Economic Growth with Locked-in Fertility: Under- and Over-Investment in Education*

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Abstract

This research theoretically analyzes the role of irreversible fertility decisions in economic growth in the presence of idiosyncratic ability shocks after childbirth. It argues that the irreversibility constraint delays the growth process by distorting the resource allocation between the quantity and quality of children. In underdeveloped stages, where family size is locked into large levels, education investment places a heavy financial burden on households. The impossibility of ex post fertility adjustment then deprives some competent children of learning opportunities. In more developed stages, by contrast, family size locked into smaller levels facilitates education investment even in some incompetent children. A redistributive policy to enhance aggregate human capital and the growth performance is proposed for each stage.

Keywords: Irreversibility; Over- and Under-Investment; Human Capital; Growth.

JEL Classification: D10; J13; J24; O15.

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1 Introduction

Nowadays, it is widely recognized that developing skills and knowledge is one of the most effective means not only to improve individual well-being but also to advance the economy as a whole. Indeed, the United Nations has set providing universal access to primary education by 2015 as one of its Millennium Development Goals. Whether developing regions make full use of this growth strategy is however hard to say, given that the net enrollment rate in their primary schools was 91 percent in 2015 (United Nations, 2015, p. 4). One of the financial hindrances to the education goal is the increase in the number of potential students, which is expected from the total fertility rate of nearly 3.0 in 2010–2015 in less developed regions excluding China (United Nations, Department of Economic and Social Affairs, 2017, p. 124).[1]

Regarding advanced countries, it appears that some of them face the opposite situation. Japanese households cannot afford to maintain replacement-level fertility partly because of their growing enthusiasm about the education of their children.[2] On the supply side, almost 40 percent of private universities did not meet their student quota in the 2017 academic year (Promotion and Mutual Aid Corporation for Private Schools of Japan, 2017, p. 2). As Clark (2012) points out, the under-enrollment will naturally urge them to accept a broader range of applicants. This leads to the declining quality of higher education. In the United States, education investment yielded negative returns for graduates from 6.5 percent ($\approx$120/1833) of colleges/universities, whereas the total fertility rate dropped to 1.88 during 2010–2015.[3]

A plausible conjecture from these observations is that the resources of those economies are allocated inadequately between the quantity and quality of labor and the bias changes its

[1]A positive effect of fertility decline on years of schooling is reported by Joshi and Schultz (2007), who assess a family planning and maternal-child health program implemented in Matlab, Bangladesh. Ashraf et al. (2013) predict, incorporating this effect into a simulation model, that a lower time path of fertility will lead to a higher path of per capita GDP in Nigeria over the 21st century.

[2]According to a questionnaire survey by National Institute of Population and Social Security Research (2017, p. 72), the average planned number of children, 1.87, is lower than the average desired number of children, 2.27, for first-married couples whose wives are between 45 and 49 years of age. For first-married couples of all age groups whose wives are below 50 years, the cost of child rearing and education is the most important reason behind this gap (ibid., p. 74).

[3]See the 2017 college ROI report by PayScale (`https://www.payscale.com/college-roi`, accessed on October 6, 2017), and also United Nations, Department of Economic and Social Affairs (2017, p. 132) for the fertility rate.

direction depending on the stage of economic development. In this sense, economies may go through a transition from under- to over-investment in education during the growth process.[4] The transition is consistent with a recent trend in returns to schooling over the last decades. Montenegro and Patrinos (2014) estimate returns to schooling from data of 139 economies, revealing a declining trend since the 1980s.

The present paper aims to shed light on the underlying mechanism of the aforementioned transition by developing a growth theory that incorporates the irreversibility of fertility decisions. This constraint restricts parental fertility adjustment in response to idiosyncratic ability shocks on their children.[5] The resulting accumulation of human capital alters technological progress, the return on education, and fertility decisions by the next generation. The irreversibility constraint therefore intervenes dynamically between population, education, and growth. This paper also examines how the government can enhance growth performance by redressing the balance of resource allocation.

The long-run, macroeconomic approach of this research distinguishes itself from the previous literature. The inequality and growth literature, which has flourished since the 1990s, affirms the possibility of under-investment in human capital in the presence of capital market imperfections (cf. Galor and Zeira, 1993; Moav, 2002; Mookherjee and Ray, 2003).[6] Other theoretical studies argue that information imperfections, along with incomplete market, may induce individuals' precautionary savings for human capital investment (cf. Gould et al., 2001; Aiyagari et al., 2002).[7] None of these findings is fully satisfactory in terms of this

[4] Under-investment in education is referred here to as a situation in which aggregate human capital is enhanced by shifting the aggregate resources for child rearing from the quantity to the quality of children. The opposite case applies to over-investment in education. The present paper defines both types of investment from the macroeconomic rather than the individual viewpoint. It is not concerned with skill mismatch between workers and their occupations [see, for example, Sicherman (1991) for this type of mismatch].

[5] Goldstein et al. (2003, p. 487, Table 2) compare mean personal ideal family size and mean personal expected family size for young women by using the Eurobarometer 2001 survey. They report that the former measure is larger than the latter by 0.2 to 0.4 points in major European countries (p. 486). A similar pattern applies to the United States (Hagewen and Morgan, 2005, p. 509, Figure 1). These disparities are consistent with this paper's assertion that some households in the developed stages are prevented from adjusting their family sizes upward.

[6] Apart from capital market imperfections, Dávila (2018) argues that the failure to internalize the externality of *aggregate* human capital brings about the social suboptimality of private investment in fertility and in education. The present paper, by contrast, attributes inefficiency in the two types of investment to the irreversibility of fertility decisions. See also Footnote 21.

[7] Gould et al. (2001) consider the eroding effect of technological progress, which is biased and random across sectors, on human capital. Aiyagari et al. (2002) highlight the lack of insurance markets for ability

paper's objective.

This research models an overlapping generations economy that features the following key elements. First, individuals derive utility from the quantity and quality of their children as well as from their own consumption, as stated by Becker and Lewis (1973). Second, in contrast to the standard literature, there is a time lag between fertility and education decisions, and idiosyncratic ability shocks occur in between. Fertility decisions are assumed to be completely irreversible for ethical, legal, and physical, and other reasons.[8] Once determined, the number of dependents is not adjustable in either direction, and such inflexibility is the source of sunk cost. Third, in line with the formulation by Galor and Moav (2000), technological progress is skill-biased in the sense that its acceleration stimulates the incentive for higher education. Forth and finally, the invention of new technology depends on the aggregate amount of human capital, which is the fruit of parental child rearing.

Taking these elements into consideration, the dynamic theory developed later demonstrates a scenario of economic development. In the early development stage, where technological progress is sluggish, education investment is not fruitful for parents whose children have average ability. Assuming that children with average ability will be born to them, all households aim to concentrate their child-rearing resources on the quantity of children.[9] While children reveal their true abilities by the time of schooling, fertility adjustment to a change in education expenses is infeasible at that time. With the locked-in fertility decision, revising the initial education plan involves an unexpected reduction in household consumption.

Accordingly, the irreversibility constraint prevents some households from coping with education costs. The resulting biased allocation of parenting resources entails under-investment in education. To complicate matters, the constraint brings about a counter effect on growth:

as well as the lack of loan markets.

[8] See, among others, Fraser (2001) and Doepke and Zilibotti (2005) for theoretical arguments underlying the irreversibility of fertility decisions. In relation to schooling, a recent study by de la Croix and Doepke (2009) focuses on the lock-in effect of fertility decisions on individuals' voting preferences to account for the differences in public education systems across countries.

[9] While this strong assumption makes a great contribution to the tractability of the dynamic model, it will not be essential for the outcome of the distorted resource allocations. See Appendix B for an extension to the expected-utility framework. Nakagawa and Sugimoto (2011) similarly analyze the lock-in effect on the education decision by assuming that adult individuals have the same expectation about their own abilities.

It increases the aggregate amount of parenting resources through the provision of education support, against the initial plan, by households whose children turn out to be significantly competent. The combination of these two opposing forces is generally ambiguous.

As a means to mitigate under-investment in education, the present paper proposes an ability-based subsidy for education financed by a universal tax on child rearing. Such a redistributive policy tends to be effective in early phases where the constraint is binding for children in the upper tail of the ability distribution. The rationale of this result is that subsidizing their skill acquisition would make a substantial contribution to the formation of aggregate human capital.

Technological progress driven by human capital accumulation eventually alters households' (ex ante) stances toward education, which is followed by a major fertility decline. Education investment in this stage is attractive even for parents whose children have average ability. With the aforementioned belief on children's abilities, all adult individuals choose smaller family sizes to cope with the cost of future education.

Since the family size is locked into small levels, cancelling the education plan certainly diminishes the utility from children while it leaves a sufficient budget for consumption. Households therefore invest in education unless their children turn out to be significantly incompetent, leading to over-investment in education. On top of that, those who cancel the education plan shift their budgets away from child rearing. Both of these effects work adversely on the accumulation of aggregate human capital. In order to mitigate the over-investment, it is useful to stimulate average fertility instead of educating low-ability children, for example, through a universal subsidy for child rearing financed by an ability-based tax on education.

The rest of this paper is organized as follows. Section 2 describes the structure of the baseline model, in which the fertility adjustment is unrestricted, and then considers optimal decisions on fertility and education. These individual choices are aggregated for an analysis of the dynamic behavior of the entire economy. Section 3 builds the mainline model by introducing the irreversibility constraint into the baseline model. Section 4 demonstrates that the constrained economy goes through a transition from under- to over-investment in education. It also investigates the workings of redistributive policies that are designed to improve the growth performance. Section 5 summarizes the discussions and presents some directions of the future research. The appendix provides the mathematical proofs of some key results and also discusses about the robustness of the model.

2 The Baseline Model: An Unlocked Economy

The economy has an overlapping-generations structure and operates over an infinite discrete time horizon, $t = 0, 1, 2 \cdots$.[10] A single homogeneous good is produced in one sector by employing human capital, and labor productivity improves through learning by doing. The economy is closed and abstracts from capital markets.

Adult individuals have all information except the abilities of the children they intend to have. Ability shocks occur after childbirth, and then parents decide whether to provide education support for their children. In making the education decision, they can adjust the number of their children as much as they want without any cost. In other words, fertility decision is "unlocked" and reversible. Because this property makes the ex ante optimization meaningless, the baseline model is essentially viewed as a perfect foresight model in which fertility and education decisions are made simultaneously.

2.1 Firms

In perfectly competitive environments, firms generate a single homogeneous good by employing human capital (i.e., efficiency units of labor) with a linear technology. The level of output per worker in period t, denoted as y_t, is determined through the production function

$$y_t = A_t H_t / N_t, \tag{1}$$

where A_t, H_t, and N_t are the levels of technology, aggregate human capital, and working population, respectively, in period t. For the sake of simplicity, the price of the final good is normalized to unity. As a result of profit maximization by price-taking firms, H_t maximizes

[10] The baseline model is an extension of the model developed by Galor and Weil (2000), who explore the mechanism underlying the demographic transition in the long-term growth process. In return for allowing the heterogeneity of individuals' abilities, the baseline model needs some modifications in, for example, the household budget constraint, the production function of individual human capital, and the creation of new technology, in order to keep its tractability.

the aggregate profit $A_t H_t - w_t H_t$, where w_t is the market wage rate per unit of human capital in period t. In the competitive labor market considered herein, w_t is adjusted so that the resulting profit is neither negative nor infinitely large, leading to $w_t = A_t$.

2.2 Households

A new generation is born at the beginning of each period and lives for two periods. Generation t, born in period $t-1$, comprises a continuum of individuals existing on the interval $[0, N_t]$.

2.2.1 Environment

Consider the lifetime of an individual $i \in [0, N_t]$ of generation t, born in period $t-1$. In the first period (childhood), the individual has no wealth and engages in skill acquisition, possibly with parental support $e_{t-1}^i \geq 0$. In the second period (adulthood or parenthood), the individual acquires $h_t^i > 0$ efficiency units of labor to earn wages, while giving birth to n_t^i units of identical children all at once.[11] Child rearing incurs a cost of $w_t(\delta + e_t^i)$ per child, where $\delta > 0$ and e_t^i are the fixed cost and the education cost, respectively.[12]

The remaining income is used up for consumption, c_t^i, so that no bequests are left to the offspring. It follows that the budget constraint is

$$c_t^i = w_t[h_t^i - n_t^i(\delta + e_t^i)]. \tag{2}$$

The utility of individual i of generation t, u_t^i, depends on not only consumption in adulthood but also aggregate income of his/her children. Each of these children, indexed by $j \in [0, N_{t+1}]$, acquires $h_{t+1}^{i,j}$ efficiency units of labor in period $t+1$.[13] With these considerations, the utility function is formulated as

$$u_t^i = (1-\alpha)\ln c_t^i + \alpha \ln \left(w_{t+1} n_t^i h_{t+1}^{i,j} \right), \tag{3}$$

where $\alpha \in (0,1)$ measures the degree of parental altruism.

[11] Siblings do not have to be born simultaneously. One may assume that when childbirth is sequential, their (identical) ability level is unveiled after the youngest child is born.

[12] Unlike in Galor and Weil's (2000) model, the costs of child rearing, δ and e_t^i, are measured not in labor time but in efficiency units of labor, h_t^i. The resulting fertility decision depends on h_t^i and, as shown later, the result facilitates the construction of a dynamical system. Moav (2005) takes a hybrid approach by measuring only the fixed cost of child rearing in time.

[13] Adult individual i in period t has a continuum of children on $[0, n_t^i]$, which is a subset of $[0, N_{t+1}]$.

