_t}{v_{t+1}^n - v_t^n} v_t^{n-1} [1 - (1 - \delta) q_{t-1}^{-n} \sum_{\tau=2}^{t} \delta^{\tau-2} q_{t-\tau}^n], \tag{16}$$

where $q_t = \frac{v_{t+1}}{v_1}$.

Proof. See the Appendix. ■

On the other hand, in the time inconsistent case, if we start from any period t , we assume without loss of generality, that the $t - k^{th}$ period constraint is the binding one. So at the t^{th} period the buyer's indifference condition can be written as

$$\frac{1 - (\frac{v_t}{v_{t+1}})^n}{1 - \frac{v_t}{v_{t+1}}} (v_t - p_t) = \delta \alpha^{k-1} (\frac{v_{t-k+1}}{v_{t+1}})^{n-1} \frac{1 - (\frac{v_{t-k}}{v_{t-k+1}})^n}{1 - \frac{v_{t-k}}{v_{t-k+1}}} (v_t - p_{t-k}), \tag{17}$$

which can be recursively substituted as

$$\frac{1-\gamma_t{}^n}{1-\gamma_t}(1-\frac{p_t}{v_t}) = \sum_{m=\frac{t}{k}}^{\frac{2}{k}} \delta^{t-m+k}\alpha^{(k-1)(t-mk+k)}(\frac{1-\gamma_{mk-k}^n}{1-\gamma_{mk-k}})(\Pi_{l=mk}^{mk-k+1}\gamma_l)(1-\Pi_{l=mk-1}^{mk-k}\gamma_l)\frac{v_{mk}}{v_t} \tag{18}$$

In the similar way as in the time consistent case, we can show here that p_t can be expressed as a function of $\gamma_t, \gamma_{t-k}, ...\gamma_1$ and v_{t+1}. So, even in the time inconsistent case we can say that p_t is a linear function of v_{t+1} given δ, α and the optimality condition.

In the initial period as $t \to T$, we have $v_{t+1} = 1$. Again, similar to the previous case we can find p_T, the price level in the starting period. We can show that p_T in the time inconsistent case is lower than the p_T in the time consistent case. This shows that the seller has to lower initial period price in order to incentivize the time inconsistent buyer to buy earlier. Since the time inconsistent buyer is naive and he believes that he would be more patient in future, so he is less likely to accept the good in the initial period. The seller thus has to lower the price in the initial period to incentivize him. This result gives the notion of a difference in intercepts of the two price paths that we get in the two cases. This result, along with another proposition that we would build, will fully characterize the change in the price path of the seller when the buyers have hyperbolic discount factor instead of an exponential one.

The buyers follow a similar strategy according to the cutoff rule and this forms the unique perfect Bayesian equilibrium in the time inconsistent case. This along with the previous result is formally stated in the following proposition :

Proposition 3: *i) In the time inconsistent case, at any period t, if the seller's posterior belief is* $[0, v_{t+1}]$, *then in the unique perfect Bayesian equilibrium the* t^{th} *period price is given by*

$$p_t = (1-\frac{1-\gamma_t}{1-\gamma_t^n}K(t))v_{t+1}\gamma_t,$$

and given a price $\widetilde{p_t}$, *all buyers with valuations* $v > v_t(\widetilde{p_t}, v_{t+1})$, *the threshold type at time period* t, *accept the price, and buyers with valuations* $v < v_t(\widetilde{p_t}, v_{t+1})$ *reject the price , where* $v_t(\widetilde{p_t}, v_{t+1})$ *is given by the equation,*

$$(1-\frac{p_t}{v_t}) = \frac{v_{t+1}-v_t}{v_{t+1}^n - v_t^n}v_{t+1}^{n-1}\sum_{m=0}^{\frac{t}{k}}(\delta\alpha^{k-1})^{m+1}\frac{1-\gamma_{\tau-mk}^n}{1-\gamma_{\tau-mk}}(\frac{q_{\tau-mk}}{q_{\tau+1}})^{n-1}(\frac{q_{\tau+1}}{q_t})^n(1-\frac{q_{\tau-mk+1}}{q_{\tau-mk-1}})$$

where $K(t) = \delta\alpha^{k-1}(\gamma_{t-k+1}\gamma_{t-k+2}...\gamma_t)^{n-1}\frac{1-\gamma_{t-k}^n}{1-\gamma_{t-k}}(1-\gamma_{t-k}\gamma_{t-k+1}...\gamma_{t-1})+$

$(\delta\alpha^{k-1})^{2/k-1}(\gamma_{t-2k+1}\gamma_{t-2k+2}\cdots\gamma_{t-k-1})^{n-1}(\gamma_{t-k}\gamma_{t-k+1}\cdots\gamma_{t})^{n}\frac{1-\gamma_{t-2k}^{n}}{1-\gamma_{t-2k}}(1-\gamma_{t-2k}\gamma_{t-2k+1}\cdots\gamma_{t-k-1})+$
...

ii) If p_T^C and p_T^I are the prices in the initial period with T periods to go in the time consistent case and time inconsistent case respectively, then $p_T^I < p_T^C$.

