Fair Division with Single-plateaued Preferences

Hans Peters*[1], Souvik Roy†[2], and Soumyarup Sadhukhan ‡[2]

[1]Department of Quantitative Economics, School of Business and Economics, Maastricht University

[2]Economic Research Unit, Indian Statistical Institute, Kolkata

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Abstract

We consider fair division problem where each agent has a single-plateaued preference over the interval $[0, 1]$. Sprumont (1991) shows that every strategy-proof, efficient, and anonymous division rule is the uniform rule. We show that if agents are allowed to have single-plateaued preferences, then every strategy-proof and efficient division rule satisfying equal treatment of equals and non-bossyness becomes a generalized uniform rule. A generalized uniform rule behaves like the uniform rule when there is an excess demand or supply. However, in all other cases, such a rule picks a selection from the plateaus of the agents.

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*Contact: h.peters@maastrichtuniversity.nl
†Corresponding Author: souvik.2004@gmail.com
‡Contact: soumyarup.sadhukhan@gmail.com

1. INTRODUCTION

We consider the problem of allotting one unit of a perfectly divisible good among a set of agents. We assume that the agents have continuous and "single-plateaued" preferences. Preferences are single-plateaued if at the top there is an indifference class and outside of the indifference class preferences strictly decrease. Single-plateaued preferences are generalization of single-peaked preferences which imply up to a certain point more is preferred to less and beyond that point vice-versa. This problem with single-peaked preferences has been studied extensively in the literature. Sprumont (1991) characterizes the uniform rule as the only rule satisfying three basic axioms such as strategy-proofness, efficiency, and anonymity. Ching (1994) shows that anonymity can be replaced by equal treatment of equals to characterize the same.[1] Other axiomatic characterization of uniform rule are given in Thomson (1994).[2]

In this paper, we consider situations where agents can have indifferences in their preferences. Most commons example of such domains are single-plateaued domains. In such domains, indifferences can occur only at the top. Such domains are well-known in social choice theory. The main objective of this paper is to characterize equal treatment of equals, non-bossy, efficient and strategy-proof division rules in this setting.

we generalize the uniform rule and define a new class of rules, called generalized uniform rules. A generalized uniform rule behaves like the uniform rule when there is an excess demand or supply. However, in all other cases, such a rule picks a selection from the plateaus of the agents.

We show that when agents have single-plateaued preferences, then a division rule satisfies strategy-proofness, efficiency, equal treatment of equals, and non-bossyness if and only if it is a generalized uniform rule. Our proof technique is independent of Sprumont's.

2. MODEL

Let $N = \{1, \dots, n\}$ be the set of agents who must share one unit of some perfectly divisible good. Each agent $i \in N$ has a preference R_i which is a complete and transitive binary relation on $[0, 1]$. For all $x, y \in [0, 1]$, xR_iy means consuming a quantity x of the good is, from i's viewpoint, at

[1]See also Ching (1992).

[2]See also Thomson (1983),Thomson (1994),Thomson (1995),Thomson (1997).

least as good as consuming a quantity y. Strict preference of R_i is denoted by P_i, indifference by I_i. We assume that R_is are continuous, i.e., for each $x \in [0,1]$, $\{y \in [0,1] \mid yR_ix\}$ and $\{y \in [0,1] | xR_iy\}$ are closed sets. We further assume preferences are single-plateaued and strictly decreasing around their plateaus, i.e., for each $i \in N$, R_i satisfies the following condition: there exists $\tau(R_i) = [l(R_i), r(R_i)]$, where $0 \leq l(R_i) \leq r(R_i) \leq 1$, such that for all $x, y \in [0,1]$

$$\begin{aligned} &x, y \in \tau(R_i) \implies xI_iy, \text{ and} \\ &y < x \leq l(R_i) \text{ or } r(R_i) \leq x < y \implies xP_iy. \end{aligned} \tag{1}$$

We denote by $\mathcal{S}$ the set of all continuous preferences satisfying (1). We let $R_N = (R_i)_{i \in N}$ denote the announced preferences of all agents and R_{-i} denote $(R_i)_{i \in N \setminus i}$ for $i \in N$. For a profile R_N, we define $\tau(R_N) = (\tau(R_1), \ldots, \tau(R_n))$.

By Δ_n we denote the n-dimensional simplex, i.e., $\Delta_n = \{\mathbf{x} = (x_1, \ldots, x_n) \mid x_i \geq 0 \text{ for all } i \in N \text{ and } \sum_{i=1}^{n} x_i = 1\}$. A division rule f is a function $f : \mathcal{S}^n \to \Delta_n$. Below we mention some desirable properties of a division function.