2.2.2 Production of Human Capital

Children may differ in the level of education and innate ability across, but not within, households, meaning that no heterogeneity exists among siblings. The labor supply in efficiency units obtained by child j, born of parent i in period t, is determined according to the production function

$$\begin{aligned} h_{t+1}^{i,j} &= h(e_t^i, a_t^i, g_{t+1}) \\ &= \begin{cases} \bar{h} - g_{t+1} & \text{if } e_t^i < \bar{e}; \\ \bar{h} - (1 - a_t^i) g_{t+1} & \text{if } e_t^i \geq \bar{e}. \end{cases} \end{aligned} \tag{4}$$

where $a_t^i \in [0, 1]$ and $e_t^i \geq 0$ denote the levels of his/her ability and education, respectively, and $g_{t+1} \geq 0$ is the rate of technology growth between periods t and $t + 1$. $\bar{h}$ is interpreted as the potential level of individual human capital.[14]

In line with the formulation by Galor and Moav (2000), the function h above satisfies three key properties for any $a_t^i \in (0, 1)$ and $g_{t+1} > 0$.[15] First, education investment has a discrete and positive impact on the formation of human capital; more precisely, $\lim_{e_t^i \to \bar{e}-0} h(e_t^i, a_t^i, g_{t+1}) < h(\bar{e}, a_t^i, g_{t+1})$. Children become either skilled or unskilled labor, depending on whether parental education support reaches a threshold level, $\bar{e} > 0$.[16] Second, the advantage of skill acquisition is to mitigate the "erosion effect" of technological progress, which makes part of acquired skills obsolete; i.e., $h_g(0, a_t^i, g_{t+1}) < h_g(\bar{e}, a_t^i, g_{t+1}) < 0$.[17] These properties bring about skill-biased technological progress: An acceleration of technological progress raises the relative skill $h(\bar{e}, a_t^i, g_{t+1})/h(0, a_t^i, g_{t+1})$, thereby making education investment more advantageous for each child. Third and finally, the supply of skilled labor is ability dependent whereas that of unskilled labor is not; i.e., $h_a(\bar{e}, a_t^i, g_{t+1}) > h_a(0, a_t^i, g_{t+1}) = 0$. This indicates that, for a given g_{t+1}, education investment is more advantageous for more competent children.

[14] Throughout the paper, one may plausibly assume that $\bar{h} \geq g_{t+1}$ to exclude an unrealistic case in which some members of generation $t + 1$ end up with a negative level of human capital.

[15] With respect to the erosion effect below, their theoretical formulation is inspired by Nelson and Phelps (1966).

[16] The discreteness of h with respect to e_t^i, which brings about a binary education choice, is not essential either under- or over-investment in education, while it contributes to the tractability of the model. It is the irreversibility of fertility decisions, not the discreteness of h, that may limit the ex post adjustment of education to the unconstrained levels.

[17] Throughout the present paper, $f_x(x, y)$ denotes the partial derivative of a function f with respect to x.

2.2.3 Optimization

In the absence of the irreversibility constraint on the fertility decision, households may adjust the quantity of children depending on their observed abilities. For this reason, it makes no sense to consider the ex ante parental decision, and the resource allocation problem facing each household is simplified to one-step optimization with no uncertainty.

Given h_t^i, a_t^i and g_{t+1}, adult individuals aim to maximize their own utility as price takers. By substituting Eqs. (2) and (4) into Eq. (3), the maximization problem faced by parent i in period t is

$$\{n_t^i, e_t^i\} = \arg\max \left\{(1-\alpha)\ln[h_t^i - n_t^i(\delta + e_t^i)] + \alpha \ln[n_t^i h(e_t^i, a_t^i, g_{t+1})]\right\}, \tag{5}$$

subject to $(n_t^i, e_t^i) \geq 0$.

First, consider the fertility decision. The objective function exhibits the logarithmic form and strict concavity with respect to n_t^i. Hence, the first-order optimality condition yields

$$n_t^i = \frac{\alpha}{\delta + e_t^i} h_t^i, \tag{6}$$

implying that a fixed fraction of labor, α, is devoted to child rearing regardless of the income level. It would be historically plausible to impose a necessary condition for sustainable population growth. That is to say, the upper bound of fertility is above the replacement level:

$$\frac{\alpha}{\delta}\bar{h} > 1. \tag{A1}$$

Next, consider the education decision. Substitution of Eq. (6) into Eq. (5) reveals that

$$e_t^i = \arg\max \frac{h(e_t^i, a_t^i, g_{t+1})}{\delta + e_t^i}, \tag{7}$$

subject to $e_t^i \geq 0$. As is evident from Eq. (4), the education choice is binary: It is rational for parents in period t to choose either $e_t^i = 0$ or $e_t^i = \bar{e}$. If $g_{t+1} = 0$, education investment is not at all productive, and thus $e_t^i = 0$ is chosen. Regarding the case with $g_{t+1} \in (0, \bar{h})$, let $\tilde{a}_t^*$ be a critical ability level above which choosing $\bar{e}$ is strictly preferable. Then, from Eq. (7),

$$\tilde{a}_t^* = \frac{\bar{e}}{\delta}\frac{\bar{h} - g_{t+1}}{g_{t+1}} \equiv \tilde{a}^*(g_{t+1}), \tag{8}$$

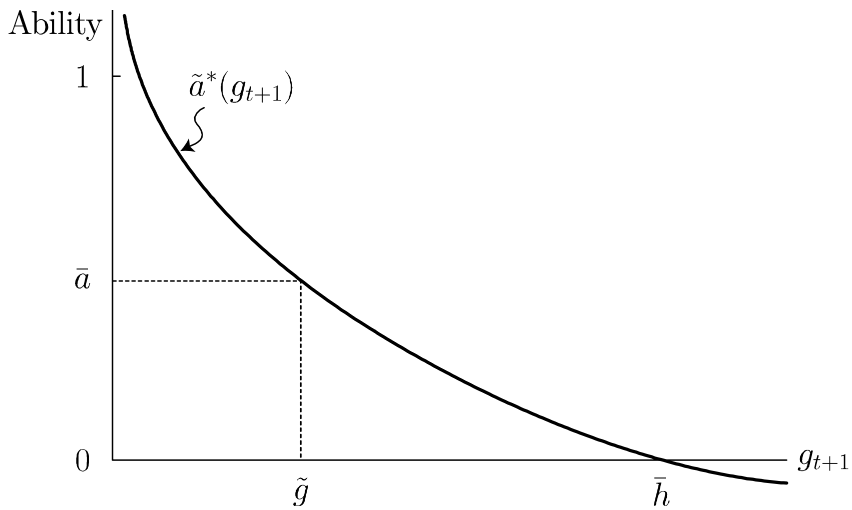

Figure 1. The Relationship between the Growth Rate of Technology and the Critical Ability Level for Education in the Unlocked Economy.

where $\tilde{a}^*(\bar{h}) = 0$ and $\tilde{a}_g^*(g_{t+1}) < 0 \; \forall g_{t+1} > 0$. Since the relative skill $h(\bar{e}, a_t^i, g_{t+1})/h(0, a_t^i, g_{t+1})$ increases strictly with the ability level a_t^i, the education decision by parent i in period t is monotonic with respect to a_t^i in a way such that

$$e_t^i = e^*(a_t^i, g_{t+1}) \equiv \begin{cases} 0 & \text{if } a_t^i \leq \tilde{a}^*(g_{t+1}); \\ \bar{e} & \text{if } a_t^i > \tilde{a}^*(g_{t+1}). \end{cases} \tag{9}$$

where $g_{t+1} \in (0, \bar{h})$.[18] Thus, in contrast to the fertility decision, the education decision is independent of the income level.

Figure 1 represents these results graphically. No one invests in education as long as technological progress is so sluggish that $\tilde{a}_t^* > 1$. A rise in g_{t+1} raises the relative skill $h(\bar{e}, a_t^i, g_{t+1})/h(0, a_t^i, g_{t+1})$ and thereby makes education investment more attractive for each parent, leading to a decline in $\tilde{a}_t^*$. Education investment begins to spread when $\tilde{a}_t^*$ falls below unity.

2.3 Macroeconomic Variables

Although siblings are identical, no genetic ability is inherited within dynasties. Technically speaking, a_t^i is identically and independently distributed across households and periods ac-

[18] For simplicity, it is assumed that individuals do not choose $e_t^i = \bar{e}$ unless it is strictly preferable to $e_t^i = 0$.

cording to a cumulative distribution function F. The function has standard properties such that $F(a) = 0 \; \forall a \leq 0$, $F(a) = 1 \; \forall a \geq 1$, and $F'(a) > 0 \; \forall a \in (0, 1)$.

2.3.1 Aggregate Human Capital and Population

Substituting Eq. (9) into Eqs. (4) and (6) gives the evolution of aggregate human capital as[19]

$$\begin{aligned} H_{t+1} &= \int_0^{N_{t+1}} h_{t+1}^j dj \\ &= \int_0^{N_t} n_t^i h_{t+1}^{i,j} di \\ &= \phi^*(g_{t+1}) H_t, \end{aligned} \tag{10}$$

where $H_0 = \int_0^{N_0} h_0^i di$ and the growth factor is defined as

$$\begin{aligned} \phi^*(g_{t+1}) &\equiv \alpha \int_0^1 \frac{h(e^*(a, g_{t+1}), a, g_{t+1})}{\delta + e^*(a, g_{t+1})} dF(a) \\ &= \alpha \left[\int_0^{\tilde{a}_t^*} \frac{\bar{h} - g_{t+1}}{\delta} dF(a) + \int_{\tilde{a}_t^*}^1 \frac{\bar{h} - (1-a) g_{t+1}}{\delta + \bar{e}} dF(a) \right]. \end{aligned} \tag{11}$$

The function has the following properties. First, $d\phi^*(g_{t+1})/dg_{t+1} < 0 \; \forall g_{t+1} > 0$ because the acceleration of technological progress delays the accumulation of human capital by making part of the acquired skills outdated and useless.[20] In addition,

$$\lim_{g_{t+1} \to 0} \phi^*(g_{t+1}) > 1; \qquad \lim_{g_{t+1} \to \infty} \phi^*(g_{t+1}) < 0. \tag{12}$$

The second property holds because if g_{t+1} is sufficiently low, no one invests in education and $\phi^*(g_{t+1})$ is close to the upper bound of the fertility rate, $\alpha \bar{h}/\delta$ in Eq. (A1). The last property is due to the linearity of the erosion effect: If g_{t+1} is sufficiently large, the adverse effect is substantial enough to generate negative human capital.

[19] In order to derive Eq. (10), note that individual i of generation t, born of individual p, acquires $h_t^i = h_t^{p,i} = h(e^*(a_{t-1}^p, g_t), a_{t-1}^p, g_t)$, which is independent of his/her child's human capital $h_{t+1}^{i,j} = h(e^*(a_t^i, g_{t+1}), a_t^i, g_{t+1})$.

[20] In Eq. (11), a marginal change in g_{t+1} through $\tilde{a}^*(g_{t+1})$ has no effect on $\phi^*(g_{t+1})$ because, in light of Eq. (8), $\tilde{a}_t^*$ is a critical value such that

$$\frac{\bar{h} - g_{t+1}}{\delta} = \frac{\bar{h} - (1 - \tilde{a}_t^*) g_{t+1}}{\delta + \bar{e}}.$$

As Lemma 1 below confirms, the unconstrained parental decision maximizes the aggregate human capital of each household, $n_t^i h_{t+1}^{i,j}$, and thus that of the economy, H_{t+1}.[21]

Lemma 1 *In the absence of the irreversibility constraint, consider individual i of generation t who aims to allocate b_t^i efficiency units of labor between n_t^i and e_t^i. Given $g_{t+1} > 0$ and $b_t^i > 0\ \forall i \in [0, N_t]$, the pair $n_t^i = b_t^i/(\delta + e_t^i)$ and $e_t^i = e^*(a_t^i, g_{t+1})$ maximizes aggregate human capital in period $t+1$.*

Proof. The resource constraint yields $n_t^i = b_t^i/(\delta + e_t^i)$. Then, in light of Eq. (10),

$$H_{t+1} = \int_0^{N_t} b_t^i \frac{h(e_t^i, a_t^i, g_{t+1})}{\delta + e_t^i} di, \tag{13}$$

where b_t^i is exogenous. Since the education decision $e_t^i = e^*(a_t^i, g_{t+1})$ satisfies Eq. (7), it maximizes the integral in Eq. (13). $\square$

Eq. (13) shows that aggregate human capital H_{t+1} depends on two factors: the amount of parenting resources, b_t^i, and the *efficiency* of its allocation between n_t^i and e_t^i. As will become apparent, these are the channels through which the irreversibility of fertility decision affects economic growth.

As with Eq. (10), the level of working population in period $t+1$ is derived as follows:

$$N_{t+1} = \int_0^{N_t} n_t^i di = \psi^*(g_{t+1}) H_t, \tag{14}$$

where

$$\begin{aligned} \psi^*(g_{t+1}) &\equiv \int_0^1 \frac{\alpha}{\delta + e^*(a, g_{t+1})} dF(a) \\ &= \frac{\alpha}{\delta} F(\tilde{a}_t^*) + \frac{\alpha}{\delta + \bar{e}} [1 - F(\tilde{a}_t^*)]. \end{aligned}$$

This is a continuous, nonincreasing function such that $\psi^*(g_{t+1}) = \alpha/(\delta + \bar{e})\ \forall g_{t+1} \geq \bar{h}$. It is strictly decreasing in g_{t+1} as long as $0 < \tilde{a}^*(g_{t+1}) < 1$. The adverse effect of technological acceleration on the working population is due to the substitution effect, i.e., the shift in parenting resources from the quantity to the quality of children.

[21] The present paper does not deal with the resource allocation problem of the social planner who takes into account the external effect of e_t^i on H_{t+1} through g_{t+1}. See also Footnote 6.

2.3.2 The Average Levels of Human Capital, Fertility, and Output

Let h_t be the average level of human capital in period t. Then Eqs. (10) and (14) yield

$$h_t \equiv \frac{H_t}{N_t} = \frac{\phi^*(g_t)}{\psi^*(g_t)} \tag{15}$$

for $t \geq 1$, and $h_0 = H_0/N_0$. This result indicates two opposing forces of technological acceleration on h_t: While a rise in g_t depresses aggregate human capital H_t through the erosion effect, it decreases the working population N_t through the substitution effect.

Let n_t be the ratio of the child to the adult population in period t. In the single-parent economy considered here, n_t is interpreted as the average fertility rate in period t. Then, it follows from Eqs. (14) and (15) that

$$n_t \equiv \frac{N_{t+1}}{N_t} = \frac{\psi^*(g_{t+1})\phi^*(g_t)}{\psi^*(g_t)} \tag{16}$$

for $t \geq 1$, and $n_0 = \psi^*(g_1)h_0$. A rise in g_{t+1} has a negative pressure on n_t through the substitution effect on N_{t+1}. Nevertheless, in light of Eq. (15), the dynamic behavior of n_t is generally unclear when g_t monotonically changes over time.

Eqs. (1) and (15) reveal that output per worker in period t, expressed as $y_t = A_t h_t$, depends on g_t as well as on A_t. Hence, y_t and A_t grow at the same rate in a steady-state equilibrium where g_t is constant.