Proof. See the Appendix.

The seller's problem is the same in both the cases. In each of the time-consistent and time-inconsistent case, the seller's value function is given by,

$$\pi_t(v_t) = Max_{p_t}[(1-(\frac{v_t}{v_{t+1}})^n)p_t + (\frac{v_t}{v_{t+1}})^n \pi_{t-1}(v_{t-1})].$$

In the event that the good gets unsold in period t, he receives a continuation payoff of $\pi_{t-1}(v_{t-1})$. The dynamic programming problems in both the cases are solved recursively through backward induction and the solutions we get are unique and interior. For example, in the time consistent case, we show that the sequence of $q_t(=\frac{v_{t+1}}{v_1})$, which are actually the indifferent buyers' valuations expressed in ratios, are an increasing solution to the difference equation :

$$q_{t+1}q_t^n - q_{t+1}^{n-1} + \frac{q_{t+1}-q_t}{q_{t+1}}[P(t)] - q_t^n q_{t-1} + q_{t-1}^{n+1} - q_t q_{t-1}^n[\frac{q_t - q_{t-1}}{q_t}(\frac{\delta P(t)}{\delta \frac{q_t}{q_{t-1}}})] + q_t^{n-1} + nq_t q_{t-1}^n = 0 \tag{19}$$

where $P(t) = \sum_{\tau=1}^{t}\delta^{t-(\tau-1)}\frac{q_\tau^n}{q_t} - \sum_{\tau=1}^{t-1}\delta^{(t-1)-(\tau-1)}\frac{q_\tau^n}{q_{t-1}}$,
with the boundary conditions $q_0 = \frac{v_1}{v_1} = 1$, and $q_1 = \frac{v_2}{v_1} = (n+1)^{\frac{1}{n}}$.

A similar equation can be derived for the time inconsistent case as well. Slight manipulations of the two difference equations show that at any time period t, the difference $(p_t - p_{t-1})$ is strictly lower for the time inconsistent case compared to the time consistent one. This implies that for any arbitrary t, the difference between two consecutive prices is strictly lower for the time inconsistent case. Since time is discrete in our model, this implies that the slope of price path for time inconsistency is strictly lower.

Earlier we have found a result that the price in the initial period is strictly lower for the time inconsistent case. So if we compare the two price paths, we can see that the time-inconsistent one has an intercept strictly smaller than the time consistent one, and moreover it is flatter. Intuitively what it means is that the seller has to lower the initial period price to incentivize the time inconsistent buyers to buy earlier. Since they are naive about their time inconsistency, they have a false notion about their own future patience level. They erroneously think that they would be more patient in future, and so they are more reluctant to buy earlier. This forces the seller to lower the first period price.

Intuition for the flatter price path is that in the subsequent periods after the initial period, the seller does lower the prices, but the amount of price decrease at any period t, gradually decreases as we go to the final period. This implies that the seller has the incentive to lower the prices, but that incentive diminishes as we move to the final period. This is due to the fact that at each period, the seller's motive is to incentivize an earlier purchase, and at the same time to disincentivize a later purchase. So at each point in time his very next price level is strictly less lower than what it was in the current period. If the time inconsistent price path would had been only a parallel shift downwards, that would mean that price decrease is the same throughout the time path. So incentive structure would then be shared equally among all the periods, and thus it would be failing to induce any effect on the buyers only apart from being inefficient and suboptimal for the seller. Thus it is the reallocation of the incentive structure among the time periods that is driving the result.

Whether the flatter curve for time inconsistency intersects the time consistent curve is ambiguous. That depends on the values of δ and α^{k-1}, i.e. the patience level and the degree of time inconsistency. If the two curves intersect at a time before the final period, then that means that the seller has to increase the subsequent prices to disincentivize a later purchase. The graph drawn in Figure 1 from our three period example has the two price paths intersecting at an interior point. The results are formalized in the following proposition which is the main result of the paper :

Proposition 4: *At any time period t, the difference between any two consecutive price levels* $(p_t - p_{t-1})$ *is strictly lower for the time inconsistent case than the time consistent one.*

Proof. See the Appendix.

