EQUIVALENCE OF LOCAL AND GLOBAL SD-STRATEGY PROOFNESS UNDER BLOCK CONNECTEDNESS *

Ujjwal Kumar[1] and Souvik Roy[1]

[1]Indian Statistical Institute, Kolkata

September, 2018

Abstract

We deal with the equivalence of local sd strategy-proofness and (global) sd strategy-proofness for random rules on domains that satisfy some block structure with respect to the alternatives. Given a partition of the set of alternatives, a preference satisfies block structure if all the alternatives in an element of the said partition always appear together. Two preferences are called local if there is a collection of pairs of adjacent blocks that flip from one preference to the other. In this setting, we provide a sufficient condition for the equivalence of local strategy-proofness and strategy-proofness for random rules. As applications of our result, we obtain that local strategy-proofness and strategy-proofness are equivalent for random rules on lexicographic multi-dimensional domains when there is exactly one component ordering and the marginal domains are unrestricted or single-peaked or single-crossing, domain for committee formation, sequentially dichotomous domains, etc.
KEYWORDS: Local sd strategy-proofness, (global) sd strategy-proofness, block connectedness, one dimensional domains, multi-dimensional lexicographic domains

JEL CLASSIFICATION CODES: D71, D82.

This is a preliminary draft. Please do not quote.

1. INTRODUCTION

1.1 BACKGROUND OF THE PROBLEM

We consider the situation where a designer has to choose a lottery over the alternatives from a feasible set of lotteries over set of alternatives based on the preferences of a group of individuals in a society. Such a procedure is called a random social choice function. A well-known desirable

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property of such functions is sd strategy-proofness. To compare lotteries, we use the stochastic dominance extension. This property ensures that the dishonest individuals in the society cannot be better off by strategically misreporting their preferences.

In the seminal papers of Gibbard (1973)-Satterthwaite (1975), it is shown that if there are at least three social outcomes and the preferences of the individuals are unrestricted, then every strategy-proof and unanimous social choice function will be dictatorial. A dictatorial social choice function is one that selects the most preferred alternative of one particular individual at every collection of reported preferences.

Domain restrictions turn out to be the most practical way to evade Gibbard-Satterthwaite (Gibbard (1973), Satterthwaite (1975)) impossibility result. Well-known domain restrictions that are studied in literature are single-peaked, single-dipped, single-crossing etc. Moulin (1980) characterize the strategy-proof and unanimous rule on single-peaked domains, Peremans and Storcken (1999) characterize those on single-dipped domains, and Saporiti (2014) on single-crossing domains.

1.2 Our Motivation

Although the sd strategy-proof rules are characterized on several well-known domains, a general characterization of those on arbitrary domains is assumed to be a hard problem. In view of this, researchers started looking at simpler (easy to check) versions of sd strategy-proofness. One such version is local sd strategy-proofness. Local sd strategy-proofness requires that an individual cannot manipulate by a 'slight' misreport of his preferences. More formally, it ensures that an individual cannot manipulate by swapping two consecutive alternatives in his preference. This raises an interesting question as to when, that is under what condition on a domain, such a simple version of sd strategy-proofness becomes equivalent to sd strategy-proofness. This is the main question we deal with in this paper.

In Sato (2013), it is shown that if a domain satisfies 'no restoration' property, then every deterministic locally strategy-proof rule is strategy-proof. In many practical scenarios, a decision maker has to take decision on multiple issues simultaneously. Examples of such situations include deciding the optimum level of budget allocations over different sectors such as health, education, defense etc. Such situations are modeled as multi-dimensional decision problem and the usefulness of such models is well-established in literature.

It is shown in Breton and Sen (1999) that if a multi-dimensional separable domain satisfies richness property, then every strategy-proof rule on it is decomposable. However, the structure of such rules on arbitrary multi-dimensional domains is still open. This motivates us to establish the equivalence of local and global sd strategy-proofness on such domains.

1.3 Our Contribution

First, we define a class of local notion. Then, we consider the equivalence of local sd strategy-proofness and (global) sd strategy-proofness for that class. We provide a sufficient condition for the said equivalence for domains whose graph falls in that class of local notion.