2.3.3 Technology

In the economy considered here, creation of new technology is a by-product of economic activities by adult individuals. Specifically, the amount of inventions in period $t+1$, $A_{t+1}-A_t$, increases proportionally with the aggregate amount of human capital in period $t+1$, H_{t+1} (cf. Jones, 1995; Nakagawa et al., 2015). Considering Eq. (10), the evolution of technology is described as

$$\begin{aligned} A_{t+1} &= A_t + \lambda H_{t+1} \\ &= A_t + \lambda \phi^*(g_{t+1}) H_t, \end{aligned} \tag{17}$$

where $\lambda > 0$ measures the degree of learning by doing and $A_0 > 0$ is historically determined. In other words, the technology level in period $t+1$ is a linear combination of the existing technology, A_t, and aggregate human capital in period $t+1$. Furthermore, note that H_t in the second line above is the source of the income effect on fertility in period t.

2.4 The Dynamical System

This section characterizes the evolution of the unlocked economy by exploring the dynamic interaction between technology and human capital. As will become apparent, the growth rate of technology monotonically converges to a positive level in the long run. In light of Eq. (17), the growth rate of technology is determined in a self-fulfilling way:

$$g_{t+1} \equiv \frac{A_{t+1} - A_t}{A_t} = \phi^*(g_{t+1}) \frac{\lambda H_t}{A_t}, \tag{18}$$

where g_{t+1} in $\phi^*(g_{t+1})$ is a perfectly forecasted value. A rise in H_t increases g_{t+1} through population expansion, whereas a rise in A_t decreases g_{t+1} through technology catch-up.

Let $x_t \equiv (A_t - A_{t-1})/A_t$ be referred to as the *innovation rate* in period t, which is the ratio of new inventions to the total amount of technologies. When x_t is high, a large proportion of present technologies are attributed to new inventions rather than to pre-existing technologies. Noting that $x_t = \lambda H_t / A_t$ from Eq. (17), one may arrange Eq. (18) to obtain

$$x_t = \frac{g_{t+1}}{\phi^*(g_{t+1})}, \tag{19}$$

where $\phi^*(g_{t+1}) > 0$ and $d\phi^*(g_{t+1})/dg_{t+1} < 0 \ \forall g_{t+1} \in (0, \bar{h}]$, recalling Eq. (11). Eq. (19) therefore shows a one-to-one positive relationship between $x_t > 0$ and $g_{t+1} \in (0, \bar{h}]$. This result is straightforward from Eq. (18), where a rise in $x_t = \lambda H_t / A_t$ occurs with the positive scale effect dominating the negative catch-up effect.

Now consider Eq. (17), $A_{t+1} = A_t + \lambda H_{t+1}$. Dividing both sides of the equation by λH_{t+1} and using the definition of g_{t+1}, one obtains $1/x_{t+1} = 1/g_{t+1} + 1$, or equivalently,

$$x_{t+1} = \frac{g_{t+1}}{1 + g_{t+1}}, \tag{20}$$

which indicates a positive and concave relationship between g_{t+1} and x_{t+1}. This result is also intuitive from Eq. (17). Since a rise in g_{t+1} implies that H_{t+1} increases more than the pre-existing technology A_t, it also increases more than their linear combination A_{t+1}, leading to a higher innovation rate in period $t+1$.

Eqs. (19) and (20) show that x_t is linked to x_{t+1} by way of g_{t+1}. These two equations therefore constitute a one-dimensional first-order autonomous system for x_t. Given the initial condition $x_0 = \lambda H_0/A_0$, the system nails down the trajectory of x_t and accordingly those of g_t and the other endogenous variables in Section 2.3.1. In view of Eq. (10), the initial quantity of aggregate human capital, H_0, depends on two exogenous factors: the initial working population, N_0, and the distribution of individual human capital h_0^i.

Eqs. (19) and (20) also reveal that the direction of growth in x_t depends on the magnitude relationship between the growth factor of aggregate human capital, $\phi^*(g_{t+1})$, and that of technology, $1+g_{t+1}$. Lemma 2 below asserts the existence of a unique, globally stable steady-state equilibrium such that x_t is constant over time.

Lemma 2 *Under Eq. (A1), there exists a unique value $\bar{g}^* > 0$ such that $\phi^*(\bar{g}^*) = 1+\bar{g}^*$ and*

$$x_{t+1} - x_t \begin{cases} < 0 & \textit{if } x_t \in (\bar{x}^*, \infty); \\ = 0 & \textit{if } x_t = \bar{x}^*; \\ > 0 & \textit{if } x_t \in (0, \bar{x}^*), \end{cases}$$

where $\bar{x}^* \equiv \bar{g}^*/\phi^*(\bar{g}^*)$.

Proof. Since $\phi^*(g_{t+1})$ in Eq. (11) is a continuous, strictly decreasing function such that $\lim_{g_{t+1}\to 0} \phi^*(g_{t+1}) > 1$, there exists a unique value $\bar{g}^* > 0$ such that

$$\phi^*(g_{t+1}) - (1 + g_{t+1}) \begin{cases} < 0 & \text{if } g_{t+1} \in (\bar{g}^*, \infty); \\ = 0 & \text{if } g_{t+1} = \bar{g}^*; \\ > 0 & \text{if } g_{t+1} \in (0, \bar{g}^*). \end{cases}$$

Hence, the result follows from Eqs. (19) and (20). □

Given these results, Figure 2 depicts the evolution of the unconstrained economy in the g-x plane. As long as the initial innovation rate $x_0 = \lambda H_0/A_0$ is positive, both x_t and the growth rate of technology, g_t, monotonically change in the same direction and converge toward their respective steady-state levels, $\bar{x}^*$ and $\bar{g}^*$. If $\bar{g}^*$ is so large that $\tilde{a}^*(\bar{g}^*) < 1$, children with $a_t^i \in (\tilde{a}^*(\bar{g}^*), 1]$ receive education in the steady-state equilibrium. That is to say, education investment prevails at least partially in the long run.[22] Recalling the proof

[22] If $\bar{g}^* > \bar{h}$, on the other hand, some low-ability children acquire negative amounts of human capital in the long-run. Excluding this unrealistic case would require an additional parameter restriction. Since this is not the mainline model, the present paper does not go into a deeper analysis of the dynamical system.

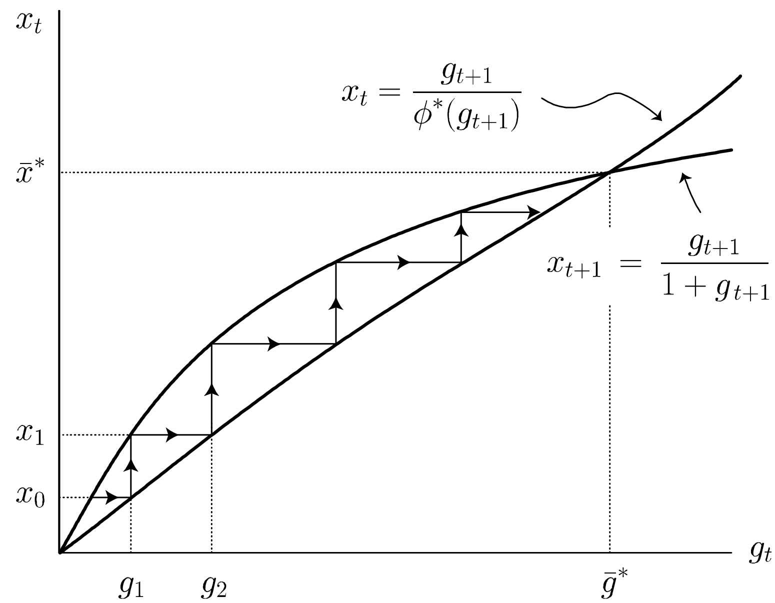

Figure 2. The Evolution of Technology for the Unlocked Economy.

of Lemma 2 and the discussion in Section 2.3.1, one finds that $\bar{g}^*$ also equals the long-run growth rate of aggregate human capital and of output per worker.

Lastly, note that the education choice given by Eq. (9) is desirable for technological progress between periods t and $t+1$. According to Lemma 1, it maximizes the growth factor of aggregate human capital for any $g_{t+1} > 0$. Hence, no other education choice can make a downward shift in the x_t curve in Figure 2 as long as the child-rearing budget $\alpha w_t h_t^i$ is unchanged.

3 The Mainline Model: A Locked-in Economy

This chapter extends the baseline model by introducing the irreversibility of fertility decisions. The "locked-in" economy herein operates in the same way as before, except that family sizes cannot be either reduced or enlarged after the occurrence of unexpected ability shocks.[23] Under such an environment, there exist households whose reactions to the shocks

[23] The assumption of perfect irreversibility in the present paper would be relaxed by allowing individuals to have children in two periods, between which unexpected ability shocks occur. The multi-period approach is taken by Iyigun (2000) for different research objectives from the present paper. The author develops a growth model with no uncertainty and demonstrates that the timing of childbearing is delayed by the accumulation of human capital.

are bound to their initial plans.

3.1 Households

Households' resource allocation problem is divided into two steps. At the time of childbirth (in ex ante optimization), parents plan for future education investment believing that their newborn children have average ability. After childbirth (in ex post optimization), they unexpectedly find the true ability levels of their children and thus may be inclined to alter their initial plans.

3.1.1 Ex Ante Optimization: Childbirth and Education Planning

An individual i of generation t (parent i in period t) decides the quantity of children, n_t^i, along with the planned level of education investment, e_t^p. This decision making builds on the belief that his/her children will have average ability $\bar{a} \in (0,1)$.[24] As a result, e_t^p coincides with the education choice for average-ability children in the unlocked economy.

Under the circumstance, one may apply Eqs. (6) and (9) to ex ante decision making. Thus, it follows that

$$n_t^i = \frac{\alpha}{\delta + e_t^p} h_t^i, \tag{21}$$

where h_t^i is the source of the income effect on the fertility decision. Regarding the education plan,

$$e_t^p = e^*(\bar{a}, g_{t+1}) = \begin{cases} 0 & \text{for } g_{t+1} \in [0, \tilde{g}]; \\ \bar{e} & \text{for } g_{t+1} \in (\tilde{g}, \infty). \end{cases} \tag{22}$$

where $\tilde{g}$ is, as indicated in Figure 1, a critical value such that $\tilde{a}^*(\tilde{g}) = \bar{a}$. In view of Eq. (8),

$$\tilde{g} = \frac{\bar{e}}{\bar{a}\delta + \bar{e}} \bar{h}. \tag{23}$$

In contrast to the fertility decision, the absence of the income effect makes education planning homogeneous within generations.

[24] While this strong assumption makes a great contribution to the tractability of the dynamic model, it will not be essential for the outcome of the distorted resource allocations. See Appendix B for an extension to the expected-utility framework.

3.1.2 Ex Post Optimization: Education Investment

After a one-time childbirth, the adult individual i in period t unexpectedly observes the ability level of his/her children, a_t^i. The gap between a_t^i and the expected level $\bar{a}$ may induce the individual to modify the education plan. Now that n_t^i in Eq. (21) is taken as given, Eqs. (5) and (22) reveal that the *actual* level of education investment, e_t^i, is such that

$$\begin{aligned} e_t^i &= \arg\max \left\{ (1-\alpha) \ln \left[1 - (\delta + e_t^i) \frac{\alpha}{\delta + e_t^p} \right] + \alpha \ln h(e_t^i, a_t^i, g_{t+1}) \right\} \\ &\equiv \arg\max V(e_t^i, a_t^i, g_{t+1}), \end{aligned} \tag{24}$$

subject to $e_t^i \geq 0$.

Eq. (24) has three notable implications. First, unlike in ex ante optimization, education investment is determined as a trade-off with consumption, c_t, not with the quantity of children, n_t. Second, there is no income effect on e_t^i. This is because a rise in h_t^i proportionally increases the quantity of children, n_t^i, with no impact on the budget constraint in Eq. (2). Third, the sunk cost of child rearing, $\alpha\delta/(\delta + e_t^p)$, is the source of the lock-in effect on education investment. If δ is sufficiently small, for example, any household choosing $e_t^p = 0$ beforehand have so many children that it cannot afford $e_t^i = \bar{e}$ for them. An extreme case such as this is beyond the scope of the present paper and thus is excluded on the assumption that[25]

$$(\delta + \bar{e}) \frac{\alpha}{\delta} < 1. \tag{A2}$$

First, consider the case with $g_{t+1} = 0$. Then, Eqs. (2) and (4) yield $V(\bar{e}, a_t^i, 0) < V(0, a_t^i, 0)$ and education investment is not attractive for any adult individual in period t. Next, turn to the case with $g_{t+1} \in (0, \bar{h})$. Then, there exists a critical ability level, $\tilde{a}_t$, for which parents in period t are indifferent between ex post education decisions; i.e., $V(\bar{e}, \tilde{a}_t, g_{t+1}) = V(0, \tilde{a}_t, g_{t+1})$. In light of Eq. (22), the critical value is given by

$$\begin{aligned} \tilde{a}_t &= \left\{ \left[\frac{\delta + e_t^p - \alpha\delta}{\delta + e_t^p - \alpha(\delta + \bar{e})} \right]^{\frac{1-\alpha}{\alpha}} - 1 \right\} \frac{\bar{h} - g_{t+1}}{g_{t+1}} \\ &\equiv \tilde{a}(g_{t+1}). \end{aligned} \tag{25}$$

[25] Eq. (A2) is not essential in the sense that its violation would merely make a wider range of under-investment in education in early development stages, with no influence on the over-investment in later stages (i.e., Lemma 6).

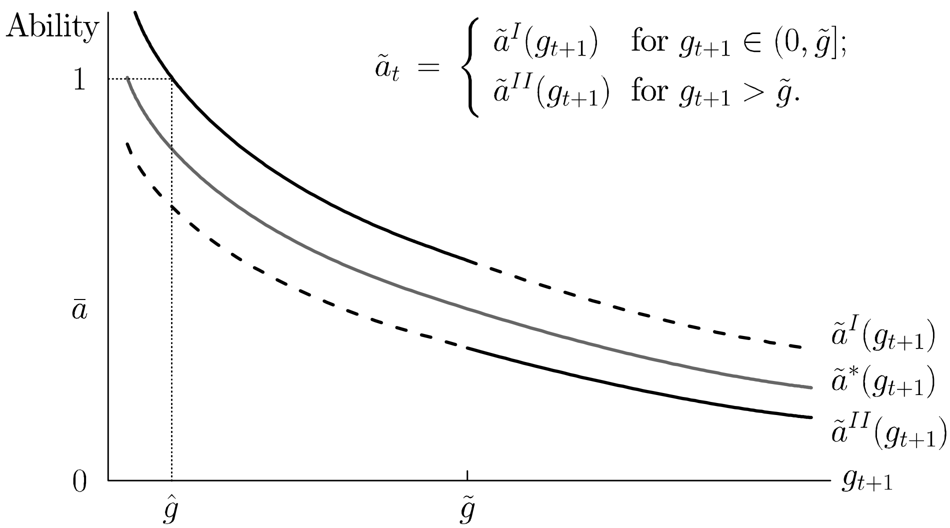

Figure 3. The Transition from Under- to Over-Investment in Education in the Locked-in Economy.

