On ‘Rusting’ Money
Silvio Gesell’s *Schwundgeld* Reconsidered

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Abstract

Silvio Gesell hypothesized that money depreciation is economically and socially beneficial, ideas that have often been contended. Here I analyze that in a Sidrauski model in which households additionally have a ‘love of wealth’-motive. It is shown Gesell’s claims may be valid in a demand-determined, short-run equilibrium and why money depreciation overcomes the zero lower bound on nominal interest rates. These results are checked against the recent demonetization episode in India. However, for a typical long-run equilibrium introducing money depreciation in isolation may be bad. But money depreciation, when coupled with expansionary monetary policy, is a necessary condition for a positive *Mundell-Tobin* effect on long-run real variables and so creates wealth in the model. It is found that this also holds in the transition to the long-run equilibrium. Hence, the spirit of Gesell’s hypotheses can be verified for a plausible, long-run environment.

KEYWORDS**:** Economic Performance, Depreciating Money, Zero Lower Bound, Demonetization, Love of Wealth

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1 Introduction

"Money is the football of economic life."

Silvio Gesell (1920)
The Natural Economic Order

In his main piece of work "The Natural Economic Order" Silvio Gesell, a German merchant and intellectual, developed various insightful arguments to improve the workings of an economy. It was first published in Bern in 1916 and received praise from economists such as Keynes (1936) in his "General Theory of Employment, Money and Interest", ch. 23, and Irving Fisher (1933) in his "Booms and Depressions".

In this paper I reconsider his idea of *Schwundgeld* (demurrage) and its consequences on economic performance. I analyze whether his key conjectures can be justified in a parsimonious, modern theoretical framework. One reason is that Gesell's claims have often been contended by arguing that they cannot be corroborated by 'state-of-the-art' theory.

Although Gesell (1920), p. 78, acknowledges that money is "the football of economic life", thus (probably) being a key driver of and so essential for any modern economy, he cautions us by arguing

> "Only money that goes out of date like a newspaper, rots like potatoes, rusts like iron, evaporates like ether, is capable of standing the test as an instrument for the exchange of potatoes, newspapers, iron and ether. For such money is not preferred to goods either by the purchaser or the seller. We then part with our goods for money only because we need the money as a means of exchange, not because we expect an advantage from possession of the money." (p. 121)

According to him placing money and commodities on equal 'physical' footing requires that money depreciates, just as normal goods do due to the wear and tear in usage or storage. In particular, he argued the face value of (paper) money depreciate at a certain percentage over a particular period of time. In order to regain the previous face value of the money (note) used, people would have to buy stamps to make up for the depreciation the monetary authority would decree for the money note.[1]

[1]Consider his example for the American currency: "This $100 note (bill) is shown as it will appear during the week August 4th-11th, thirty-one ten-cent stamps ($3.10) having been attached to it by its various holders on the dated spaces provided for the purpose, one stamp for each week since the beginning of the year. In the course of the year 52 ten cent stamps ($5.20) must be attached to the $100 note, or in other words it depreciates 5.2% annually at the expense of its holders." Gesell (1920), p. 121/2.

The introduction of such a monetary arrangement would then influence the economy in ways about which he formed, among others, the following four hypotheses.

Gesell Conjecture 1 (GC1) *The introduction of, and, when present, an increase in, the money depreciation rate leads to a higher velocity of money in circulation.*

> "Everyone of course tries to avoid the expense of stamping the notes by passing them on - by purchasing something, by paying debts, by engaging labour, or by depositing the notes in the bank, which must at once find borrowers for the money, if necessary by reducing the rate of interest on its loans. In this way the circulation of money is subjected to pressure." Gesell (1920), p. 123.

Gesell Conjecture 2 (GC2) *Money depreciation coupled with expansionary monetary policy stimulates aggregate demand and through that output and employment.*

> "In all conceivable circumstances, in fair weather and in foul, demand will then exactly equal: - The quantity of money circulated and controlled by the State. Multiplied by: The maximum velocity of circulation possible with the existing commercial organisation. What is the effect upon economic life? The effect is that we now dominate the fluctuations of the market; that the Currency Office, by issuing and withdrawing money, is able to tune demand to the needs of the market; that demand is no longer controlled by the holders of money, by the fears of the middle classes, the gambling of speculators or the tone of the Stock Exchange, but that its amount is determined absolutely by the Currency Office. The Currency Office now creates demand, just as the State manufactures postage stamps, or as the workers create supply." Gesell (1920), p. 127.

Gesell Conjecture 3 (GC3) *A money depreciation rate is welfare enhancing.*

> "The elimination of interest is the natural result of the natural order of things when undisturbed by artificial interference. Everything in the nature of men as in the nature of economic life urges the continual increase of so-called real capital - an increase which continues even after the complete disappearance of interest. The sole disturber of the peace in this natural order we have shown to be the traditional medium of exchange. The unique and characteristic advantages of this medium of exchange permit the arbitrary postponement of demand, without direct loss to its possessor; whereas supply, on account of the physical characteristics of the wares, punishes delay with losses of all kinds. In defence of their economic welfare both the individual and the community have been and are at enmity with interest; and they would long ago have eliminated interest if their power had not been trammelled by money." Gesell (1920), p. 190.

Gesell Conjecture 4 (GC4) *A money depreciation rate benefits workers relatively more than capital owners.*

"By the laws of free competition the manufacturer's profit must be reduced to the level of a technician's salary - an unpleasant result for many manufacturers whose success was mainly due to their commercial ability. With Free-Money, creative power has become unnecessary in commerce, for the difficulties which called for the comparatively rare and therefore richly rewarded commercial talent have disappeared. And someone must benefit by the reduction of the manufacturer's profit. Either goods must become cheaper, or, to put it the other way about, wages must rise. There is no other possibility." Gesell (1920), p. 135.

As pointed out above, the present paper complements research that has used modern economic theory to investigate whether the Gesell-hypotheses can be replicated in standard model frameworks. One finds that the results of previous research are mixed.

For example, Rösl (2006) finds that only the first hypothesis can be derived in a Sidrauski (1967), that is, in a money-in-the-utility set-up. He concludes that Gesell neglected an analysis of the long run and any possible effects on capital accumulation so that the other three hypotheses turn out be non-valid in his model.

In turn, Menner (2011), for example, uses an elaborate and involved New Keynesian DSGE model to find that "*inflation and 'Gesell taxes' maximize steady state capital stock, output, consumption, investment and welfare at moderate levels. ... In a recession scenario a Gesell tax speeds up the recovery in a similar way as a large fiscal stimulus but avoids 'crowding out' of private consumption and investment.*" Thus, he finds support for the Gesell hypotheses at moderate levels in his business cycle model of the third-generation of monetary search models.

The present paper uses an alternative micro-founded and simple general equilibrium model to analyze whether depreciation of money is socially beneficial. Doing this we will abstract from fiscal policy, as Gesell did not consider the interaction of fiscal and monetary policy in detail.[2]

In the paper the basic Sidrauski-framework is changed in an important way. Apart from the motive to derive utility from money it is assumed that agents also derive utility from their wealth.[3] People are taken to be rational and are not fooled by money

[2]If one likes, the results here may also interpreted as holding relative to some given and constant fiscal policy operating in the background, and Ricardian Equivalence holds. Furthermore, another word of caution should be mentioned. I will not address the historical and the more recent empirical experiences that, mostly, local experiments using money depreciation have produced. Of course, the most famous one is the Wörgel experiment from 1932 to 1933 which was stopped by the Austrian National Bank in September 1933. The interested reader will find a plethora of empirical evidence on whether money depreciation and Gesell's ideas in general work or not in the literature. Here the focus is on theory.

[3]This has been done, for example, by Weber (1930) and Pigou (1941) who argue that individuals

illusion. Thus, the agents only consider real, physical capital as wealth.

Here I relate to this more general concept as 'love of wealth' as in Rehme (2011). These motives are important for deriving non-degenerate short-run relationships between the nominal interest rate and consumption (an IS and LM curve in a "nominal interest rate and consumption"-space). A similar approach based on 'love of wealth' has recently been presented by Michaillat and Saez (2014).

In this framework the paper analyzes two approaches to capture Gesell's ideas. The first one concentrates on textbook-like short-run, demand determined equilibria of the IS-LM-AS-AD variety, which are based on the micro-foundations of optimal behaviour, that is, the demand of the agents. The link to supply is assumed to be Keynes's "principle of effective demand".

The second uses a standard Ramsey-Cass-Koopmans framework where markets are assumed to clear at each point in time, and demand equals supply. It turns out that this yields interesting insights about money depreciation for the steady state of an economy and its transitional dynamics.

The following results are obtained for the first approach. In a short-run, demand-determined equilibrium where the (physical) capital stock, the inflation rate, transfers, and money supply are fixed, but real factor prices are flexible, Gesell's hypotheses GC1 - GC4 are all valid, given the (demand) micro-foundations of the model and given that the micro-foundations feature direct utility derived from money transactions and 'love of wealth', where only physical capital is considered to be the true source of wealth.

A key assumption for the derivation of this result is that the marginal productivity theory of distribution does not necessarily hold in the short run. Importantly, when the inflation rate is given, and the Fisher relationship holds, the real interest rate moves in the same direction as the nominal interest rate in any short-run equilibrium.[4]

The details for this are presented in the main text. Thus, the real interest rate is determined by other factors than technology in the short-run.

derive utility from the mere possession of wealth and not simply its expenditure. Later Kurz (1968) provided a thorough analysis of an optimal growth model where wealth features in utility. Furthermore, Zou (1994), Bakshi and Chen (1996) and Carroll (2000) relate to Max Weber and argue that the dependence of utility on wealth captures the spirit of capitalism. More generally, it captures 'love of wealth' as argued in Rehme (2017).

[4]Notice that the "Fisher relationship" captures that the nominal interest rate is (approximately) the sum of the real interest rate and (expected) inflation. This should not be equated with the "Fisher effect" which states that the real interest rate is independent of the rate of inflation. For this clarification see, for example, Ahmed and Rogers (1996). For textbook models where the real and the nominal interest rate move in the same direction in the short run, see, for example, Blanchard (2017), ch. 6 and 16.

Gesell's ideas have been important in discussions about overcoming the zero lower bound that has played such an important role after the Great Recession. One argument has been to make nominal interest rates negative to combat what is called a "liquidity trap". For a good survey on this, its relation to Gesell's ideas and their historic precursors see Ilgmann and Menner (2011) and Svensson and Westermark (2016).

In the present paper it turns out that many different combinations of money depreciation and money supply policies can sustain a "liquidity trap", that is, a situation with a short-run equilibrium, zero nominal interest rate. These monetary policy combinations are shown to have non-negligible effects for distribution, that is, the rewards to labour and capital.

In various model variants Buiter and Panigirtzoglou (2003) and other contributions by W. Buiter have shown that money depreciation may be used to make the short-run equilibrium interest rate negative and pull an economy out of a "liquidity trap". In this paper I basically find the same so complementing their/his results. But the model structure here is quite different and simple. Given the present model's microfoundations this result follows straightforwardly and easily.

Another application of the model for the short run is the recent episode of demonetization in India where the 500 and 1000 rupee notes (INR) were declared invalid in a surprise move by the Indian Prime Minister. In India cash is by far the most important medium for economic transactions. Overnight this affected 86.9 percent of the value of total currency in circulation.

The model predicts that such a demonetization leads to lower consumption, aggregate demand, and lower wages, but higher real interest rates. Thus, the measure does not seem to be good for workers in the short run. These findings are broadly in line with the empirical evidence documented by the Reserve Bank of India's Monetary Policy Department (MPD) (2016).

Next, the analysis is extended to the long run. In an optimal growth framework these results then emerge. In the steady state inflation depends on the sum of the money growth and depreciation rate. It turns out that the model dichotomizes into a monetary and real sector, if there is no money depreciation. If the latter is present, the model features non-superneutrality. Thus, in the model money depreciation is a *necessary* condition for particular forms of a *Mundell-Tobin* effect. That effect is present if inflation leads people to hold less money and more real capital, implying a lower real interest rate.

More precisely, it is found that the introduction of or an increase in money depreciation in isolation reduces the steady state capital stock (wealth), consumption, income and welfare. It also implies a higher return to capital, but lower steady state wage rate. Thus, more money depreciation seems to destroy wealth and implies lower wages. The only hypothesis that is validated is that higher money depreciation implies a higher velocity of money, [GC1].

Some authors have stopped here to argue that money depreciation is generally a bad idea, because it just destroys long-run wealth, instead of fostering it. However, in light of the quotes above that view does not do justice to Gesell's thinking. He was not arguing solely about money depreciation. Of course, he knew that the monetary authority is also issuing new and withdrawing old money.

Here it turns out that, for a given positive money depreciation rate, an increase in the money growth rate produces a *Mundell-Tobin* effect. Thus, a higher money growth increases steady state inflation, but also the steady state capital stock, output, and consumption. It implies a higher long-run wage rate and a lower return to capital. The consequences for the holdings of real money balances and so for total welfare are not unambiguously clear. But the velocity of money increases. However, the partial welfare channels through consumption and wealth work clearly in a positive direction.

Hence, the conjectures GC1 and GC2 can be validated for the long run. But given the *necessary* nature of money depreciation for these results one may argue that GC3 and GC4 are also not too far off their marks. In terms of the economic effects the conjectures ultimately wish to capture they are not really wrong because of the possibility of a positive *Mundell-Tobin* effect which would indeed support GC3 and GC4.

The analysis of the transitional dynamics reveals that the speed of convergence increases if money depreciation increase, and decreases if the money growth rate is raised. That complements Fischer (1979) who finds that more money growth speeds up convergence when utility is non-logarithmic and the steady state features asymptotic superneutrality. Here the steady state generally features non-superneutrality, utility is logarithmic, and convergence is slower when the money growth rate increases.

A simulation exercise based on some standard calibration values reveals that the response of the key variables to permanent changes in the monetary policy variables is the same in the transition as in steady state. That also holds for the jump variables, namely, initial money holdings and consumption.

Furthermore, for temporary changes in the policy variables one obtains the temporary responses that, again, qualitatively equal those for the steady state.

Summarizing all these findings yields that the present model-framework is indeed capable to verify most of Gesell's claims. In the short-run, demand-determined equilibrium all claims can verified. For the long-run equilibrium two claims follow directly, and the other two indirectly, because money depreciation is a necessary condition for a positive *Mundell-Tobin* effect.

The paper is organized as follows. Section 2 presents the model and 3 analyzes the demand-determined (short-run) equilibrium, an overcoming of the zero lower bound on nominal interest rates and applies the model to the Indian demonetization episode. Section 4 derives and analyzes the long-run equilibrium and section 5 the transitional dynamics. Section 6 concludes.

2 The Model

To simplify the algebra the model is set in continuous time. For all variables that are continuous functions of time I use the subscript t to denote their dependence on time. Thus, we define $h_t \equiv h(t)$ for some variable h depending on time. Furthermore, the change of a variable h over time, i.e. $\frac{dh_t}{dt}$, is denoted by $\dot{h}_t$.

By assumption the economy is populated by many, price-taking households. The aggregate resource constraint of the households is given by

$$C_t + \dot{K}_t + \frac{\dot{M}_t}{P_t} + \sigma \cdot \frac{M_t}{P_t} = w_t N_t + r_t K_t + X_t \tag{1}$$

where C_t and K_t denote aggregate real consumption and the aggregate real capital stock, respectively. M_t represents the aggregate nominal money holdings and P_t the price level. N_t denotes population and w_t the real wage rate. r_t denotes the real rate of return on capital, net of depreciation of physical capital K_t. The lump-sum (real) transfers of the government that are granted to the households are denoted by X_t.

Thus, the right hand side of the budget constraint in (1) captures aggregate income, consisting of total wage ($w_t N_t$) and capital income ($r_t K_t$) as well as government transfers (X_t).

The left hand side, in turn, captures aggregate spending. Thus, income is spend on consumption (C_t), and investment in new capital ($\dot{K}_t$) and acquisitions of new, real money holdings ($\frac{\dot{M}_t}{P_t}$).

The aggregate budget constraint in equation (1) corresponds to the conventional set-up of a Sidrauski (1967), money-in-the-utility-function model. The novel feature and, for this paper, crucial difference is the term $\sigma \cdot \frac{M_t}{P_t}$. It captures the *Gesell tax*, that is, the idea of "*rusting money*". That can be interpreted as a depreciation on the circulating real money holdings of the households and is tantamount to a tax on them.

Sometimes it is argued that the *Gesell* tax is simply a another form of an inflation rate that most people also consider a tax on money holdings. But notice that the *Gesell* tax is directly determined by a political entity such as e.g. a central bank, and not, like the inflation rate (tax), indirectly by the workings of markets.

Now consider a representative agent economy, and define per capita consumption c_t, real money balances m_t, as well as the per capita capital stock k_t and transfers x_t as follows

$$c_t \equiv \frac{C_t}{N_t}, \quad m_t \equiv \frac{M_t}{P_t N_t}, \quad k_t \equiv \frac{K_t}{N_t}, \quad \text{and} \quad x_t \equiv \frac{X_t}{N_t}.$$

Division of equation (1) by N_t and using our definitions then yields

$$c_t + \frac{\dot{K}_t}{N_t} + \frac{\dot{M}_t}{P_t N_t} + \sigma m_t = w_t + r_t k_t + x_t.$$

It is not difficult to verify that $\frac{\dot{K}_t}{N_t} = \dot{k}_t + n_t k_t$ and $\frac{\dot{M}_t}{P_t N_t} = \dot{m}_t + \pi_t m_t + n_t m_t$ where $\pi_t \equiv \frac{\dot{P}_t}{P_t}$ represents the rate of inflation and $n_t = \frac{\dot{N}_t}{N_t}$ the population growth rate. Then the budget constraint of the representative household is given by

$$c_t + \dot{k}_t + n_t k_t + \dot{m}_t + \pi_t m_t + n_t m_t + \sigma m_t = w_t + r_t k_t + x_t.$$

Again, the right hand side corresponds to the household's income and the left hand side captures the household's expenditure. Notice that σm_t can be regarded as an outlay for the household. The longer the household holds real money balances m_t, the more is foregone (a form of expenditure) in terms of real income. For a similar set-up see, for example, Rösl (2006). It captures what is called the *Gesell tax*.

Building on, for example, Blanchard and Fischer (1989), ch. 4.5, and the Rösl set-up we now denote real per capita resources by a_t where $a_t \equiv k_t + m_t$. Thus, the household has real resources in the form of physical capital and real money balances. It follows that $\dot{a}_t = \dot{k}_t + \dot{m}_t$. After collecting terms and rearrangement one then obtains[5]

$$\dot{a}_t = [(r_t - n_t)a_t + w_t + x_t] - [c_t + (r_t + \pi_t + \sigma)m_t] \,. \tag{2}$$

Thus, the change in real per capita resources $\dot{a}_t$ depends on the household's income from capital and real money balances $(r_t - n_t)a_t$, labour income w_t and transfers x_t. Consumption then consists of the consumption of goods c_t and the expenses for using money services. The latter depend on the user cost of money $(r_t + \pi_t + \sigma)m_t$. Here we employ the Fisher relation that *nominal* interest rates i_t equal the *real* interest rate r_t plus the inflation rate π_t. The user cost of holding money, thus, depends on the nominal interest rate i_t and the depreciation of money σ.

To simplify the algebra consider an economy with no population growth $n_t = 0$ and a population set to $N_t = 1$ for all t. One easily verifies that the paper's qualitative results do not depend on these assumptions. Furthermore, notice that by assumption r_t represents the rate of return of physical capital *net* of depreciation. This will become clearer when presenting the firms' problem.[6]

As an important departing point from a standard Sidrauski model the representative household is now taken to "love wealth". By assumption the household is forward looking and not fooled by money illusion. Thus, only *physical* capital is considered to be "wealth" that directly bears on welfare. Hence, k_t features in the utility function as, for example, in Kurz (1968).[7]

However, the household also values the fact that real money balances facilitate exchange and so transactions. Thus, (real) money balances are also taken to bear on

[5] The steps are

$$\begin{aligned} c_t + (\dot{k}_t + \dot{m}_t) + (n_t k_k + n_t m_t) + \pi_t m_t + \sigma m_t &= w_t + r_t k_t + x_t \\ c_t + \dot{a}_t + a_t n_t + \pi_t m_t + \sigma m_t &= w_t + r_t k_t + r_t m_t - r_t m_t + x_t \\ c_t + \dot{a}_t + a_t n_t + \pi_t m_t + \sigma m_t &= w_t + r_t a_t - r_t m_t + x_t \end{aligned}$$

and so

$$\dot{a}_t = w_t + r_t a_t + x_t - a_t n_t - (r_t + \pi_t + \sigma)m_t - c_t.$$

Rearrangement yields equation (2).

[6] Otherwise, some slight adjustments of the model after reintroduction of a positive n would also serve the purpose of working with a net return on capital, because n can also be interpreted as a factor that corresponds to some form of social depreciation rate in a simple Solow model. This argument can be found in almost any elementary textbook on macroeconomics.