As an application, first we consider one dimensional domains. Cho(2016) shows that the complete domain ($\mathcal{D}_C$), the domain of single peaked preferences, the single-crossing domain and the single-dipped domain all are RLGE domains w.r.t the adjacency graph. We obtain this as a corollary of our main Theorem 3.1.

Next, we consider multi-dimensional lexicographic domain and explore the equivalence of local and global sd strategy-proofness on such domains. We show that if there is exactly one admissible preference over components, then local and global sd strategy-proofness are equivalent for the multidimensional domain if the marginal domains are unrestricted or single-peaked or single-crossing. We further show that if the cardinality of marginal domains is either 1 or 2, then local and global sd strategy-proofness are equivalent on the multi-dimensional lexicographic domains that are obtained by these collection of marginal domains and *any* collection of component orderings.

2. Model

Let $X = \{a, b, \dots\}$ be a finite set of alternatives with $|X| \geq 2$, and $I = \{1, \dots, N\}$ be a finite set of voters with $|I| = N \geq 1$.[1] A **preference** P is an antisymmetric, complete and transitive binary relation over X, i.e., a *linear order*. Given $a, b \in X$, aPb is interpreted as "a is strictly preferred to b" according to P, and in particular, $aP!b$ denotes that aPb and there exists no $c \in X$ such that aPc and cPb. Let $r_k(P)$, $k = 1, \dots, m$, denote the kth ranked alternative in preference P. Let $L(a, P) = \{x \in X | aPx\}$ denote the **low contour set** of a at P. Given two distinct preferences P and P', let $P \triangle P' = \{(a, b) \in X^2 | aPb \text{ and } bP'a\}$ be the set containing each pair of alternatives

[1] We include the case of a singleton voter just for the convenience of simplicity.

whose relative rankings are disagreed by P and P'.[2] Correspondingly, $|P \triangle P'|$ denotes the **Kemeny distance** between P and P' , and we say that alternative a **overtakes** b from P to P' if $(a, b) \in P \triangle P'$. Also for any two preferences P and P' and $B \subseteq X$, by $P |_B = P' |_B$, we mean xPy if and only if $xP'y$ for all $x, y \in B$. Let $\mathcal{D}$ be the set of admissible preferences, referred to as a preference **domain**.[3] In particular, if the domain contains all linear orders, it is referred to as **the complete domain**, denoted $\mathcal{D}_C$, and otherwise, it is a *restricted domain*. A preference profile is a N-tuple of N voter's preferences, denoted $(P_i, P_{-i}) \equiv (P_{\hat{I}}, P_{-\hat{I}}) \in \mathcal{D}^N$, where $i \in I$ and $\hat{I} \subsetneq I$ is nonempty.[4] Let $\Delta(X)$ denote the *lottery space* over X. An element $\varepsilon \in \Delta(X)$ is a *lottery* or *probability distribution* over X. Let ε_a denote the probability of a in the lottery ε. A **Random Social Choice Function** (or **RSCF**) is a map $\varphi : \mathcal{D}^N \to \Delta(X)$ associating each preference profile to a social lottery. For instance, let $\varphi_a(P)$ denote the probability of a in the social lottery $\varphi(P)$. We adopt *the (first-order) stochastic dominance* introduced by Gibbard to establish the axioms of local strategy-proofness (w.r.t. a graph) and strategy-proofness.

2.1 Local strategy-proofness and strategy-proofness

We locate all preferences of a domain on an exogenous graph. Accordingly, let $G = \langle \mathcal{D}, \mathcal{E} \rangle$ denote a (undirected) **graph** where the vertex set is $\mathcal{D}$, and a pair of distinct preferences $P, P' \in \mathcal{D}$ forms an edge if $(P, P') \in \mathcal{E}$. When $(P, P') \in \mathcal{E}$, we say that P and P' are **neighbors**. A **path** $\pi \equiv (P^1, \ldots, P^t)$, $t \geq 2$, in $G(\mathcal{D})$ is a sequence of consecutively neighbored preferences, i.e., $(P^k, P^{k+1}) \in \mathcal{E}$, $k = 1, \ldots, t-1$. Let $\Pi(P, P')$ denote the set of all paths from P to P' in G.[5] Given a path $\pi = (P^1, \ldots, P^l)$, we denote $\pi|_{[P^s, P^t]} = (P^s, P^{s+1}, \ldots, P^t)$ the subpath of π from P^s to P^t where $1 \leq s < t \leq l$.