Since the ratio $V(\bar{e}, a_t^i, g_{t+1})/V(0, a_t^i, g_{t+1})$ strictly increases with a_t^i, the ex post education decision by parent i in period t is

$$e_t^i = e(a_t^i, g_{t+1}) \equiv \begin{cases} 0 & \text{if } a_t^i \leq \tilde{a}(g_{t+1}); \\ \bar{e} & \text{if } a_t^i > \tilde{a}(g_{t+1}), \end{cases}$$

where $g_{t+1} \in (0, \bar{h})$. Thus, unlike in the ex ante case, the ex post education decision is heterogeneous across the members of each generation. As shown later, the function $\tilde{a}(g_{t+1})$, depicted by Figure 3, is strictly decreasing and discontinuous at $\tilde{g}$.

3.2 Macroeconomic Variables

By analogy to the baseline model in Section 2.3, the evolution of the working population is $N_{t+1} = \psi(g_{t+1})H_t$, where

$$\psi(g_{t+1}) \equiv \frac{\alpha}{\delta + e^*(\bar{a}, g_{t+1})}. \tag{26}$$

In view of Eq. (22), this is a step function that drops once for all at $g_{t+1} = \tilde{g}$. The fall is due to the trade-off relationship of fertility with the education plan, not with the actual education spending.

The dynamics of aggregate human capital is given by $H_{t+1} = \phi(g_{t+1})H_t$, where

$$\begin{aligned}\phi(g_{t+1}) &\equiv \alpha \int_0^1 \frac{h(e(a, g_{t+1}), a, g_{t+1})}{\delta + e^*(\bar{a}, g_{t+1})} dF(a) \\ &= \psi(g_{t+1}) \left[\bar{h} - g_{t+1} + g_{t+1} \int_{\tilde{a}(g_{t+1})}^1 a dF(a) \right]. \end{aligned} \tag{27}$$

In comparison with Eq. (11) for the baseline case, the effect of a change in g_{t+1} on the growth factor $\phi(g_{t+1})$ is complicated and may not necessarily be negative. The ambiguity is due to the irreversibility constraint that severs the trade-off relationship between the quantity and quality of children.[26]

It follows from Eqs. (26) and (27) that the average level of human capital in period $t \geq 1$ is given by

$$h_t = \bar{h} - g_t + g_t \int_{\tilde{a}(g_t)}^1 a dF(a), \tag{28}$$

and $h_0 = H_0/N_0$. This indicates two opposing forces of technology acceleration on h_t: While a rise in g_t depresses human capital directly through the erosion effect, it increases the proportion of skilled workers in generation t. Despite the complexity, y_t and A_t grow at the same rate in a steady-state equilibrium where g_t and thus h_t are constant, as in the baseline model. Finally, the average fertility rate in period t is expressed as

$$n_t = \psi(g_{t+1})h_t. \tag{29}$$

A rise in g_{t+1} over $\tilde{g}$ triggers education savings and thereby depresses n_t through $\psi(g_{t+1})$, and a change in g_t ambiguously affects n_t through h_t. Considering these properties, fertility dynamics display no general trend when g_t rises monotonically over time.

3.3 The Dynamical System

The evolution of the economy is described in the same way as the baseline model, with the growth factor of aggregate human capital being the only difference. By replacing $\phi^*(g_{t+1})$ in Eq. (19) with $\phi(g_{t+1})$ in Eq. (27) and by using Eq. (20), one obtains

$$\begin{aligned} x_t &= \frac{g_{t+1}}{\phi(g_{t+1})}; \\ x_{t+1} &= \frac{g_{t+1}}{1 + g_{t+1}}, \end{aligned} \tag{30}$$

[26] In the presence of the irreversible fertility decision, there is no quantity-quality trade-off as in Eq. (7); accordingly, a change in the critical ability level $\tilde{a}(g_{t+1})$ has an influence on $\phi(g_{t+1})$. See also Footnote 20.

where the initial innovation rate $x_0 = \lambda H_0 / A_0$ is determined by three exogenous factors: A_0, N_0, and the distribution of h_0^i.

These two equations constitute a one-dimensional first-order autonomous system for x_t. The nature of the dynamical system depends on whether or not technology grows faster than aggregate human capital. This question is investigated below with regard to two possible cases of g_{t+1}.

3.3.1 The Case of $0 < g_{t+1} \leq \tilde{g}$

According to Eq. (22), households' education plan in this case is

$$e_t^p = e^*(\bar{a}, g_{t+1}) = 0, \tag{31}$$

leading to $n_t^i = \alpha h_t^i / \delta$ from Eq. (21). Households have no education plan and invest all child-rearing resources in the quantity of children. Given the initial plan, Eq. (25) reveals that the critical ability level for the ex post education decision is

$$\tilde{a}(g_{t+1}) = \kappa^I \cdot \frac{\bar{h} - g_{t+1}}{g_{t+1}} \equiv \tilde{a}^I(g_{t+1}), \tag{32}$$

where

$$\kappa^I \equiv \left[\frac{(1-\alpha)\delta}{\delta - \alpha(\delta + \bar{e})} \right]^{\frac{1-\alpha}{\alpha}} - 1.$$

Since $\kappa^I > 0$ under Eq. (A2), it follows that $\tilde{a}^I(\hat{g}) = 1$, $\tilde{a}^I(\bar{h}) = 0$, and $\tilde{a}_g^I(g_{t+1}) < 0 \ \forall g_{t+1} > 0$, where $\hat{g} \equiv \kappa^I \bar{h} / (1 + \kappa^I)$. The last property means that education investment prevails with the acceleration of technological progress, as it works in favor of skilled workers.

Substitution of $e^*(\bar{a}, g_{t+1})$ and $\tilde{a}(g_{t+1})$ from Eqs. (31) and (32), respectively, into Eq. (27) reveals that the growth factor of aggregate human capital is, for $g_{t+1} \in (0, \tilde{g}]$,

$$\begin{aligned} \phi(g_{t+1}) &= \frac{\alpha}{\delta} \left[\bar{h} - g_{t+1} + g_{t+1} \int_{\tilde{a}^I(g_{t+1})}^1 a dF(a) \right] \\ &\equiv \phi^I(g_{t+1}), \end{aligned} \tag{33}$$

where $\lim_{g_{t+1} \to 0} \phi^I(g_{t+1}) = \alpha \bar{h} / \delta > 1$ under the demographic condition in Eq. (A1). Hence, if g_{t+1} is sufficiently small, $\phi(g_{t+1}) > 1 + g_{t+1}$ and then Eq. (30) yields $x_{t+1} > x_t$.

To analyze an economy that passes through the entire development process, we suppose that there is no steady-state equilibrium in which no one saves for children's education. Such

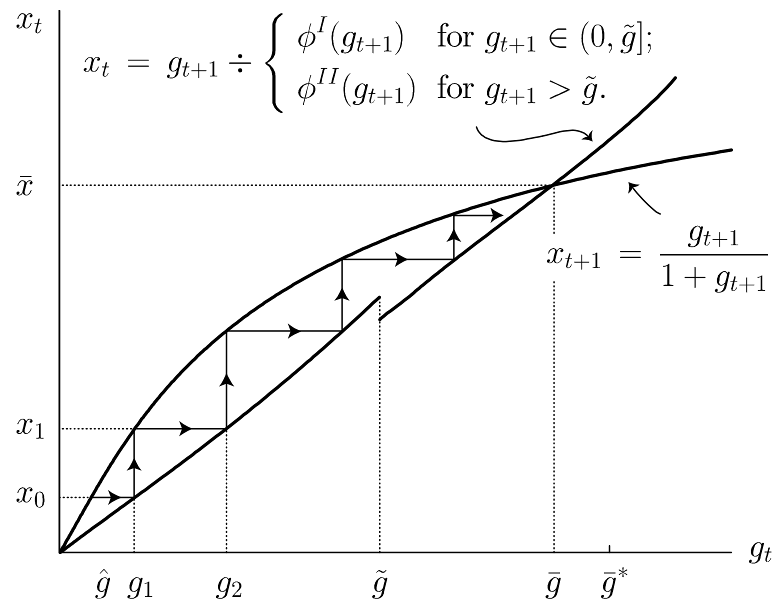

Figure 4. The Evolution of Technology for the Locked-in Economy.

a poverty trap is ruled out by assuming that

$$\phi(g_{t+1}) > 1 + g_{t+1} \quad \forall g_{t+1} \in (0, \tilde{g}]. \tag{A3}$$

Recalling the definition of $\tilde{g}$ in Eq. (23), the inequality above holds if Eq. (A1) is satisfied and if $\bar{e}$ is sufficiently small.[27]

The resulting evolution of the economy is illustrated by Figure 4. As can be seen, g_t exhibits monotonic growth and eventually exceeds the critical level for education planning, $\tilde{g}$. The monotonicity of technology growth seems to be inconsistent with the productivity slowdown experienced by advanced economies in recent years. More important, however, from the viewpoint of this paper's objective, is the result that g_{t+1} remains above $\tilde{g}$ after a certain period of time.

3.3.2 The Case of $g_{t+1} > \tilde{g}$

In light of Eqs. (21) and (22), households' planned level of education investment is

$$e_t^p = e^*(\bar{a}, g_{t+1}) = \bar{e}, \tag{34}$$

[27] Assuming a sufficiently small $\bar{e}$ is compatible with Eq. (A2).

leading to $n_t^i = \alpha h_t^i/(\delta + \bar{e})$ from Eq. (21). All households plan to invest in education by choosing smaller family sizes for a given amount of income. Then Eq. (25) shows that the critical ability level for the ex post education decision is

$$\tilde{a}(g_{t+1}) = \kappa^{II} \cdot \frac{\bar{h} - g_{t+1}}{g_{t+1}} \equiv \tilde{a}^{II}(g_{t+1}), \tag{35}$$

where

$$\kappa^{II} \equiv \left[\frac{(1-\alpha)\delta + \bar{e}}{(1-\alpha)(\delta + \bar{e})}\right]^{\frac{1-\alpha}{\alpha}} - 1 > 0.$$

Substitution of $e^*(\bar{a}, g_{t+1})$ and $\tilde{a}(g_{t+1})$ from Eqs. (34) and (35), respectively, into Eq. (27) reveals that the growth factor of aggregate human capital is, for $g_{t+1} > \tilde{g}$,

$$\begin{aligned} \phi(g_{t+1}) &= \frac{\alpha}{\delta + \bar{e}}\left[\bar{h} - g_{t+1} + g_{t+1}\int_{\tilde{a}^{II}(g_{t+1})}^{1} a dF(a)\right] \\ &\equiv \phi^{II}(g_{t+1}), \end{aligned} \tag{36}$$

where $\phi^{II}(g_{t+1}) = 0$ if and only if $g_{t+1} = \bar{h}/(1-\bar{a})$. This property, along with Eq. (A3) and the continuity of $\phi^{II}(g_{t+1})$, ensures the existence of a nontrivial steady-state equilibrium in which g_t stays at $\bar{g} \equiv \min\{g \in \mathbb{R}_{++} | \phi(g) = 1 + g\} > \tilde{g}$. The steady-state condition in the curly braces requires that technology and aggregate human capital grow at the same rate.

Given the results so far, Figure 4 shows that g_t monotonically converges toward $\bar{g}$ as long as g_1 falls on the interval $(0, \bar{g})$.[28] Otherwise, g_t may converge to a higher level because the dynamical system may exhibit multiplicity of nontrivial steady-state equilibria. In any of the equilibria, technology, aggregate human capital, and output per worker grow at the same rate. The main result is summarized below.

Lemma 3 *Consider the locked-in economy characterized by Eqs. (A2)–(A3). Given $g_1 \in (0, \bar{g})$, the growth rate of technology monotonically converges toward $\bar{g}(> \tilde{g})$.*

A few technical remarks need to be made regarding Figure 4. First, the diagram represents the case in which $\phi^I(\tilde{g})$ is smaller than $\phi^{II}(\tilde{g})$. While their quantitative relationship is generally ambiguous, this case is likely to occur unless $\bar{e}$ is above a certain level and the ability distribution is left-skewed. In Stage II, a high education cost discourages fertility

[28] Recalling that $x_0 = g_1/\phi(g_1)$, one can set g_1 on $(0, \bar{g})$ by choosing the initial condition x_0 appropriately.

whereas a left-skewed ability distribution limits the spread of education investment. Both of these effects depress the growth factor $\phi^{II}(\tilde{g})$.

Second, in order to encompass the opposite case, $\phi^{I}(\tilde{g}) > \phi^{II}(\tilde{g})$, $\phi(\tilde{g})$ needs to be a linear combination of $\phi^{I}(\tilde{g})$ and $\phi^{II}(\tilde{g})$ so that any x_t on the interval $(\tilde{g}/\phi^{I}(\tilde{g}), \tilde{g}/\phi^{II}(\tilde{g}))$ has a corresponding value of g_{t+1}, which is equal to $\tilde{g}$.[29] Then, the education cost $\bar{e}$ needs to be sufficiently small so that Eq. (A3) is satisfied, as in the first case. This is because, in light of Eqs. (23) and (36), $\tilde{g}$ and $\phi^{II}(\tilde{g})$ respectively go to zero and to $\alpha\bar{h}/\delta > 1$ as $\bar{e}$ approaches zero.

Third, the curve $g_{t+1}/\phi(g_{t+1})$ is positively sloped on $\mathbb{R}_{++}\backslash\{\tilde{g}\}$, so that x_t has a one-to-one relationship with g_{t+1} in most circumstances. While the monotonicity of the curve is not essential for Lemma 3, a sufficient condition for this property is that $aF'(a)$ is small enough for any $a \in (0,1)$. This requires that the ability distribution is not heavily concentrated, especially around the upper tail. Fourth and finally, the nontrivial steady-state equilibrium is unique if the function F fulfills a similar condition. These results are asserted by Lemmas 8 and 9 in the Appendix.

4 Analysis

This section demonstrates a scenario of economic development in the presence of the irreversibility constraint on the fertility decision. As will become clear, the lock-in effect on the growth performance is equivocal in the early stages of development, whereas it is necessarily negative in the later stages.