Definition 2.1. (**Efficiency**) For all $R_N \in \mathcal{S}^n$ and all $\mathbf{x}, \mathbf{y} \in \Delta_n$, $\mathbf{x}_i R_i \mathbf{y}_i$ for all $i \in N$ and $\mathbf{x}_j P_j \mathbf{y}_j$ for some $j \in N$ imply $f(R_N) \neq \mathbf{y}$.

REMARK 2.1. Note that efficiency implies the following: for all $R_N \in \mathcal{S}^n$,

$$\sum_{i=1}^{n} r(R_i) \leq 1 \implies f_i(R_N) \geq r(R_i) \text{ for all } i \in N,$$

$$\sum_{i=1}^{n} l(R_i) \geq 1 \implies f_i(R_N) \leq l(R_i) \text{ for all } i \in N, \text{ and}$$

$$\sum_{i=1}^{n} l(R_i) \leq 1 \leq \sum_{i=1}^{n} r(R_i) \implies f_i(R_N) = c_i \text{ such that } c_i \in [l(R_i), r(R_i)] \text{ for all } i \in N \text{ and } \sum_{i=1}^{n} c_i = 1.$$

Definition 2.2. (**Strategy-proofness**) For all $i \in N$, all $R_N \in \mathcal{S}^n$, and all $R'_i \in \mathcal{S}$, we have

$$f_i(R_N) R_i f_i(R'_i, R_{-i}).$$

Definition 2.3. (**Equal treatment of equals**) For all $i, j \in N$, all $R_N \in \mathcal{S}^n$ with $R_i = R_j$, we have

$$f_i(R_N) = f_j(R_N)$$

.

Definition 2.4. (**Non-bossyness**) For all $i \in N$, all $R_N \in \mathcal{S}^n$, and all $R_i' \in \mathcal{S}$, $f_i(R_N) = f_i(R_i', R_{-i})$ implies $f(R_N) = f(R_i', R_{-i})$.

Let $U = \{\tilde{R}_N \in \mathcal{S}^n \mid \sum_{i=1}^{n} l(\tilde{R}_i) \leq 1 \leq \sum_{i=1}^{n} r(\tilde{R}_i)\}$. A function $g : U \to \Delta_n$ is called a $*$ function if

(i) $g_i(\tilde{R}_N) \in [l(\tilde{R}_i), r(\tilde{R}_i)]$ for all $i \in N$,

(ii) $g_i(\tilde{R}_N) = g_j(\tilde{R}_N)$ whenever $\tau(\tilde{R}_i) = \tau(\tilde{R}_j)$, and

(iii) for $(\tilde{R}_i', \tilde{R}_{-i}) \in U$, $g_i(\tilde{R}_N) = g_i(\tilde{R}_i', \tilde{R}_{-i})$ implies $g(\tilde{R}_N) = g(\tilde{R}_i', \tilde{R}_{-i})$

Definition 2.5. (**Generalized Uniform Rule**) An allocation rule is called generalized uniform rule if for all $i \in N$,

$$f_i(R_N) = \begin{cases} \min\{l(R_i), \lambda(R_N)\} \text{ if } \sum_{i=1}^{n} l(R_i) > 1, \\ \max\{r(R_i), \mu(R_N)\} \text{ if } \sum_{i=1}^{n} r(R_i) < 1, \text{ and} \\ g_i(R_N) \text{ if } \sum_{i=1}^{n} l(R_i) \leq 1 \leq \sum_{i=1}^{n} r(R_i). \end{cases}$$

where

(i) $\lambda(R_N)$ solves the equation $\sum_{i=1}^{n} \min\{l(R_i)\lambda(R_N)\} = 1$,

(ii) $\mu(R_N)$ solves the equation $\sum_{i=1}^{n} \max\{r(R_i), \mu(R_N)\} = 1$, and

(iii) g is a $*$ function.

2.1 Results

Theorem 2.1. *A strategy-proof and efficient division rule f satisfies equal treatment of equals and non-bossyness if and only if f is a generalized uniform rule.*

Proof. (If part) It is easy to verify that every generalized uniform rule satisfies efficiency, equal treatment of equals and non-bossyness. We show that every generalized uniform rule is also strategy-proof. Consider $R_N \in \mathcal{S}^n$. Note that if $\sum_{i=1}^{n} l(R_i) \leq 1 \leq \sum_{i=1}^{n} r(R_i)$ then by definition