[7] The expression *love of wealth* is based on Plutarch's (46 AD - 120 AD) essay "Περὶ φιλοπλουτίας" ("De Cupiditate Divitiarum" or "On the Love of Wealth") in his *Moralia*.

welfare as in Sidrauski (1967). Although both money and capital feature directly in utility, they do so for different reasons. Money is valued because it facilitates exchange, whereas physical capital is valued as an expression of wealth.[8]

The household's problem is then taken to be to maximize the functional

$$W = \int_0^\infty \varphi(c_t, m_t, k_t)\, e^{-\rho t} dt, \tag{3}$$

where $\varphi(c_t, m_t, k_t)$ is period utility depending on consumption, real money balances and physical capital. Welfare is discounted at the (positive) rate of time preference ρ, capturing how patient households are, and convergence of the utility functional.

One needs to put more structure on these preferences, because Kurz (1968) has analyzed a neoclassical growth model with physical capital in the utility function (i.e. preferences with "love of wealth") and shown that the dynamic properties of such a model are extremely cumbersome to analyze. Furthermore, no clear results appear to be obtainable if allowing for the more general setups.[9]

In order to derive clear predictions that also allow for an analysis of transitional dynamics, and building on previous own work, cf. Rehme (2011), we now make the following assumptions about the period utility function $\varphi(c_t, m_t, k_t)$.

1. $\varphi(c_t, m_t, k_t)$ is taken to be separable in c_t, m_t and k_t. In particular, assume that $\partial^2\varphi(\cdot)/\partial i\, \partial j = 0$ for all $i, j = c_t, m_t, k_t$ and $i \neq j$.

2. $\varphi(c_t, m_t, k_t)$ is increasing and concave in each (own) argument, that is, $\partial\varphi(\cdot)/\partial i > 0$ and $\partial^2\varphi(\cdot)/\partial i^2 < 0$ for all $i = c_t, m_t, k_t$.[10]

3. $\varphi(c_t, m_t, k_t)$ satisfies the Inada conditions for each (own) argument, that is,

$$\lim_{i\to 0} \varphi(\cdot)/\partial i \to \infty \quad \text{and} \quad \lim_{i\to\infty} \varphi(\cdot)/\partial i \to 0 \quad \text{where } i = c_t, m_t, k_t.$$

Thus, as in Sidrauski (1967) period utility depends on (per capita) consumption c and real money balances m. What is different is that the agent additionally derives welfare from (per capita) wealth (capital).[11] Thus, $\varphi(\cdot)$ is also a function of k. That

[8]For a clarification why money may be taken to feature directly in utility cf. Feenstra (1986).

[9]In fact, the more general a setup, the more empty the content of a model may often be.

[10]A positive marginal utility of wealth $\partial\varphi(\cdot)/\partial k_t > 0$ is necessary for a non-degenerate IS curve.

[11]The question arises whether it is relative wealth (status concerns) or absolute wealth that matters for individuals. The former played a great role for preferences according to, for example, Smith (1759),

captures that many people value wealth and capital per se. For instance, many people like to look at and visit impressive buildings, e.g. the Eiffel Tower, the Empire State Building or the like, and derive utility from that.[12]

The separability assumption is often invoked. It means that the decision on one of the variables does not depend on any of the other variables. Thus, the agent focuses only on one variable when making plans. That, of course, does not imply that the optimal choice is independent of the other variables, because they are linked through the budget constraint. On separation approaches in economic modelling see, for example, Blackorby, Primont, and Russel (2008) or Acemoglu (2009), ch. 10.1.

Taking welfare to be increasing in wealth is perhaps more problematic. Clearly, there are cases where additional capital may be valued less. An example may be an additional nuclear power plant. However, k is an index of all sorts of capital stocks. Most evidence would suggest that people generally like wealth and especially more of it. Otherwise, they would not do the things one can observe to increase their wealth. Of course, this is a perennial phenomenon. Thus, drawing on this "stylized fact" may justify the assumption that welfare is increasing in wealth, $\varphi_k > 0$.

Assuming that the welfare gain becomes smaller as wealth increases captures the observation that very rich people often say that an additional "yacht" may not make them much happier, especially in comparison to the first one they already own.

The Inada conditions on welfare's reaction on the effects of wealth when there is hardly any or too much capital are not really necessary for most of the analysis below, but can be rationalized on quite intuitive grounds. For example, $\lim_{k\to 0} \varphi_k = \infty$ would say that one is really craving for wealth if one does not have any. In turn, $\lim_{k\to\infty} \varphi_k = 0$ would imply that Bill Gates does not really care if he gets an additional computer.

A simple and convenient period utility function that satisfies all these requirements is the logarithmic one. So we invoke

Assumption 1 *Period utility $\varphi(c_t, m_t, k_t)$ is separable and logarithmic in each argu-*

the latter according to Plutarch. In the present model the distinction does not matter, as is shown in Appendix A. In this context, the reader may find the relevant passage of Adam Smith, as presented by Corneo and Jeanne (2001a) in Appendix F. But notice that the paper allows for comparisons, too, because below we will conduct comparative static exercises by which one contrasts different policies and their effects on economic performance.

[12] Also, many firms offer guided tours through their often very impressive plants of production such as e.g. the Boeing assembly halls in Seattle or Volkswagen's "Auto Manufaktur" in Dresden. Clearly, marvelling at buildings from outside means that these buildings or plants have a public good nature. However, visiting them usually requires a fee to be paid so that buildings and guided tours then have a private good nature.

ment and given by

$$\varphi(c_t, m_t, k_t) = \ln c_t + \delta \ln m_t + \beta \ln k_t \quad \textit{where} \quad \delta, \beta > 0. \tag{4}$$

The parameter δ measures how people value the transaction services real money balanced render, and β captures "love of wealth". The assumption that δ and β are positive means that the model is structurally different from the more conventional set-ups of "money-in-the-utility-function"-models without "love of wealth".

From the logarithmic utility set-up it is immediate that relative wealth, for instance, the logarithm of the ratio of individual to total (aggregate) wealth would be separable in the two concepts. If the representative individual takes total wealth as given, then both approaches, that is, working with relative or absolute wealth would not make a difference for the individual's decision and would yield similar results. As argued above I follow Plutarch here.

Let $[h_t]_{t=0}^{+\infty}$ denote the continuous time path of variable h_t and use the following definitions: $k_t \equiv (1 - z_t)\, a_t$ and $m_t \equiv z_t\, a_t$ where a_t is an indicator of the total real resources of the household, and z_t denotes the share of the real resources held in terms of real money balances. These definitions serve to facilitate the analysis, and i.a. imply

$$\begin{aligned} \varphi(c_t, m_t, k_t) &= \ln c_t + \delta \ln[z_t \cdot a_t] + \beta \ln[(1 - z_t) \cdot a_t] \\ &= \ln c_t + (\delta + \beta) \ln a_t + \delta \ln z_t + \beta \ln[(1 - z_t)]. \end{aligned} \tag{5}$$

We can then formulate the representative household's problem as the maximization of intertemporal welfare based on (5) subject to the flow budget constraint in (2). Thus, the household's problem is

$$\begin{aligned} &\max_{c_t, z_t} \int_0^{\infty} [\ln c_t + (\delta + \beta) \ln a_t + \delta \ln z_t + \beta \ln(1 - z_t)]\, e^{-\rho t} dt \\ s.t. \qquad &\dot{a}_t = [r_t\, a_t + w_t + x_t] - [c_t + (r_t + \pi_t + \sigma) z_t\, a_t]\,. \end{aligned}$$

Here consumption c_t and real money balances m_t in terms of per capita resources a_t, that is, z_t are the control variables, and a_t is the state variable. The household takes the paths of the real interest rate, the wage rate, the inflation rate and government transfers $[r_t, w_t, \pi_t, x_t]_{t=0}^{+\infty}$ and the (constant) policy parameter σ as given. Recall that

$n_t = 0, \forall t$, (no population growth) has been assumed. Furthermore, the household takes as given his initial level of real resources, a_0.

To solve the consumer's problem we set up the current-value Hamiltonian

$$\begin{aligned}\mathcal{H} &= \{\ln c_t + (\delta+\beta)\ln a_t + \delta \ln z_t + \beta \ln(1-z_t)\} \\ &\quad + \mu_t \left[r_t\, a_t + w_t + x_t - c_t - (r_t + \pi_t + \sigma) z_t\, a_t)\right]\end{aligned}$$

where μ_t is the current-value costate variable.[13] The necessary first order conditions for this maximization problem are

$$\frac{1}{c_t} - \mu_t = 0 \tag{6}$$

$$\frac{\delta}{z_t} - \frac{\beta}{1-z_t} - \mu_t \cdot a_t(r_t + \pi_t + \sigma) = 0 \tag{7}$$

$$-\left[\frac{\delta+\beta}{a_t} + r_t\, \mu_t - \mu_t(r_t + \pi_t + \sigma)\cdot z_t\right] = -\rho\mu_t + \dot{\mu}_t \tag{8}$$

where we also require that equation (2) holds (with $n_t = 0$) and the transversality condition is satisfied, i.e.

$$\lim_{t\to\infty} \mu_t \cdot a_t \cdot e^{-\rho t} = 0. \tag{9}$$

Recalling the definition of z_t with $m_t \equiv z_t \cdot a_t$ and $k_t \equiv (1-z_t)\cdot a_t$ and using equation (6) one can simplify equation (7) to

$$\begin{aligned}\frac{\delta}{z_t \cdot a_t} &= \frac{\beta}{(1-z_t)\cdot a_t} + \mu_t \cdot (r_t + \pi_t + \sigma) \\ \frac{\delta c_t}{m_t} &= \frac{\beta c_t}{k_t} + (r_t + \pi_t + \sigma) \qquad (10)\end{aligned}$$

which implicitly describes the demand for real money balances m as is shown below.[14]

The equations (6) and (8) with $z_t = m_t/a_t$ imply that

$$\frac{\dot{c}_t}{c_t} = \frac{(\delta+\beta)c_t}{a_t} + r_t - \frac{(r_t + \pi_t + \sigma)m_t}{a_t} - \rho, \tag{11}$$

[13] For what is to follow we now use subscripts, except subscript t, to denote partial derivatives.

[14] Notice that for a conventionally shaped LM curve below one has to invoke the (mild) assumption that $\delta/\beta > m/k$. See also section 3.

whereby consumption growth depends on “love of wealth” and the preference for money holdings. Unlike in conventional models the stocks of money and physical capital, which feature in a_t, bear on the growth rate of consumption. Notice that in this model there is, in general, a wedge between the real interest rate r_t and the time preference rate ρ. It is not difficult to see that in a steady state when $\dot{c}_t = 0$ the real interest rate r_t will in general not be equal to the time preference rate.

Using equation (10) where

$$\frac{\delta c_t}{m_t} - \frac{\beta c_t}{k_t} = (r_t + \pi_t + \sigma)$$

the expression for the consumption growth rate in equation (11) boils down to

$$\begin{aligned}
\frac{\dot{c}_t}{c_t} &= \frac{(\delta+\beta)c_t}{a_t} + r_t - \left(\frac{\delta c_t}{m_t} - \frac{\beta c_t}{k_t}\right)\frac{m_t}{a_t} - \rho \\
&= \frac{\beta c_t}{a_t} + \left(\frac{\beta c_t}{k_t}\right)\frac{m_t}{a_t} + r_t - \rho = \frac{\beta c_t}{a_t}\left[\frac{k_t + m_t}{k_t}\right] + r_t - \rho.
\end{aligned}$$

Thus, the growth rate of consumption is given by

$$\frac{\dot{c}_t}{c_t} = \beta\left(\frac{c_t}{k_t}\right) + r_t - \rho \tag{12}$$

which shows that “love of wealth”, i.e. β is an important determinant of the consumption growth rate. In particular, if β is zero, we are back to the conventional and simplest money-in-the-utility-model, where the economy *dichotomizes* into a real and nominal sector. This is because, if that is the case, in steady state $r_t = \rho$. But here with a β that is taken to be non-zero, consumption growth depends on how people value capital.

3 A demand-determined (short-run) equilibrium

In this section we take the representative household’s optimum to describe the microfoundations of aggregate demand. These foundations are, of course, based on the preferences postulated in assumption 1. Thus, suppose the short run is described by the demand side of the economy so that Keynes’s “Principle of Effective Demand” would hold. To fix ideas assume that the capital stock, the real money supply and prices (inflation rate) are fixed in the short run, possibly at their steady state levels. We can then conduct a simple thought experiment which is similar to a conventional, textbook-like

IS-LM analysis. In order to do this we now drop time subscripts for variables in steady state and proceed as follows.

Equation (10) describes the choice of z_t and so implicitly the choice of real money balances m, and yields the model's demand for money

$$m^d = \frac{\delta \cdot c}{i + \sigma + \beta \cdot \frac{c}{k}} \tag{13}$$

when assuming that the Fisher relation $i = r + \pi$ holds.

Notice that money demand here depends negatively on the nominal interest rate, but the latter can be zero and money would still be demanded, when β and σ are non-zero. Furthermore, it is not difficult to verify that money demand depends positively on consumption c, which reflects the transaction motive of money demand.

We follow Gesell as closely as possible below and assume that the money market is in equilibrium. To this end money supply, m^s, is taken to be exogenously determined by the monetary authority, and, importantly, taken to equal money demand m^d. For simplicity use m to convey this from now on. Thus, $m = m^s = m^d$ is assumed.

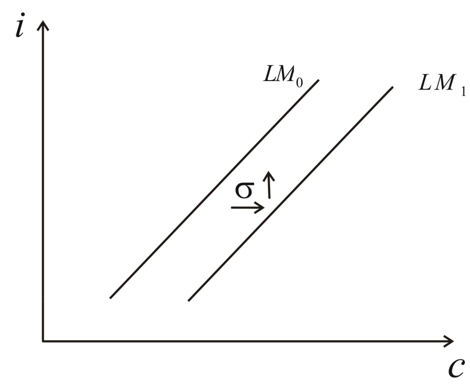
But from equation (10) one then also obtains a quasi LM-curve in consumption c and the nominal interest rate i. To get a rather conventionally shaped relationship between c and i assume that $\delta/\beta > m/k$, then the quasi-LM curve is given by

$$LM: \qquad c = (i + \sigma)\left[\frac{\delta}{m} - \frac{\beta}{k}\right]^{-1}. \tag{14}$$

Similarly to a textbook LM curve one gets $dc/di_{|LM} > 0$ and $dc/dm_{|LM} > 0$. Thus, in a $(c, i)-$space the LM has positive slope in terms of the nominal interest rate i and is shifted to the right when money supply increases. Importantly for this paper, the LM is also shifted to the right in a $(c, i)-$space if σ is increased, that is, $dc/d\sigma_{|LM}$ for a given nominal interest rate.

Result 1 (LM Curve) *Based on the household's optimality conditions, equation (14) describes a LM curve in $(c, i)-$space for a given capital stock k, and fixed money supply m and inflation rate π. It expresses consumption as a function of the nominal interest rate $i = r + \pi$, depends on real money balances m and the money depreciation rate σ. It describes equilibrium in the money market. An increase in the* Gesell *tax σ or in real money balances m shifts the LM curve to the right in a $(c, i)-$space, for a given nominal interest rate.*

Figure 1: The LM Curve



Next, we consider equation (11) which, as one should recall, is entirely based on the demand side of the economy, i.e. the households' optimality conditions. In steady state that equation reduces to

$$(\delta+\beta)\, c=(r+\pi+\sigma)m-ra+\rho a$$

and, after some manipulation, can be rearranged to yield[15]

$$IS: \qquad c=\left(\frac{1}{\delta+\beta}\right)\left[(\pi+\sigma)(m+k)-(i+\sigma)k+\rho(m+k)\right]. \tag{15}$$

This equation amounts to a quasi-IS curve that has a negative slope with respect to i in a $(c,i)-$plane. Thus, $dc/di_{|IS}<0$. Furthermore, as wealth considerations play a role, i.e. $\beta>0$, it turns out that the IS schedule also depends on real money balances. That is so, because through the introduction of preferences for wealth (physical capital) the model also implies a *Pigou* effect whereby money positively bears on (real) consumption, i.e., $dc/dm_{|IS}>0$.[16]

[15]From $(\delta+\beta)\, c=(r+\pi+\sigma)m-ra+\rho a$ one gets that

$$\begin{aligned}(\delta+\beta)\, c &= (r-r)m+(\pi+\sigma)m-rk+\rho(m+k)\\ &= (\pi+\sigma)m-rk+(\pi+\sigma)k-(\pi+\sigma)k+\rho(m+k)\\ &= (\pi+\sigma)(m+k)-(r+\pi+\sigma)k+\rho(m+k)\end{aligned}$$

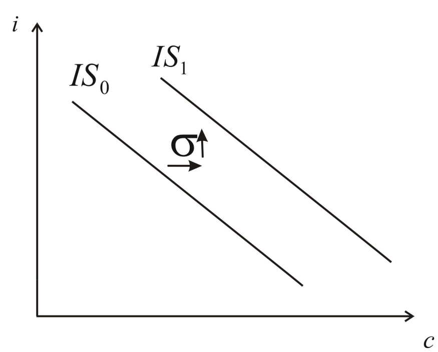
which becomes equation (15) by the Fisher relationship $i=\pi+r$.

[16]Pigou (1943) argues that output and employment can be stimulated by increasing consumption due to a rise in real money balances. Later Patinkin (1948) coined the term for this effect after Arthur Cecil

The same holds for an increase in the inflation rate π, that is, $dc/d\pi_{|IS} > 0$. One also verifies that $dc/d\sigma_{|IS} > 0$. Thus, apart from an increase in real money balances m, an increase in the *Gesell tax* (an increase in σ) also shifts the IS curve to the right - for a given nominal interest rate.

Result 2 (IS Curve) *Based on the household's optimality conditions, equation (15) describes an IS curve in $(c, i)-$space for a given capital stock k, and fixed money supply and inflation rate. It expresses consumption as a function of the nominal interest rate $i = r + \pi$, and depends on real money balances m and the money depreciation rate σ. It describes equilibrium in the goods market. The IS curve features a* Pigou *effect. An increase in real money balances or in the inflation rate raises consumption and shifts the IS curve to the right for a given i. An increase in the* Gesell Tax *shifts the IS curve to the right in a $(c, i)-$plane for a given nominal interest rate.*

Figure 2: The IS Curve



3.1 The short-run, demand-determined equilibrium

As is well known from elementary macroeconomics, the intersection of the LM and IS curves describes a short-run, demand-determined equilibrium. From now on let a variable $h = h(t)$ in short-run equilibrium be denoted by $\widehat{h}$.

Then solving equation (14) for the nominal interest rate i plus σ, inserting the result into the *IS* equation (15) and rearrangement yields the aggregate (short-run) demand

Pigou, one of the teachers of John Maynard Keynes.

for goods $\widehat{c}$ given by[17]

$$\widehat{c} = \frac{(\pi + \sigma + \rho) \cdot m}{\delta}. \tag{16}$$

Using equation (14) one verifies that the (short-run) equilibrium nominal interest rate satisfies

$$\widehat{i} = (\pi + \sigma + \rho) \left[1 - \left(\frac{m}{k} \right) \left(\frac{\beta}{\delta} \right) \right] - \sigma. \tag{17}$$

where the expression in square bracket is non-negative by assumption.

One can calculate the velocity of money as the ratio of $\widehat{c}$ to real money balances m. As both quantities are expressed relative to the price level, the velocity of money (in terms of consumption) in a short-run equilibrium is then given by

$$\widehat{\nu} \equiv \frac{\widehat{c}}{m} = \frac{(\pi + \sigma + \rho)}{\delta}, \tag{18}$$

which is clearly increasing in σ, and captures Gesell's idea that controlling the velocity of money has a direct bearing on aggregate (real) demand.

The velocity of money is usually larger than one which I assume to be the case.

Assumption 2 *The velocity of money, in terms of consumption, is taken to be larger than one, that is, $\nu > 1$ and, thus, δ to be sufficiently smaller than $\rho + \pi + \sigma$.*

---

[17]From equation (14) we get $c\ [\delta/m - \beta/k] = i + \sigma$. Substituting this in equation (15) implies

$$\begin{aligned} c\,(\delta + \beta) &= [(\pi + \sigma)(m + k) - c\,[\delta/m - \beta/k]\,k + \rho(m + k)] \\ &= (\rho + \pi + \sigma)(m + k) - c\,[\delta(k/m) - \beta] \\ c\ [(\delta + \beta) + \delta(k/m) - \beta] &= (\rho + \pi + \sigma)(m + k) \\ c\ [\delta + \delta(k/m)] &= (\rho + \pi + \sigma)(m + k) \\ c\ \delta \left[\frac{m + k}{m} \right] &= (\rho + \pi + \sigma)(m + k) \end{aligned}$$

Rearrangement then yields equation (16), that is, the expression for $\widehat{c}$.