Let the economy start with a development stage characterized by a high fertility rate and a limited spread of education. As mentioned in the introduction, the focus here is not on underdeveloped stages in which some households rely on child labor (cf. Footnote 4). In

[29] More precisely, $\phi(\tilde{g})$ is modified to

$$\phi(\tilde{g}) = [1 - p(x_t)]\phi^{I}(\tilde{g}) + p(x_t)\phi^{II}(\tilde{g}),$$

where $p(x_t)$ is a single-valued function such that $p(\tilde{g}/\phi^{I}(\tilde{g})) = 0$, $p(\tilde{g}/\phi^{II}(\tilde{g})) = 1$, and $p'(x_t) > 0$ $\forall x_t \in (\tilde{g}/\phi^{I}(\tilde{g}), \tilde{g}/\phi^{II}(\tilde{g}))$. One may view $p(x_t)$ as the probability with which each household observing x_t chooses $e_t^p = \bar{e}$ when it is indifferent between the two ex ante choices, 0 and $\bar{e}$.

light of Eqs. (23) and (32), this assumes the following relationship between $\hat{g}$, $\tilde{g}$, and g_1:

$$\hat{g} < g_1 < \tilde{g}, \tag{A4}$$

where $\hat{g} < \tilde{g}$ if the education cost $\bar{e}$ is sufficiently small.[30]

Let $\tilde{t}$ be the period after which g_{t+1} exceeds the critical level for education planning, $\tilde{g}$, for the first time; i.e., $g_{t+1} \leq \tilde{g}\ \forall t \leq \tilde{t}$ and $g_{t+1} > \tilde{g}\ \forall t > \tilde{t}$. Then, the development process is divided into two stages.

Stage I, defined as the time interval $[0, \tilde{t}]$, involves *under*-investment in education, i.e., a biased allocation of parenting resources toward the quantity of children. Households make fertility decisions with no prospect of future education investment. The family size is locked into relatively large levels, which would scale up the potential cost of education. Thus, households adhere to the initial plan unless their children are unexpectedly and significantly competent.

Stage II, defined as the subsequent periods, is characterized by *over*-investment in education, i.e., a biased allocation of parenting resources toward the quality of children. Since fertility decisions are made in prospect of future education investment, the family size is locked into relatively small levels. In this situation, households invest in education as planned unless their children are unexpectedly and significantly incompetent.

4.1 Stage I: Under-Investment in Education

In Stage I, where $0 \leq t \leq \tilde{t}$ and $\hat{g} < g_{t+1} \leq \tilde{g}$, all households aim to concentrate their resources on the quantity, rather than the quality, of children at the time of childbirth.

4.1.1 The Lock-in Effect on the Growth Process

Figure 5 graphically represents the lock-in effects on the ex post parental decisions in Stage I. Recall that $\tilde{a}^*(g_{t+1})$ is the critical ability level in the unconstrained case. The downward arrow indicates that the irreversibility constraint prevents education investment by households

[30] Recalling that $\hat{g} \equiv \kappa^I \bar{h}/(1+\kappa^I)$ and $\tilde{g} \equiv \bar{e}\bar{h}/(\bar{a}\delta+\bar{e})$, one may rewrite the condition $\hat{g} < \tilde{g}$ as $\bar{a} < \bar{e}/(\delta\kappa^I)$, where $0 < \bar{a} < 1$ by assumption and $\bar{e}/(\delta\kappa^I) < 1$ from Lemma 4. This condition is satisfied if $\bar{e} > 0$ is sufficiently small because, in light of L'Hôpital's rule, $\bar{e}/(\delta\kappa^I)$ approaches unity as $\bar{e}$ goes to zero. Assuming such a small $\bar{e}$ is consistent with Eqs. (A2) and (A3). See also Footnote 28 for setting the value of g_1.

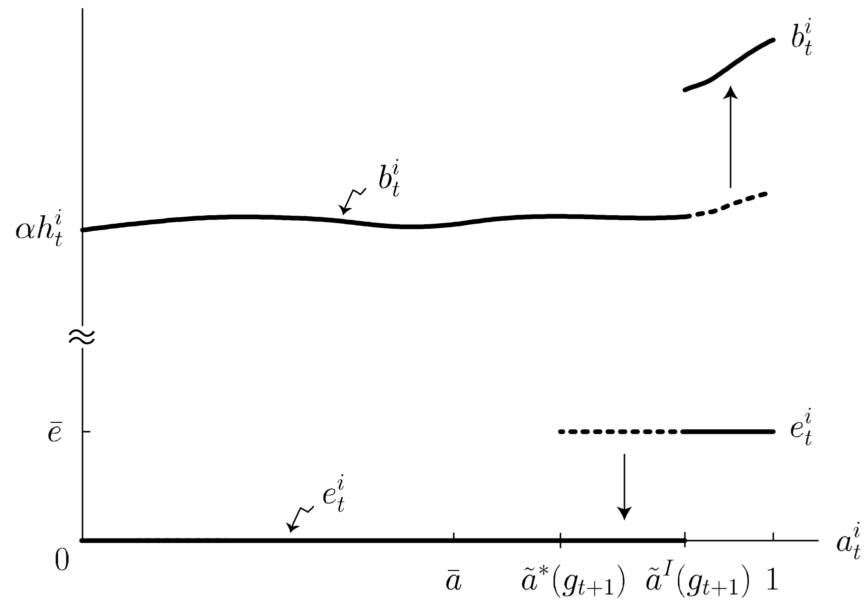

Figure 5. The Lock-in Effects on Education and Parenting Resources in Stage I.

receiving ability shocks in the range $(\tilde{a}^*(g_{t+1}), \tilde{a}^I(g_{t+1})]$. If fertility decision were reversible, they would reduce their family sizes to finance the cost of schooling.[31] However, such an adjustment is in fact infeasible and education investment would incur a fall in consumption. The ability shocks to those households are not large enough for them to make such sacrifices.

When $\tilde{a}^I(g_{t+1}) > \tilde{a}^*(g_{t+1})$, the economy in period t suffers from *under*-investment in education or, equivalently, of *over*-investment in the quantity of children. As Lemma 1 suggests earlier, the unconstrained parental choice is optimal for the formation of aggregate human capital, H_{t+1}, regardless of the child-rearing budget b_t^i. The irreversibility constraint prevents some households from making the choice and thereby distorts resource allocation between investment in the quantity and in the quality of labor.[32]

The existence of the interval $(\tilde{a}^*(g_{t+1}), \tilde{a}^I(g_{t+1})]$ is assured by Lemma 4 below, according to which Figure 3 is depicted.

Lemma 4 *Under Eq. (A2),* $\tilde{a}^I(g_{t+1}) > \tilde{a}^*(g_{t+1}) > 0 \ \forall g_{t+1} \in (0, \bar{h})$.

[31] The discontinuity of $h(e_t^i, a_t^i, g_{t+1}^i)$ with respect to e_t^i is not essential for the lock-in effect on education decisions. If the function h was alternatively continuous with respect to e_t^i, the reaction function $e(a_t^i, g_{t+1}^i)$ would also be continuous with respect to a_t^i. Then, the irreversibility constraint would make the education reaction less sensitive to ability shocks.

[32] Technically speaking, the education decision for $a_t^i \in (\tilde{a}^*(g_{t+1}), \tilde{a}^I(g_{t+1})]$ does not maximize the fraction in Eq. (13).

Proof. See Appendix A. □

Despite the inefficient resource allocation, the irreversibility constraint has an ambiguous effect on the growth process in Stage I because it also increases child-rearing expenses, b_t^i, for some households. This counterforce is represented by the upward arrow in Figure 5. Households who receive a shock $a_t^i \in [0, \tilde{a}^I(g_{t+1})]$ follow the quantity-oriented plan in Eq. (31) and thus spend a fixed fraction of income on child rearing; i.e., $b_t^i = \alpha h_t^i$ as in the unlocked economy in Section 2. By contrast, those with $a_t^i \in (\tilde{a}^I(g_{t+1}), 1]$ invest in education against the initial plan. Because their family sizes cannot be reduced accordingly, the upward revision of the education plan results in more than αh_t^i efficiency units of labor devoted to their children. More precisely, $b_t^i = \alpha h_t^i(\delta + \bar{e})/\delta$ from Eq. (24). Such a self-sacrifice is made only if the observed ability level is sufficiently large.

To summarize, the increase in child-rearing expenses has a positive impact on aggregate human capital, thereby negating the adverse effect of under-investment in education. The resulting quantitative relationship of $\phi^I(g_{t+1})$ to $\phi^*(g_{t+1})$, which reflects the lock-in effect on the growth process, is generally ambiguous in Stage I. Then the following proposition is established.

Proposition 1 (The Lock-in Effect in Stage I) *Under Eqs.* (A2)–(A4)*, the irreversibility constraint on fertility decisions has an ambiguous effect on the growth process in Stage I.*

4.1.2 A Redistribution Policy

This subsection examines the possibility of a redistribution policy that reallocates parenting resources between the quantity and quality of children and thereby enhances the growth performance in Stage I.

Suppose that the government of the economy temporarily imposes a tax $w_t\delta_t^g$ on raising one child, while it provides a subsidy $-w_t e_t^g$ for sending one child for higher education, where $\delta_t^g \geq 0$ and $e_t^g \leq 0$. That is to say, the taxation is an obligation for all households (and thus is similar to imposing a poll tax), whereas the subsidy is targeted only at investors in child education. The policy scheme is announced by the time when individuals give birth to children. While the government fully knows about the entire economy, it cannot identify the ability levels of unborn children.

Under the circumstances, the aggregate amount of parenting resources, B_t, is expressed as

$$\begin{aligned} B_t &\equiv \int_0^{N_t} b_t^i di \\ &= \{\delta + \delta_t^g + [1 - F(\tilde{a}_t)](\bar{e} + e_t^g)\} \int_0^{N_t} n_t^i di, \end{aligned} \tag{37}$$

where $\bar{e} + e_t^g$ is the subsidized education cost and $1 - F(\tilde{a}_t)$ is the fraction of households spending on education in period t. The government budget is supposed to be balanced and accordingly[33]

$$e_t^g = \frac{-\delta_t^g}{1 - F(\tilde{a}_t)} \leq 0. \tag{38}$$

With the balanced budget, the redistribution policy has no direct influence on B_t and is controlled by an exogenous change in δ_t^g.

Turning to the individual optimization, Eq. (31) is modified to[34]

$$e_t^p = 0 \quad \text{and} \quad n_t^i = \frac{\alpha}{\delta + \delta_t^g} h_t^i. \tag{39}$$

As expected intuitively, a rise in δ_t^g increases the fixed cost of child rearing, thereby locking n_t^i into a smaller level. In view of Eq. (25), the critical ability level for the ex post education decision is, in Stage I,

$$\tilde{a}_t = \left\{ \left[\frac{(1-\alpha)(\delta + \delta_t^g)}{(1-\alpha)(\delta + \delta_t^g) - \alpha(\bar{e} + e_t^g)} \right]^{\frac{1-\alpha}{\alpha}} - 1 \right\} \frac{\bar{h} - g_{t+1}}{g_{t+1}}, \tag{40}$$

where e_t^g is given by Eq. (38). Eq. (40) is reduced to Eq. (32) if $\delta_t^g = 0$. Thus, the critical ability level $\tilde{a}_t$ in Eq. (40) is expressed as a single-valued function such that, for $g_{t+1} > 0$ and $\delta_t^g = 0$,

$$\tilde{a}_t = \tilde{a}^I(g_{t+1}; \delta_t^g).$$

[33]Section 4 assumes that δ_t^g is not large enough to cause fundamental changes in the economy. Executing the policy in period t does not alter the development stage in the same period, so that $0 < F(\tilde{a}_t) < 1$ in Eq. (38).

[34]The policy analysis here assumes that δ_t^g is not large enough to alter the ex ante education decision e_t^p and thus the development stage.

where $\tilde{a}^I(g_{t+1};0) = \tilde{a}^I(g_{t+1})$.

Now, consider a redistribution policy that increases δ_t^g marginally at $\delta_t^g = 0$. The policy effects on the critical ability level are summarized by Lemma 5 below, in which $\tilde{a}_\delta^I(g_{t+1};0)$ denotes the derivative $\partial\tilde{a}^I(g_{t+1};\delta_t^g)/\partial\delta_t^g$ evaluated at $\delta_t^g = 0$.[35]

Lemma 5 *Under Eq. (A2),*

(a) $\tilde{a}_\delta^I(g_{t+1};0) < 0 \ \forall g_{t+1} \in (\hat{g}, \bar{h})$;

(b) $\tilde{a}_\delta^I(g_{t+1};0) \to -\infty$ *as* $g_{t+1} \to \hat{g} + 0$.

Proof. See Appendix A. □

Recalling Eq. (38), one can interpret these properties as follows. First, a rise in δ_t^g leads to a spread of education because the associated decrease in e_t^g lowers the hurdle to education investment for each household. As a result, a larger part of households choose $e_t^i = e^*(a_t^i, g_{t+1})$ and the policy mitigates under-investment in education in period t.[36] Second, the policy effect becomes infinitely large as g_{t+1} approaches $\hat{g}$ and thus as $\tilde{a}^I(g_{t+1})$ approaches 1. This is due to a certain amount of education subsidies provided to a small part of households.

The growth factor of aggregate human capital in Eq. (33) is now replaced with

$$\phi^I(g_{t+1};\delta_t^g) \equiv \frac{\alpha}{\delta + \delta_t^g}\left[\bar{h} - g_{t+1} + g_{t+1}\int_{\tilde{a}_t}^1 adF(a)\right],$$

where $\tilde{a}_t = \tilde{a}^I(g_{t+1};\delta_t^g)$ and $\phi^I(g_{t+1};0) = \phi^I(g_{t+1})$. The fraction $\alpha/(\delta + \delta_t^g)$ above is associated with fertility, whereas the terms in the square brackets indicate the average level of human capital in period $t+1$, h_{t+1} from Eq. (28). The aforementioned policy is effective if $\phi_\delta^I(g_{t+1};0) > 0$. With this condition, the increasing δ_t^g is expected to shift the $g_{t+1}/\phi^I(g_{t+1})$ curve in Figure 4 downward. The curve shifts back later as long as the policy is executed temporarily. The resulting increase in g_{t+1} expedites the transition to Stage II.[37]

[35] This notation applies to other functions in what follows.

[36] Since $\tilde{a}^*(g_{t+1})$ in Lemma 4 is the critical ability level for the education decision in the unlocked economy, it is immune from any policies executed by the locked-in economy. This is also the case for Lemma 6 and Figures 5-6 below. By contrast, $\tilde{a}^*(g_{t+1})$ used to define $\tilde{g}$ in Eq. (8) is the critical ability level in ex ante optimization and thus is under an influence of those policies.