Once the graph is fixed, we identify the neighbors of each preference, and then systematically weaken the requirement of strategy-proofness: Each voter cannot strictly benefit from misrepresenting preferences neighbor to her sincere one, and we ignore possible manipulations via preferences beyond the neighborhood.

[2] Henceforth, in order to avoid confusion, whenever we write $(a, b) \in P \triangle P'$, it denotes aPb and $bP'a$.

[3] For simplicity, we assume that the preference domain is identical for all voters.

[4] Throughout this paper, we *only* attach subscripts $i, j, h, \hat{I}$ (respectively, $-i, -j, -h, -\hat{I}$) to preferences to emphasize that they are possessed by these voters (respectively, complementary voters). In this paper, since many additional subscripts and superscripts will be added to preferences and profiles, for notational simplicity, we sometimes write a profile (P, P_{-i}) without the subscript "i".

[5] We allow that a preference appears multiple times in a path. For simplicity, we call (P) a **singleton path** and let $\Pi(P, P) = \{(P)\}$ be a set containing a singleton path. To avoid confusion, whenever we write $\Pi(P, P')$, we assume that each path of $\Pi(P, P')$ starts from P and ends at P'.

Definition 2.1. Fix a graph $G = \langle \mathcal{D}, \mathcal{E} \rangle$. Formally, an RSCF $\phi : \mathcal{D}^N \to X$ is **locally sd-strategy-proof** if for all $i \in I$, $P, P' \in \mathcal{D}$ with $(P, P') \in \mathcal{E}$ and $P_{-i} \in \mathcal{D}^{N-1}$, $\phi(P, P_{-i})$ stochastically dominates $\phi(P', P_{-i})$ according to P, i.e., $\sum_{k=1}^{t} \phi_{r_k(P)}(P, P_{-i}) \geq \sum_{k=1}^{t} \phi_{r_k(P)}(P', P_{-i})$, $t = 1, \ldots, |X|$. Moreover, an RSCF $\varphi : \mathcal{D}^N \to X$ is **sd-strategy-proof** if for all $i \in I$, $P, P' \in \mathcal{D}$ and $P_{-i} \in \mathcal{D}^{N-1}$, $\varphi(P, P_{-i})$ stochastically dominates $\varphi(P', P_{-i})$ according to P.

We study strategy-proof SCFs which says that regardless of others' reporting of preferences, a voter cannot strictly benefit by misrepresenting her true preference.

Definition 2.2. Fix a graph $G = \langle \mathcal{D}, \mathcal{E} \rangle$. Domain $\mathcal{D}$ is a **random-local-global-equivalence** (or **RLGE**) domain if every locally sd-strategy-proof RSCF is sd-strategy-proof.

For a given domain $\mathcal{D}$, we define the following class of graphs :

Definition 2.3. A block $B \subseteq X$ in a preference P is defined as a set of contiguous alternatives i.e. B is a block in P if there does not exist $x \in X \setminus B$ and $y, z \in B$ such that $yPxPz$.

Definition 2.4. A pair of disjoint blocks (A, B) is called adjacent in a preference P if for all $x \in A, y \in B, z \in X \setminus (A \cup B)$ we have either $zPxPy$ or $xPyPz$.

For any preference P and P' in $\mathcal{D}$, $(P, P') \in \mathcal{E}$ **implies** there are disjoint pairs of adjacent blocks $(A_1, B_1), \ldots, (A_k, B_k)$ in P such that $(B_1, A_1), \ldots, (B_k, A_k)$ are adjacent in P' and for all $x \in X$ and all $y \in \cup_i (A_i \cup B_i)$, xPy if and only if $xP'y$. In such situations, we say P' is $(A_1, B_1), \ldots, (A_k, B_k)$ flip of P and we write $P' = P[(A_1, B_1), \ldots, (A_k, B_k)]$.

3. The Theorem

For the class of graphs defined above the following theorem holds.