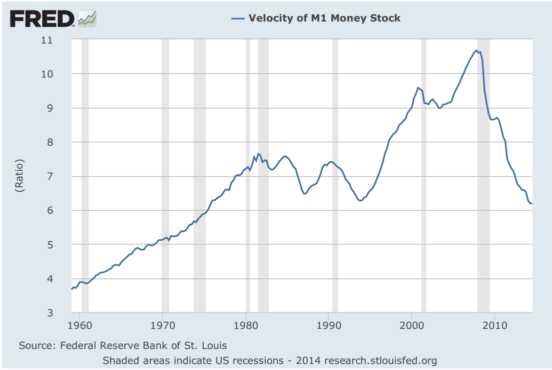
To obtain the expression for $\widehat{i}$ substitute the last expression for c_t in equation (14) to get

$$\begin{aligned} \frac{(\pi + \sigma + \rho) \cdot m}{\delta} &= (i + \sigma) \left[\frac{\delta}{m} - \frac{\beta}{k} \right]^{-1} \\ (\pi + \sigma + \rho) \cdot \frac{m}{\delta} \cdot \left[\frac{\delta}{m} - \frac{\beta}{k} \right] &= (i + \sigma). \end{aligned}$$

From this equation (17) and so the expression for $\widehat{i}$ follows in a straightforward way.

Thus, the ratio of consumption - or more conventionally GDP - to money aggregates like M_0 (base money) or M_1 is taken to be a value in excess of one. As an example consider the velocity of M_1 in the U.S. between 1960 and today.[18]

Figure 3: Velocity of M_1 in the U.S.



Source: http://research.stlouisfed.org/fred2/categories/32242

From the graph the velocity of M_1 has consistently been larger than one over the period considered. In this model M refers to M_0 (base money). It is well known that the velocity of M_0 is usually higher than the one for M_1, because $M_0 < M_1$. As aggregate consumption corresponds to roughly 60 percent of GDP in most, especially OECD countries, it is safe to say that empirically the ratio of M_0 to aggregate consumption is also larger that one. This holds no matter whether we look at the steady state or shorter time spans.

Given the expressions for a demand-determined equilibrium various comparative static investigations are then possible. As the paper's focus is on Gesell's conjectures, I concentrate on the effects on the short-run equilibrium if σ or m is changed. For now assume that the inflation rate is non-negative, that is, $\pi \geq 0$.

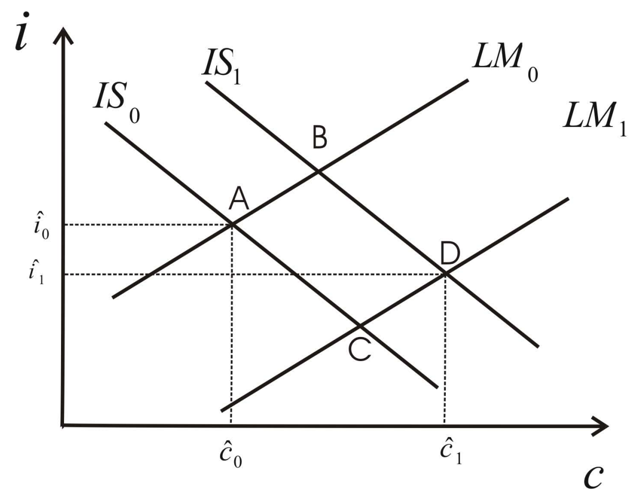
From equations (16) and (17) aggregate demand for goods (in short run-equilibrium) is increased and the short-run equilibrium nominal interest rate falls, when the Gesell tax (given real money balances) or real money balances (given money depreciation) rise. Thus, when the inflation rate is non-negative, we have

$$d\widehat{c}/d\sigma > 0, \quad d\widehat{i}/d\sigma < 0 \quad \text{and} \quad d\widehat{c}/dm > 0, \quad d\widehat{i}/dm < 0. \tag{19}$$

[18]*Money Velocity*: Velocity is a ratio of nominal GDP to a measure of the money supply (M1 or M2). It can be thought of as the rate of turnover in the money supply, that is, the number of times one dollar is used to purchase final goods and services included in GDP. Source: http://research.stlouisfed.org/fred2/categories/32242

That means the (negative) nominal (short-run equilibrium) interest rate reaction to a positive change in the Gesell tax (σ) is larger in absolute value for the LM shift than the absolute (but positive) shift in the IS curve. This follows because $di/d\sigma_{|IS} = m/k$ and $di/d\sigma_{|LM} = -1$, and by the assumption that $m < k$.

Figure 4: The Short-Run, Demand-Determined Equilibrium



Thus, if the economy's short-run equilibrium is initially at the point A, an increase in σ or in real money balances will move the LM *and* the IS curve to the right, to end up at a point like D with a higher $\widehat{c}$ *and* a lower nominal interest rate $\widehat{i}$ in the new short-run, demand-determined equilibrium.

Proposition 1 *Suppose the capital stock, prices, the inflation rate, and the transfers are fixed in the short run. Then an increase in the Gesell Tax σ, for a given nominal money supply,*

1. *increases the velocity of money $\widehat{\nu}$, and*

2. *increases short-run, aggregate consumption $\widehat{c}$, and*

3. *implies to a lower short-run nominal interest rate $\widehat{i}$*

in a (quasi-) IS-LM environment in a $(c, i)-$plane.

By similar arguments we also obtain that, for a given σ and $\pi \geq 0$, an increase in real money balances, m, increases short-run, aggregate consumption, $\widehat{c}$, and implies a lower short-run nominal interest rate, $\widehat{i}$.

So far we have ignored that the household's budget constraint, that is, equation (2) must also be satisfied. We consequently need that

$$r \cdot k + w + x - c - (\pi + \sigma) \cdot m = 0.$$

For convenience denote variables that are fixed in the short run by an upper bar.[19]

Assume that in a demand-determined (short-run) equilibrium the sum of wages and capital income equals output, called $\widehat{y}$, which equals aggregate supply. Then $\widehat{y} = r \cdot \overline{k} + w$. Given the determination of consumption by the IS-LM apparatus and in light of the budget constraint equation (2) we get

$$\widehat{c} + (\overline{\pi} + \sigma) \cdot \overline{m} - \overline{x} = \widehat{y}(r, w, \overline{k}) = r \cdot \overline{k} + w \tag{20}$$

where the left hand side denotes aggregate demand (net of fixed and given transfers $\overline{x}$) and the right hand side is a quasi aggregate supply relationship that depends on the fixed capital stock $\overline{k}$, and the factor prices r and w.

If the factor prices are taken to vary freely and are not tied to marginal productivity remuneration, but some other exogenous process that is independent of k, it is indeed possible that the left hand side of the equation, that is, aggregate demand, called ad, determines the right hand side of the equation.

Letting $ad \equiv \widehat{c} + (\overline{\pi} + \sigma) \cdot \overline{m} - \overline{x}$ denote aggregate demand, we have in a (short-run) demand-determined equilibrium that

$$ad(\sigma; \overline{m}, \overline{\pi}, \overline{x}) \equiv \widehat{c} + (\overline{\pi} + \sigma) \cdot \overline{m} - \overline{x} = \widehat{y}(r, w; \overline{k}).$$

As a consequence we can then define the following.

Definition 1 *Based on the household's optimality conditions in equations (10), (11), and (2), a short-run, demand-determined equilibrium is given when aggregate demand* $ad(\sigma; \overline{m}, \overline{\pi}, \overline{x})$ *equals aggregate output (supply),* $\widehat{y}(r, w; \overline{k})$*, for a given capital stock,*

[19]Recall that the IS-LM apparatus holds for a simultaneous equilibrium in the goods and money market. In that sense a given supply money makes it an exogenous variable for most of the analysis in this part of the paper.

given real money balances and inflation rate. For flexible factor prices r and w, the intersection of IS and LM determines aggregate demand $ad(\dots)$ and with it output $\widehat{y}(r, w; \overline{k})$ so that the equilibrium is demand-determined.

Whatever the values of the fixed variables and the parameters may be, the factor prices are able to equilibrate short-run demand and "supply" in such a world. Notice that we have not invoked the marginal productivity theory of distribution in which case the rewards would ultimately be functions of k. Instead, here we think of r and w determined by (e.g. market) forces outside the model, but still assume that they equilibrate demand and supply in the way required by the model. If that is the case, $ad(\cdot)$ indeed determines "supply" $\widehat{y}(r, w; \overline{k})$.[20]

Proposition 2 *Suppose the capital stock, output prices, the inflation rate, the transfers, and the money supply are fixed, but real factor prices are flexible in the short run. Then a short-run, demand-determined equilibrium, when the inflation rate is non-negative, is characterized by*

$$ad(\sigma; \overline{m}, \overline{\pi}, \overline{x}) = \widehat{y}(r, w; \overline{k}).$$

An increase in the Gesell Tax σ or in real money balances then increases short-run, aggregate demand, ad, and consequently short-run output and supply, $\widehat{y}$.

The properties easily follow from equation (20). From the proposition we can also deduce the following. If σ rises, it follows from Proposition 1 that the (short-run) equilibrium nominal interest rate $\widehat{i}$ falls. If the inflation rate is fixed in the short run, then the real interest rate r would have to fall. This follows from the Fisher relation $i = r + \pi$. If we assume that the factor prices are free to move in the short run, then Proposition 2 implies that the wage rate w must rise when σ increases. Thus, a higher σ implies a lower r, for a given $\overline{\pi}$, and higher ad so a higher $\widehat{y}$ and a higher w. Hence, for a given capital stock, labour input and inflation rate, the wage earners would benefit from an increase in the *Gesell* tax.

Corollary 1 *For fixed capital, labour input and inflation rate, the wage earners may benefit from an increase in the* Gesell *tax or in real money balances in the short-run,*

[20]It is interesting to note that there may be many different combinations of w and r that can equilibrate ad and $\widehat{y}$. Hence, under the assumptions made many different distributional arrangements for the rewards to capital and labour are feasible, and so the income distribution would in general not be determinate.

demand-determined equilibrium environment. The capital owners may earn less in such an environment.

Of course, that begs the question if the factor prices are really more flexible than output prices, which determine the inflation rate π. Clearly, this distributional implication may not hold if the inflation rate is not fixed in the short run.

Lastly, the welfare implications in the short-run, demand determined environment are considered. Clearly, if the money supply and capital stock are fixed in the short run, period (short-run) welfare from equation (4) is given by

$$\widehat{\varphi}(\widehat{c}, \overline{m}, \overline{k}) = \ln \widehat{c} + \delta \ln \overline{m} + \beta \ln \overline{k}.$$

But then one easily verifies that $d\widehat{\varphi}/d\sigma = (d\widehat{c}/d\sigma)/\widehat{c} > 0$, because $d\widehat{c}/d\sigma > 0$. Thus, period welfare would rise with an increase in σ.

Proposition 3 *Suppose the capital stock, the inflation rate, transfers, and money supply are fixed, but real factor prices are flexible in the short run. Then a short-run, demand-determined equilibrium is characterized by period welfare*

$$\widehat{\varphi}(\widehat{c}, \overline{m}, \overline{k}) = \ln \widehat{c} + \delta \ln \overline{m} + \beta \ln \overline{k} \quad \textit{with} \quad \frac{d\widehat{\varphi}}{d\sigma} = \frac{d\widehat{c}/d\sigma}{\widehat{c}} > 0, \frac{d\widehat{\varphi}}{d\overline{m}} = \frac{d\widehat{c}/d\overline{m}}{\widehat{c}} + \frac{\delta}{\overline{m}} > 0$$

that is, period welfare is higher, when the Gesell *tax or real money balances are higher in a (short-run) demand-determined equilibrium.*

The most interesting implication of the propositions for the short run is that Gesell's conjectures are true in the environment developed in this section. Thus,

Theorem 1 *In a short-run, demand-determined equilibrium where the capital stock, the inflation rate, transfers, and the money supply are fixed, but real factor prices are flexible and the inflation rate is non-negative, Gesell's hypotheses GC1 - GC4 are all generically valid, given the (demand) micro-foundations in equations (2), (4), (6), (7), (8), and (9), and given that the microfoundations feature direct utility derived from money and "love of wealth" where physical capital is considered to be the true source of wealth.*

This result is striking and in contrast to some contributions in the literature. Clearly, the theorem is based on the non-implausible assumptions invoked here. Notice that the theorem is about the short run. However, Gesell's ideas have occupied the imagination of researchers and policy makers alike in the years right after the Great Recession. It has been and, somehow still, is being felt that money depreciation may be one way out of important crisis problems, in the short and in the longer run.

3.2 Liquidity trap and the zero lower bound on nominal interest rates

Recently, it has been an important question what monetary policy can accomplish, if the nominal interest rate is at its zero lower bound, that is, if it takes on a value close to zero. As mentioned above there has been renewed interest in Gesell's ideas. In order to shed some led onto why Gesell's ideas may be relevant in the current situation consider money demand and short-run equilibrium again.[21]

Money Demand Conditions

Consider a situation where the nominal interest rate is at the zero lower bound. Let us again concentrate on equation (7), which describes the optimal choice (demand) of money holdings of the private sector. For simplicity continue to use m_t to denote real money balances demanded and supplied. Then

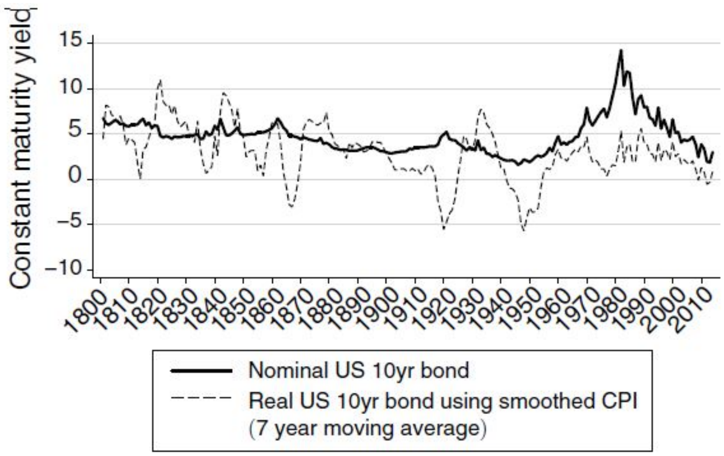
$$\frac{\delta}{z_t} - \frac{\beta}{1-z_t} - \mu_t \cdot a_t(r_t + \pi_t + \sigma) = 0. \tag{7}$$

So far we have concentrated on an interior solution implying that the equation above is satisfied as an equality. Suppose that that is not the case. In particular, suppose that the nominal interest rate is at its zero lower bound with $i_t = r_t + \pi_t = 0$.

By implication the real interest rate r_t, the inflation rate π_t or both might in principle be negative. But in the short-run equilibrium the inflation rate is (exogenously) given by assumption so we take the real interest rate r_t to adjust when $i_t = 0$. Thus, the real interest rate may be negative. There is, for example, evidence for the U.S. that negative real interest rates are far from unrealistic as is shown e.g. by Eichengreen (2015), Figure 1, which I represent here for convenience.

[21] The following analysis is also interesting for another reason. Gesell advocated a "free money" and "free land" economy. For those the interest rate would eventually have to abolished and any form of credit would be free of interest according to his *utopia*.

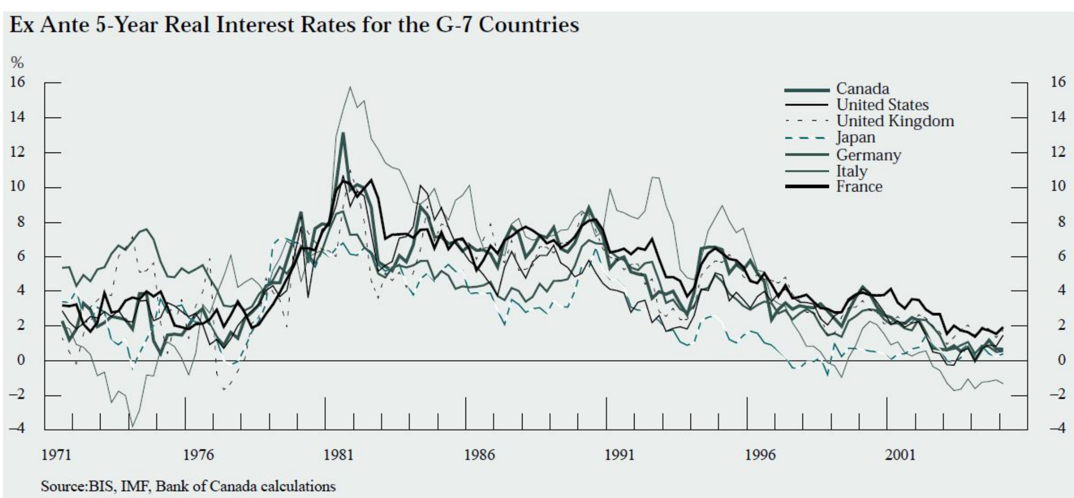
Figure 5: Long-Run US Interest Rates



Source: Eichengreen (2015), p. 66, Figure 1

More evidence for a range of countries can also be found in Desroches and Francis (2006-2007), which I also re-render here.[22]

Figure 6: Interest Rates for the G-7 Countries



Source: Desroches and Francis (2006-2007), p. 4, Box 1, Chart B1

Now, for the ensuing analysis recall that $\mu_t = 1/c_t$ and $a_t = k_t + m_t$ where in this section now $m_t = m_t^d$. We can then investigate various cases.

[22]Some more recent evidence for the G-7 countries is provided by Yi and Zhang (2017), Figure 1.

Case 1: Suppose $i = 0$, $\sigma = 0$ and $\beta = 0$. Then the left hand side of equation (7) becomes $\frac{\delta}{z_t} > 0$ so that $z_t \to 1$ is optimal. Given that $m_t = z_t\, a_t = z_t\, (k_t + m_t)$ we need that $m_t \to \infty$ for $m_t/(k_t + m_t) \to 1$. Thus, people would demand an infinite amount of money balances and hoard cash. This is the conventional result following from the Sidrauski model. The common explanation is that in a situation where the opportunity cost of holding money is nil, people would hold all their resources in the form of real money balances. That is usually associated with the notion of a "liquidity trap".[23]

Case 2: Suppose $i = 0$, $\sigma = 0$ and $\beta > 0$. Then equation (7) may yield an interior solution satisfying

$$\frac{\delta}{z_t} = \frac{\beta}{1 - z_t} \Leftrightarrow \frac{m_t}{m_t + k_t} = \frac{\delta}{\beta + \delta} \Leftrightarrow \frac{k_t}{m_t} = \frac{\beta}{\delta}.$$

The important implication here is that $z_t < 1$ is optimal and so the presence of a "love of wealth"-motive (β) makes a liquidity trap less likely. That should be clear from the motive itself. If people value (physical) capital they will not try to get rid of all their capital in order to hoard only cash.

In fact, pushing the argument further reveals that when the "love of wealth"-motive is extremely strong ($\beta \to \infty$) then people would want to get rid of all their money balances and only hold physical capital k_t.[24] This seems to be not too unrealistic in view of the flight into real assets, that is, assets other than money which has been observed in many economies in the aftermaths of the Great Recession.[25]

Case 3: Suppose $i = 0, \sigma > 0, \beta > 0$. Then equation (7) must satisfy

$$\frac{\delta}{z_t} - \frac{\beta}{1 - z_t} - \left(\frac{k_t + m_t}{c_t}\right) \cdot \sigma = 0$$

[23]The term and concept of a "liquidity trap" was well known by British economists before Keynes's publication of the "General Theory of Employment, Money and Interest", who actually never used the term himself. For details on that and some clarifications on misconceptions in current discourse on the phenomenon of a "liquidity trap" see Barens (2011).

[24]This may be the case in an interior equilibrium with $z_t = \frac{\delta}{\beta+\delta}$ or as a boundary solution $\frac{\delta}{z_t} - \frac{\beta}{1-z_t} < 0$ of equation (7) when $\beta \to \infty$. Recently it has been argued that such a flight into real assets happened in the period where the nominal interest rate has indeed been at the zero lower bound. Thus, the model may provide a micro-founded explanation for this behaviour.

[25]It should be borne in mind, though, that this only holds if the money balances demanded actually satisfy the condition $k_t/m_t = \beta/\delta$ where by assumption k_t is fixed in the short run, that is, $k_t = \overline{k}$.

which also implies an optimal z_t less than one.

When one takes the total differential of the left hand side with respect to z_t and σ one obtains that $dz_t/d\sigma < 0$. Thus, a higher σ lowers the ratio $m_t/(k_t + m_t) = z_t$, and so either k_t is higher or m_t lower than the optimal z_t in case 2. Notice also that one gets a form of a (degenerate) LM-curve despite the fact that the nominal interest rate is at its lower bound, i.e. $i_t = 0$.[26]

The upshot of that is that the *Gesell* tax may stimulate investment in assets other than money, at least from the consumer's perspective and when the nominal interest rate is at its zero lower bound. In that sense the introduction of such a tax may stimulate investments in physical assets, especially if the "love of wealth" motive is not strong.

Proposition 4 *Suppose the short-run, nominal interest rate is at the zero lower bound $i_t = 0$. Then the household's optimality conditions for money balances demanded imply the following.*

When $\sigma = 0$, $i_t = 0$ and $\beta = 0$ the households all hoard cash when $\sigma = 0$ and $\beta = 0$, which corresponds to a "liquidity trap".