[37] Nevertheless, output per capita may not necessarily increase because, as shown by Eq. (28), the acceleration of technology growth may have an ambiguous effect on $y_t = A_t h_t$.

A simple calculation reveals that the policy effect is decomposed into two conflicting factors:

$$\phi_{\delta}^{I}(g_{t+1};0) = -\frac{1}{\delta}\phi^{I}(g_{t+1}) - \frac{\alpha}{\delta} g_{t+1}\tilde{a}^{I}(g_{t+1})F'(\tilde{a}^{I}(g_{t+1}))\tilde{a}_{\delta}^{I}(g_{t+1};0), \tag{41}$$

where the first and second terms on the right side, respectively, indicate the negative effect on the quantity and the positive effect on the quality of labor. Recalling Eq. (13), one finds that the sign of $\phi_{\delta}^{I}(g_{t+1};0)$ is not necessarily positive: While the increase in δ_t^g improves under-investment in education, it has an ambiguous impact on the amount of child-rearing expenses, b_t^i.[38]

We now take a closer look at Eq. (41). The quantity effect is limited because $\phi^{I}(g_{t+1}) < \alpha\bar{h}/\delta \ \forall g_{t+1} > 0$, whereas the quality effect may or may not be. It follows from Lemma 5 that as long as $\lim_{a\to 1} F'(a) > 0$, the quality effect is the determining factor when g_{t+1} is sufficiently close to $\hat{g}$. Under the circumstances, the critical value $\tilde{a}^{I}(g_{t+1})$ approaches the upper tail of the ability distribution, so that the education subsidy allows competent children to become skilled labor. The condition for $F'(a)$ above ensures the existence of those beneficiaries. Considering that g_{t+1} grows monotonically over time, one finds that the policy is likely to be effective in the early stages of development.

4.2 Stage II: Over-Investment in Education

When g_{t+1} exceeds $\tilde{g}$, the allocation of parenting resources in period t switches from a quantity-biased to a quality-biased approach. In Stage II, where $t > \tilde{t}$ and $g_{t+1} > \tilde{g}$, all households prepare for future education by choosing small family sizes.

4.2.1 The Lock-in Effect on the Growth Process

The irreversibility constraint induces some households to invest in education, thereby shifting macroeconomic resource allocation from the quantity to the quality of children. The lock-in effect is graphically represented by Figure 6. The upward arrow shows that the constraint affects the ex post decision of parents with $a_t^i \in (\tilde{a}^{II}(g_{t+1}), \tilde{a}^{*}(g_{t+1})]$. While those households spend on education as planned in Eq. (34), they would not carry out the plan if the family

[38] The policy has two opposing effects on B_t in Eq. (37) and thus on b_t^i of some households. As shown by Lemma 5 and Eq. (39), it increases the fraction of households investing in education, $1 - F(\tilde{a}_t)$, while decreasing the quantity of children, n_t^i.

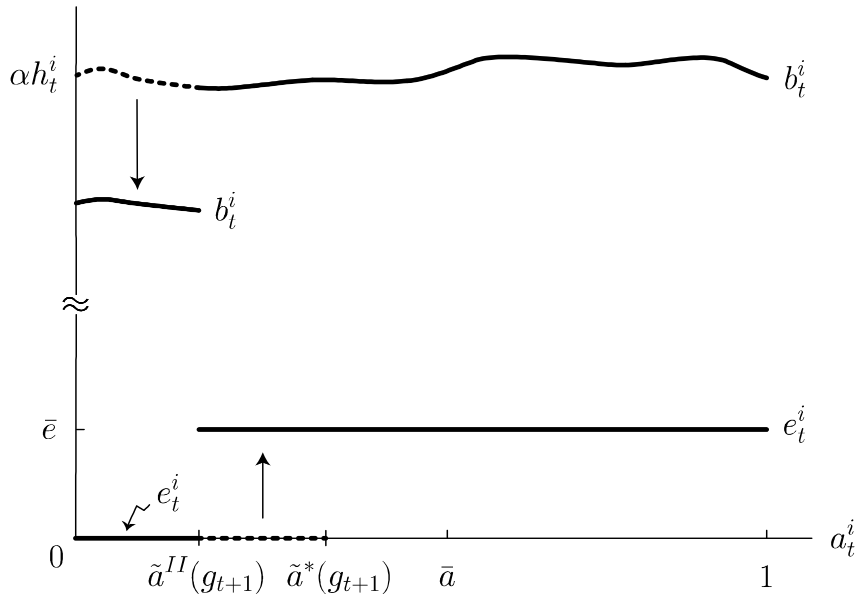

Figure 6. The Lock-in Effects on Education and Parenting Resources in Stage II.

size could be enlarged after receiving the negative ability shocks. Such fertility adjustment is in fact infeasible, and education investment does not place a heavy burden on the small households in terms of consumption. This is why parents are inclined to invest in education unless observed abilities are significantly poor [i.e., $a_t^i \leq \tilde{a}^{II}(g_{t+1})$].

Lemma 6 below, on which Figure 3 is based, ensures the existence of those constrained households. As follows from Lemma 1, their education and fertility decisions, $e_t^i > e^*(a_t^i, g_{t+1})$ and $n_t^i = b_t^i/(\delta + e_t^i)$, are unfavorable for aggregate human capital H_{t+1}. In this sense, the economy suffers from *over*-investment in education or, equivalently, of *under*-investment in fertility.[39]

Lemma 6 $0 < \tilde{a}^{II}(g_{t+1}) < \tilde{a}^*(g_{t+1})\ \forall g_{t+1} \in (0, \bar{h})$.

Proof. See Appendix A. □

There is another channel through which the irreversibility constraint retards human capital accumulation in Stage II. Unlike in the previous stage, it decreases the child-rearing

[39] In this situation, education investment is not productive sufficiently for some workers. Since those workers earn wages appropriate to their skill levels, they are categorized as "apparently over-educated workers" in Chevalier (2003). The author defines apparently over-educated workers as graduates being satisfied with a non-graduate job and genuinely over-educated workers as those who are not. Using data on UK graduates from 1985 and from 1990, he reports that 483 out of 4844 graduates fall into the former category (p. 514, Table 1).

budget for some households. This effect is illustrated by the downward arrow in Figure 6. Households with $a_t^i \in [0, \tilde{a}^{II}(g_{t+1})]$ find that observed abilities are too low to carry out the education plan in Eq. (34). Because the number of their children cannot be increased correspondingly, their ex post decision leads to a smaller fraction of income spent for their children; i.e., $b_t^i < \alpha h_t^i$ from Eq. (24). In contrast, those who observe $a_t^i \in (\tilde{a}^{II}(g_{t+1}), 1]$ follow the initial plan and allocate αh_t^i efficiency units of labor to child rearing as in the unlocked economy.

All things considered, the lock-in effect on the growth process is necessarily negative in Stage II; i.e., $\phi^{II}(g_{t+1}) < \phi^*(g_{t+1})\ \forall g_{t+1} > \tilde{g}$. Figure 4 illustrates the locked-in economy converging toward the lower steady-state equilibrium, $\bar{g} < \bar{g}^*$. Since technology A_t, aggregate human capital H_t and output per worker y_t grow at the same rate in the steady state, the following proposition is now derived.

Proposition 2 *Under Eqs.* (A2)–(A4), *the irreversibility constraint on fertility decisions decelerates the growth process in Stage II and lowers the long-run growth rate of output per worker.*

4.2.2 A Redistribution Policy

This subsection designs another type of redistribution policy, which may improve over-investment in education and stimulate fertility with no direct influence on aggregate resources for child rearing, B_t.

Throughout Stage II, the government permanently provides a subsidy $-w_t\delta_t^g$ for raising one child (i.e., parenting support), while imposing a tax $w_t e_t^g$ on sending a low-ability child to higher education, where $\delta_t^g \leq 0$ and $e_t^g \geq 0$.[40] In order to encourage child bearing, suppose that the education tax is targeted at households whose children are below average in ability.[41]

Under the circumstances, B_t is expressed by replacing $1 - F(\tilde{a}_t)$ in Eq. (37) with $F(\bar{a}) -$

[40] In reality, one may interpret the education tax considered here as the ability-based provision of a public scholarship.

[41] More generally, one may choose any ability level on $[\tilde{a}_t, \bar{a}]$ as the critical level for the education tax. Alternatively, if an ability level on $(\bar{a}, 1]$ is chosen as the critical level, all adult individuals in Stage II expect to be subject to taxation at the time of childbirth. Such a policy would reduce rather than increase the quantity of children they intend to raise. For this reason, this section does not deal with an unconditional education policy as in Stage I.

$F(\tilde{a}_t)$. The balanced budget constraint faced by the government yields

$$e_t^g = \frac{-\delta_t^g}{F(\bar{a}) - F(\tilde{a}_t)} \geq 0, \tag{42}$$

where the denominator is the fraction of households paying the education tax.[42] The ability distribution is fully known to the government as in Stage I. Increasing the subsidy (i.e., decreasing δ_t^g) requires a heavier burden of the taxation. Also note that, with δ_t^g held constant, e_t^g goes to infinity as $\tilde{a}_t$ approaches $\bar{a}$. This result reflects the situation in which a small fraction of households bear the cost of providing a certain amount of subsidies.

In the beginning of each period in Stage II, the policy scheme is announced and none of the adult individuals, who expect their children to be born with average ability, expects to be subject to taxation. Thus, the ex ante decisions given by Eq. (34) are modified to

$$e_t^p = \bar{e} \quad \text{and} \quad n_t^i = \frac{\alpha}{\delta + \delta_t^g + \bar{e}} h_t^i.$$

Decreasing δ_t^g from zero lightens the financial burden of child rearing, thereby locking n_t^i into a higher level.

In order to analyze the ex post education decision, note that $\tilde{a}_t < \bar{a}$ in Stage II. Namely, households who are indifferent in their education decisions observe an ability level below average. Because they would be subject to taxation if they invested in education, Eq. (25) reveals that the critical ability level for this stage is rewritten as

$$\tilde{a}_t = \left\{ \left[\frac{(1-\alpha)(\delta + \delta_t^g) + \bar{e}}{(1-\alpha)(\delta + \delta_t^g + \bar{e}) - \alpha e_t^g} \right]^{\frac{1-\alpha}{\alpha}} - 1 \right\} \frac{\bar{h} - g_{t+1}}{g_{t+1}}, \tag{43}$$

where e_t^g is given by Eq. (42). Eq. (43) is reduced to Eq. (35) if $\delta_t^g = 0$. Thus, the critical ability level $\tilde{a}_t$ in Eq. (43) is expressed as a single-valued function such that, for $g_{t+1} > 0$ and $\delta_t^g = 0$,

$$\tilde{a}_t = \tilde{a}^{II}(g_{t+1}; \delta_t^g),$$

where $\tilde{a}^{II}(g_{t+1}; 0) = \tilde{a}^{II}(g_{t+1})$.

Now, consider a redistribution policy that decreases δ_t^g marginally at $\delta_t^g = 0$. The resulting policy effects on $\tilde{a}_t$ are summarized by Lemma 7 below.

[42] The critical ability level for education, $\tilde{a}_t$, is smaller than $\bar{a}$ as depicted in Figure 6. Eqs. (22) and (8) respectively show that $\tilde{a}^*(\tilde{g}) = \bar{a}$ and $d\tilde{a}^*(g_{t+1})/dg_{t+1} < 0 \ \forall g_{t+1} > 0$. It then follows from Lemma 6 that $\tilde{a}_t = \tilde{a}^{II}(g_{t+1}) < \bar{a}$ in Stage II, where $t > \tilde{t}$ and $g_{t+1} > \tilde{g}$.

Lemma 7

(*a*) $\tilde{a}_\delta^{II}(g_{t+1};0) < 0 \; \forall g_{t+1} \in (\tilde{g}, \bar{h})$;

(*b*) $\tilde{a}_\delta^{II}(g_{t+1};0) \to 0$ *as* $g_{t+1} \to \bar{h}$.

Proof. See Appendix A. □

The results of Lemma 7 are understood intuitively by using Eq. (42). First, the decreasing δ_t^g limits the fraction of households investing in education, $1 - F(\tilde{a}_t)$, because the increased potential cost of education, $\bar{e} + e_t^g$, lifts the hurdle to schooling. The policy mitigates over-investment in education by reducing the gap between $\tilde{a}^{II}(g_{t+1};\delta_t^g)$ and the critical ability level for the unlocked economy, $\tilde{a}^*(g_{t+1})$. Second, the policy effect does not become infinitely large as g_{t+1} approaches $\bar{h}$ and thus as $\tilde{a}^{II}(g_{t+1})$ approaches zero. This is in part because the tax burden on each tax payer, e_t^g, is lightened in the situation where the total cost of the subsidy is borne by most households of below-average children.

Eq. (36) is now modified to

$$\phi^{II}(g_{t+1};\delta_t^g) \equiv \frac{\alpha}{\delta + \delta_t^g + \bar{e}} \left[\bar{h} - g_{t+1} + g_{t+1} \int_{\tilde{a}_t}^1 a dF(a) \right],$$

where $\tilde{a}_t = \tilde{a}^{II}(g_{t+1};\delta_t^g)$ and $\phi^{II}(g_{t+1};0) = \phi^{II}(g_{t+1})$. The terms in the square brackets represent the average level of human capital h_{t+1}. The redistribution policy is effective provided that $\phi_\delta^{II}(g_{t+1};0) < 0$. In this case, the decreasing δ_t^g is expected to cause a downward shift in the $g_{t+1}/\phi^{II}(g_{t+1})$ curve from Figure 4, leading to a rise in g_{t+1}. The permanent shift enhances the steady-state value $\bar{g}$, which coincides with the long-run growth rate of output per worker.[43]

Since the policy affects the costs of child rearing, its influence on the growth factor $\phi^{II}(g_{t+1})$ is decomposed into two components:

$$\phi_\delta^{II}(g_{t+1};0) = -\frac{1}{\delta + \bar{e}} \phi^{II}(g_{t+1}) - \frac{\alpha}{\delta + \bar{e}} g_{t+1} \tilde{a}^{II}(g_{t+1}) F'(\tilde{a}^{II}(g_{t+1})) \tilde{a}_\delta^{II}(g_{t+1};0), \tag{44}$$

where the first and second terms on the right side, respectively, have negative and positive signs. They respectively indicate the positive effect on the quantity and the negative effect on the quality of labor. The sign of $\phi_\delta^{II}(g_{t+1};0)$ is not necessarily negative even though

[43] Its short-term effect on output per worker, y_t, is less clear-cut because accelerated technological progress has two opposing effects on h_t [cf. Eq. (28)].

the decrease in δ_t^g mitigates over-investment in education. The ambiguity stems from the associated change in b_t^i in Eq. (13).