2.4 Effect of Competition

Suppose the number of buyers increases. The force of increased competition among the buyers acts against the force of time-inconsistency. When buyers are time-inconsistent, they falsely believe that they will be more patient in the future, and thus they delay their purchase. However, buyer-competition induces them to buy earlier. Consider a three-period version of the earlier model with n buyers. In the first period, the seller faces the following constraint:

$$\frac{1-v_3^n}{1-v_3}(v_3-p_3) \geq \delta v_3^{n-1}\frac{1-(\frac{v_2}{v_3})^n}{1-\frac{v_1}{v_2}}(v_3-p_2) \tag{20}$$

$$\frac{1-v_3^n}{1-v_3}(v_3-p_3) \geq \delta\alpha v_2^{n-1}\frac{1-(\frac{v_1}{v_2})^n}{1-\frac{v_1}{v_2}}(v_3-v_1). \tag{21}$$

Consider the RHS of both the constraints. If $\alpha = 1$, for given n, the second constraint becomes the binding one and hyperbolic discounting has effects on the optimal price path of the seller. But, given $\alpha = 1$, if n increases, it raises the competition among the buyers and induces them to accept much higher prices. Thus as n increases, equilibrium v_3 increases. With very large n, the first constraint again becomes the binding one, and the problem coincides with that of time-consistent buyers.

2.5 Discussion

The paper shows the impact of time-inconsistency of buyer behaviors on the seller's price mechanism in a revenue management literature. An obvious extension would be to relax the assumption of fixed deadline and consider the same problem in an infinite horizon setting. That model would be comparable to the one in the durable goods monopoly literature. Relaxing the assumption of scarcity of goods would imply that the inherent competition among the buyers is no more driving them to accept earlier. The only thing that is acting against their incentive to wait is their impatience. If time-inconsistency has a similar bite in that framework, that would make the result stronger and establish the impact of time inconsistency more strongly.

Finally one could also relax the assumption of uniform distribution of buyer valuations and deal in a more general distribution framework. That could identify what other equilibria arise from this framework so that one could see if any of these other equilibria has any relevance in the real world.

References

[1] Alderighi, Marco, A. Cento and C. Piga. 2011. "A case study of pricing strategies in European airline markets: The London – Amsterdam route." *Journal of Air Transport Management*, 17(6), 369-373.

[2] Ausubel, Lawrence M. and R.J. Deneckere. 1989. "Reputation in bargaining and durable goods monopoly." *Econometrica*, 57(3), 511-531.

[3] Banerjee, Shraman. 2017. "Treating Symmetric Buyers Asymmetrically". *SSRN Working Paper No.* 3069512.

[4] Benizon, Uri, A. Rapoport and J. Yagil. 1989. "Discount Rates Inferred from Decisions", *Management Science*, 35 (3), 270-284.

[5] Board, Simon, and A. Skrzypacz.. 2016. "Revenue Management with Forward-looking Buyers." *Journal of Political Economy,* 124.4,1046-1087.

[6] Chapman, Gretchen B. 1996. "Temporal Discounting and Utility for Health and Money," *Journal of Experimental Psychology:Learning, Memory, Cognition* 22 (3), 771–791.

[7] Chapman, Gretchen B. and A. S. Elstein. 1995. "Valuing the Future: Temporal Discounting of Health and Money," *Med. Decision Making* 15 (4), 373–386.

[8] Chen, C. 2012. "Name Your Own Price at Priceline.com: Strategic Bidding and Lockout Periods," *Review of Economic Studies*, 79(4), 1341-1369.

[9] Gershkov, A. and B. Moldovanu. 2010. "Efficient Sequential Assignment with Incomplete Information". *Games and Economic Behavior*, 68(1), 144-154.

[10] Gul, Faruk , H. Sonnenschein and R. Wilson. 1986. "Foundations of Dynamic Monopoly and the Coase Conjecture." *Journal of Economic Theory*, 39(1),155-190.

[11] Hörner, Johannes, and L. Samuelson. 2011. "Managing Strategic Buyers," *Journal of Political Economy*, 119, 379-425.

[12] Laibson, D. 1997. "Golden Eggs and Hyperbolic Discounting", *Quarterly Journal of Economics*, 62 (2), 443-477.

[13] O'Donaghue, T. and M. Rabin. 1999. "Doing it Now or Later", *American Economic Review*, 89 (1), 103-125.

[14] O'Donaghue, T. and M. Rabin. 1999. "Incentives for Procrastinators", *Quarterly Journal of Economics*, 114 (3), 769-816.

[15] O'Donaghue, T. and M. Rabin. 2001. "Choice and Procrastination", *Quarterly Journal of Economics*, 116 (1), 121-160.

[16] Pai, Mallesh and R. Vohra. 2013. "Optimal Dynamic Auctions and Simple Index Rules", *Mathematics of Operations Research*, 38(4), 682-697.

[17] Pavan, Alessandro., I. Segal, and J. Toikka. 2009. "Dynamic Mechanism Design: Revenue Equivalence, Profit Maximization, and Information Disclosure". *Discussion paper*, Northwestern University and Stanford University.