With "love of wealth" ($\beta > 0$) an interior solution is possible where there is no hoarding of cash and households hold money and physical capital.

If "love of wealth" is very strong (β is extremely large), people may move all their investments into physical capital and will get rid of all their money holdings.

The same effect may hold for a sufficiently large Gesell *tax σ.*

Implication I

Notice that there are combinations for the monetary policy variables m and σ so that the short-run nominal interest rate is indeed zero in a short-run equilibrium. Then $i = \widehat{i} = 0$ in equation (17) implies that

$$(\pi + \sigma + \rho)\left((1 - \left(\frac{m}{k}\right)\left(\frac{\beta}{\delta}\right)\right) = \sigma \tag{21}$$

must hold in the general case where β and σ are non-zero. That requires particular combinations for the monetary policy variables m^s and σ to sustain a zero, short-run equilibrium nominal interest rate. With that in mind we now analyze the consequences for consumption and other real variables based on the cases considered above.

[26]Clearly, from equation (14) the LM curve is degenerate in this case, but there still is a money demand equation as can be gleaned from equation (13).

Case 1: If $\widehat{i} = 0$, $\beta = 0$, and $\sigma = 0$, money demand is infinite, not well-defined and the LM-curve is a flat line. However, the model features a Pigou effect, the IS curve can, therefore, be shifted to the right, that is, consumption can be increased when the money supply m^s is increased. In that sense monetary policy can be used to stimulate real demand and activity, even though $\beta = \sigma = 0$ and π is given. However, the demand for money will always be larger than the supply of it. Consequently, there is no equilibrium in the money market. One may argue that that puts pressure on prices and may cause inflation to rise. These results are not very surprising.

Case 2: If $\widehat{i} = 0$, $\beta > 0$, and $\sigma = 0$, then by equations (7), (16) and (17) as well as concentrating on an interior solution, and under the assumption that the money supply m^s equals money demand m^d, we know that $m = \overline{k} \cdot (\delta/\beta)$ would have to hold. Based on that it is not difficult to verify that

$$\widehat{c}_{|\sigma=0,\widehat{i}=0} = \frac{(\overline{\pi} + \rho)\overline{k}}{\beta} = \frac{(\overline{\pi} + \rho)m}{\delta}.$$

For a given capital stock and inflation rate, consumption can then not be stimulated by monetary policy. Furthermore, in an interior money market equilibrium the money balances, and especially money supply, must satisfy $m = m^s = m^d = \overline{k} \cdot (\delta/\beta)$.

Case 3: If $\widehat{i} = 0$, $\sigma > 0$, and $\beta > 0$, we can substitute for $(\pi + \sigma + \rho)$ from equation (21) in equation (16) to obtain

$$\widehat{c}_{|\sigma>0,\widehat{i}=0} = m \cdot \sigma \left[\delta - \frac{m}{k} \cdot \beta\right]^{-1}.$$

Given that only certain combinations of m^s and σ sustain $\widehat{i} = 0$, it is an interesting question whether these combinations have any real effects. It turns out that $dm/m = -\left[(\sigma/(\pi + \rho + \sigma)\right] d\sigma/\sigma$ must hold when $\widehat{i} = 0$. See Appendix B. Thus, for example, a one-percent-increase in σ requires a corresponding $[(\sigma/(\pi + \rho + \sigma)]$ percent decrease in money supply m^s to keep $\widehat{i}$ at the zero lower bound.

In the appendix it is then shown that the introduction of money depreciation coupled with a corresponding lower money stock when $\widehat{i} = 0$ does not bear on consumption in equilibrium when the economy's interest rate is at the zero lower bound.

However, aggregate demand may change. Recall that $ad(\sigma; m, \pi, x) = \widehat{c} + (\overline{\pi} + \sigma)m_1 - \overline{x}$ when $\sigma > 0$ with $m_1 < m_0$, where m_0 denotes the money balances when

$\sigma = 0$. A higher σ does not imply a higher $\widehat{c}$, but it implies a larger $(\overline{\pi} + \sigma)m_1$ by the following arguments.

Taking logarithms one obtains $\ln(\overline{\pi}+\sigma)+\ln m_1$. The differential for this is $d\sigma/(\pi+\sigma) + dm_1/m_1$. For this expression to be positive it must be that

$$\left(\frac{\sigma}{\pi+\sigma}\right)\frac{d\sigma}{\sigma} + \frac{dm_1}{m_1} > 0$$

where $dm_1/m_1 = -(d\sigma)/\sigma \cdot \sigma/(\pi + \rho + \sigma)$. Making the substitution in the last inequality and rearranging implies for a positive change in σ that

$$\left[\frac{\sigma}{\pi+\sigma} - \frac{\sigma}{\pi+\rho+\sigma}\right]\frac{d\sigma}{\sigma} > 0$$

must hold. Indeed it does, because the expression in square brackets is positive. But then $d(ad)/d\sigma > 0$, which implies a higher $\widehat{y}(r, w; \overline{k})$.

By the Fisher relation we have $\widehat{i} = \pi + r = 0$, that is, $\pi = -r$ when the nominal interest is at the zero lower bound for a given inflation rate. This ties down the (short-run equilibrium) real interest rate. Hence, an increase in σ implies an increase in the wage rate w. In that sense the introduction of money depreciation has important distributional consequences when the nominal interest rate is at the zero lower bound.

Proposition 5 *If the short-run, nominal interest rate is at the zero lower bound, then a positive* Gesell *tax σ must be matched by a corresponding lower money supply to maintain $\widehat{i}_t = 0$ and does not have effects on real equilibrium consumption, but has positive effects on overall aggregate demand. In a short-run demand-determined equilibrium a higher σ with a corresponding lower m implies a relatively higher aggregate demand with a higher wage rate w and an unaltered real interest rate r.*

3.3 Overcoming the zero lower bound on nominal interest rates

It has recently been argued that a way out of the zero-lower-bound-problem is to reduce the nominal interest rates to negative values. From the analysis up to now that is a trivial consequence of the model.

Equation (19) tells us that an increase in m^s or in σ would increase real consumption (increase the demand for goods) and lower the nominal interest rate, given that the inflation rate is non-negative. But the latter may not always be the case. In particular in the aftermaths of the Great Recession it was feared, and sometimes observed, that there was deflation, $\pi < 0$.

Thus, reconsider the equations (16), (17) and (19) which are given by

$$d\widehat{c}/d\sigma = \frac{m}{\delta} > 0 \quad , \qquad d\widehat{i}/d\sigma = -\left(\frac{m}{k}\right)\left(\frac{\beta}{\delta}\right) < 0$$
$$d\widehat{c}/dm = \frac{\pi + \rho + \sigma}{\delta} \quad , \qquad d\widehat{i}/dm = -(\pi + \sigma + \rho)\left[\left(\frac{1}{k}\right)\left(\frac{\beta}{\delta}\right)\right].$$

One easily verifies that if deflation, $\pi < 0$, is strong so that $(\pi + \rho + \sigma) < 0$, then a change in m would produce an effect that may be unwanted, namely it would decrease consumption and raise the interest rate. This policy option may not be that attractive, especially if the monetary authority is not certain how strong deflation really is.

Thus, the other monetary policy instrument namely σ may be the more attractive to use, because an increase in money depreciation unambiguously raises consumption, aggregate demand by equation (20) and lowers the nominal interest rate in a short-run equilibrium, irrespective of what the inflation rate is.

If the equilibrium interest rate $\widehat{i}$ falls, then the real interest must fall too, when the inflation rate, be it positive or negative, is given. Thus, the equilibrium condition then implies that the wage rate must increase. Hence, again, this policy is good for the workers' income.

Proposition 6 *Suppose the short-run, nominal interest rate is at the zero lower bound $\widehat{i} = 0$. Then, irrespective of the inflation rate π, an increase in the* Gesell *tax, σ, makes the short-run, nominal interest rate negative, $\widehat{i} < 0$, increases equilibrium consumption, $\widehat{c}$ and aggregate demand, ad. The real interest rate falls and the wage rate increases for equilibrium to hold.*

Thus, the model highlights arguments brought forward to combat the liquidity-trap-situation that most economists agree has been around in recent years in, for example, the Unites States, Europe and Japan, in order to stimulate the real activity of the economy and raise welfare. Whether in reality the effects on the factor income distribution are as predicted by the model here, cannot easily be ascertained. This is so, because direct money depreciation was not used as a policy instrument.

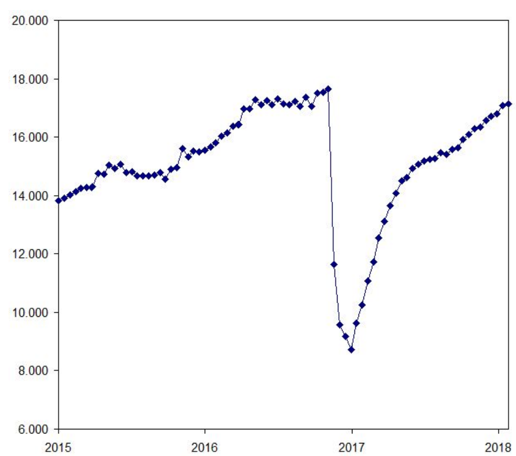
Lastly one easily figures out the (similar) effects of changes in m on the short-run equilibrium and the factor income distribution. However, it is still an unresolved empirical question whether labour has really benefitted more than capital when the economy is being pulled out of a "liquidity trap".

3.4 Demonetization

The model also allows for predictions on the (short-run) effects of demonetization. For example, in the recent demonetization episode in India, 500 and 1000 rupee notes (INR) were declared invalid in a surprise move communicated on television on 8 November 2016 by the Indian Prime Minister Narendra Modi. The Reserve Bank of India (RBI) set a period of fifty days until 30 December 2016 to deposit the demonetized banknotes as credit in bank accounts.[27]

The policy objective of the measure was to combat corruption, black and counterfeit money as well as terror financing. The demonetization affected 86.9 percent of the value of total currency in circulation. Note that currency (cash) is by far the most important medium of monetary exchange in India.[28]

Figure 7: Demonetization in India 2016



Currency in Circulation | Billion Rupees | Weekly | last date: 2 Feb. 2018

Source: Reserve Bank of India Data Warehouse

[27] The banknotes could also be exchanged at bank branches up to a limit that varied over the days. Initially, the limit was 4,000 INR per person from 8 to 13 November, then 4,500 INR per person from 14 to 17 November, and in the end 2,000 INR per person from 18 November. All exchange of banknotes was abruptly actually stopped from 25 November 2016 onwards.

[28] For example, some figures suggest that 99 percent of consumer transactions in India are carried out in cash and that the currency to GDP ratio is very high, but not the highest in international comparison. For instance, Japan's ratio is higher than India's. For that reason it is a natural experiment with empirical consequences that allows for thinking about the this issue using economic theory.

Demonetization can reasonably be identified with a reduction in the circulation of (base) money, that is, a decrease in m^s.

From the theoretical arguments above it follows what one should expect in this case.[29] When, as in India for years before and after 2016, the inflation rate is positive, lower m^s implies reduced consumption (lower real demand for goods), $\widehat{c}$, and generally a decrease in aggregate demand, $\widehat{ad}$, coupled with a higher nominal interest rate, $\widehat{i}$, in a short-run equilibrium. Through the Fisher relation and for a given inflation rate, this means that the real interest rate $\widehat{r}$ increases, but the real wage rate, $\widehat{w}$, decreases.

Proposition 7 *The short-run effects of a demonetization that decreases base money for a given inflation rate, lowers consumption and aggregate demand. It implies higher nominal and real interest rates, but a lower wage rate. It would benefit capital owners relatively more than workers.*

The proposition captures very interesting aspects of (surprise) demonetization on an economy. But, of course, the model is too coarse to capture all ramifications of a policy measure that has had consequences on an economy as large as the Indian one.

A first, quasi-official assessment is provided by of the Reserve Bank of India's (RBI's) Monetary Policy Department (MPD) (2016). According to that there were important short-run negative effects after the policy announcement on some key sectors of the Indian economy, namely organized manufacturing (e.g. less vehicle, including three-wheelers sales, see Table 1), a drop in investment proposals, construction and other sectors. See p. 2-15 in that report.

As regards the effects on the nominal interest rate, the model may not do so well. Most evidence in the report shows that after the Indian demonetization move nominal interest rates (for most financial markets) fell. That would contradict the theoretical prediction. Of course there are many indicators of "the" nominal interest rate in reality.

A drop in investment proposals may also be due to higher interest rates for business credits. Furthermore, stock market indices for India showed declines in sectoral performances, especially for the realty (property) sector after the policy move. See Table 16, p. 30. That might be an indication of lower nominal interest rates as well. However, the evidence for the other financial market is not so clear.

[29]Notice that a conventional IS-LM model would yield similar results. But as can be gleaned from figure 4 the effects on consumption would be stronger and the effects on the nominal interest rate smaller in the present set-up than in a textbook IS-LM model.

But, as can be inferred from Figure 3, demonetization was quickly followed by remonetization, that is, by injections for new 500 and 2000 INR notes into the economy. As a consequence most of the negative effects abided and the report of the Monetary Policy Department (MPD) (2016) concludes that, all in all, the negative effects were "modest" over the short time span from November 2016 to February 2017.

Importantly, remonetization is running in the opposite direction of demonetization. Thus, we would expect the opposite of the short-run effects captured by Proposition 7.

Finally, it ought to be recognized that it is still an unresolved issue whether the policy objective to combat non-legal activities and transactions was successfully achieved by the Indian demonetization episode.

4 The long-run general equilibrium

We now focus on the more conventional approach to let supply and demand forces interact equilibratingly with each other at each point in time. This changes some of the previous insights in important ways. In particular, we now let the factor markets be determined by marginal productivity considerations. It turns out that depreciation of money has important implications for the accumulation of physical capital and the long-run position of, that is, the steady state of the economy.

In order to close the model for the long-run equilibrium we now put structure on policy. To that end assume that the new issuance of money $\dot{M}_t$ depends on a constant fraction θ of the outstanding stock of nominal money M_t, plus the cost to be borne by replacing the "rotten" money due to "rusting", that is, $\sigma\, M_t$. In this paper θ and σ are (constant) policy variables of the monetary authority.

Thus, $\dot{M}_t = \theta M_t + \sigma M_t$ and so the (gross) *issuance of money* is $(\theta + \sigma) M_t$. Letting $d\left(M_t/P_t\right)/dt \equiv \dot{m}_t$ yields

$$\dot{m}_t = \frac{\dot{M}_t}{P_t} - \left(\frac{\dot{P}_t}{P_t}\right)\left(\frac{M_t}{P_t}\right).$$

As $m_t \equiv M_t/P_t$ one obtains $\dot{M}_t/P_t = \dot{m}_t + \pi_t m_t$ where $\pi_t \equiv \dot{P}_t/P_t$. But then $\frac{\dot{M}_t}{P_t} = \frac{(\theta+\sigma)M_t}{P_t} = (\theta + \sigma)\, m_t$ so that real money balances change according to

$$\dot{m}_t = (\theta + \sigma - \pi_t)\, m_t. \tag{22}$$

Thus, (real) money growth is determined by $\theta + \sigma - \pi_t$, where θ and σ are controlled by the government. Furthermore, the monetary authority raises seigniorage by its *issuance of money* $(\theta+\sigma)M_t$ which, in this representative agent economy, is rebated lump-sum and in real terms to the household. Thus, $x_t = (\theta + \sigma)m_t$.

In order to obtain clear-cut results the analysis is now restricted to satisfy the following criteria.

Assumption 3 *The aggregate technology is Cobb-Douglas and given by* $Y = F(K, N) = K^{\alpha}N^{1-\alpha}$ *where* $0 < \alpha < 1$. *Thus,* $y = f(k) = k^{\alpha}$ *where* $y = Y/N$ *and* $k = K/N$.

Assumption 4 *Firms are price takers and maximize profits.*

These assumptions form the basis for the marginal productivity theory of factor remuneration to hold in the ensuing analysis.

Definition 2 *A long-run general equilibrium consists of paths for consumption, real money balances, nominal money balances, the physical capital stock, the nominal interest rate, the real interest rate, the inflation rate, and government transfers* $[c_t, m_t, M_t, k_t, i_t, r_t, \pi_t, x_t]_{t=0}^{+\infty}$ *such that*

1. *money demand is described by equation (7);*
2. *investment decisions satisfy equation (8);*
3. *the transversality (9) condition is satisfied;*
4. *households obey their budget constraints (2)*
5. *real money balances supplied evolve according to* $\dot{m}_t = (\theta + \sigma - \pi_t)m_t$, *where* θ *and* σ *are determined by the monetary authority;*
6. *the lump-sum transfers to the household are equal to the seigniorage from money issue so that* $x_t = (\theta + \sigma)m_t$;
7. *production uses a constant returns to scale technology, and firms maximize profits where* $w_t = f(k_t) - r_t k_t$ *and* $r_t = f'(k_t)$ *and the factor markets clear;*
8. *the money and asset markets clear;*
9. *prices are flexible and monetary policy works, i.e. the money supply is exogenous.*

We continue to focus on the factor rewards to labour, w_t, and capital, r_t to measure distribution. This is compatible with Gesell's reasoning as can reasonably be inferred from his writings.

It is important to notice that now the marginal productivity theory of distribution holds (Definition 2, point 7). That rules out negative real interest in any long-run equilibrium, i.e. a situation when market eventually clear. As the present model is of the standard variety, this also holds at any point in time given concurrent optimizing behaviour of the agents. It is not clear whether Gesell was thinking along those lines, though. But here we are interested in the validity of his hypotheses in the light of contemporaneous modelling.

The general equilibrium satisfying the definition above is characterized by a system of one static and three differential equations. In particular, the equilibrium is described by the differential equation (22)

$$\frac{\dot{m}_t}{m_t} = \theta + \sigma - \pi_t$$

where $\theta + \sigma = \pi$ in steady state, and the differential equation equation (12)

$$\frac{\dot{c}_t}{c_t} = \beta\left(\frac{c_t}{k_t}\right) + r_t - \rho,$$

as well as the dynamic budget constraint in equation (2) with $n_t = 0$,

$$\begin{aligned} \dot{a}_t &= [r_t \cdot a_t + w_t + x_t] - [c_t + (r_t + \pi_t + \sigma) \cdot m_t] \quad \text{and } a_t \equiv k_t + m_t \\ \dot{k}_t + \dot{m}_t &= [r_t \cdot (k_t + m_t) + w_t + x_t] - [c_t + (r_t + \pi_t + \sigma) \cdot m_t] \end{aligned}$$

In equilibrium seigniorage revenue paid out to the household is $x_t = (\theta_t + \sigma)\, m_t$. Furthermore, $\dot{m}_t = (\theta_t + \sigma - \pi_t)\, m_t$ as well as $r_t k_t + w_t = y_t = f(k_t) = k_t^{\alpha}$ hold in equilibrium. Thus, the last dynamic equation becomes

$$\begin{aligned} \dot{k}_t + (\theta_t + \sigma - \pi_t) \cdot m_t &= [r_t \cdot (k_t + m_t) + w_t + (\theta_t + \sigma) \cdot m_t] - [c_t + (r_t + \pi_t + \sigma) \cdot m_t] \\ \frac{\dot{k}_t}{k_t} &= \frac{f(k_t)}{k_t} - \frac{c_t}{k_t} - \sigma \cdot \frac{m_t}{k_t}. \end{aligned}$$

Finally, the static optimality condition in equation (10) requires

$$\frac{\delta c_t}{m_t} = \frac{\beta c_t}{k_t} + (r_t + \pi_t + \sigma) \quad \Leftrightarrow \quad \pi_t = \frac{\delta c_t}{m_t} - \frac{\beta c_t}{k_t} - (r_t + \sigma). \tag{23}$$

We drop the time subscript from now on when it is clear that a variable depends on time, and index variables in steady state by *.

If we substitute the expression of π from the last equation into the expression for the growth rate of real money balances one verifies that the equilibrium is characterized by a system of three dynamic equations.

$$\begin{aligned}
\frac{\dot{k}}{k} &= \frac{f(k)}{k} - \frac{c}{k} - \sigma \cdot \frac{m}{k}, && \text{(24a)}\\
\frac{\dot{c}}{c} &= \beta\left(\frac{c}{k}\right) + r - \rho, && \text{(24b)}\\
\frac{\dot{m}}{m} &= \theta + \sigma - \frac{\delta c}{m} + \frac{\beta c}{k} + (r + \sigma) && \text{(24c)}
\end{aligned}$$

where $f(k) = k^\alpha$ and $r = f'(k) = \alpha k^{\alpha-1}$.