In order to investigate Eq. (44) further, let g_{t+1} approach $\bar{h}$ from below. Then, the quantity effect remains positive because $\phi^{II}(\bar{h}) > 0$. On the other hand, $\tilde{a}^{II}(g_{t+1})$ becomes so small that the redistribution policy merely deprives incompetent children of access to education. From Lemma 7, the resulting quality effect is limited and less significant than the quantity effect unless the ability distribution is concentrated at the bottom level (more precisely, unless $\lim_{a \to 0} F'(a) = \infty$). The policy then boosts aggregate human capital H_{t+1} and, thanks to the accompanying rise in the wage rate $w_{t+1} = A_{t+1}$, even those low ability children may benefit from the policy.

Recalling Lemma 3 about the convergence of g_t to $\bar{g}$, the discussion above concludes that the redistribution policy is successful at least in the long run if $\bar{h}$ is sufficiently close to $\bar{g}$.[44]

5 Concluding Remarks

This theoretical research has elucidated the role of irreversible fertility decisions in economic growth from the long-run perspective. In the presence of unexpected ability shocks on children, the irreversibility constraint affects the formation of aggregate human capital through parental decisions on child rearing, and its qualitative effect varies with the stage of economic development.

In the underdeveloped stage, parents have no education plan and concentrate their child-rearing resources on the quantity of children. Once the family size is fixed after childbirth, investing in education against the initial plan incurs an unexpected reduction in consumption. The initial plan is therefore executed by all households except those who find their children significantly competent. As a consequence, the irreversibility constraint not only increases aggregate resources for child rearing but also prevents skill acquisition of relatively competent children. While their overall effect is generally ambiguous, the latter indicates that the

[44] There is a set of structural parameters that make such a case feasible. The steady-state condition, $\phi^{II}(g_{t+1}) = 1 + g_{t+1}$, reveals that a nontrivial steady-state equilibrium occurs at $g_{t+1} = \bar{h}$ if $\bar{h} = (\delta + \bar{e})/[\alpha\bar{a} - (\delta + \bar{e})]$. The implied condition, $\delta + \bar{e} < \alpha\bar{a}$, is compatible with Eqs. (A1)–(A4) and also with the sufficient conditions for the uniqueness of the equilibrium, provided by Lemma 9 in the Appendix. If the steady-state equilibrium is unique, one can make $\bar{g}$ slightly smaller than $\bar{h}$ by changing $\phi^{II}(g_{t+1})$ marginally through α, δ, and $\bar{e}$.

government can enhance aggregate human capital by altering the resource allocation between the quantity and quality of children. One possibility is an ability-based education subsidy financed by universal taxation on child rearing. Such a redistribution policy tends to be effective in early periods when the constraint is binding for children in the upper tail of the ability distribution, as assisting their skill acquisition makes a significant contribution to aggregate human capital.

Technological progress fueled by human capital accumulation eventually alters households' (ex ante) stances toward education, which is followed by a major fertility decline. Now that family sizes are locked into smaller levels, investing in children's quality does not place a heavy financial burden on their parents, and such investment is necessary to make up for the reduced investment in children's quantity. The initial education plan is therefore executed by all but households who find their children in the bottom of the ability distribution. As a consequence, the irreversibility constraint unambiguously depresses the formation of human capital by decreasing aggregate resources for child rearing and by causing over-investment in education. The countermeasure proposed by this paper, which would be effective in highly advanced economies, is to stimulate average fertility instead of educating low-ability children. Since the policy has an ambiguous influence on those children, who are induced to become unskilled labor, numerical analysis will be necessary for further assessment.

While the central thesis of the present research is intuitive, the theory developed above builds on several simplifying assumptions to be discussed. The first assumption is that individuals have no retirement period. If they lived on public pensions after retirement, the pension benefits they receive would depend on the proportion of the senior to the working population. This would be another reason for under-investment in the quantity of children to be reformed. The second is that siblings are identical within households, so that parents treat their children equally. Introducing sibling heterogeneity is expected to weaken the lock-in effects because parents would allocate education budgets according to their children's abilities. The third is that fertility choices are continuous. In order to cope with the education cost for children, parents are able to reduce their family sizes as much as they want. In reality, however, it is impossible to invest in education for less than one unit of children. With the discreteness of fertility choices, the irreversibility constraint would lock family sizes into higher levels and possibly mitigate over-investment in education in developed stages. The fourth is that the economy is not exposed to any demographic changes attributable to, for instance, immigration or emigration. The former would increase the working population, whereas the latter would be associated with a brain drain. It is worth investigating how they affect the macroeconomic problem of resource allocation between the quantity and quality of labor. These issues should be addressed in future research.

Appendix

A Proofs

Lemma 8 *Under Eq. (A2), $\phi^I(g_{t+1})/g_{t+1}$ and $\phi^{II}(g_{t+1})/g_{t+1}$ are strictly decreasing in g_{t+1} on $\mathbb{R}_{++}$ if*

$$aF'(a) < 1/\kappa^I \qquad \forall a \in (0,1).$$

Proof. Eq. (32) shows that $0 < \tilde{a}^I(g_{t+1}) < 1$ if and only if $g_{t+1} \in (\hat{g}, \bar{h})$. Then it follows from Eq. (33) that for any $g_{t+1} \in \mathbb{R}_{++} \backslash \{\hat{g}, \bar{h}\}$,

$$\frac{d[\phi^I(g_{t+1})/g_{t+1}]}{dg_{t+1}} = -\frac{\alpha}{\delta}\frac{\bar{h}}{g_{t+1}^2}[1 - \kappa^I \cdot \tilde{a}^I(g_{t+1})F'(\tilde{a}^I(g_{t+1}))],$$

where $\kappa^I > 0$ under Eq. (A2). The derivative above has a negative sign for any $g_{t+1} \in (\hat{g}, \bar{h})$ if $aF'(a)$ is smaller than $1/\kappa^I$ for any $a \in (0,1)$. The negative sign also holds for any $g_{t+1} \in \mathbb{R}_{++} \backslash [\hat{g}, \bar{h}]$, because in this case $\tilde{a}^I(g_{t+1}) > 1$ or $\tilde{a}^I(g_{t+1}) < 0$ and thus $F'(\tilde{a}^I(g_{t+1})) = 0$. On the other hand, Eq. (35) reveals that $0 < \tilde{a}^{II}(g_{t+1}) < 1$ if and only if $g_{t+1} \in (\breve{g}, \bar{h})$, where $\breve{g}$ is defined as a critical value such that $\tilde{a}^{II}(\breve{g}) = 1$. Then, it follows from Eq. (36) that for any $g_{t+1} \in \mathbb{R}_{++} \backslash \{\breve{g}, \bar{h}\}$,

$$\frac{d[\phi^{II}(g_{t+1})/g_{t+1}]}{dg_{t+1}} = -\frac{\alpha}{\delta + \bar{e}}\frac{\bar{h}}{g_{t+1}^2}[1 - \kappa^{II} \cdot \tilde{a}^{II}(g_{t+1})F'(\tilde{a}^{II}(g_{t+1}))],$$

where $\kappa^{II} < \kappa^I$. In a similar fashion, one finds that the derivative above has a negative sign for any $g_{t+1} \in \mathbb{R}_{++} \backslash \{\breve{g}, \bar{h}\}$ under the same assumption. Lemma 8 is established by these results, along with the continuity of $\phi^I(g_{t+1})$ and $\phi^{II}(g_{t+1})$ on $\mathbb{R}_{++}$. □

Lemma 9 *Under Eq. (A3), the nontrivial steady-state equilibrium of the locked-in economy is unique if*

$$aF'(a) < \frac{1}{\kappa^{II}}\left(1 - \frac{\delta + \bar{e}}{\alpha\bar{h}}\right) \qquad \forall a \in (0, \bar{a}).$$

Proof. Eq. (A3) ensures that no steady-state equilibrium occurs if $g_{t+1} \in (0, \tilde{g}]$, where $\tilde{g}$ is a positive value such that $\tilde{a}^*(\tilde{g}) = \bar{a}$ [cf. Eq. (23)]. Hence, a nontrivial steady-state equilibrium occurs if and only if $\phi^{II}(g_{t+1}) = 1 + g_{t+1}$ and $g_{t+1} > \tilde{g}$. First, consider the case in which $\tilde{g} < g_{t+1} < \bar{h}$. Since $0 < \tilde{a}^*(g_{t+1}) < \bar{a}$ from Eq. (8), Lemma 6 reveals that $0 < \tilde{a}^{II}(g_{t+1}) < \bar{a}(< 1)$. Then, Eq. (36) yields

$$\phi_g^{II}(g_{t+1}) = \frac{1}{g_{t+1}}\left\{\phi^{II}(g_{t+1}) - \frac{\alpha\bar{h}}{\delta + \bar{e}}\left[1 - \kappa^{II} \cdot \tilde{a}^{II}(g_{t+1})F'(\tilde{a}^{II}(g_{t+1}))\right]\right\}.$$

Using the assumption for $F(a)$ above, one can show that $\phi_g^{II}(g_{t+1}) < 1$ in any nontrivial steady-state equilibrium in this case. Second, suppose that $g_{t+1} > \bar{h}$. Since $\tilde{a}^{II}(g_{t+1}) < 0$, Eq. (36) reveals that

$$\phi_g^{II}(g_{t+1}) = \frac{\alpha}{\delta + \bar{e}}(\bar{a} - 1) < 0.$$

Third, consider the case $g_{t+1} = \bar{h}$. Noting the continuity of $\phi_g^{II}(g_{t+1})$ on $(\tilde{g}, \bar{h})$ and $(\bar{h}, \infty)$ respectively, one finds that both one-sided limits of $\phi_g^{II}(g_{t+1})$ is less than unity if a steady-state equilibrium occurs at $g_{t+1} = \bar{h}$. These results prove the uniqueness of the equilibrium. □

Proof of Lemma 4. Consider an adult individual i in period t in the unlocked economy. Eq. (6) implies that given $g_{t+1} > 0$ and $e_t^i = \bar{e}$, the individual expects to obtain more utility by choosing $n_t^i = \frac{\alpha}{\delta+\bar{e}}h_t^i$ than by choosing $n_t^i = \frac{\alpha}{\delta}h_t^i$, which is a feasible choice under Eq. (A2). Applying this result to Eq. (5) reveals that $\kappa^I > \bar{e}/\delta$. Then, the lemma follows from Eqs. (8) and (32). □

Proof of Lemma 5. Applying the implicit function theorem to Eq. (40) yields

$$\tilde{a}_\delta^I(g_{t+1}; 0) = -\frac{(1-\alpha)(1+\kappa^I)}{\delta - \alpha(\delta + \bar{e})}\left[\frac{\bar{e}}{\delta} + \frac{1}{1 - F(\tilde{a}^I(g_{t+1}))}\right]\frac{\bar{h} - g_{t+1}}{g_{t+1}}.$$

where $\tilde{a}^I(g_{t+1})$ is given by Eq. (32). The results (a) and (b) are obtained respectively by noting that $\tilde{a}^I(g_{t+1}) < 1$ and $F(\tilde{a}^I(g_{t+1})) < 1$ $\forall g_{t+1} > \hat{g}$ and that $F(\tilde{a}^I(g_{t+1})) \to 1$ as $g_{t+1} \to \hat{g} + 0$. □

Proof of Lemma 6. Consider an adult individual i in period t in the unlocked economy. Eq. (6) implies that given $g_{t+1} > 0$ and $e_t^i = 0$, the individual expects to obtain more utility by choosing $n_t^i = \frac{\alpha}{\delta} h_t^i$ than by choosing $n_t^i = \frac{\alpha}{\delta+\bar{e}} h_t^i$. Applying this result to Eq. (5) reveals that $\kappa^{II} < \bar{e}/\delta$. Then, the lemma follows from Eqs. (8) and (35). □

Proof of Lemma 7. Applying the implicit function theorem to Eq. (43) yields

$$\tilde{a}_\delta^{II}(g_{t+1}; 0) = -\frac{1+\kappa^{II}}{\delta+\bar{e}} \left[\frac{(1-\alpha)\bar{e}}{\delta+\bar{e}-\alpha\delta} + \frac{1}{F(\bar{a}) - F(\tilde{a}^{II}(g_{t+1}))} \right] \frac{\bar{h} - g_{t+1}}{g_{t+1}},$$

where $\tilde{a}^{II}(g_{t+1})$ is given by Eq. (35). The results (a) and (b) are obtained respectively by noting that $\tilde{a}^{II}(g_{t+1}) < \bar{a}$ and $F(\tilde{a}^{II}(g_{t+1})) < F(\bar{a})$ $\forall g_{t+1} > \tilde{g}$ and that $F(\tilde{a}^{II}(g_{t+1})) \to 0$ as $g_{t+1} \to \bar{h}$. □

B The Model with Expected Utility

This appendix generalizes the model developed in this paper by relaxing the assumption about expectation. At the time of childbirth, individuals aim to maximize their utility expected from the ability distribution that is, for simplicity, uniform over $[0, 1]$. The other aspects of the economy are the same as those of the locked-in economy. This framework allows us to solve the optimization problem backwardly.