Theorem 3.1. *A domain $\mathcal{D}$ is an RLGE domain if for all $P, P' \in \mathcal{D}$, there exist a path $\pi = (P^1, \ldots, P^l)$ such that for all $1 \leq k \leq l-1$ and for all $1 \leq j \leq m_k$, $P^k \mid_{A_j^k \cup B_j^k} = P^1 \mid_{A_j^k \cup B_j^k}$ where $P^{k+1} = P^k[(A_1^k, B_1^k), \ldots, (A_{m_k}^k, B_{m_k}^k)]$.*

The proof of this theorem is relegated to Appendix A

4. Applications

4.1 Domains in the one-dimensional voting model

We define the graph in the following manner and call it adjacency graph.

Definition 4.1. $P, P' \in \mathcal{E}$ if there exists a single pair of **adjacent** alternatives x, y which are switched between the preferences: xPy and $yP'x$.

By $P \sim P'$ we mean P and P' are local preferences.

Lemma 4.1. *The complete domain ($\mathcal{D}_C$), the domain of single peaked preferences, the single-crossing domain and the single-dipped domain all are RLGE domains w.r.t the adjacency graph.*

The proof of this lemma is relegated to Appendix B

Some of these results are shown by Carroll (2012) and Cho (2016).

4.2 Lexicographically separable domains

We next turn to preference domains studied in the multidimensional voting models. First, we assume that the alternative set can be decomposed as a Cartesian product, i.e., $A = \times_{s \in M} A_s$ where $M = \{1, 2, \ldots, m\}$ is a finite set of components with $m \geq 2$, and for each component $s \in M$, the component set A^s contains finitely many elements and $|A_s| \geq 2$. Thus, an alternative is a assembling of m elements from all component sets, and we hence write $a \equiv (a_1, \ldots, a_m) \equiv (a_S, a_{-S}) \in \times_{s \in M} A_s$ where $S \subsetneq M$ is not empty.

We start the investigation from *lexicographically separable preferences*. First, a **lexicographic order**, i.e., a linear order over M, is fixed to characterize an agent's attitude towards all components. Second, on each component set, a linear order is independently specified, which is referred to as a **marginal preference**. Last, a lexicographically separable preference over A is established such that given two distinct alternatives, according to the most important disagreed component, the alternative owning a better element is always preferred.

Definition 4.2. A preference P is **lexicographically separable** if there exists a (unique) lexicographic order $P_{|0}$ and a (unique) marginal preference $P_{|s}$ for each $s \in M$ such that for all $a, b \in A$, we have $\left[a_s P_{|s} b_s \text{ and } a_\tau = b_\tau \text{ for all } \tau P_{|0} s\right] \Rightarrow [aPb]$.

Evidently, a lexicographically separable preference P can be uniquely represented by an $m+1$-tuple of the lexicographic order $P_{|0}$ and marginal preferences $P_{|1}, \ldots, P_{|m}$, i.e., $P = (P_{|0}; P_{|1}, \ldots, P_{|m})$. Then, one would observe that a pair of two distinct lexicographically separable preferences has the minimum Kemeny distance if and only if they disagree exactly on either the lexicographic orders or one component's marginal preferences, and moreover, the disagreement presents in the form of adjacency. Formally, we say that a pair of lexicographically separable preferences $P = (P_{|0}; P_{|1}, \ldots, P_{|m})$ and $P' = (P'_{|0}; P'_{|1}, \ldots, P'_{|m})$ is **lexicographically adjacent**, denoted $P \simeq P'$, if there exists a unique $s \in \{0, 1, \ldots, m\}$ such that $P_{|s} \sim P'_{|s}$ and $P_{|\tau} = P'_{|\tau}$ for all $\tau \in \{0, 1, \ldots, m\} \backslash \{s\}$.

Let $\mathcal{D}_{LS}$ denote the lexicographically separable domain containing all lexicographically separable preferences with exactly one lexicographic order say($P_{|0}$) and marginals $\mathcal{D}_s, s \in M$. Accordingly, we construct the **lexicographic adjacency graph** (or **LA-graph**) $\langle \mathcal{D}_{\mathrm{LS}}, \mathcal{E}_{\simeq} \rangle$ such that $(P, P') \in \mathcal{E}_{\simeq}$ if and only if $P \simeq P'$.