Consequently the steady state where $\dot{k} = \dot{m} = \dot{c} = 0$ is given by $\pi^* = \theta + \sigma$ and

$$\begin{aligned}
f(k^*) &= c^* + \sigma \cdot m^*, && \text{(25a)}\\
\beta\left(\frac{c^*}{k^*}\right) &= \rho - r^*, && \text{(25b)}\\
\frac{\delta c^*}{m^*} &= \frac{\beta c^*}{k^*} + (r^* + \theta + 2\sigma). && \text{(25c)}
\end{aligned}$$

Clearly, equation (25b) only makes sense if $\rho > r^*$, that is, when there is "love of wealth", and so $\beta > 0$. As the rate of time preference is an essentially unobservable variable, assume that indeed $\rho > r^*$. Below it will be shown that ρ may not have to assume extremely unreasonable values to satisfy the condition. See footnote 34.

4.1 Steady State Analysis

Now we turn to the comparative static properties of the steady state. One readily verifies that

$$\pi^* = \theta + \sigma.$$

Hence, with flexible prices π adjusts so that the equality holds in a steady state. That means inflation is determined by the money growth rate θ and the money depreciation rate σ, which are constant and under the control of the monetary authority.

From equation (25a) we get

$$\sigma m^* + c^* = f(k^*) = y^* = w^* + r^* k^* \tag{26}$$

because the technology features constant returns to scale so that factor payments exhaust output.

The left hand side of this equation can be interpreted as expenditures and the right hand side the income of the household in steady state. Again note that apart from expenditures on real consumption the household must also 'buy' stamps for maintaining the face value of money. That outlay is captured by the amount σm^*.

From equation (25c) we then get that

$$\frac{\delta \cdot c^*}{m^*} = \frac{\beta \cdot c^*}{k^*} + (r^* + \pi^* + \sigma) \quad \text{i.e.} \quad \frac{k^*}{m^*} = \frac{\beta}{\delta} + \frac{(r^* + \pi^* + \sigma) \cdot k^*}{c^* \cdot \delta}, \tag{27}$$

and from equation (25b) it follows that in steady state

$$\frac{c^*}{k^*} = \frac{\rho - r^*}{\beta}.$$

Substituting the last expression in equation (27) and rearranging implies

$$m^* = \left(\frac{\delta k^*}{\beta}\right)\left[\frac{\rho - r^*}{\rho + \theta + 2\sigma}\right] = \frac{\delta c^*}{\rho + \theta + 2\sigma} \tag{28}$$

which captures the *demand for real money balances* in steady state.

From the budget constraint in steady state (26) we have $c^*/k^* = y^*/k^* - \sigma \cdot m^*/k^*$. Substituting for m^*/k^* from equation (27) yields

$$\frac{c^*}{k^*} = \frac{y^*}{k^*} - \sigma \cdot \left[\frac{\beta}{\delta} + \frac{(r^* + \pi^* + \sigma) \cdot k^*}{c^* \cdot \delta}\right]^{-1}.$$

Now invoke $y/k = k^\alpha/k = k^{\alpha-1} = \alpha/\alpha \cdot k^{\alpha-1} = r/\alpha$, which holds at any point in time, plus the result that $c^*/k^* = (\rho - r^*)/\beta$. Then we get

$$\begin{aligned}
\frac{\rho - r^*}{\beta} &= \frac{r^*}{\alpha} - \sigma \cdot \left[\frac{\beta}{\delta} + \frac{(r^* + \pi^* + \sigma)}{\delta} \cdot \frac{\beta}{\rho - r^*}\right]^{-1} \\
\frac{\rho - r^*}{\beta}\left[\frac{\beta}{\delta} + \frac{(r^* + \pi^* + \sigma)}{\delta} \cdot \frac{\beta}{\rho - r^*}\right] &= \frac{r^*}{\alpha}\left[\frac{\beta}{\delta} + \frac{(r^* + \pi^* + \sigma)}{\delta} \cdot \frac{\beta}{\rho - r^*}\right] - \sigma \\
\frac{\rho - r^*}{\delta} + \frac{r^* + \pi^* + \sigma}{\delta} + \sigma &= \frac{r^*}{\alpha}\left[\frac{\beta}{\delta} + \frac{(r^* + \pi^* + \sigma)}{\delta} \cdot \frac{\beta}{\rho - r^*}\right] \\
\rho + \pi^* + \sigma + \delta\sigma &= \frac{r^*}{\alpha}\left[\beta + (r^* + \pi^* + \sigma) \cdot \frac{\beta}{\rho - r^*}\right].
\end{aligned}$$

The last equation can be rearranged to yield

$$\rho + \pi^* + (1 + \delta) \cdot \sigma = \frac{r^*}{\alpha} \cdot \beta \cdot \left[\frac{\rho - r^* + r^* + \pi^* + \sigma}{\rho - r^*}\right] = \frac{r^*}{\alpha} \cdot \frac{\beta}{\rho - r^*} \cdot [\rho + \pi^* + \sigma].$$

For convenience rearrange the last expression to obtain

$$\Delta = \beta \quad \text{where} \quad \Delta \equiv \left(1 + \frac{\delta\sigma}{\rho + \pi^* + \sigma}\right) \cdot \frac{\rho - r^*}{r^*} \cdot \alpha \quad \text{and} \quad \pi^* = \theta + \sigma, \tag{29}$$

which implicitly defines the capital stock in steady state, that is, k^*, as a function the model's parameters, that is, $k^* = k^*(\sigma, \beta, \delta, \rho, \theta, \alpha)$.

From that we obtain an important result. If $\sigma = 0$, then k^* would be independent of monetary variables and the model would dichotomize into a monetary and real sector. To see this consider equation (29) to find that k^* would then be independent of θ and σ. Furthermore, given that, c^* and y^* would also be independent of σ and θ.[30]

In contrast, if σ is non-zero, then one easily verifies that the steady state capital stock depends on the money growth rate θ and the money depreciation rate σ. Thus, the model is then *not* super-neutral.[31]

Proposition 8 *Without a Gesell tax, that is, when $\sigma = 0$, the model's steady state dichotomizes into a monetary and real sector. Monetary variables would then be neutral and superneutral in a long-run equilibrium. In contrast, if $\sigma \neq 0$, the model implies*

[30] As argued above the model also dichotomizes when $\beta = 0$. This is the world that Rösl (2006) analyzed. Clearly und rather unsurprisingly, neutrality and superneutrality are then a feature of such a model. For this reason, amongst others, a positive β is one constitutional feature of the present model.

[31] Recall that non-neutrality implies that money supply variables bear on long-run real variables like the steady state capital stock. Non-superneutrality means that the rate of money supply growth has an effect on real variables. See, for example, Ahmed and Rogers (1996) for a clarifying study of this issue.

non-superneutrality.

For the rest of the paper assume that σ is non-zero. The economy does not dichotomize in that case and has, in general, a non-superneutral long-run equilibrium. As a consequence the model features some form of a *Mundell-Tobin* effect.

Recall that Tobin (1965) and Mundell (1963) argued that monetary variables, in particular realized or expected inflation, may have an effect on the real variables, especially on the (long-run) real interest rate of an economy. The effect is usually taken to be positive, because it is argued that higher inflation causes people to hold less money and more real capital. That would then imply a lower real interest rate.[32]

In this model $\pi^* = \theta + \sigma$ in steady state which, according to equation (29), bears on k^* and so the long-run real interest rate r^*. Thus, it is through θ and σ that the model features Mundell-Tobin effects. However, the effects of θ and σ will be shown to be different. When any (positive) change in the variables leads to a higher real interest rate, I call that a *reverse Mundell-Tobin* effect.

We now analyze the comparative static properties of the steady state values of k, m, and c, and other variables of interest. I analyze the effects on k in more details in the main text and present the derivation for the other variables in the appendix.

For a change in σ on k note that

$$\Delta_r = -\left(1 + \frac{\delta\sigma}{\rho + \pi^* + \sigma}\right) \cdot \frac{\alpha\rho}{(r^*)^2} < 0. \tag{30}$$

As $r = \alpha k^{\alpha-1}$ we have $r_k < 0$. But then $\Delta_k = \Delta_r \cdot r_k > 0$ by equation (29), where again subscripts denote partial derivatives.

Furthermore, it turns out that, if $\delta > 0$,

$$\Delta_\sigma = \left(\frac{\delta(\rho + \theta + 2\sigma) - 2\delta\sigma}{(\rho + \theta + 2\sigma)^2}\right) \cdot \left(\frac{\rho - r^*}{r^*}\right) \cdot \alpha > 0.$$

Then we have that $\Delta_k \cdot dk + \Delta_\sigma \cdot d\sigma = 0$ has to hold from equation (29). But consequently we get $dk/d\sigma = -\Delta_\sigma/\Delta_k < 0$, that is, a higher money depreciation rate implies a *lower* steady state capital stock. Thus, households choose to hold less physical capital which implies some form of a *reverse Mundell-Tobin* effect. Higher σ may require more outlays for money holdings. These more "expensive" money

[32] Fischer (1988), p. 296/7 explains where the differences in the respective contributions of Tobin and Mundell lie. See also Temple (2000) for a more recent literature survey on the interaction of inflation and economic growth.

holdings also make it more costly to hold physical capital. Holding less capital, in turn, entails a higher long-run real interest rate r^*, that is, it makes physical capital more "expensive". Hence, raising σ appears to 'destroy' long-run wealth, that is, it implies a smaller, long-run physical capital stock.

For the effect of "love of wealth" β one easily verifies that $dk/d\beta = 1/\Delta_k > 0$ so that an increase in the 'love of wealth' raises the long-run capital stock.

Valuing monetary transactions more (larger δ) implies

$$\Delta_\delta = \left(\frac{\sigma}{\rho+\theta+2\sigma}\right) \cdot \frac{\rho - r^*}{r^*} \cdot \alpha > 0$$

so that $dk/d\delta = -\Delta_\delta/\Delta_k < 0$. Clearly, if people derived more utility from money transactions (higher δ) they might wish to hold more money, but in the model they definitely want to have less physical capital, implying a higher real interest rate r^*.

The effect of more impatience (larger ρ) depends on

$$\begin{aligned}
\Delta_\rho &= \frac{\alpha}{r^*}\left[1+\frac{\delta\sigma}{\rho+\theta+2\sigma}\right] - \alpha\left(\frac{\rho-r^*}{r^*}\right)\left[\frac{\delta\sigma}{(\rho+\theta+2\sigma)^2}\right] \\
&= \frac{\alpha}{r^*}\left[1+\frac{\delta\sigma}{\rho+\theta+2\sigma} - \frac{(\rho-r^*)\delta\sigma}{(\rho+\theta+2\sigma)^2}\right] \\
&= \frac{\alpha}{r^*}\left[1+\frac{\delta\sigma(\rho+\theta+2\sigma)-(\rho-r^*)\delta\sigma}{(\rho+\theta+2\sigma)^2}\right] = \frac{\alpha}{r^*}\left[1+\frac{\delta\sigma(\theta+2\sigma)+r^*\delta\sigma}{(\rho+\theta+2\sigma)^2}\right] > 0
\end{aligned}$$

so that $dk/d\rho = -\Delta_\rho/\Delta_k < 0$. Thus, when the representative household is more impatient, there will be less physical capital in steady state.

For the impact of the money growth rate θ I find

$$\Delta_\theta = -\left(\frac{\delta\sigma}{(\rho+\theta+2\sigma)^2}\right)\left(\frac{\rho-r^*}{r^*}\right) \cdot \alpha < 0$$

from which it follows that $dk/d\theta = -\Delta_\theta/\Delta_k > 0$ so that k^* would be larger.

Thus, with a positive money depreciation rate a higher money growth rate implies a positive *Mundell-Tobin* effect. This is because for a given positive σ an increase in θ entails a higher steady state inflation rate π^*. But a higher θ has just been found to raise the long-run capital stock, coupled with a lower real interest rate. Therefore, this captures the main point of a positive *Mundell-Tobin* effect.

The parameter α represents the elasticity of output with respect to capital, but also the capital share, since it is assumed that firms are profit maximizers under conditions of perfect competition. As $r = \alpha k^{\alpha-1}$ we can express $\Delta = \beta$ in equation (29) as

$$\Delta \;=\; \left(1+\tfrac{\delta\sigma}{(\rho+\theta+2\sigma)}\right)(\rho - r^*)\cdot(k^*)^{1-\alpha} = \left(1+\tfrac{\delta\sigma}{(\rho+\theta+2\sigma)}\right)(\rho\cdot(k^*)^{1-\alpha}-\alpha) = \beta.$$

Then it follows that

$$\Delta_\alpha = -\left(1+\frac{\delta\sigma}{(\rho+\theta+2\sigma)}\right)\left(\rho\cdot\ln k^*\cdot(k^*)^{1-\alpha}\right)$$

which is negative as long as $\ln k^*$ is larger than zero.[33] I assume this to be true, because it only depends on mild theoretical assumptions and very plausible values for the capital-labour ratio, often shown in the empirical literature.[34] As a consequence, $dk/d\alpha = -\Delta_\alpha/\Delta_k > 0$ so that a higher capital share implies a higher steady state capital stock.

Summarizing these findings, the model features the following properties of the steady state capital stock

$$k^* = k^*(\underset{(-)}{\sigma},\underset{(+)}{\beta},\underset{(-)}{\delta},\underset{(-)}{\rho},\underset{(+)}{\theta},\underset{(+)}{\alpha}). \tag{31}$$

Clearly as $y = f(k)$ is monotonically increasing in k, the properties of $k^*(\cdot)$ carry over to steady state output $y^* = f(k^*(\cdot))$, and - in our Cobb-Douglas world - also to the wage rate $w^* = f(k^*) - f'(k^*)\cdot k^* = (1-\alpha)f(k^*)$ and the real interest rate $r^* = f'(k^*)$. The latter immediately follows from assumption 3.

From equation (12) consumption in steady state is given by

$$c^* = \left(\frac{\rho - r^*}{\beta}\right)\cdot k^*. \tag{32}$$

In Appendix C.1 the reaction of steady state consumption is analyzed and found to be characterized by

$$c^* = c^*(\underset{(-)}{\sigma},\underset{(+)}{\beta},\underset{(-)}{\delta},\underset{(-)}{\rho},\underset{(+)}{\theta},\underset{(+)}{\alpha}). \tag{33}$$

[33] Note that $k^{1-\alpha} = e^{(1-\alpha)\ln k}$ and $dk^{1-\alpha}/d\alpha = -\ln k\cdot e^{(1-\alpha)lnk} = -\ln k\cdot k^{1-\alpha}$.

[34] Clearly the model requires $\rho > r^*$ and so $\rho > \alpha(k^*)^{\alpha-1}$ and $k^* > (a/\rho)^{1/(1-\alpha)}$. Thus, as long as α is larger than ρ then the condition $k^* > 1$ is met. It is conventionally assumed that α is around $1/3$ and $\rho << 0.33$.

Two results are noteworthy here. The monetary policy variables θ and σ have opposite effects on steady state consumption. A higher money depreciation rate lowers it, whereas a higher money growth rate raises it. This is probably less surprising if one notes that higher θ raises income and capital, but σ does not. Actually, more money depreciation is 'bad' for capital as well as income, and it competes through money depreciation outlays with consumption. The second interesting finding is that more 'love of wealth' makes more consumption possible in steady state. Even though higher β may seem to be only conducive to more investment, it leads to more steady state capital and income, making a higher level of steady state consumption possible. A related finding is presented in Rehme (2017), and analyzed there in more detail.

From equation (28) the demand for real balances in steady state is given by

$$m^* = \frac{\delta c^*}{\rho + \theta + 2\sigma}.$$

As $\nu \equiv c/m$ it follows that in steady state ν^* is increasing in σ and in θ. In that sense the short-run and long-run effects of monetary policy on the velocity of money are very similar.

Next, in Appendix C.2 the reaction of steady state real money balances is analyzed. The findings there can be summarized by

$$m^* = m^*(\underset{(-)}{\sigma}, \underset{(+)}{\beta}, \underset{(?)}{\delta}, \underset{(-)}{\rho}, \underset{(?)}{\theta}, \underset{(+)}{\alpha}). \tag{34}$$

Interestingly, households hold less money in steady state when money depreciation is increased. This is due to the fact that a higher σ implies a higher velocity of money so that households need to hold less money in a long-run equilibrium to conduct their monetary transactions.

In turn, the effect of θ is not unambiguously clear and depends on the parameter values of the model. If δ and/or σ are sufficiently small, then a higher money growth rate is coupled with less money holdings, but a higher velocity of money.

From equations (31), (33), and (34) and the expression of the welfare function in equation (4) the reactions of the steady state variables and welfare to changes in the variables of interest here yield the following.

Proposition 9 *Given everything else, the introduction of a positive, previously non-existent Gesell tax, which is kept in place forever, implies a higher velocity of money*

ν^*, a lower capital stock k^*, lower consumption c^*, and less holdings of real money balances m^* and so lower welfare $\varphi^*(c^*, m^*, k^*)$ in steady state. The steady state return on capital r^* rises so that some form of a reverse Mundell-Tobin *effect is present.*

Thus, when looking at the effect of money depreciation on long-run outcomes *in isolation*, it seems that it would be a 'bad' policy option to introduce a depreciation rate on money holdings. Only **[GC1]** is validated. However, the introduction of a Gesell tax may not be too 'bad' an option because of the following.

Proposition 10 *Given everything else and conditional on a positive (possibly very small) Gesell tax, a higher rate of money growth θ that is kept in place forever, implies a* Mundell-Tobin *effect. The capital stock k^*, output y^*, consumption c^*, and the velocity of money ν^* would be higher and the long-run real interest rate r^* lower and the wage rate w^* higher. Steady state real money balances m^* may be higher or lower, depending on the parameter values of the model. The effect on long-run welfare is in general not unambiguously clear. For sufficiently high values of σ and/or δ, an increase in θ may raise k^*, c^* and m^* and long-run welfare $\varphi^*(c^*, m^*, k^*)$.*

Those findings would lend clear support of the *Gesell Conjecture* 1, and 2, **[GC1]**, and **[GC2]**. Note that the proposition requires a positive Gesell tax. The latter is, thus, a *necessary* condition for any *Mundell-Tobin* effect to work. In order to see this more clearly consider the effects of joint variations in σ and θ on steady state k^*. They can be determined from the differential $\Delta_k \cdot dk + \Delta_\sigma \cdot d\sigma + \Delta_\theta \cdot d\theta = 0$ using equation (29). We know that $\Delta_k > 0$. Thus, the reaction of k is, for example, positive, if $\Delta_k \cdot dk > 0$. But that requires that $-\Delta_\sigma \cdot d\sigma - \Delta_\theta \cdot d\theta > 0$, that is,

$$-\left(\frac{\delta(\rho+\theta+2\sigma) - 2\delta\sigma}{(\rho+\theta+2\sigma)^2}\right) Q \cdot d\sigma + \left(\frac{\delta\sigma}{(\rho+\theta+2\sigma)^2}\right) Q \cdot d\theta > 0$$

where $Q = \alpha\left(\frac{\rho - r^*}{r^*}\right)$ and the expressions for $\Delta_i, i = \sigma, \theta$ follow from above. This holds, if both σ and θ are changed. Again we see that, if σ is zero, θ does not affect k^*.

For a non-zero σ, and simultaneous changes in both policy variables simplification yields that a positive effect on steady state k is present if

$$\sigma \cdot d\theta > (\rho + \theta) \cdot d\sigma.$$

As a higher σ lowers k^* whereas a higher θ raises it, the change in θ must be sufficiently strong, that is, that it must obey $d\theta/d\sigma > (\rho + \theta)/\sigma$ to have an overall positive effect on k^*.

Result 3 *In general monetary policy conducted through changes in σ and/or θ has ambiguous effects on the steady state capital stock k^*. If the relative changes in the two monetary policy variables satisfy $d\theta/d\sigma > (\rho + \theta)/\sigma$, that is, if the change in θ is sufficiently strong and positive, given that money depreciation is present or its change is positive and given, then the long-run capital stock k^* can be increased and the long-run real interest rate r^* decreased.*

Thus, by a right combination of σ and θ the monetary authority can generate a Mundell-Tobin effect with a higher long-run physical capital stock and lower real interest rate. This appears to be in line with Gesell's idea that expansionary monetary policy increases real activity. Only here it is found that simply focusing on money depreciation alone may not be enough for generating a positive effect on real variables. Although a necessary condition in this model, money depreciation has to be coupled with (new) money creation, that is, it must be accompanied by the injection of "new money" into the economy to have any positive effect on real variables in the long run.

5 Transitional Dynamics

The dynamic system of the equations in (24) can be log-linearized in a standard way to yield insights about the transitional dynamics and convergence properties of the system. The technical details for that are presented in Appendix E.

As the dynamics of the system is essentially governed by the same variables as in the standard Sidrauski model, one can employ the same arguments as in, for example, Blanchard and Fischer (1989), Appendix B of chapter 4, and Fischer (1979).

Thus, note that the capital stock is given, but the money stock and consumption can jump at any point in time. As a consequence if the system is to have a (locally) unique stable path, it must have two positive roots (or a pair of complex roots with positive real part) and one negative root. If that is the case, the jump variables take on (initial) values that make the system converge. The analysis of the roots that govern the speed of convergence of the system is presented in the appendix. For the present model that implies the following result.