[18] Phelps, E. S. and R.A. Pollak. 1968. "On Second-Best National Saving and Game-Equilibrium Growth", *Review of Economic Studies*, 35 (2), 85-199.

[19] Sarafidis, Y. 2006. "Solution Concepts for Games with Time Inconsistent Players", *Working Paper*

[20] Skreta, Vasiliki. 2015. "Optimal Auction Design under Non-Commitment," *Journal of Economic Theory*, 159, 854–890.

[21] Strotz, R. H. 1956. "Myopia and Inconsistency in Utility Maximization", *Review of Economic Studies*, 23 (3), 165-180.

[22] Talluri, Kalyan T. and Garrett J. van Ryzin. 2005. "The Theory and Practice of Revenue Management." *Springer*, New York.

[23] Thaler, Richard H.1981. "Some Empirical Evidence on Dynamic Inconsistency," *Economic Letters* 8, 201–207.

Appendix: Proofs

A.1 : Proof of Proposition 2

In the time consistent case, the marginal buyers' indifference condition in the t^{th} period :

$$\frac{1-(\frac{v_t}{v_{t+1}})^n}{1-\frac{v_t}{v_{t+1}}}(v_t-p_t)=\delta(\frac{v_t}{v_{t+1}})^{n-1}\frac{1-(\frac{v_{t-1}}{v_t})^n}{1-\frac{v_{t-1}}{v_t}}(v_t-p_{t-1}) \tag{22}$$

Taking $\gamma_t=\frac{v_t}{v_{t+1}}$, this can be rewritten as :

$$\begin{aligned}\frac{1-{\gamma_t}^n}{1-\gamma_t}(v_t-p_t) &= \delta\gamma_t^{n-1}\frac{1-\gamma_{t-1}}{1-\gamma_{t-1}}(v_t-p_{t-1}) \\ &= \delta\gamma_t^{n-1}\frac{1-\gamma_{t-1}}{1-\gamma_{t-1}}[(v_t-v_{t-1})+(v_{t-1}-p_{t-1})] \end{aligned}\tag{23}$$

Recursively substituting, we get,

$$\begin{aligned}\frac{1-{\gamma_t}^n}{1-\gamma_t}(v_t-p_t) &= \sum_{\tau=1}^{t-1}\delta^{t-\tau}(1-\gamma_\tau^n)(\Pi_{l=\tau+1}^t\gamma_l^{n-1})v_{\tau+1} \\ &\Rightarrow \frac{1-{\gamma_t}^n}{1-\gamma_t}(1-\frac{p_t}{v_t})=\sum_{\tau=1}^{t-1}\delta^{t-\tau}(1-\gamma_\tau^n)(\Pi_{l=\tau+1}^t\gamma_l^{n-1})\frac{v_{\tau+1}}{v_t}\end{aligned}\tag{24}$$

We can expand $\sum_{\tau=1}^{t-1}\delta^{t-\tau}(1-\gamma_\tau^n)(\Pi_{l=\tau+1}^t\gamma_l^{n-1})$ as

$$\begin{aligned}&\delta\gamma_t^{n-1}-\delta\gamma_t^{n-1}\gamma_{t-1}^n+\delta^2\gamma_t^{n-1}\gamma_{t-1}^n-\delta^2\gamma_t^{n-1}\gamma_{t-1}^n\gamma_{t-2}^n+.... \\ =\ &\delta\gamma_t^{n-1}(1-\gamma_{t-1}^n+\delta\gamma_{t-1}^n-\delta\gamma_{t-1}^n\gamma_{t-2}^n+\delta^2\gamma_{t-1}^n\gamma_{t-2}^n-....) \\ =\ &\delta\gamma_t^{n-1}[1-\gamma_{t-1}^n(1-\delta)-\delta\gamma_{t-1}^n\gamma_{t-2}^n(1-\delta)-\delta^2\gamma_{t-1}^n\gamma_{t-2}^n\gamma_{t-3}^n(1-\delta)-....] \\ =\ &\delta\gamma_t^{n-1}[1-(1-\delta)\sum_{\tau=1}^{t-1}\delta^{l-1}\Pi_{l=\tau}^{t-1}\gamma_l^n]\end{aligned}\tag{25}$$

Rearranging, we get the desired result :

$$p_t=1-\delta\frac{1-\gamma_t}{1-{\gamma_t}^n}\gamma_t^{n-1}[1-(1-\delta)(\gamma_t^n+\delta\gamma_{t-1}^n\gamma_{t-2}^n+...)]. \tag{26}$$