Theorem 4.1. *If the marginals, $\mathcal{D}_s$ is either unrestricted(complete) or single peaked or dipped or single crossing, for all $s \in M$. Then the domain $\mathcal{D}_{\mathrm{LS}}$ is an RLGE domain.*

The proof of this theorem is relegated to Appendix C

Let the domain for committee formation $\mathcal{D}_{CF}$ denote the lexicographically separable domain containing all lexicographically separable preferences with all possible lexicographic order and $|A_s| \leq 2$, for all $s \in M$.

Accordingly, we construct the **lexicographic adjacency graph** (or **LA-graph**) $\langle \mathcal{D}_{\mathrm{CF}}, \mathcal{E}_{\simeq} \rangle$ such that $(P, P') \in \mathcal{E}_{\simeq}$ if and only if $P \simeq P'$.

Theorem 4.2. *The committee formation domain $\mathcal{D}_{\mathrm{CF}}$ is an RLGE domain.*

The proof of this theorem is relegated to Appendix D

A. Proof of Theorem 3.1

Proof. Let φ be a locally strategy-proof RSCF on $\mathcal{D}$. We show that φ is strategy-proof.

Consider $P, P' \in \mathcal{D}$. We show

$$\varphi_{U(x,P)}(P) \geq \varphi_{U(x,P)}(P') \text{for all} x \in X. \tag{1}$$

Let $\pi = (P^1, \ldots, P^l)$ be a path in $G(\mathcal{D})$ such that for all $1 \leq k \leq l-1$ and for all $1 \leq j \leq m_k$, $P^k \mid_{A_{k_j} \cup B_{k_j}} = P^1 \mid_{A_{k_j} \cup B_{k_j}}$. Fix $x \in X$. In order to show (1), it is enough to show that $\varphi_{U(x,P)}(P^k) \geq \varphi_{U(x,P)}(P^{k+1})$ for all $k \in \{1, \ldots, l-1\}$.

Take $k \in \{1, \ldots, l-1\}$ and suppose $P^{k+1} = P^k[(A_1^k, B_1^k), \ldots, (A_{m_k}^k, B_{m_k}^k)]$. We show that $\varphi_{U(x,P)}(P^k) \geq \varphi_{U(x,P)}(P^{k+1})$. Note that since $P^{k+1} = P^k[(A_1^k, B_1^k), \ldots, (A_{m_k}^k, B_{m_k}^k)]$, we must have

$$\varphi_{(A_l^k \cup B_l^k)}(P^k) = \varphi_{(A_l^k \cup B_l^k)}(P^{k+1}) \text{ for all } l \in \{1, \ldots, m_k\}. \tag{2}$$

Let $U_1(x,P) \subseteq U(x,P)$ be such that for all $y \in U_1(x,P), yP^kz$ for all $z \in A_1^k$. Also let $U_{m_k+1}(x,P) \subseteq U(x,P)$ be such that for all $y \in U_{m_k+1}(x,P), zP^ky$ for all $z \in B_{m_k}^k$. Note that for all $y \in X$ with $yP^kA_1^k$ or $B_{m_k}^kP^ky, U(y,P^k) = U(y,P^{k+1})$. Therefore, by local strategy-proofness between P^k and P^{k+1},

$$\varphi_y(P^k) = \varphi_y(P^{k+1}) \text{for all } y \text{ with } yP^kA_1^k. \tag{3}$$

This in particular means $\varphi_{U_1(x,P)}(P^k) = \varphi_{U_1(x,P)}(P^{k+1})$ and $\varphi_{U_{m_k+1}(x,P)}(P^k) = \varphi_{U_{m_k+1}(x,P)}(P^{k+1})$.

For $l \in \{2, \ldots, m_k\}$, let $U_l(x,P) \subseteq U(x,P)$ be such that for all $y \in U_l(x,P), wP^kyP^kz$ for all $z \in A_l^k$ and for all $w \in B_{l-1}^k$.