Proposition 11 *Given a positive money depreciation rate, an increase in σ speeds up, an increase in θ lowers the speed of convergence to the steady state.*

Interestingly, that is a complement of the result in Fischer (1979), who shows that more money growth would lead to faster convergence, when the utility function is *non-logarithmic* and the steady state features asymptotic superneutrality.

In turn, in this paper the presence of (positive) money depreciation entails that the steady state is non-superneutral, but convergence is slower, if the money growth rate θ is increased and we have a *logarithmic* utility function.[35]

5.1 Numerical simulation

The model is calibrated along some commonly observed magnitudes. The resulting system is then solved for those values. As a starting value assume that the initial capital stock, which is a given (state) magnitude, is taking a value of 5, that is, by assumption $k_0 = 5$. For the other parameters of the model consider the following values.

Table 1: Simulation

α	β	δ	ρ	θ	σ
0.33	0.1	0.02	0.1	0.01	0.01

The major reason for working with these values is that they command wide support in the literature. For example, the value for α is pretty standard and that for δ is almost the same as in Walsh (2010), p. 72. The money growth rate implied by Walsh is roughly equivalent to $\theta = 0.01$ for quarterly U.S. data on money supply M1, but in the model here I take the sum of θ and σ to equal the long-run inflation rate, which many people consider to be around two percent.

An exception may be the value of ρ which is taken to be a lot higher than what is conventionally used in empirical work. However, when one reminds oneself that the time preference rate is an important and somehow pervasive, but, nevertheless, ultimately quite unobservable concept, I assume a value of 10 percent, because it will make the other calibrated values correspond to ranges one finds in the literature.

[35] Recall that if $\sigma = 0$, then r^* is independent of σ and θ. So the latter variables would not impinge on convergence it that case. A similar result for logarithmic utility can be found in Fischer (1979).

Furthermore, note that β is also very difficult to measure. Even some data of the World Value Service are not clearly established to be good measures of the "love of wealth", although the latter has clearly been identified by hermeneutic thinking (e.g. in philosophy, psychology, history and sociology among others) to be an important deep fundamental for social and, particularly, economic relationships. Here I calibrate β so the long-run interest rate assumes a reasonable value.

With that in mind the parameter values generate the following steady state magnitudes of the variables of interest.

Table 2: Simulated Steady State Values

k^*	y^*	k^*/y^*	r^*	m^*	k^*/m^*	c^*	$v = c^*/m^*$	π^*
9.016	2.081	4.332	0.077	0.320	28.200	2.078	6.500	0.020

These numbers imply a steady state inflation rate $\pi^* = \sigma + \theta$ of two percent. The steady state capital stock and output are then calculated as $k^* = 9.016$ and $y^* = 2.081$.[36] That implies a capital-output ratio of about four which seems realistic for many countries. Then the steady state (i.e. long-run) return on capital is about 7.7 percent which is broadly in line with many findings in the literature. See, for example, Jordá, Knoll, Kuvshinov, Schularick, and Taylor (2017), Table 11, for recent evidence.

Furthermore, the implied velocity of money in circulation is around 6.5 for measures such as $v = c^*/m^*$ or $v_1 = y^*/m^*$ which one approximately finds as a period average, for example, for the United States for the period 1960-2015. See figure 3.

From equation (44) in the appendix we get the following numerical representation of the calibrated, log-linearized system

$$\begin{pmatrix} \frac{d\ln k}{dt} \\ \frac{d\ln c}{dt} \\ \frac{d\ln m}{dt} \end{pmatrix} = \begin{pmatrix} 0.0769 & -0.2305 & -0.0004 \\ -0.0743 & 0.0230 & 0.0000 \\ -0.0744 & -0.1069 & 0.1300 \end{pmatrix} \times \begin{pmatrix} d\ln k \\ d\ln c \\ d\ln m \end{pmatrix} + \begin{pmatrix} -0.035\, d\sigma \\ 0 \\ 2\, d\sigma + 1\, d\theta \end{pmatrix}$$

where $d\sigma$ and $d\theta$, our variables of interest here, denote the differentials of σ and θ which are constants.

[36]The simulation, and the numerical convergence analysis were carried out in MATHEMATICA. The code used for the results and graphs below is available upon request.

From that one obtains

$$\begin{pmatrix} d\ln k \\ d\ln c \\ d\ln m \end{pmatrix} = \xi_1 \begin{pmatrix} -0.713 \\ -0.469 \\ -0.496 \end{pmatrix} e^{\lambda_1 \cdot t} + \begin{pmatrix} -0.045 d\sigma + 0.0041 d\theta \\ -0.144 d\sigma + 0.0131 d\theta \\ -15.529 d\sigma - 7.6792 d\theta \end{pmatrix} \tag{35}$$

as the solution to the system. The derivation can be found in Appendix E.1. Here $\lambda_1 = -0.084$ is the only negative root of the system for the given parameter values. Its associated eigenvector is $(-0.713, -0.469, 0.496)$ and ξ_1 is a constant that needs to be definitized.

As c and m are jump variables we concentrate on the first component of this system, i.e., the equation for the capital stock k to determine the constant ξ_1 from initial conditions. Thus, we solve for ξ_1 when $t = 0$, that is, we solve

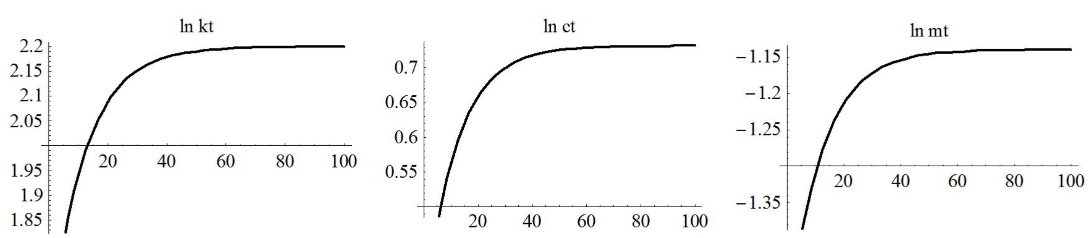
$$(d\ln k)_{t=0} = \ln k_0 - \ln k^* = \xi_1 \cdot (-0.713) \cdot e^{\lambda_1 \cdot 0} - (0.045) \cdot d\sigma + (0.0041) \cdot d\theta$$

for ξ_1 with $e^{\lambda_1 \cdot 0} = 1$. This yields the definitized constant

$$\xi_1^* = \frac{(\ln k_0 - \ln k^*) + 0.045 \cdot d\sigma - 0.0041 \cdot d\theta}{-0.713} \tag{36}$$

where k_0 and k^* are predetermined (non-jump) variables which are constant like the chosen values of $d\sigma$ and $d\theta$. Hence, ξ_1^* is the constant sought after. Clearly, ξ_1^* is also important for the paths of the jump variables c and m and it depends on $d\sigma$ and $d\theta$. The paths of k_t, c_t and m_t in natural logarithms are presented in the next figure, and those for the levels are presented in the appendix.

Figure 8: The paths of k_t, c_t and m_t in natural logarithms



We now conduct the following experiment for ξ when each policy variable θ and σ has a value of one percent so that the steady-state inflation rate is two percent, i.e. around $\sigma = \theta = 0.01$. The experiment is to increase the variables by one percentage point. For instance, we look at the system if σ is raised from one to two percentage points, given θ. The same is done for θ. A final experiment is to consider a joint increase of one percentage point each, given that they were one percent.

Table 3: Changes in σ and θ and the resulting ξ_1^*

Case	$d\sigma$	$d\theta$	$\xi^*_{1\|\text{Case}}$
0	0.00	0.00	0.827303
1	0.01	0.00	0.826675
2	0.00	0.01	0.827360
3	0.01	0.01	0.826732

The changes are taken around $\sigma = \theta = 0.01$.

From the table the differences are small. But it can be verified that

$$\xi^*_{1|2} > \xi^*_{1|0} > \xi^*_{1|3} > \xi^*_{1|1},$$

which one may have expected from the theoretical predictions.

First consider the equation for the capital stock when $\lambda_1 = -0.084$. Given that $d\ln k = \ln k_t - \ln k^*$, we obtain from equation (35) that at any point in time t

$$\begin{aligned} \ln k_t - \ln k^* &= \xi_1^* \cdot (-0.713) \cdot e^{-0.084 \cdot t} - 0.045 \cdot d\sigma + 0.0041 \cdot d\theta \\ &= \left(\tfrac{(\ln k_0 - \ln k^*) + 0.045 d\sigma - 0.0041 d\theta}{-0.713}\right) (-0.713)\, e^{-0.084 \cdot t} \\ &\quad -0.045 \cdot d\sigma + 0.0041 d\theta \\ \ln k_t &= \left(1 - e^{-0.084 \cdot t}\right) \ln k^* + e^{-0.084 \cdot t} \ln k_0 \\ &+ \left(e^{-0.084 \cdot t} - 1\right) (0.045 \cdot d\sigma - 0.0041 \cdot d\theta)\,. \end{aligned}$$

From these relationships one readily obtains that for any $t > 0$

$$(\ln k_t)_{|2} > (\ln k_t)_{|0} > (\ln k_t)_{|3} > (\ln k_t)_{|1}.$$

Hence, at a long-run equilibrium with σ and θ at one percent each, an increase in σ, or in θ, or in both implies that an isolated increase in σ of one percentage point forever, given no change in θ, leads to a lower path of capital at each point in time where $t > 0$

in comparison to the initial long-run equilibrium. An isolated increase in θ, given no change in σ, in turn, implies a higher path of capital for each $t > 0$.

Thus, an increase in θ implies a higher steady-state capital stock, but that requires a positive (non-zero) σ. Note, however, that a simultaneous positive change in *both* variables is not necessarily augmenting capital as the values for $(\ln k_t)_{|3}$ reflect.

For consumption and the changes considered one obtains the following

$$(d\ln c)_{|i} = (\ln c_t)_{|i} - \ln c^* \quad = \quad \xi^*_{1|i} \cdot (-0.469) \cdot e^{-0.084 \cdot t} - 0.144 \cdot d\sigma + 0.0131 \cdot d\theta$$

where $i = 0, 1, 2, 3$ reflects the changes of $d\sigma$ and $d\theta$ contemplated in Table 3 and where initial consumption $(c_0)_{|i}$ jumps to a value that satisfies this equation.

Calculating the differences $(d\ln c)_{|i} - (d\ln c)_{|0}$ for $i = 1, 2, 3$ then reveals that

$$(\ln c_t)_{|2} > (\ln c_t)_{|0} > (\ln c_t)_{|3} > (\ln c_t)_{|1}.$$

So an increase in θ that is in place forever is 'good' for consumption at each point in time, but again requires a non-zero money depreciation rate σ. That also holds for initial consumption c_0.

On the other hand a higher σ entails that initial consumption is lower than the value of steady state consumption without money depreciation. Furthermore, no matter what initial consumption is, consumption at t will decrease from its initial value. From that one also verifies that, if you keep $d\sigma > 0$ in place forever, the new long run value of consumption is lower.

Next, turn to real money balances that are also a jump variable in this model. From the arguments above one readily gets that

$$(d\ln m)_{|i} = (\ln m_t)_{|i} - \ln m^* = \xi^*_{1|i} \cdot (-0.469) \cdot e^{-0.084 \cdot t} - 15.529 \cdot d\sigma - 7.6792 \cdot d\theta.$$

Then it is not difficult to verify that

$$(\ln m_t)_{|0} > (\ln m_t)_{|2} > (\ln m_t)_{|1} > (\ln m_t)_{|3}.$$

In a long-run equilibrium the policy changes contemplated would, thus, imply less money holdings at each point in time.

The result is not difficult to justify because in the model an increase in θ and σ increases the velocity of money ν and as a consequence people want to hold less real money balances at each point in time.

Summarizing these effects for permanent policy changes yields the following.

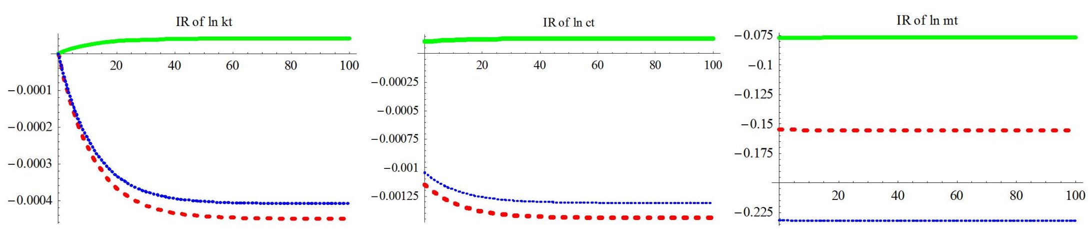
Result 4 *The model's simulation yields that for the policy experiments considered that at each point in time*

$$
\begin{aligned}
(\ln k_t)_{|2} > (\ln k_t)_{|0} > (\ln k_t)_{|3} > (\ln k_t)_{|1} \\
(\ln c_t)_{|2} > (\ln c_t)_{|0} > (\ln c_t)_{|3} > (\ln c_t)_{|1} \\
(\ln m_t)_{|0} > (\ln m_t)_{|2} > (\ln m_t)_{|1} > (\ln m_t)_{|3}.
\end{aligned}
$$

The state variable k as well as the jump variables m and c exhibit the same reactions for the policy changes at each (finite) point in time as the steady state reactions.

Hence, for a given money depreciation rate σ a higher money growth rate θ implies less money holdings (less monetization), more consumption, and generally a higher capital stock at each point in time. Lastly, note that the effects on transitional welfare are obvious. The following figure presents these effects as deviations from the path, where policy is not changed, that is, the paths presented in figure 8.

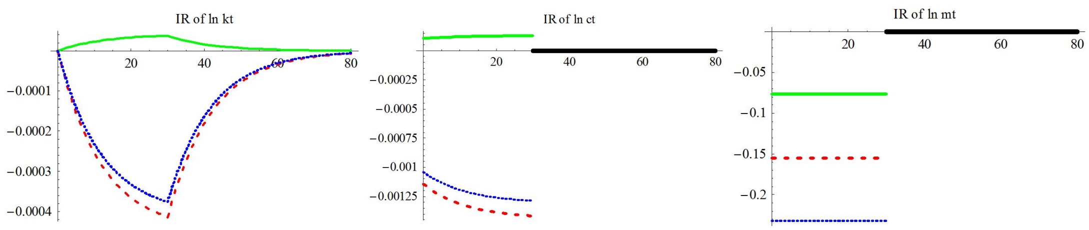
Figure 9: Permanent policy changes



Changes: $d\sigma$ - red dashed line, $d\theta$ - solid green line, $d\sigma + d\theta$ - dotted blue line

Furthermore, one verifies that temporary changes in θ and/or σ produce the effects presented above. This is visualized in the following graph for a transitory change lasting 30 time periods. Again the reaction are presented as deviations from the original path in figure 8.

Figure 10: Short-run policy changes lasting the period $t \in [0, 30]$



Changes: $d\sigma$ - red dashed line, $d\theta$ - solid green line, $d\sigma + d\theta$ - dotted blue line
Plotted as deviations from the benchmark log-linear model.

Thus, initial consumption and money holdings jump down after the changes involving σ, they jump up when θ is raised in isolation. They would then pursue a path getting to the new steady state, if the policy changes were kept in place forever. But when the changes are transitory and revoked, consumption and money balances jump back to their pre-disturbance path. The natural logarithm of the state variable k_t declines first and then converges to the unperturbed path after the changes involving changes in σ are revoked. For isolated changes in θ (with no changes in $\sigma > 0$) these effects work in the opposite direction as is obvious from the graphs.

Lastly notice that, for example, a temporary drastic negative change in θ may well describe the Indian demonetization experience. Lower θ implies a lower capital stock, higher real interest rate, lower consumption, and lower real money according to the model. All this has more or less been observed in India, but has been a temporary phenomenon. When remonetization finally got under way, θ was increased again and things operated in reverse. The open question is still whether the policy change has really been neutral for the Indian economy in the long run.

6 Conclusion

About one hundred years ago Silvio Gesell argued that money should 'rot' as any other good does. He argued that depreciation of money (cash) in circulation would be stimulative for economic performance and be socially beneficial.

In this paper I question the claim that his ideas for an unconventional monetary policy cannot really be verified in modern economic theory frameworks. To this end I focus on four hypotheses Gesell made and analyze these using standard contemporaneous macroeconomic theory. The following findings of the paper are then noteworthy.

First, in a short-run, IS-LM-AS-AD-like demand-determined equilibrium where the (physical) capital stock, the inflation rate, transfers, and money supply are fixed, but real factor prices are flexible, Gesell's hypotheses are all valid, given the (demand) micro-foundations of the model which feature utility directly derived from money and 'love of wealth', and physical capital is taken to be the true source of wealth.

This short-run analysis also implies that money depreciation can be a policy option to overcome the zero lower bound problem of nominal interest rates. Furthermore, an interpretation of the economic effects of the recent demonetization episode in India is possible from the model.

Second, for the long run it is shown that the steady state inflation equals the money growth and depreciation rate. The economy dichotomizes into a monetary and real sector, if there is no money depreciation. If the latter is present, the model features non-superneutrality. Money depreciation is a necessary condition for particular forms of a *Mundell-Tobin* effect.

Third, raising money depreciation in isolation lowers the steady state capital stock (wealth), consumption, income and welfare. It also implies a higher return to capital, but lower steady state wage rate. Thus, more money depreciation seems to destroy wealth and is not 'good' for labour. Higher money depreciation only implies a higher velocity of money.

Fourth, Gesell did not consider money depreciation as the only monetary policy tool. Here I find that, for a given positive money depreciation rate, an increase in the money growth rate produces a *Mundell-Tobin* effect. Thus, a higher money growth increases steady state inflation, but also the steady state capital stock, output, and consumption. It implies a higher long-run wage rate and a lower return to capital. The consequences for the holdings of real money balances and so for total welfare are not unambiguously clear. But the velocity of money increases. However, the partial welfare channels through consumption and wealth work clearly in a positive direction. Hence, the conjectures are broadly validated for the long-run in this model.

Fifth, the transitional dynamics reveal that the speed of convergence increases if money depreciation is raised, and decreases if the money growth rate is higher. In the present model Fischer (1979) is complemented, because here the steady state generally features non-superneutrality, the utility function is logarithmic, and convergence is slower when the money growth rate increases.

A simulation exercise reveals that the response of key variables to permanent changes in the monetary policy variables is qualitatively the same in the transition as in steady state. That also holds for the jump variables, namely, initial money holdings and consumption. Furthermore, for temporary changes in the policy variables one obtains that qualitatively the temporary responses, again, basically equal those for the steady state.

Hence, in the present model-framework most of Gesell's claims can be verified. In the short-run, demand-determined equilibrium all claims can verified. For the long-run equilibrium two claims follow directly, and the other two indirectly, because money depreciation is a necessary condition for a positive *Mundell-Tobin* effect.

Of course, the analysis faces several caveats. The setup of the model is simple. Alternative utility and production functions might imply more complicated equilibria or the lack thereof. The introduction of fiscal policy may make the results less clean. 'Love of wealth' was captured by a constant. This begs the question how changes over time in the 'love of wealth' may bear on the optimal paths. These and other extensions of the model are left for further research.

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A Absolute and relative wealth

Suppose relative wealth of an individual i is given by

$$x_i = \frac{k_i}{\sum k_j}.$$

Then relative wealth is the (absolute) level of wealth k_i in relation (relative) to total wealth (wealth of all people). If that individual's preferences are $u^i(c_i, x_i)$, then consumption c_i and relative wealth x_i would matter for person i's welfare.

If there are many people, $j = 1, \ldots N$ where $i \in [1, N]$ with N very large, the effect of changes of k_i by individual i has no discernible bearing on total wealth $\sum k_j$ where the summation is from 1 to N.[37]

If the utility function of individual i is logarithmic,

$$u^i = \ln c_i + \gamma \ln x_i = \ln c_i + \gamma \ln k_i - \gamma \ln \sum k_j,$$

then the decisions of individual i about c_i and k_i would not have an effect on $\gamma \ln \sum k_j$ which would be a *datum* for individual i. That follows from the assumption that there are many people. Those arguments justify what is mentioned in the text.