$f_i(R_N) \in [l(R_i), r(R_i)]$ for all $i \in N$. This means no agent will manipulate at R_N. So without loss of generality we assume that $\sum_{i=1}^{n} l(R_i) > 1$. It follows from Sprumont (1991) that no agent $i \in N$ can not manipulate at R_N via R_i' where $\sum_{j \neq i} l(R_j) + l(R_i') > 1$ or $\sum_{j \neq i} r(R_j) + r(R_i') < 1$. So consider $R_i' \in \mathcal{S}$ such that $\sum_{j \neq i} l(R_j) + l(R_i') \leq 1 \leq \sum_{j \neq i} r(R_j) + r(R_i')$. Since $\sum_{j \neq i} l(R_j) + l(R_i') \leq 1 < \sum_{j=1}^{n} l(R_j)$, there must exits $c \in [l(R_i'), l(R_i)]$ such that $\sum_{j \neq i} l(R_j) + c = 1$. By the definition of generalized uniform rule $l(R_i) \geq f_i(R_N) \geq c$. To show that f is not manipulable at R_N via R_i', it is enough to show $c \geq f_i(R_i', R_{-i})$. Take R_i'' such that $\tau(R_i'') = c$. By the definition of generalized uniform rule $f_i(R_i'', R_{-i}) = c$ and $f_j(R_i'', R_{-i}) = l(R_j)$ as $\sum^{j \neq i} l(R_j) + l(R_j'') = 1$. This means if $f_i(R_i', R_{-i}) > c$ then there exists $j \neq i$ such that $f_j(R_i', R_{-i}) < l(R_j)$. However, this is a contradiction to the definition of f since $\sum_{j \neq i} l(R_j) + l(R_i') \leq 1 \leq \sum_{j \neq i} r(R_j) + r(R_i')$. This completes the proof of the If part.

(Only-if part) Suppose f satisfies strategy-proofness, efficiency, equal treatment of equals and non-bossyness. We show f is a generalized uniform rule. Let $R_N \in \mathcal{S}^n$ such that $\sum_{i=1}^{n} l(R_i) > 1$. Consider $\hat{R}_N \in \hat{\mathcal{S}}^n$ such that $\tau(\hat{R}_i) = l(R_i)$ for all $i \in N$. This means $\sum_{i=1}^{n} \tau(\hat{R}_i) > 1$. We first derive $f(\hat{R}_N)$ and then show $f(\hat{R}_N) = f(R_N)$. We first prove a lemma.

Lemma 2.1. *For all $R_N \in \hat{\mathcal{S}}^n$, all $i \in N$, and all $R_i' \in \hat{\mathcal{S}}$,*

(i) if $\tau(R_i) < f_i(R_N)$ and $\tau(R_i') \leq f_i(R_N)$, then $f_i(R_N) = f_i(R_i', R_{-i})$;

(ii) if $\tau(R_i) > f_i(R_N)$ and $\tau(R_i') \geq f_i(R_N)$, then $f_i(R_N) = f_i(R_i', R_{-i})$.

Proof. Fix $R_N \in \hat{\mathcal{S}}^n$, $i \in N$, $R_i' \in \hat{\mathcal{S}}$. Suppose, with out loss of generality, that $\tau(R_i) < f_i(R_N)$ and $\tau(R_i') \leq f_i(R_N)$. The first inequality and efficiency imply for all $j \in N$, $\tau(R_j) \leq f_j(R_N)$. This means $\tau(R_i') + \sum_{j \neq i} \tau(R_j) \leq \sum_{k=1}^{n} f_k(R_N) = 1$. By efficiency, for all $\tau(R_i') \leq f_i(R_i', R_{-i})$. Suppose by contradiction $f_i(R_N) \neq f_i(R_i', R_{-i})$. Consider the following two cases. (i) If $f_i(R_N) < f_i(R_i', R_{-i})$, then $\tau(R_i') \leq f_i(R_N) < f_i(R_i', R_{-i})$ which implies by single-peakedness $f_i(R_N) P_i' f_i(R_i', R_{-i})$. (ii) If $f_i(R_i', R_{-i}) < f_i(R_N)$, let $R_i'' \in \hat{\mathcal{S}}$ such that $\tau(R_i) = \tau(R_i'')$ and $f_i(R_i', R_{-i}) P_i'' f_i(R_N)$. Since $\tau(R_i) = \tau(R_i'')$, by strategy-proofness and efficiency, $f_i(R_i'', R_{-i}) = f_i(R_N)$. This means $f_i(R_i', R_{-i}) P_i'' f_i(R_i'', R_{-i})$. In both case we have a contradiction on strategy-proofness. ■

We now prove a lemma which gives a structure of the outcome at the profile R_N.

Lemma 2.2. *Suppose $\tau(\hat{R}_i) < \tau(\hat{R}_j)$ for some $i, j \in N$. Then one of the following must be true:*

(i) $f_k(\hat{R}_N) = \tau(\hat{R}_k)$ *for all* $k \in \{i, j\}$.

(ii) $f_i(\hat{R}_n) = \tau(\hat{R}_i)$ *and* $\tau(\hat{R}_i) < f_j(\hat{R}_N) < \tau(\hat{R}_j)$.

(iii) $f_i(\hat{R}_N) = f_j(\hat{R}_N) < \tau(\hat{R}_i)$.