Claim A.1. *$\varphi_{U_l(x,P)}(P^k) = \varphi_{U_l(x,P)}(P^{k+1})$ for all $l \in \{2, \ldots, m_k\}$.*

Proof. We prove this by induction. First we show that $\varphi_{U_2(x,P)}(P^k) = \varphi_{U_2(x,P)}(P^{k+1})$. By equation 2, it must be that $\varphi_{(A_1^k \cup B_1^k)}(P^k) = \varphi_{(A_1^k \cup B_1^k)}(P^{k+1})$. Also note that for all $y \in U_2(x,P), U(y,P^k) = U(y,P^{k+1})$. This together with the fact that $\varphi_{(A_1^k \cup B_1^k)}(P^k) = \varphi_{(A_1^k \cup B_1^k)}(P^{k+1})$ and using local strategy-proofnes from P^k to P^{k+1} implies that $\varphi_y(P^k) = \varphi_y(P^{k+1})$ for all $y \in U_2(x,P)$, which in particular means $\varphi_{U_2(x,P)}(P^k) = \varphi_{U_2(x,P)}(P^{k+1})$.

Now let $\varphi_{U_l(x,P)}(P^k) = \varphi_{U_l(x,P)}(P^{k+1})$ for all $2 \leq l \leq r$. We prove $\varphi_{U_{r+1}(x,P)}(P^k) = \varphi_{U_{r+1}(x,P)}(P^{k+1})$ which completes the proof of the claim by induction. Let $b \in B_r^k$ such that zP^kb for all $z \in B_r^k \setminus \{b\}$. Then by 2 and the assumption of induction hypothesis that $\varphi_{U_l(x,P)}(P^k) = \varphi_{U_l(x,P)}(P^{k+1})$ for all $2 \leq l \leq r$, we have

$$\varphi_{U(b,P^k)}(P^k) = \varphi_{U(b,P^k)}(P^{k+1}). \tag{4}$$

Now note that for all $y \in U_{r+1}(x,P), U(y,P^k) = U(y,P^{k+1})$. This together with equation 4 and local strategy-proofness from P^k to P^{k+1}, implies that $\varphi_y(P^k) = \varphi_y(P^{k+1})$ for all $y \in U_{r+1}(x,P)$, which in particular means $\varphi_{U_{r+1}(x,P)}(P^k) = \varphi_{U_{r+1}(x,P)}(P^{k+1})$. This completes the proof of the claim. ■

For $l \in \{1,\dots,m_k\}$, let us define $\hat{U}_l(x,P) \subseteq U(x,P)$ be such that $\hat{U}_l(x,P) = U(x,P) \cap (A_l^k \cup B_l^k)$.

Claim A.2. *$\varphi_{\hat{U}_l(x,P)}(P^k) \geq \varphi_{\hat{U}_l(x,P)}(P^{k+1})$ for all $l \in \{1,\dots,m_k\}$.*

Proof. Take any $r \in \{1,\dots,m_k\}$. Since $P^k \mid_{A_r^k \cup B_r^k} = P^1 \mid_{A_r^k \cup B_r^k}$, there must exist $x_r \in A_r^k \cup B_r^k$ such that $\hat{U}_r(x,P) = U(x_r,P^k) \cap (A_r^k \cup B_r^k)$. Because $U(x_r,P^k) = [U(x_r,P^k) \cap (A_r^k \cup B_r^k)] \cup [U(x_r,P^k) \setminus (A_r^k \cup B_r^k)]$, by the previous claim and 2, we have $\varphi_{U(x_r,P^k)\setminus(A_r^k \cup B_r^k)}(P^k) = \varphi_{U(x_r,P^k)\setminus(A_r^k \cup B_r^k)}(P^{k+1})$. By local strategy-proofness from P^k to P^{k+1}, $\varphi_{U(x_r,P^k)}(P^k) \geq \varphi_{U(x_r,P^k)}(P^{k+1})$. This together with the fact that $\varphi_{U(x_r,P^k)\setminus(A_r^k \cup B_r^k)}(P^k) = \varphi_{U(x_r,P^k)\setminus(A_r^k \cup B_r^k)}(P^{k+1})$ implies $\varphi_{\hat{U}_r(x,P)}(P^k) \geq \varphi_{\hat{U}_r(x,P)}(P^{k+1})$. This completes the proof of the claim. ■

Since, $U(x,P) = U_{m_k+1}(x,P) \cup [\cup_{l=1}^{m_k}(U_l(x,P) \cup \hat{U}_l(x,P))]$, by A.1 and A.2 it follows that $\varphi_{U(x,P)}(P^k) \geq \varphi_{U(x,P)}(P^{k+1})$. This completes the proof of the theorem.