For proving the second part of proposition 5, we can again rearrange the buyers' indifference condition as :

$$
\begin{aligned}
1-\frac{p_t}{v_t} &= \delta\frac{1-\gamma_t}{1-{\gamma_t}^n}\gamma_t^{n-1}[1-(1-\delta)\sum_{\tau=1}^{t-1}\delta^{l-1}\Pi_{l=\tau}^{t-1}\gamma_l^n] \\
&= \delta\frac{v_{t+1}-v_t}{v_{t+1}^n-v_t^n}v_t^{n-1}[1-(1-\delta)((\frac{q_{t-2}}{q_{t-1}})^n+\delta(\frac{q_{t-2}}{q_{t-1}})^n(\frac{q_{t-3}}{q_{t-2}})^n+....+\delta^{t-2}(\frac{1}{q_{t-1}})^n)] \\
&= \delta\frac{v_{t+1}-v_t}{v_{t+1}^n-v_t^n}v_t^{n-1}[1-(1-\delta)((\frac{q_{t-2}}{q_{t-1}})^n+\delta(\frac{q_{t-3}}{q_{t-1}})^n+\delta^2(\frac{q_{t-4}}{q_{t-1}})^n+....+\delta^{t-2}(\frac{1}{q_{t-1}})^n)] \\
&= \delta\frac{v_{t+1}-v_t}{v_{t+1}^n-v_t^n}v_t^{n-1}[1-(1-\delta)q_{t-1}^{-n}\sum_{\tau=2}^{t}\delta^{\tau-2}q_{t-\tau}^n]. \qquad (27)
\end{aligned}
$$

Thus combining these we construct the perfect Bayesian equilibrium , which consists of the pricing strategy of the seller and the cutoff rule for the buyers.

A.2 : Proof of Proposition 3

In the case of time inconsistency, the marginal buyers' indifference condition at t^{th} period is given by :

$$
\begin{aligned}
\frac{1-(\frac{v_t}{v_{t+1}})^n}{1-\frac{v_t}{v_{t+1}}}(v_t-p_t) &= \delta\alpha^{k-1}(\frac{v_t-k+1}{v_{t+1}})^{n-1}\frac{1-(\frac{v_{t-k}}{v_{t-k+1}})^n}{1-\frac{v_{t-k}}{v_{t-k+1}}}(v_t-p_{t-k}) \\
&= \delta\alpha^{k-1}(\gamma_{t-k+1}\gamma_{t-k+2}....\gamma_t)\frac{1-\gamma_{t-k}^n}{1-\gamma_{t-k}}(v_t-p_{t-k}) \\
&= \delta\alpha^{k-1}(\gamma_{t-k+1}\gamma_{t-k+2}....\gamma_t)\frac{1-\gamma_{t-k}^n}{1-\gamma_{t-k}}[(v_t-v_{t-k})+(v_{t-k}-p_{t-k})] \qquad (28)
\end{aligned}
$$

$$
\Rightarrow \quad \frac{1-\gamma_t^n}{1-\gamma_t}(v_t-p_t)=\delta\alpha^{k-1}(\gamma_{t-k+1}\gamma_{t-k+2}....\gamma_t)\frac{1-\gamma_{t-k}^n}{1-\gamma_{t-k}}[(1-\gamma_{t-k}\gamma_{t-k+1}...\gamma_{t-1})v_t+(v_{t-k}-p_{t-k})] \qquad (29)
$$

By recursive substitution, we get

$$
\begin{aligned}
\frac{1-\gamma_t^n}{1-\gamma_t}(v_t-p_t) &= \delta\alpha^{k-1}(\gamma_{t-k+1}\gamma_{t-k+2}....\gamma_t)\frac{1-\gamma_{t-k}^n}{1-\gamma_{t-k}}(1-\gamma_{t-k}\gamma_{t-k+1}...\gamma_{t-1})v_t+(\delta\alpha^{k-1})^2 \\
&\quad (\gamma_{t-2k+1}\gamma_{t-2k+2}....\gamma_t)\frac{1-\gamma_{t-2k}^n}{1-\gamma_{t-2k}}(1-\gamma_{t-2k}\gamma_{t-2k+1}...\gamma_{t-k-1})v_{t-k}+.... \\
&= \sum_{m=\frac{t}{k}}^{\frac{2}{k}}(\delta\alpha^{k-1})^{t-mk+k}\frac{1-\gamma_{t-mk-k}^n}{1-\gamma_{t-mk-k}}(\Pi_{l=mk}^{mk-k+1}\gamma_l)(1-\Pi_{l=mk-1}^{mk-k}\gamma_l)v_{mk} \qquad (30)
\end{aligned}
$$

$$\Rightarrow 1-\frac{p_t}{v_t}=\frac{1-\gamma_t}{1-\gamma_t^n}[\sum_{m=\frac{t}{k}}^{\frac{2}{k}}(\delta\alpha^{k-1})^{t-mk+k}\frac{1-\gamma_{t-mk-k}^n}{1-\gamma_{t-mk-k}}(\Pi_{l=mk}^{mk-k+1}\gamma_l)(1-\Pi_{l=mk-1}^{mk-k}\gamma_l)v_{mk}]$$