B The Zero Lower Bound on Nominal Interest Rates

In short-run equilibrium the nominal interest rate is at the zero lower bound, $\widehat{i} = 0$, when

$$(\pi + \sigma + \rho) \left((1 - \left(\frac{m}{k}\right) \left(\frac{\beta}{\delta}\right) \right) = \sigma. \tag{21}$$

The total differential of equation (21) with respect to σ and m yields

$$\left(1 - \left(\frac{m}{k}\right)\left(\frac{\beta}{\delta}\right)\right) d\sigma - (\pi + \sigma + \rho)\left(\frac{1}{k}\right)\left(\frac{\beta}{\delta}\right) dm = d\sigma,$$

which can be simplified to

$$-\frac{d\sigma}{\sigma}\left(\frac{\sigma}{\pi + \rho + \sigma}\right) = \frac{dm}{m} \quad \text{or} \quad \frac{d\sigma}{\sigma} = -\left(\frac{\pi + \rho + \sigma}{\sigma}\right)\frac{dm}{m}.$$

This relationship upholds $\widehat{i} = 0$ when one of the policy instruments is changed. Thus, a one-percent-increase in one instrument requires a corresponding percentage-decrease in the other one. Now recall

$$\widehat{c}_{|\sigma=0,\widehat{i}=0} = \frac{(\overline{\pi} + \rho) \cdot k}{\beta} = \frac{(\overline{\pi} + \rho) \cdot m_0}{\delta} \quad \text{and} \quad \widehat{c}_{|\sigma>0,\widehat{i}=0} = m_1 \cdot \sigma \left[\delta - \frac{m_1}{k} \cdot \beta\right]^{-1}.$$

where the indexation $m_i, i = 0, 1$ expresses the fact that m will be lower when $\sigma > 0$, that

[37] This is almost always assumed in this literature. See, for example, Corneo and Jeanne (2001b).

is, $m_1 < m_0$. I want to check whether $\widehat{c}_{|\sigma=0,\widehat{i}=0} \gtreqless \widehat{c}_{|\sigma>0,\widehat{i}=0}$. To this end let us suppose $\widehat{c}_{|\sigma=0,\widehat{i}=0} \leq \widehat{c}_{|\sigma>0,\widehat{i}=0}$. Then

$$\frac{(\overline{\pi}+\rho)k}{\beta} \leq m_1 \cdot \sigma \left[\delta - \frac{m_1}{k} \cdot \beta\right]^{-1},$$

where, of course, $k = \overline{k}$. Then rearrangement implies

$$\begin{aligned} \frac{(\overline{\pi}+\rho)k}{\beta} \cdot \left[\delta - \frac{m_1}{k} \cdot \beta\right] &\leq m_1 \cdot \sigma \\ \left(\frac{k}{m_1}\right)\left(\frac{\delta}{\beta}\right) &\leq \frac{\pi+\rho+\sigma}{\pi+\rho}. \end{aligned}$$

This inequality also holds when one takes logarithms. Thus, the claim would have to be that

$$\ln k - \ln m_1 + \ln\left(\frac{\delta}{\beta}\right) \leq \ln(\pi+\rho+\sigma) - \ln(\pi+\rho).$$

Taking the total differential of this expression yields that

$$-\frac{dm_1}{m_1} \leq \frac{d\sigma}{(\pi+\rho+\sigma)} = \frac{d\sigma}{\sigma}\left(\frac{\sigma}{\pi+\rho+\sigma}\right)$$

would have to hold. But as can be ascertained from above, both sides of this inequality are equal when $\widehat{i} = 0$ is upheld. Hence, the introduction of money depreciation coupled with a lower money stock when $\widehat{i} = 0$ does not bear on consumption in equilibrium when the economy's interest rate is at the zero lower bound. This is the argument presented in the main text.

C Comparative Statics

C.1 The effects on steady state consumption c^*

Recall that in the steady state $c^* = \left(\frac{\rho - r^*}{\beta}\right) \cdot k^*$. The effects of c^* are then determined as follows.

C.1.1 The sign of $dc^*/d\sigma$

Notice that

$$\frac{dc^*}{d\sigma} = \frac{1}{\beta}\left(-r_k^* \cdot \frac{dk^*}{d\sigma} \cdot k^* + (\rho - r^*) \cdot \frac{dk^*}{d\sigma}\right).$$

As $r_k^* < 0$ and $dk^*/d\sigma < 0$, it follows that $dc^*/d\sigma < 0$.

C.1.2 The sign of $dc^*/d\beta$

It is not difficult to verify that

$$\frac{dc^*}{d\beta} = \frac{1}{\beta}\left(-r_k^* \cdot \frac{dk^*}{d\beta} \cdot k^* + (\rho - r^*) \cdot \frac{dk^*}{d\beta} - \frac{(\rho - r^*) \cdot k^*}{\beta}\right). \tag{37}$$

From equation (29) and (30) we know that

$$\Delta_k = \Delta_r \cdot r_k^* = -\frac{\alpha \cdot x \cdot \rho \cdot r_k^*}{(r^*)^2} \quad \text{where} \quad x \equiv 1 + \frac{\delta\sigma}{\rho + \pi + \sigma}. \tag{38}$$

Furthermore $dk/d\beta = 1/\Delta_k = 1/(\Delta_k \cdot r_k^*)$. Then equation (37) can be rearranged as

$$\begin{aligned} \frac{dc^*}{d\beta} &= \frac{dk^*/d\beta}{\beta}\left[-r_k^* \cdot k^* + (\rho - r^*) - \frac{(\rho - r^*) \cdot k^*}{\beta \cdot (dk^*/d\beta)}\right] \\ &= \frac{dk^*/d\beta}{\beta}\left[-r_k^* \cdot k^* + (\rho - r^*) - \frac{(\rho - r^*) \cdot k^*}{\beta}\left(\frac{\alpha \cdot x \cdot \rho \cdot r_k^*}{(r^*)^2}\right)\right]. \end{aligned}$$

It turns out that the expression in square bracket is positive by the following arguments. We have that $\beta = (x \cdot (\rho - r^*) \cdot \alpha)/r^*$, $r^* = \alpha^(k^*)^{\alpha-1}$, and $r_k^* = \alpha(\alpha - 1)(k^*)^{\alpha-2}$. Making the substitutions for β and r_k^* where appropriate above yields

$$\begin{aligned} &\left[-\alpha(\alpha-1)(k^*)^{\alpha-2} \cdot k^* + (\rho - r^*)\right. \\ &\left. + (\rho - r^*) \cdot k\left(\frac{\alpha \cdot x \cdot \rho \cdot \alpha(\alpha-1)(k^*)^{\alpha-2}}{(r^*)^2}\right) \cdot \frac{r^*}{x \cdot (\rho - r^*) \cdot \alpha}\right] \\ \Leftrightarrow \quad &\left[-\alpha(\alpha-1)(k^*)^{\alpha-1} + (\rho - r^*) + \left(\frac{\rho}{r^*}\right)\alpha(\alpha-1)(k^*)^{\alpha-1}\right]. \end{aligned}$$

When using the substitutions again one finds that the expression in square brackets boils down to

$$-(\alpha - 1) \cdot r^* + (\rho - r^*) + \frac{\rho}{r^*} \cdot (\alpha - 1) \cdot r^* = \alpha \cdot (\rho - r^*)$$

which is positive because $\rho > r^*$ in the model. Hence, $dc^*/d\beta > 0$.

C.1.3 The signs of $dc/d\delta$ and $dc/d\rho$

Recall that $r_k^* < 0$ and $\rho > r^*$. Then it follows that

$$\frac{dc^*}{dj} = \frac{1}{\beta}\left(-r_k^* \cdot \frac{dk^*}{dj} \cdot k^* + (\rho - r^*) \cdot \frac{dk^*}{dj}\right) < 0 \quad \text{where} \quad j = \delta, \rho$$

because $dk^*/dj < 0$ for $j = \delta, \rho$.

C.1.4 The signs of $dc^*/d\theta$ and $dc^*/d\alpha$

Recall that $r_k^* < 0$ and $\rho > r^*$. Then it follows that

$$\frac{dc^*}{di} = \frac{1}{\beta}\left(-r_k^* \cdot \frac{dk^*}{di} \cdot k^* + (\rho - r^*) \cdot \frac{dk^*}{di}\right) > 0 \quad \text{where} \quad i = \theta, \alpha$$

because $dk^*/di > 0$ for $i = \theta, \alpha$.

C.2 The effects on steady state real money balances m^*

Recall $m^* = \dfrac{\delta c^*}{\rho + \theta + 2\sigma}$, the effects of which are then determined as follows.

The sign of $dm^*/d\sigma$.

$$\frac{dm^*}{d\sigma} = \frac{\delta(dc^*/d\sigma)(\rho + \theta + 2\sigma) - 2\delta c^*}{(\rho + \theta + 2\sigma)^2} < 0 \quad \text{because} \quad \frac{dc^*}{d\sigma} < 0.$$

The sign of $dm^*/d\beta$

$$\frac{dm^*}{d\beta} = \frac{\delta(dc^*/d\beta)}{\rho + \theta + 2\sigma} > 0 \quad \text{because} \quad \frac{dc^*}{d\beta} > 0.$$

The sign of $dm^*/d\delta$. I want to show that

$$\frac{dm^*}{d\delta} = \frac{c^* + \delta(dc^*/d\delta)}{\rho + \theta + 2\sigma} \gtreqless 0.$$

To that end recall that

$$c^* = \left(\frac{\rho - r^*}{\beta}\right) k^* \quad \text{and} \quad \frac{dc^*}{d\delta} = \frac{1}{\beta}\left(-r_k^* \cdot k^* \cdot \frac{dk^*}{d\delta} + (\rho - r^*) \cdot \frac{dk^*}{d\delta}\right)$$

and $-r_k^* \cdot k^* = (1 - \alpha) r^*$. So we get

$$\begin{aligned}\frac{dm^*}{d\delta} = \frac{c^* + \delta(dc^*/d\delta)}{\rho + \theta + 2\sigma} &= \frac{\left[(\rho - r^*)k^* + \delta\left[(1-\alpha)r^* + (\rho - r^*)\right]\frac{dk^*}{d\delta}\right]}{\beta(\rho + \theta + 2\sigma)} \\ &= \frac{\left[(\rho - r^*)k^* + \delta\left[(\rho - \alpha r^*)\right]\frac{dk^*}{d\delta}\right]}{\beta(\rho + \theta + 2\sigma)}\end{aligned}$$

where we know that

$$\frac{dk^*}{d\delta} = -\frac{\Delta_\delta}{\Delta_k} = -\frac{\left(\frac{\sigma}{\rho+\theta+2\sigma}\right) \cdot \frac{\rho - r^*}{r^*} \cdot \alpha}{-\left(1 + \frac{\delta\sigma}{\rho+\pi+\sigma}\right) \cdot \frac{\alpha \cdot \rho}{(r^*)^2} \cdot r_k^*} = \frac{\sigma(\rho - r^*) \cdot (r^*/\rho)}{(\rho + \theta + (2+\delta)\sigma) \cdot r_k^*}$$

Making the substitution above yields

$$\left[(\rho - r^*)k^* + \delta\left[(\rho - \alpha r^*)\right]\left\{\frac{\sigma(\rho - r^*)\cdot(r^*/\rho))}{(\rho+\theta+(2+\delta)\sigma)\cdot r_k^*}\right\}\right]\cdot B \tag{39}$$

where $B \equiv [\beta\cdot(\rho+\theta+2\sigma)]^{-1} > 0$. Pulling out $(\rho - r^*)$ the expression in square brackets is positive, zero or negative if

$$k^* \gtreqless -\delta\left[(\rho - \alpha r^*)\right]\left\{\frac{\sigma\cdot(r^*/\rho)}{(\rho+\theta+(2+\delta)\sigma)\cdot r_k^*}\right\}. \tag{40}$$

As $r_k^* = (\alpha - 1)\cdot r^*\cdot(k^*)^{-1}$ the inequality boils down to

$$\begin{aligned}(1-\alpha)r^*k^* &\gtreqless \frac{\delta\left[(\rho - \alpha r^*)\right]\cdot\sigma\cdot(r^*/\rho)\cdot k^*}{(\rho+\theta+(2+\delta)\sigma)} \\ (1-\alpha)r^*k^*\cdot(\rho+\theta+(2+\delta)\sigma) &\gtreqless \delta\cdot\left[(\rho - \alpha r)\right]\cdot\sigma\cdot(r^*/\rho)\cdot k^*.\end{aligned}$$

No clear relationship can be established for this inequality. For example, if δ or σ are very low ($\delta, \sigma \to 0$), then the inequality is positive and $dm/d\delta > 0$ would follow. In turn, if, for example, σ is very large (e.g. $\sigma \to \infty$) then $dm/d\delta > 0$ would be implied. Hence, the sign of $dm/d\delta$ is generally not unambiguously clear.

The sign of $dm^*/d\rho$.

$$\frac{dm^*}{d\rho} = \frac{\delta(dc^*/d\rho)(\rho+\theta+2\sigma) - \delta c^*}{(\rho+\theta+2\sigma)^2} < 0 \quad \text{because} \quad \frac{dc^*}{d\rho} < 0.$$

The sign of $dm^*/d\theta$. We have

$$\frac{dm^*}{d\theta} = \frac{\delta(dc^*/d\theta)(\rho+\theta+2\sigma) - \delta c^*}{(\rho+\theta+2\sigma)^2},$$

where the sign of that expression depends on the sign of $(dc^*/d\theta)(\rho+\theta+2\sigma) - c^*$ for non-zero δ, and

$$c^* = \left(\frac{\rho - r^*}{\beta}\right)k^* \quad , \quad \frac{dc^*}{d\theta} = \frac{1}{\beta}\left(-r_k^*\cdot k^*\cdot\frac{dk^*}{d\theta} + (\rho - r^*)\cdot\frac{dk^*}{d\theta}\right) = \left(\frac{\rho - \alpha r^*}{\beta}\right)\frac{dk^*}{d\theta} \quad \text{and}$$

$$\frac{dk^*}{d\theta} = -\frac{\Delta_\theta}{\Delta_k} = \frac{\left(\frac{\delta\sigma}{(\rho+\theta+2\sigma)^2}\right)\cdot\frac{\rho - r^*}{r^*}\cdot\alpha}{-\left(1+\frac{\delta\sigma}{\rho+\theta+2\sigma}\right)\cdot\frac{\alpha\cdot\rho}{(r^*)^2}\cdot r_k^*} = \frac{\delta\sigma(\rho - r^*)\cdot r^*k^*}{(\rho+\theta+(2+\delta)\sigma)\cdot\rho\cdot(1-\alpha)r^*}$$

where again I have used that $-r_k^*\cdot k^* = (1-\alpha)r^*$. Making the appropriate substitutions yields after simplification that the sign of $(dc^*/d\theta)(\rho+\theta+2\sigma) - c^*$ depends on whether

$$\left(\frac{(\rho - r^*)k^*}{\beta}\right)\left[\frac{(\rho - \alpha r^*)\delta\sigma(\rho+\theta+2\sigma)}{(\rho+\theta+(2+\delta)\sigma)\rho(1-\alpha)} - 1\right] \gtreqless 0.$$

The sign of the expression in square bracket depends on the model's parameters. For example, if σ or δ are sufficiently small, the expression in square brackets is negative, if they are sufficiently large, it is positive. Hence, the sign of $dm^*/d\theta$ is not unambiguously clear.

D Long-run welfare effects

Long-run period welfare is given by φ^* and by equations (28) and (32) amounts to

$$\begin{aligned}\varphi^* &= \ln c^* + \delta \ln m^* + \beta \ln k^* \\ &= \left(\ln\left(\frac{\rho - r^*}{\beta}\right) + \ln k^*\right) \\ &\quad +\delta\left(\ln\left(\frac{\delta}{\rho+\theta+2\sigma}\right) + \ln\left(\frac{\rho - r^*}{\beta}\right) + \ln k^*\right) + \beta \ln k^*.\end{aligned}$$

Collecting terms then reveals that long-run period welfare is

$$\varphi^* = (1+\delta+\beta)\ln k^* + (1+\delta)\ln\left(\frac{\rho - r^*}{\beta}\right) - \delta \ln\left(\frac{\rho+\theta+2\sigma}{\delta}\right)$$

where $v = \frac{\rho+\theta+2\sigma}{\delta}$ equals the velocity of money. Notice it has a negative effect on long-run welfare in this model.

D.1 The effect of σ and θ

As c^*, m^* and k^* all depend negatively σ if follows that $d\varphi^*/d\sigma < 0$.

For the money growth rate θ one calculates

$$\frac{d\varphi^*}{d\theta} = (1+\delta+\beta)\cdot\frac{dk^*}{d\theta}\cdot\frac{1}{k^*} + (1+\delta)\left(-r_k\cdot\frac{\beta}{\rho-r^*}\right)\cdot\frac{dk^*}{d\theta} - \frac{\delta}{\rho+\theta-2\sigma}.$$

From the main text we know that

$$\frac{dk^*}{d\theta} = -\frac{-\left(\frac{\delta\sigma}{(\rho+\theta+2\sigma)^2}\right)\left(\frac{\rho-r^*}{r^*}\right)\cdot\alpha}{-\left(1+\frac{\delta\sigma}{\rho+\theta+2\sigma}\right)\left(\frac{\alpha\rho}{(r^*)^2}\right)\cdot r_k}.$$

I want to check whether $\frac{d\varphi^*}{d\theta} > 0$. This boils down to analyze whether

$$\left(\tfrac{\delta\sigma}{(\rho+\theta+2\sigma)^2}\right)\left(\tfrac{\rho-r^*}{r^*}\right)\cdot\alpha\cdot\left[(1+\delta+\beta)\cdot\tfrac{1}{k^*} + (1+\delta)\left(-r_k\cdot\left(\tfrac{\beta}{\rho-r^*}\right)\right)\right] >$$
$$\left(\tfrac{\delta}{\rho+\theta-2\sigma}\right)\cdot(-1)\cdot\left(1+\tfrac{\delta\sigma}{\rho+\theta+2\sigma}\right)\left(\tfrac{\alpha\rho}{(r^*)^2}\right)\cdot r_k.$$

Cancellation by common terms then yields

$$\sigma\cdot(\rho-r^*)\cdot\left[(1+\delta+\beta)\cdot\tfrac{1}{k^*} + (1+\delta)\left(-r_k\cdot\left(\tfrac{\beta}{\rho-r^*}\right)\right)\right] >$$
$$(\rho+\theta+(2+\delta)\sigma)\left(\tfrac{\rho}{r^*}\right)\cdot(-r_k).$$

Note that $-r_k = \alpha(\alpha-1)k^{\alpha-2} = (\alpha-1)\cdot r^* \cdot (k^*)^{-1}$. Substituting this in and rearrangement yields

$$\sigma \cdot (\rho - r^*) \cdot \left[(1+\delta+\beta) + (1+\delta)\left((1-\alpha)\cdot r^* \cdot \left(\tfrac{\beta}{\rho - r^*}\right)\right)\right] > \\ (\rho+\theta+(2+\delta)\sigma)\cdot\rho\cdot(1-\alpha).$$

It is not difficult to see that the last inequality does not always hold and depends in an important way on the parameters of the model. For example, if σ is very low, it does not hold. It may hold for sufficiently large values of it, though. It may also hold, if β is sufficiently large. But in general, no clear overall relationship between w^* and θ holds.

But it is definitely so that the partial effects of θ on welfare through the consumption and capital channel raise welfare derived from them, that is, they raise welfare conditionally. In the model the impact of the velocity of money and its reaction to changes in θ are so large that the other partial effect are outweighed.