Proof. Suppose (i) does not hold, we show either (ii) or (iii) holds. Since (i) does not hold, there exists $k \in \{i, j\}$ such that $\tau(\hat{R}_k) \neq f_k(\hat{R}_N)$. We first show if $k = i$ then we can also take $k = j$, i.e., $\tau(\hat{R}_j) \neq f_j(\hat{R}_N)$. Suppose not and $\tau(\hat{R}_j) = f_j(\hat{R}_N)$. Note that by efficiency $f_i(\hat{R}_N) < \tau(\hat{R}_i)$. This means by Lemma 2.1, $f_i(\hat{R}_j, \hat{R}_{-i}) = f_i(\hat{R}_N)$. Since f is non-bossy it must be that $f_j(\hat{R}_j, \hat{R}_{-i}) = f_j(\hat{R}_N)$, but this is a contradiction to the fact that f satisfies equal treatment of equals as by our assumption $f_i(\hat{R}_N) < f_j(\hat{R}_N)$. So, $f_j(\hat{R}_N) < \tau(\hat{R}_j)$. Now if $f_i(\hat{R}_N) = \tau(\hat{R}_i)$, then using a similar argument we can show that $\tau(\hat{R}_i) < f_j(\hat{R}_N) < \tau(\hat{R}_j)$ which implies (ii). If $f_i(\hat{R}_N) < \tau(\hat{R}_i)$, then suppose $f_i(\hat{R}_N) \neq f_j(\hat{R}_N)$. Since $\tau(\hat{R}_i) < \tau(\hat{R}_j)$, by efficiency $f_i(\hat{R}_N) < f_j(\hat{R}_N)$. Combining all these observations and applying Lemma 2.1, we get $f_i(\hat{R}_j, \hat{R}_{-i}) = f_i(\hat{R}_N)$. Since f is non-bossy it must be that $f_j(\hat{R}_j, \hat{R}_{-i}) = f_j(\hat{R}_N)$, but this is a contradiction to the fact that f satisfies equal treatment of equals as $f_i(\hat{R}_N) < f_j(\hat{R}_N)$. This proves $f_i(\hat{R}_N) = f_j(\hat{R}_N) < \tau(\hat{R}_i)$. ■

Without loss of generality we assume that $\tau(R_1) \leq \cdots \leq \tau(R_n)$. If all $\tau(R_i)$s are equal then by equal treatment of equals we get $f_i(\hat{R}_N) = \dfrac{1}{n}$. So, we assume there exists $i \in N$ such that $\tau(\hat{R}_i) < \tau(\hat{R}_{i+1})$. This means by Lemma 2.2 there exists $k \in \{0, 1, \ldots, n-1\}$ such that $f_i(\hat{R}_N) = \tau(\hat{R}_i)$ for all $i \in \{1, \ldots, k\}$ and $\tau(\hat{R}_k) < f_i(\hat{R}_N) < \tau(R_{k+1})$ for all $i \in \{k+1, \ldots, n\}$. This in particular means $f_i(\hat{R}_N) = \min\{\tau(\hat{R}_i), \lambda(\hat{R}_N)\}$ where $\lambda(R_N)$ solves the equation $\sum_{i=1}^{n} \min\{\tau(\hat{R}_i), \lambda(\hat{R}_N)\} = 1$.

For the preference profile $(R_i, \hat{R}_{-i})$, $\sum_{k \neq i} \tau(\hat{R}_i) + l(R_i) > 1$, by efficiency this implies $f_i(R_i, \hat{R}_{-i}) \leq l(R_i)$. As $\tau(\hat{R}_i) = l(R_i)$, by strategy-proofness $f_i(R_i, \hat{R}_{-i}) = f_i(\hat{R}_N)$. This together with non-bossyness imply $f_k(R_i, \hat{R}_{-i}) = f_k(\hat{R}_N)$ for all $k \in N$. Continuing in this way we can show that $f(R_N) = f(\hat{R}_N)$. This proves f is a generalized uniform rule for all $R_N \in \mathcal{S}^n$ with $\sum_{i=1}^{n} l(R_i) > 1$. Similarly, we can show this for the profiles $R_N \in \mathcal{S}^n$ $\sum_{i=1}^{n} r(R_i) < 1$.

Now consider profiles $R_N \in \mathcal{S}^n$ such that $\sum_{i=1}^{n} l(R_i) \leq 1 \leq \sum_{i=1}^{n} r(R_i)$. By efficiency, $f_i(R_N) \in [l(R_i), r(R_i)]$. Since f satisfies equal treatment of equals and non-bossyness it is easy to see that we can get a $*$ function such that $f(R_N) = g(R_N)$ for all R_N with $\sum_{i=1}^{n} l(R_i) \leq 1 \leq \sum_{i=1}^{n} r(R_i)$. This completes the proof of the only-if part. ■

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