$$\Rightarrow 1-\frac{p_t}{v_{t+1}}\frac{1}{\gamma_t}=\frac{1-\gamma_t}{1-\gamma_t^n}[\sum_{m=\frac{t}{k}}^{\frac{2}{k}}(\delta\alpha^{k-1})^{t-mk+k}\frac{1-\gamma_{t-mk-k}^n}{1-\gamma_{t-mk-k}}(\Pi_{l=mk}^{mk-k+1}\gamma_l)(1-\Pi_{l=mk-1}^{mk-k}\gamma_l)v_{mk}]$$

$$\Rightarrow p_t=(1-\frac{1-\gamma_t}{1-\gamma_t^n}[\sum_{m=\frac{t}{k}}^{\frac{2}{k}}(\delta\alpha^{k-1})^{t-mk+k}\frac{1-\gamma_{t-mk-k}^n}{1-\gamma_{t-mk-k}}(\Pi_{l=mk}^{mk-k+1}\gamma_l)(1-\Pi_{l=mk-1}^{mk-k}\gamma_l)v_{mk}])v_{t+1}\gamma_t \tag{31}$$

This can be rewritten as :

$$\frac{1-\gamma_t^n}{1-\gamma_t}(v_t-p_t)=\delta\alpha^{k-1}(\frac{q_{t-k}}{q_t})^{n-1}\frac{1-\gamma_{t-k}^n}{1-\gamma_{t-k}}(1-\frac{q_{t-k+1}}{q_{t-1}})+$$
$$(\delta\alpha^{k-1})^2(\frac{q_{t-2k}}{q_{t-k+1}})^{n-1}(\frac{q_{t-k+1}}{q_t})^n\frac{1-\gamma_{t-2k}^n}{1-\gamma_{t-2k}}(1-\frac{q_{t-2k+1}}{q_{t-k-1}})+...$$
$$=\sum_{m=0}^{\frac{t}{k}}(\delta\alpha^{k-1})^{m+1}\frac{1-\gamma_{\tau-mk}^n}{1-\gamma_{\tau-mk}}(\frac{q_{\tau-mk}}{q_{\tau+1}})^{n-1}(\frac{q_{\tau+1}}{q_t})^n(1-\frac{q_{\tau-mk+1}}{q_{\tau-mk-1}})$$
$$\Rightarrow 1-\frac{p_t}{v_t}=\frac{v_{t+1}-v_t}{v_{t+1}^n-v_t^n}v_{t+1}^{n-1}[\sum_{m=0}^{\frac{t}{k}}(\delta\alpha^{k-1})^{m+1}\frac{1-\gamma_{\tau-mk}^n}{1-\gamma_{\tau-mk}}(\frac{q_{\tau-mk}}{q_{\tau+1}})^{n-1}(\frac{q_{\tau+1}}{q_t})^n(1-\frac{q_{\tau-mk+1}}{q_{\tau-mk-1}})] \tag{32}$$

Combining, we get the perfect Bayesian equilibrium which consists of the pricing strategy of the seller, and the cutoff rule for the buyers.

For proving (ii) of proposition 6, we require to find the limiting value of p_t (which we denote as p_T) when $t\to T$, for each of the time consistent and time inconsistent case. In the time consistent case,

$$p_T=v_T-\delta\frac{1-v_T}{1-v_T^n}v_T^n[1-(1-\delta)(v_{T-1}^n+\delta v_{T-1}^n v_{T-2}^n+...)]. \tag{33}$$

We get a similar expression for time inconsistent case. The proof is straightforward. Since we know that in the time inconsistent case, the k^{th} period constraint is the binding one, the right hand side of the expression is lower than that of the time consistent case, and hence p_T is lower for the time inconsistent buyers.

A.3 Proof of Proposition 4

In the time inconsistent case, the seller's problem is

$$
\begin{aligned}
Max^n_{v_t}\pi_t(v_t) &= Max_{v_t}[(1-(\frac{v_t}{v_{t+1}})^n)p_t + (\frac{v_t}{v_{t+1}})^n\pi_{t-1}(v_{t-1})] \\
&= Max_{\gamma_t}[(1-\gamma_t^n)(p_t - v_t) + (1-\gamma_t^n)v_t + \gamma_t^n\pi_{t-1}(v_{t-1})] \\
&= Max_{\gamma_t}[-(1-\gamma_t)\sum_{m=0}^{\frac{t-2}{k}}(\delta\alpha^{k-1})^{m+1}\frac{1-\gamma^n_{t-mk-k}}{1-\gamma_{t-mk-k}}(\Pi_{l=t}^{t-mk-k+1}\gamma_l) \\
&\quad (1-\Pi_{l=t-mk-1}^{t-mk-k}\gamma_l)v_{t-mk} + (1-\gamma_t^n)v_t + \gamma_t^n\pi_{t-1}(v_{t-1})]
\end{aligned}
\tag{34}
$$