■

B. Proof of Lemma 4.1

Proof. Let $\mathcal{D}$ be a domain that is either a complete domain ($\mathcal{D}_C$) or a domain of single peaked preferences or a single-crossing domain or a single-dipped domain. Then for any two preferences $P, P' \in \mathcal{D}$, there exixts a path $\pi = (P^1,\dots,P^l)$ from P to P' having no (a,b) restoration for all $a,b \in X$. Take $k \in \{1,\dots,l-1\}$. Let $P^{k+1} = P^k[(a_k,b_k)]$.

Claim B.1. $P^k \mid_{\{a_k,b_k\}} = P^1 \mid_{\{a_k,b_k\}}$

Proof. Suppose not. Then it must be the case that $a_k P^k b_k$ and $b_k P^1 a_k$. Since $P^{k+1} = P^k[(a_k,b_k)]$, it must be that $b_k P^{k+1} a_k$. This together with the facts that $a_k P^k b_k$ and $b_k P^1 a_k$, it folows that π has an (b_k,a_k) restoration which is a contradiction. Hence this completes the proof of the claim. ■

The claim implies that $\mathcal{D}$ satisfies the sufficiency condition stated in Theorem 3.1. This implies $\mathcal{D}$ is RLGE which completes the proof of the lemma.

■

C. Proof of Theorem 4.1

Proof. Let $P, P' \in \mathcal{D}_{LS}$. It is enough to show that there exists a path π that satisfies the sufficiency condition in Theorem 3.1. Since $P, P' \in \mathcal{D}_{LS}$, let $P = (P_{|_0}, P_1, \ldots, P_m)$ and $P' = (P_{|_0}, P'_1, \ldots, P'_m)$. Given that marginals $\mathcal{D}_s$ is either unrestricted(complete) or single peaked or dipped or single crossing, for all $s \in M$. So for all $l \in M$, there exists a path π_l from P_l to P'_l in $G(\mathcal{D}_l)$ having no (a, b) restoration for all $a, b \in A_l$. Without loss of generality we can assume $1P_{|_0}2P_{|_0} \ldots P_{|_0}m$. Define the path $\pi = ((\pi_1, P_{|_0}, P^2, \ldots, P_m), (\pi_2, P_{|_0}, P'_1, P^3, \ldots, P_m), \ldots, (\pi_m, P_{|_0}, P'_1, \ldots, P'_{m-1}))$. Since π_l is a path without restoration for all $l \in M$ and there is exactly one component ordering, it follows that the path π satisfies the sufficiency condition stated in Theorem 3.1. This completes the proof of the Theorem. ■

D. Proof of Theorem 4.2

Proof. Let $P, P' \in \mathcal{D}_{CF}$. It is enough to show that there exists a path π that satisfies the sufficiency condition in Theorem 3.1. Note that since $1 \leq |A_s| \leq 2$ for all $s \in M$, it must be that $1 \leq |\mathcal{D}_s| \leq 2$ for all $s \in M$. Therefore, let $\mathcal{D}_s = \{P_s, P'_s\}$ for all $s \in M$. Since $P, P' \in \mathcal{D}_{CF}$, let $P = (P_{|_0}, P_1, \ldots, P_m)$ and $P' = (P'_{|_0}, P'_1, \ldots, P'_m)$. Without loss of generality we can assume that $1P'_{|_0}2 \ldots P'_{|_0}m$. Define the path $\pi = ((\pi_1, P_{|_0}, P_2, \ldots, P_m), (\pi_0(1), P'_1, P_2, \ldots, P_m), \ldots, (\pi_0(m), P'_1, \ldots, P'_m))$. Since $1 \leq |\mathcal{D}_s| \leq 2$ for all $s \in M$ and along the path π, first the component 1 overtakes other components and come to the top and then the component 2 overtakes and comes to the second top and so on, it follows that the path π satisfies the sufficiency condition stated in Theorem 3.1. This completes the proof of the Theorem. ■

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