B.1 Ex Post Optimization

Consider the education decision of individual i of generation t. The individual has n_t^i units of children whose ability level turned out to be $a_t^i \in [0, 1]$. In view of Eqs. (2)–(4), the critical ability level $\tilde{a}_t^i$, for which the individual is indifferent between $e_t^i = 0$ and $e_t^i = \bar{e}$, satisfies

$$(1-\alpha) \ln \frac{h_t^i - n_t^i(\delta+\bar{e})}{h_t^i - n_t^i\delta} = \alpha \ln \frac{\bar{h} - g_{t+1}}{\bar{h} - (1-\tilde{a}_t^i) g_{t+1}},$$

where $0 < n_t^i < h_t^i/(\delta+\bar{e})$ and $0 < g_{t+1} < \bar{h}$ by assumption. The equation above implies that $\tilde{a}_t^i$ is given by

$$\tilde{a}(n_t^i, g_{t+1}) \equiv \left\{ \left[\frac{h_t^i - n_t^i\delta}{h_t^i - n_t^i(\delta+\bar{e})} \right]^{\frac{1-\alpha}{\alpha}} - 1 \right\} \frac{\bar{h} - g_{t+1}}{g_{t+1}}, \tag{45}$$

where $\tilde{a}(n_t^i, g_{t+1}) > 0$, $\tilde{a}_n(n_t^i, g_{t+1}) > 0$, $\tilde{a}_g(n_t^i, g_{t+1}) < 0$, $\tilde{a}(0, g_{t+1}) = 0$, and $\tilde{a}(n_t^i, \bar{h}) = 0$ for any n_t^i and g_{t+1} in the range assumed above. Then, the optimal education choice is expressed as

$$e_t^i = \begin{cases} 0 & \text{if } a_t^i \le \tilde{a}(n_t^i, g_{t+1}); \\ \bar{e} & \text{if } a_t^i > \tilde{a}(n_t^i, g_{t+1}). \end{cases}$$

B.2 Ex Ante Optimization

Consider the optimal fertility choice of the individual when a_t^i is not observable yet. n_t^i is chosen so as to maximize his/her expected utility by taking into account the ex post decision along with Eqs. (2)–(4). It follows that

$$\begin{aligned} n_t^i = \arg\max \Bigg\{ & \alpha \ln n_t^i + \int_0^{\tilde{a}_t^i} \big[(1-\alpha)\ln(h_t^i - n_t^i\delta) + \alpha \ln(\bar{h} - g_{t+1})\big]\, dF(a) \\ & + \int_{\tilde{a}_t^i}^1 \big[(1-\alpha)\ln(h_t^i - n_t^i(\delta+\bar{e})) + \alpha \ln(\bar{h} - (1-a)g_{t+1})\big]\, dF(a) \Bigg\}, \end{aligned} \tag{46}$$

where $\tilde{a}_t^i = \tilde{a}(n_t^i, g_{t+1})$ and $0 < g_{t+1} < \bar{h}$. The optimization problem is divided into several cases depending on whether $\tilde{a}_t^i$ is greater than unity.

In order to facilitate the analysis below, let $\hat{n}_t^i$ be the critical value of n_t^i such that $\tilde{a}(n_t^i, g_{t+1}) = 1$. Using Eq. (45), one finds that

$$\hat{n}_t^i = \frac{(1-\lambda)h_t^i}{(1-\lambda)\delta + \bar{e}}, \quad \text{where } \lambda \equiv \left(\frac{\bar{h} - g_{t+1}}{\bar{h}}\right)^{\frac{\alpha}{1-\alpha}}.$$

Note that $0 < \hat{n}_t^i < 1/(\delta + \bar{e})$ and $\partial \hat{n}_t^i / \partial g_{t+1} > 0\ \forall g_{t+1} \in (0, \bar{h})$. Moreover, $\hat{n}_t^i = \alpha h_t^i/\delta$ if $g_{t+1} = \hat{g}$ because, in view of Eqs. (32) and (45), $\hat{g}$ is a critical value on $(0, \bar{h})$ such that $\tilde{a}^I(\hat{g}) = \tilde{a}(\alpha h_t^i/\delta, \hat{g}) = 1$.

Case 1: $g_{t+1} \in (0, \hat{g}]$

First, consider the optimal fertility choice on the interval $[\hat{n}_t^i, \infty)$. Since $\tilde{a}(n_t^i, g_{t+1}) \ge 1$ in Eq. (46), the first-order condition is simplified to

$$D(n_t^i) \equiv \frac{\alpha}{1-\alpha}\frac{1}{n_t^i} - \frac{\delta}{h_t^i - n_t^i\delta} = 0,$$

where $D(n_t^i)$ is a strictly decreasing function such that $D(\hat{n}_t^i) \ge 0$ with equality if and only if $g_{t+1} = \hat{g}$ (and thus $\hat{n}_t^i = \alpha h_t^i/\delta$). Thus, the optimal choice on this interval is $n_t^i = \alpha h_t^i/\delta \ge \hat{n}_t^i$, with equality if and only if $g_{t+1} = \hat{g}$.

Second, consider the interval $(0, \hat{n}_t^i]$. Noting that $0 < \tilde{a}(n_t^i, g_{t+1}) \leq 1$ under the circumstances, differentiating the objective function in Eq. (46) with respect to n_t^i and arranging the result yield $G(\hat{n}_t^i, g_{t+1}) \geq 0$, with equality if and only if $g_{t+1} = \hat{g}$, where

$$\begin{aligned} G(n_t^i, g_{t+1}) \quad \equiv \quad & \frac{\alpha}{1-\alpha}\frac{1}{n_t^i} - \tilde{a}(n_t^i, g_{t+1})\frac{\delta}{h_t^i - n_t^i\delta} \\ & -[1 - \tilde{a}(n_t^i, g_{t+1})]\frac{\delta + \bar{e}}{h_t^i - n_t^i(\delta + \bar{e})}. \end{aligned}$$

In what follows, the second-order condition is imposed to ensure the uniqueness of the solution. That is to say, for any (n_t^i, g_{t+1}) such that $0 < \tilde{a}(n_t^i, g_{t+1}) \leq 1$ and $G(n_t^i, g_{t+1}) = 0$,

$$\begin{aligned} G_n(n_t^i, g_{t+1}) = & -\frac{\alpha}{1-\alpha}\frac{1}{(n_t^i)^2} \\ & + \tilde{a}_n(n_t^i, g_{t+1})\frac{\bar{e}h_t^i}{(h_t^i - n_t^i\delta)[h_t^i - n_t^i(\delta + \bar{e})]} \\ & - \tilde{a}_t^i\frac{\delta^2}{(h_t^i - n_t^i\delta)^2} - (1 - \tilde{a}_t^i)\frac{(\delta + \bar{e})^2}{[h_t^i - n_t^i(\delta + \bar{e})]^2} \\ & < 0. \end{aligned} \tag{47}$$

Then, $G(n_t^i, g_{t+1}) > 0 \ \forall n_t^i \in (0, \hat{n}_t^i)$ and thus $n_t^i = \hat{n}_t^i$ is the optimal choice on this interval.[45]

Given the two results above, consider the entire interval $(0, \infty)$. Since the objective function in Eq. (46) is continuous at $\hat{n}_t^i$, one finds that $n_t^i = \alpha h_t^i/\delta$ is the globally optimal solution for $g_{t+1} \in (0, \hat{g}]$.

Case 2: $g_{t+1} \in (\hat{g}, \bar{h})$.

First, consider the optimal fertility choice on the interval $[\hat{n}_t^i, \infty)$. Since $\tilde{a}(n_t^i, g_{t+1}) \geq 1$, differentiating the objective function in Eq. (46) with respect to n_t^i and arranging the result reveal that $D(n_t^i) < 0$ on this interval. Hence, the optimal choice is $n_t^i = \hat{n}_t^i$.

Second, consider the interval $(0, \hat{n}_t^i]$. Since $0 < \tilde{a}(n_t^i, g_{t+1}) \leq 1$, the optimality condition is $G(n_t^i, g_{t+1}) = 0$, where $G(\cdot)$ is a continuous function such that $G(\hat{n}_t^i, g_{t+1}) < 0$ and $G(n_t^i, g_{t+1}) \to \infty$ as $n_t^i \to 0$. This first-order condition is therefore satisfied by a value of n_t^i on $(0, \hat{n}_t^i)$. One may find the value by guessing that n_t^i is proportional to h_t^i; i.e., $n_t^i = \gamma_t h_t^i$.

[45] The second-order condition in Eq. (47) is satisfied when $\bar{e}$ is sufficiently small. Such a restriction on $\bar{e}$ is compatible with the other key assumptions (cf. Footnotes 30 and 44).

Then, the first-order condition is simplified to

$$\frac{\alpha}{1-\alpha}\frac{1}{\gamma_t} - \tilde{a}_t^i \frac{\delta}{1-\gamma_t\delta} - (1-\tilde{a}_t^i)\frac{\delta+\bar{e}}{1-\gamma_t(\delta+\bar{e})} = 0,$$

where $\tilde{a}_t^i = \tilde{a}(\gamma_t h_t^i, g_{t+1})$ is independent of h_t^i [cf. Eq. (45)]. Then, the second-order condition in Eq. (47), along with the property that $\tilde{a}_g(n_t^i, g_{t+1}) < 0$, ensures the one-to-one negative relationship of γ_t to $g_{t+1} \in [\hat{g}, \bar{h}]$. In particular, note that $\gamma_t = \alpha/\delta$ if $g_{t+1} = \hat{g}$ and that $\gamma_t = \alpha/(\delta+\bar{e})$ if $g_{t+1} = \bar{h}$.[46]

With the two results above, consider the entire interval $(0, \infty)$. Because the objective function is continuous at $\hat{n}_t^i$, $n_t^i = \gamma_t h_t^i < \alpha h_t^i/\delta$ is the globally optimal solution for $g_{t+1} \in (\hat{g}, \bar{h})$.

Summary

Given the analysis so far, one finds that the optimal fertility choice is expressed as a continuous function of g_{t+1} kinked at $\hat{g}$. More precisely,

$$n_t^i = \gamma(g_{t+1}) h_t^i, \tag{48}$$

where $\gamma(g_{t+1}) = \alpha/\delta \ \forall g_{t+1} \in (0, \hat{g}]$, $\gamma'(g_{t+1}) < 0 \ \forall g_{t+1} \in (\hat{g}, \bar{h})$, and $\gamma(\bar{h}) = \alpha/(\delta+\bar{e})$.

B.3 Comparison with the Unlocked Economy

As shown below, the central result of the present paper—the emergence of under- and over-investment in education—is retained in the expected-utility framework considered here.

In order to prove the result, it is necessary to reconstruct $\tilde{a}(g_{t+1})$ in Eq. (25). Substituting Eq. (48) into Eq. (45) yields

$$\tilde{a}_t^i = \kappa(g_{t+1})\frac{\bar{h} - g_{t+1}}{g_{t+1}} \equiv \tilde{a}(g_{t+1}), \tag{49}$$

where

$$\kappa(g_{t+1}) \equiv \left[\frac{1-\gamma(g_{t+1})\delta}{1-\gamma(g_{t+1})(\delta+\bar{e})}\right]^{\frac{1-\alpha}{\alpha}} - 1.$$

In view of Eqs. (32) and (35), $\kappa(g_{t+1})$ is a continuous function such that $\kappa(g_{t+1}) = \kappa^I$ $\forall g_{t+1} \in (0, \hat{g}]$, $\kappa'(g_{t+1}) < 0 \ \forall g_{t+1} \in (\hat{g}, \bar{h})$, and $\kappa(\bar{h}) = \kappa^{II}$. Thus, unlike in Figure 3, the

[46] The first and second properties are, respectively, obtained by noting that $\tilde{a}(\alpha h_t^i/\delta, \hat{g}) = 1$ and that $\tilde{a}(\gamma_t h_t^i, \bar{h}) = 0 \ \forall \gamma_t \in (0, 1/(\delta+\bar{e}))$.

expected-utility framework makes the function $\tilde{a}(g_{t+1})$ continuous at any point on $(0, \bar{h})$. On the other hand, as in the diagram, this function has a decreasing property and $\tilde{a}(\hat{g}) = 1$.

Correspondingly, redefine $\tilde{g}$ as a critical level of g_{t+1} for which $\tilde{a}(g_{t+1}) = \tilde{a}^*(g_{t+1})$. In other words,

$$\kappa(g_{t+1}) = \frac{\bar{e}}{\delta} \quad \text{for } g_{t+1} = \tilde{g},$$

using Eqs. (8) and (49). Since $\kappa^I > \bar{e}/\delta > \kappa^{II}$ as implied by Lemmas 4 and 6, the properties of $\kappa(g_{t+1})$ ensure that $\tilde{g}$ exists uniquely and

$$\tilde{a}(g_{t+1}) \begin{cases} > \tilde{a}^*(g_{t+1}) & \text{for } g_{t+1} \in (0, \tilde{g}); \\ = \tilde{a}^*(g_{t+1}) & \text{for } g_{t+1} = \tilde{g}; \\ < \tilde{a}^*(g_{t+1}) & \text{for } g_{t+1} \in (\tilde{g}, \bar{h}). \end{cases}$$

Consistent with Section 4, this indicates that the economy goes through under- and then over-investment in education as g_{t+1} increases.

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Figure Captions

Figure 1. Education Decisions in the Unlocked Economy.
Notes: The diagram depicts the negative relationship between the growth rate of technology, g_{t+1}, and the critical ability level for education, $\tilde{a}^*(g_{t+1})$. A fall in $\tilde{a}^*(g_{t+1})$ implies a higher

ratio of households investing in education in period t. The spread of education is attributed to the skill-biased technological progress.

Figure 3. The Transition from Under- to Over-Investment in Education in the Unlocked Economy.
Notes: The diagram depicts the relationship between the growth rate of technology and the critical ability level for education in the locked-in economy, $\tilde{a}_t$, in comparison with the one for the unconstrained case, a_t^*. $\tilde{a}_t$ decreases as g_{t+1} increases, and its quantitative relationship with a_t^* reverses when g_{t+1} crosses over $\tilde{g}$.

Figure 2. The Evolution of Technology for the Unlocked Economy.
Notes: For any $x_0 > 0$, the innovation rate, x_t, and the growth rate of technology, g_t, monotonically converge toward their respective steady-state levels, $\bar{x}^*$ and $\bar{g}^*$.

Figure 4. The Evolution of Technology for the Locked-in Economy.
Notes: The growth rate of technology, g_t, monotonically increases over time as long as the initial innovation rate x_0 is sufficiently small. It eventually exceeds the critical level for the fertility decision, $\tilde{g}$, and converges towards the steady-state level $\bar{g}$.

Figure 5. The Lock-in Effects on Education and Parenting Resources in Stage I.
Notes: The diagram depicts the lock-in effects on the amount of efficient labor devoted to children, b_t^i, and on the education decision, e_t^i. As the upward and downward arrows respectively indicate, the irreversibility constraint induces households on the interval $(\tilde{a}^I(g_{t+1}), 1]$ to spend more for their children, whereas it prevents those on $(\tilde{a}^*(g_{t+1}), \tilde{a}^I(g_{t+1})]$ from investing in child education.

Figure 6. The Lock-in Effects on Education and Parenting Resources in Stage II.
Notes: In contrast to Figure 5**,** the diagram shows that the irreversibility constraint induces households on the interval $[0, \tilde{a}^{II}(g_{t+1})]$ to spend less for their children, whereas it induces those on $(\tilde{a}^{II}(g_{t+1}), \tilde{a}^*(g_{t+1})]$ to invest in education.