Let $\mu_t = \frac{\pi_t(v_t)}{v_t}$. Therefore, we can write,

$$
\mu_t = -(1-\gamma_t)[\delta\alpha^{k-1}(\gamma_{t-k+1}\gamma_{t-k+2}\cdots\gamma_t)^{n-1}\frac{1-\gamma^n_{\tau-mk}}{1-\gamma_{\tau-mk}}(1-\gamma_{t-k}\gamma_{t-k+1}\cdots\gamma_{t-1}) + (\delta\alpha^{k-1})^2
$$

$$
(\gamma_{t-2k+1}\gamma_{t-2k+2}\cdots\gamma_{t-k-1})^{n-1}\frac{1-\gamma^n_{\tau-mk}}{1-\gamma_{\tau-mk}}(1-\gamma_{t-2k}\gamma_{t-2k+1}\cdots\gamma_{t-k-1}) + ...] + 1 - \gamma_t^n + \gamma_t^n\mu_{t-1}
\tag{35}
$$

Let the multiplicand of $-(1-\gamma_t)$ be denoted as $X(t)$.

The first order condition w.r.t. γ_t gives:

$$
-(1-\gamma_t)[\delta\alpha^{k-1}(\gamma_{t-k+1}\gamma_{t-k+2}\cdots\gamma_t)^{n-1}(n-1)\gamma_t^{n-1}\frac{1-\gamma^n_{\tau-mk}}{1-\gamma_{\tau-mk}}(1-\gamma_{t-k}\gamma_{t-k+1}\cdots\gamma_{t-1})
\tag{36}
$$

$$
+(\delta\alpha^{k-1})^2(\gamma_{t-2k+1}\gamma_{t-2k+2}\cdots\gamma_{t-k-1})^{n-1}n\gamma_t^{n-1}\frac{1-\gamma^n_{\tau-mk}}{1-\gamma_{\tau-mk}}(1-\gamma_{t-2k}\gamma_{t-2k+1}\cdots\gamma_{t-k-1}) + ...]
$$

$$
+X(t) - n\gamma_t^{n-1} + n\gamma_t^{n-1}\mu_{t-1} = 0
\tag{37}
$$

Now, if $q_t = \frac{v_{t+1}}{v_1}$,

$$
\begin{aligned}
\mu_t q_t^{n+1} &= (q_{t-1}-q_t)[\delta\alpha^{k-1}q_t q^{n-1}_{t-k}\frac{1-(\frac{q_{t-k-1}}{q_{t-k}})^n}{1-\frac{q_{t-k-1}}{q_{t-k}}}(1-\frac{q_{t-k-1}}{q_{t-1}}) + (\delta\alpha^{k-1})^2 \\
&\quad q_t q^{n-1}_{t-k-1}\frac{1-(\frac{q_{t-2k-1}}{q_{t-2k}})^n}{1-\frac{q_{t-2k-1}}{q_{t-2k}}}(1-\frac{q_{t-2k-1}}{q_{t-k-1}}) + ...] + q_t^{n+1} - q_t q^n_{t-1} + q_t q^n_{t-1}\mu_{t-1}
\end{aligned}
\tag{38}
$$

$$
\Rightarrow \varepsilon_t = q_t^{n+1} - q_t q^n_{t-1} + q_t q^n_{t-1}\mu_{t-1}
\tag{39}
$$

where

$$\varepsilon_t = \mu_t q_t^{n+1} - (q_{t-1} - q_t)[\delta\alpha^{k-1} q_t q_{t-k}^{n-1} \frac{1 - (\frac{q_{t-k-1}}{q_{t-k}})^n}{1 - \frac{q_{t-k-1}}{q_{t-k}}} (1 - \frac{q_{t-k-1}}{q_{t-1}}) + (\delta\alpha^{k-1})^2 q_t q_{t-k-1}^{n-1} \frac{1 - (\frac{q_{t-2k-1}}{q_{t-2k}})^n}{1 - \frac{q_{t-2k-1}}{q_{t-2k}}} (1 - \frac{q_{t-2k-1}}{q_{t-k-1}}) + ...] \quad (40)$$

If

$$Y(t) = [\delta\alpha^{k-1} q_t q_{t-k}^{n-1} \frac{1 - (\frac{q_{t-k-1}}{q_{t-k}})^n}{1 - \frac{q_{t-k-1}}{q_{t-k}}} (1 - \frac{q_{t-k-1}}{q_{t-1}}) + (\delta\alpha^{k-1})^2 q_t q_{t-k-1}^{n-1} \frac{1 - (\frac{q_{t-2k-1}}{q_{t-2k}})^n}{1 - \frac{q_{t-2k-1}}{q_{t-2k}}} (1 - \frac{q_{t-2k-1}}{q_{t-k-1}}) + ...]$$

$$\varepsilon_t = q_t^{n+1} - q_t q_{t-1}^n + q_t q_{t-1}^n [\frac{q_{t-1} - q_t}{q_t} (\delta\alpha^{k-1} q_{t-k}^{n-1} (n-1) \frac{1 - (\frac{q_{t-k-1}}{q_{t-k}})^n}{1 - \frac{q_{t-k-1}}{q_{t-k}}} (1 - \frac{q_{t-k-1}}{q_{t-1}}) + (\delta\alpha^{k-1})^2 n \frac{q_{t-k-1}}{q_{t-1}} q_{t-2k} \frac{1 - (\frac{q_{t-2k-1}}{q_{t-2k}})^n}{1 - \frac{q_{t-2k-1}}{q_{t-2k}}} (1 - \frac{q_{t-2k-1}}{q_{t-k-1}}) + ...] - q_t^{n-1} Y(t) + n q_{t-1}^{n-1} \quad (41)$$

Similarly, the first order condition can be written as :

$$n\varepsilon_{t-1} = n q_{t-1}^{n-1} + \frac{q_{t-1} - q_t}{q_t} [\delta\alpha^{k-1} q_{t-k}^{n-1} (n-1) \frac{1 - (\frac{q_{t-k-1}}{q_{t-k}})^n}{1 - \frac{q_{t-k-1}}{q_{t-k}}} (1 - \frac{q_{t-k-1}}{q_{t-1}}) + (\delta\alpha^{k-1})^2 n \frac{q_{t-k-1}}{q_{t-1}} q_{t-2k} \frac{1 - (\frac{q_{t-2k-1}}{q_{t-2k}})^n}{1 - \frac{q_{t-2k-1}}{q_{t-2k}}} (1 - \frac{q_{t-2k-1}}{q_{t-k-1}}) + ...] \quad (42)$$

Eliminating ε_t and ε_{t-1} from the above equations, we get a polynomial in q_{t+1}, which we denote as P. We omit the actual expression of P as it is excessively messy. If we put $q_{t+1} = q_t$, this gives the value of the polynomial P at the point q_t, and we denote it by $P(q_t)$.

$$P(q_t) = q_t^{n-1} - q_t^{n+1} + q_t q_{t-1}^n W(t) + q_t^{n-1} Y(t) - n q_t q_{t-1}^{n-1},$$

where

$$Y(t) = [\delta\alpha^{k-1} q_t q_{t-k}^{n-1} \frac{1-(\frac{q_{t-k-1}}{q_{t-k}})^n}{1-\frac{q_{t-k-1}}{q_{t-k}}}(1-\frac{q_{t-k-1}}{q_{t-1}}) + (\delta\alpha^{k-1})^2 q_t q_{t-k-1}^{n-1} \frac{1-(\frac{q_{t-2k-1}}{q_{t-2k}})^n}{1-\frac{q_{t-2k-1}}{q_{t-2k}}}(1-\frac{q_{t-2k-1}}{q_{t-k-1}}) + \quad (43)$$

$$W(t) = [\delta\alpha^{k-1} q_{t-k}^{n-1}(n-1) \frac{1-(\frac{q_{t-k-1}}{q_{t-k}})^n}{1-\frac{q_{t-k-1}}{q_{t-k}}}(1-\frac{q_{t-k-1}}{q_{t-1}}) + (\delta\alpha^{k-1})^2 n \frac{q_{t-k-1}}{q_{t-1}} q_{t-2k} \frac{1-(\frac{q_{t-2k-1}}{q_{t-2k}})^n}{1-\frac{q_{t-2k-1}}{q_{t-2k}}}(1-\frac{q_{t-2k-1}}{q_{t-k-1}}) + ...] \quad (44)$$

Through a similar exercise we can find a corresponding value of $P(q_t)$ for time consistent case. The expression is quite similar to the one derived, only the expressions $Y(t)$ and $W(t)$ change accordingly, since in the time consistent case, the very next period constraint binds.

If we apply Descarte's rule of sign to both the expressions we can see that there are more than one real roots in both the case. Twice differentiation with q_t ensures that the expression is more convex in the time inconsistent cases. Thus for time inconsistency, the expression intersects the $x - axis$ at a point where the distance from that point to q_t is lower than the corresponding distance for time consistent case. That point of intersection is the root of the equation $P = 0$, where P is a function of q_{t+1}. So, $q_{t+1} - q_t$ is lower in the time inconsistent case. This proves the result.