E Analysis of the Transitional Dynamics

The dynamic system of the equations in (24) can be formulated in (natural) logs as

$$\frac{d\ln k}{dt} = e^{(\alpha-1)\ln k} - e^{\ln(c/k)} - \sigma e^{\ln(m/k)}, \tag{41a}$$

$$\frac{d\ln c}{dt} = \beta e^{\ln(c/k)} + \alpha e^{(\alpha-1)\ln k} - \rho, \tag{41b}$$

$$\frac{d\ln m}{dt} = \theta + 2\sigma - \delta e^{\ln(c/m)} + \beta e^{\ln(c/k)} + \alpha e^{(\alpha-1)\ln k}. \tag{41c}$$

In steady state $\frac{d\ln k}{dt} = \frac{d\ln c}{dt} = \frac{d\ln m}{dt} = 0$ so that

$$e^{(\alpha-1)\ln k^*} = e^{\ln(c^*/k^*)} + \sigma e^{\ln(m^*/k^*)}, \tag{42a}$$

$$\beta e^{\ln(c^*/k^*)} + \alpha e^{(\alpha-1)\ln k^*} = \rho, \tag{42b}$$

$$\delta e^{\ln(c^*/m^*)} = \theta + 2\sigma + \beta e^{\ln(c^*/k^*)} + \alpha e^{(\alpha-1)\ln k^*}. \tag{42c}$$

From these equations it then follows that in steady state

$$e^{\ln(c^*/m^*)} = \frac{\rho+\theta+2\sigma}{\delta} \quad \text{and} \quad \sigma e^{\ln(m^*/k^*)} = e^{(\alpha-1)\ln k^*} - e^{\ln(c^*/k^*)} = \frac{r^*}{\alpha} - \frac{\rho - r^*}{\beta}.$$

where $f(k^*)/k^* = (k^*)^{\alpha-1} = r^*/\alpha$ and

$$r^* = \alpha(k^*)^{\alpha-1} = \alpha e^{(\alpha-1)\ln k^*}. \tag{43}$$

Now we linearize the system in (41) to get

$$\begin{pmatrix} \frac{d\ln k}{dt} \\ \frac{d\ln c}{dt} \\ \frac{d\ln m}{dt} \end{pmatrix} = \boldsymbol{\Delta} \times \begin{pmatrix} d\ln k \\ d\ln c \\ d\ln m \end{pmatrix}$$

where $d\ln j = \ln j - \ln j^* = \ln(j/j^*)$ for $j = k, c, m$, and the star * denotes variables that are in their steady state. $\mathbf{\Delta}$ is defined as

$$\mathbf{\Delta} \equiv \begin{pmatrix} \Delta_{1k} & \Delta_{1c} & \Delta_{1m} \\ \Delta_{2k} & \Delta_{2c} & \Delta_{2m} \\ \Delta_{3k} & \Delta_{3c} & \Delta_{3m} \end{pmatrix}_{k^*, c^*, m^*}$$

and represents the Jacobian of the system, evaluated in steady state equilibrium. Its elements are given by

$$\begin{aligned}
&\Delta_{1k} = (\alpha-1)e^{(\alpha-1)\ln k^*} + e^{\ln(c^*/k^*)} + \sigma e^{\ln(m^*/k^*)}, && \Delta_{1c} = -e^{\ln(c^*/k^*)}, && \Delta_{1m} = -\sigma e^{\ln(m^*/k^*)}, \\
&\Delta_{2k} = -\beta e^{\ln(c^*/k^*)} + \alpha(\alpha-1)e^{(\alpha-1)\ln k^*}, && \Delta_{2c} = \beta e^{\ln(c^*/k^*)}, && \Delta_{2m} = 0, \\
&\Delta_{3k} = -\beta e^{\ln(c^*/k^*)} + +\alpha(\alpha-1)e^{(\alpha-1)\ln k^*}, && \Delta_{3c} = -\delta e^{\ln(c^*/m^*)} + \beta e^{\ln(c^*/k^*)}, && \Delta_{3m} = \delta e^{\ln(c^*/m^*)}.
\end{aligned}$$

Using the information about the steady state values yields the following.

$$\begin{aligned}
\Delta_{1k} &= (\alpha-1)e^{(\alpha-1)\ln k^*} + e^{\ln(c^*/k^*)} + \sigma e^{\ln(m^*/k^*)} \\
&= \alpha e^{(\alpha-1)\ln k^*} - e^{(\alpha-1)\ln k^*} + e^{\ln(c^*/k^*)} + \sigma e^{\ln(m^*/k^*)} = r^*
\end{aligned}$$

because $e^{(\alpha-1)\ln k^*} = e^{\ln(c^*/k^*)} + \sigma e^{\ln(m^*/k^*)}$ in steady state and $\alpha e^{(\alpha-1)\ln k^*} = r^*$.

$$\Delta_{1c} = -e^{\ln(c^*/k^*)} = \frac{r^* - \rho}{\beta} < 0$$

on account of equation (42b) and (43). Furthermore,

$$\Delta_{1m} = -\sigma e^{\ln(m^*/k^*)} = -e^{(\alpha-1)\ln k^*} + e^{\ln(c^*/k^*)} = -\frac{r^*}{\alpha} + \frac{\rho - r^*}{\beta} = \frac{\rho - r^*(1+\frac{\beta}{\alpha})}{\beta} < 0,$$

i.e. for a positive money depreciation rate Δ_{1m} is negative.[38]

Next, we have

$$\begin{aligned}
\Delta_{2k} &= -\beta e^{\ln(c^*/k^*)} + \alpha(\alpha-1)e^{(\alpha-1)\ln k^*} = -(\rho - r^*) + (\alpha-1)r^* = \alpha r^* - \rho < 0 \\
\Delta_{2c} &= \beta e^{\ln(c^*/k^*)} = \rho - r^* > 0 \quad \text{and} \quad \Delta_{2m} = 0.
\end{aligned}$$

For the effect on money growth we get

$$\begin{aligned}
\Delta_{3k} &= -\beta e^{\ln(c^*/k^*)} + +\alpha(\alpha-1)e^{(\alpha-1)\ln k^*} = \Delta_{2k} = \alpha r^* - \rho < 0 \\
\Delta_{3c} &= -\delta e^{\ln(c^*/m^*)} + \beta e^{\ln(c^*/k^*)} = -\delta\left[\frac{\rho + \theta + 2\sigma}{\delta}\right] + \rho - r^* = -(r^* + \theta + 2\sigma) \\
\Delta_{3m} &= \delta e^{\ln(c^*/m^*)} = \rho + \theta + 2\sigma.
\end{aligned}$$

All of this and the definition $\mathbf{\Delta}$ imply that the log-linearized system is given by

[38] Note that for $\sigma = 0$, that is, when the model dichotomizes we would, of course, have $\Delta_{1m} = 0$.

$$
\begin{pmatrix} \frac{d\ln k}{dt} \\ \frac{d\ln c}{dt} \\ \frac{d\ln m}{dt} \end{pmatrix} = \begin{pmatrix} \Delta_{1k} & \Delta_{1c} & \Delta_{1m} \\ \Delta_{2k} & \Delta_{2c} & \Delta_{2m} \\ \Delta_{3k} & \Delta_{3c} & \Delta_{3m} \end{pmatrix}_{k^*, c^*, m^*} \times \begin{pmatrix} d\ln k \\ d\ln c \\ d\ln m \end{pmatrix}
$$

$$
= \begin{pmatrix} r^* & \dfrac{r^* - \rho}{\beta} & \dfrac{\rho - r^*(1 + \frac{\beta}{\alpha})}{\beta} \\ \alpha r^* - \rho & \rho - r^* & 0 \\ \alpha r^* - \rho & -(r^* + \theta + 2\sigma) & \rho + \theta + 2\sigma \end{pmatrix} \times \begin{pmatrix} d\ln k \\ d\ln c \\ d\ln m \end{pmatrix}.
$$

In order to analyze the question how the log-linearized system reacts to a change in monetary policy, that is, to changes in the Gesell Tax and the money growth rate one verifies that the complete linearized system is really given by[39]

$$
\begin{pmatrix} \frac{d\ln k}{dt} \\ \frac{d\ln c}{dt} \\ \frac{d\ln m}{dt} \end{pmatrix} = \mathbf{\Delta} \times \begin{pmatrix} d\ln k \\ d\ln c \\ d\ln m \end{pmatrix} + \begin{pmatrix} \frac{\rho - r^*\left(\frac{\beta+\alpha}{\alpha}\right)}{\sigma\beta} \\ 0 \\ 2 \end{pmatrix} \times d\sigma + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times d\theta \tag{44}
$$

where $d\sigma$ and $d\theta$ are scalars that represent the differential of σ and θ, respectively, and the entries of the column vector v represent the response of the (log-linearized) differential system to a change in σ when the transpose of v is given by $v' \equiv (\frac{\rho - r^*\left(\frac{\beta+\alpha}{\alpha}\right)}{\sigma\beta}, 0, 2)'$ and in steady state $\Delta_{1\sigma} = e^{\ln(m^*/k^*)}$ and $\sigma e^{\ln(m^*/k^*)} = \frac{r^*}{\alpha} - \frac{\rho - r^*}{\beta}$.[40]

In turn, the the entries of the column vector w represent the response of the (log-linearized) differential system to a change in θ when the transpose of w is given by $w' \equiv (0, 0, 1)'$

We can then express the system in (44) in compact form as

$$
\mathbf{J}' = \mathbf{\Delta} \boldsymbol{J} + \mathbf{g} \quad \text{where} \quad \mathbf{J}' = \begin{pmatrix} \frac{d\ln k}{dt} \\ \frac{d\ln c}{dt} \\ \frac{d\ln m}{dt} \end{pmatrix}, \mathbf{J} = \begin{pmatrix} d\ln k \\ d\ln c \\ d\ln m \end{pmatrix}, \text{ and}
$$

$$
\mathbf{g} = \begin{pmatrix} \frac{\rho - r^*\left(\frac{\beta+\alpha}{\alpha}\right)}{\sigma\beta} \\ 0 \\ 2 \end{pmatrix} \times d\sigma + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times d\theta = \begin{pmatrix} \frac{\rho - r^*\left(\frac{\beta+\alpha}{\alpha}\right)}{\sigma\beta} \; d\sigma \\ 0 \\ 2\, d\sigma + d\theta \end{pmatrix}.
$$

This is a nonhomogeneous differential equation system. The homogeneous part is $\mathbf{J}' = \mathbf{\Delta} \boldsymbol{J}$ and depends in an important way on the Jacobian $\mathbf{\Delta}$. In turn, the term $\mathbf{g}$ makes the system nonhomogeneous.

First we solve the homogeneous part $\mathbf{J}' = \mathbf{\Delta} \boldsymbol{J}$, that is $\mathbf{J}' - \mathbf{\Delta} \boldsymbol{J} = \mathbf{0}$, by employing the

[39]Here the assumption is, of course, that the initial values are close to the steady state. Although log-linear approximations are widely used in macroeconomics, the requirement that they apply only as approximations in the neighborhood of the steady sate can be regarded a disadvantage. See, for example, Barro and Sala–i–Martin (2004), p. 111.

[40]Again note that this only holds for a non-zero σ. If $\sigma = 0$, then in view of equation (28) we would have $v' \equiv (\frac{\delta}{\beta}\left[\frac{\rho - r^*}{\rho + \theta + 2\cdot 0}\right], 0, 2)'$ and then there is no effect of a change in θ on k in the transition.

guess $\mathbf{J} = \boldsymbol{x}e^{\lambda t}$. From this we get[41]

$$\mathbf{J}' = \lambda \boldsymbol{x}e^{\lambda t} = \boldsymbol{\Delta}\boldsymbol{x}e^{\lambda t}, \quad \text{hence} \quad \lambda \boldsymbol{x} = \boldsymbol{\Delta}\boldsymbol{x}.$$

For a nontrivial solution we need the eigenvalues (roots) and the eigenvectors of this three-dimensional system. The general solution of the homogeneous system is given by

$$\boldsymbol{J_h} = \xi_1\, \boldsymbol{x^{(1)}}e^{\lambda_1 \cdot t} + \xi\,_2\boldsymbol{x^{(2)}}e^{\lambda_2 \cdot t} + \xi_3\, \boldsymbol{x^{(3)}}e^{\lambda_3 \cdot t}.$$

For $i = 1, 2, 3$ the roots of the system are given by λ_i, the eigenvectors by $\boldsymbol{x^{(i)}}$, and the arbitrary constants by ξ_i.

As the dynamics of the system is essentially governed by the same variables as in the standard Sidrauski model, we can employ the same arguments as in, for example, Blanchard and Fischer (1989), Appendix B of chapter 4, and Fischer (1979). Hence, we note that the capital stock is given, but the money stock and consumption can jump at any point in time. As a consequence if the system is to have a (locally) unique stable path, it must have two positive roots (or a pair of complex roots with positive real part) and one negative root. If this is the case the jump variables will take on (initial) values that make the system converge.[42]

In fact, we can determine this more rigorously for the present model by following arguments. It is well known that the product of the roots is equal to the determinant of $\boldsymbol{\Delta}$. Calculating the determinant then yields

$$\begin{aligned} |\boldsymbol{\Delta}| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \quad &= \quad \rho^2 r^* - \frac{\rho^2 r^*}{\alpha} + 2\rho r^* \sigma - \frac{2\rho^2 r^* \sigma}{\alpha} + \rho r^* \theta - \frac{\rho^2 r^* \theta}{\alpha} \\ &= \quad -\frac{(1-\alpha)\rho\, r^*(\rho + 2\sigma + \theta)}{\alpha} < 0. \end{aligned}$$

Thus, either all three roots are negative, or there are two positive and one negative root. If the system features saddle path stability, the latter is true and we additionally should have that the trace of $\boldsymbol{\Delta}$ which equals the sum of the eigenvalues be non-negative. The latter is easily calculated as

$$tr(\boldsymbol{\Delta}) = \lambda_1 + \lambda_2 + \lambda_3 = 2\rho + 2\sigma + \theta$$

which is indeed positive. Hence, at least one root is positive. With $|\boldsymbol{\Delta}| < 0$ and $tr(\boldsymbol{\Delta}) > 0$ the system is, therefore, saddle-path stable, because for our 3×3 system at least one root is positive so that there can only be one negative root. In summary, there will be two positive and one negative eigenvalue in the system.

One can also calculate the eigenvalues of the system. They are given by

$$\lambda_1, \lambda_2, \lambda_3 = \left\{ -\frac{-\alpha\rho \pm \sqrt{\alpha\,\rho} \cdot \sqrt{\alpha\rho + 4(1-\alpha)\, r^*}}{2\alpha}, \rho + 2\sigma + \theta \right\}.$$

[41] In this section I follow the solution method presented in Kreyszig (2006), ch. 4.

[42] If the system had, for example, three negative roots, then starting from any value of c and m. the system would - locally - converge. There would be nothing to tie down the money stock or the level of consumption c. See Blanchard and Fischer (1989), p. 204.

Let λ_1 denote the negative root. Given the parameters it satisfies

$$\lambda_1 = -\frac{-\alpha\rho + \sqrt{\alpha\,\rho}\cdot\sqrt{\alpha\rho + 4(1-\alpha)\,r^*}}{2\alpha} < 0.$$

It is important to note that the negative root is governing the speed of convergence of the system. The more negative the negative eigenvalue λ_1 is, the faster is the speed at which the system converges to its steady state. In this context, it is not difficult to verify that $d\lambda_1/d\sigma < 0$ and $d\lambda_1/d\theta > 0$. That means that as you increase σ, the root λ_1 will be more negative and so the convergence to the steady state will be faster, whereas an increase in θ is associated with a less negative root, implying that convergence will be slower. From that Proposition 11 in the main text follows in a straightforward manner.

Notice that we cannot have a convergent system when any of the roots is positive and the associated eigenvector $\boldsymbol{x^{(i)}}$ non-zero. One way to rule out explosive paths is to set the arbitrary constant associated with a positive root equal to zero. In our context, $\lambda_1 < 0$, and $\lambda_2, \lambda_3 > 0$, and $\xi_1 \neq 0$, but then we need that $\xi_2 = \xi_3 = 0$ to rule out explosive behaviour. As a consequence the solution to the homogenous system boils down to $\boldsymbol{J_h} = \xi_1\,\boldsymbol{x^{(1)}}e^{\lambda_1\cdot t}$.

For a particular solution of the nonhomogeneous system above and since the vector $\mathbf{g}$ is constant, we try a constant column vector $\boldsymbol{J_p} = \boldsymbol{a}$ with components α_1, a_2 and a_3.[43] As a consequence, $\boldsymbol{J'_p} = \mathbf{0}$ and substitution in the system $\mathbf{J}' = \boldsymbol{\Delta J} + \mathbf{g}$ yields $\boldsymbol{\Delta a} + \mathbf{g} = \mathbf{0}$. Solving for the components of $\mathbf{a}$, we get the following system under the assumptions made so far

$$\boldsymbol{J} = \boldsymbol{J_h} + \boldsymbol{J_p} = \xi_1\,\boldsymbol{x^{(1)}}e^{\lambda_1\cdot t} + \mathbf{a}.$$

The last step then is to use the inial conditions to definitize the constant ξ_1. Let $\tilde{\xi}_1$ denote the definitzed constant and let $\tilde{\xi}_1 \cdot \boldsymbol{x^{(1)}} \equiv \widetilde{\boldsymbol{x}}^{(1)}$. Then the solution of our system is given by

$$\boldsymbol{J} = \boldsymbol{J_h} + \boldsymbol{J_p} = \widetilde{\boldsymbol{x}}^{(1)}e^{\lambda_1\cdot t} + \mathbf{a}.$$

The numerical simulation below clarifies the procedure in more detail.

E.1 Numerical simulation

From the values in Tables 1 and 2 one gets the following numerical representation of the system in (44),

$$\begin{pmatrix} \frac{d\ln k}{dt} \\ \frac{d\ln c}{dt} \\ \frac{d\ln m}{dt} \end{pmatrix} = \begin{pmatrix} 0.0769 & -0.2305 & -0.0004 \\ -0.0743 & 0.0230 & 0.0000 \\ -0.0744 & -0.1069 & 0.1300 \end{pmatrix} \times \begin{pmatrix} d\ln k \\ d\ln c \\ d\ln m \end{pmatrix} + \begin{pmatrix} -0.035\,d\sigma \\ 0 \\ 2\,d\sigma + 1\,d\theta \end{pmatrix}$$

where $d\sigma$ and $d\theta$, our variables of interest in this section, denote the differentials of σ and θ which are constants. The numerical convergence analysis was carried out in MATHEMATICA. The code used for the results is available upon request.

The 3×3 matrix represents the Jacobian $\boldsymbol{\Delta}$. The roots λ of the homogenous part $\boldsymbol{J_h}$ are given by $(-0.0837, 0.1300, 0.1838)$.

[43] In this paragraph I closely follow Kreyszig (2006), p. 133.

As outlined above I concentrate on the negative root and call it λ_1. Thus, $\lambda_1 = -0.084$. Associated with λ_1 is the eigenvector $(-0.713, -0.469, 0.496)$. Hence, the general solution for our system is given by[44]

$$\boldsymbol{J_h} = \xi_1\, \boldsymbol{x^{(1)}} e^{\lambda_1 \cdot t} = \xi_1 \begin{pmatrix} -0.713 \\ -0.469 \\ -0.496 \end{pmatrix} e^{\lambda_1 \cdot t}.$$

For the particular solution $\boldsymbol{J_p}$ one obtains

$$\boldsymbol{J_p} = \begin{pmatrix} -0.045 d\sigma + 0.0041 d\theta \\ -0.144 d\sigma + 0.0131 d\theta \\ -15.529 d\sigma - 7.6792 d\theta \end{pmatrix}$$

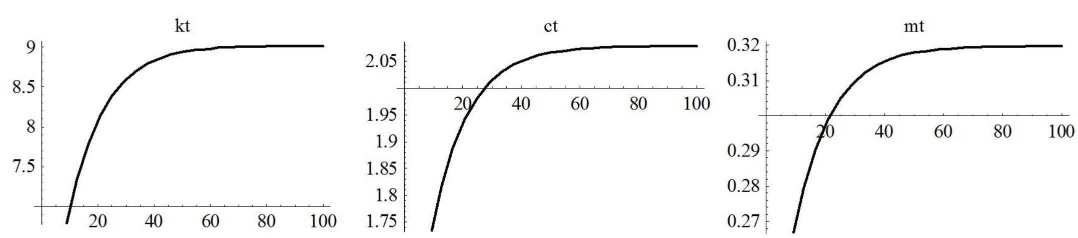
that solves $\boldsymbol{\Delta a} + \mathbf{g} = \mathbf{0}$ when looking at changes in σ and θ. From that we get

$$\begin{pmatrix} d\ln k \\ d\ln c \\ d\ln m \end{pmatrix} = \boldsymbol{J} = \boldsymbol{J_h} + \boldsymbol{J_p} = \xi_1 \begin{pmatrix} -0.713 \\ -0.469 \\ -0.496 \end{pmatrix} e^{\lambda_1 \cdot t} + \begin{pmatrix} -0.045 d\sigma + 0.0041 d\theta \\ -0.144 d\sigma + 0.0131 d\theta \\ -15.529 d\sigma - 7.6792 d\theta \end{pmatrix}$$

as the solution to the system which features in the main text as equation (35).

With the definitized constant $\xi_1^* = 0.827$ the graphs of the variables of interest in levels are presented in the following figure.

Figure 11: The paths of k_t, c_t and m_t in levels



[44] Notice that with these values we get $|\boldsymbol{\Delta}| = -0.002$ and $tr(\boldsymbol{\Delta}) = 0.23$.

F Quotes

- "The material part of money has for economic life about the same importance that the leather of a football has for the players. The players do not concern themselves with the material of the ball, or with its ownership. Whether it is battered or dirty, new or old, matters little; so long as it can be seen, kicked or handled the game can proceed. It is the same with money. Our aim in life is an unceasing, restless struggle to possess it, not because we need the ball itself, the money-material, but because we know that others will strive to regain possession of it, and to do so must make sacrifices. In football the sacrifices are hard knocks, in economic life they are wares, that is the only difference. Lovers of epigram may find pleasure in the following: **Money is the football of economic life.**" Gesell (1920), p. 78.

- "Money requires the State, without a State money is not possible; indeed the foundation of the State may be said to date from the introduction of money. Money is the most natural and the most powerful cement of nations. The Roman Empire was held together more by the Roman currency than by the Roman legions. When the gold and silver mines became exhausted, and coins could no longer be struck, the Roman Empire fell asunder." Gesell (1920), p. 81.

- (*"Usually when a German wants anything he also wants the opposite.", Bismarck. (Gesell (1920), p. 82.)

- "This revenue of the currency administration is an accidental by-product of the reform, and is comparatively insignificant. The disposal of this revenue will be specially provided for by law. (*For other methods of applying the principle of Free-Money see page 245.) p.124"

- "In all conceivable circumstances, in fair weather and in foul, demand will then exactly equal: - The quantity of money circulated and controlled by the State. Multiplied by: The maximum velocity of circulation possible with the existing commercial organisation. What is the effect upon economic life ? The effect is that we now dominate the fluctuations of the market; that the Currency Office, by issuing and withdrawing money, is able to tune demand to the needs of the market; that demand is no longer controlled by the holders of money, by the fears of the middle classes, the gambling of speculators or the tone of the Stock Exchange, but that its amount is determined absolutely by the Currency Office. The Currency Office now creates demand, just as the State manufactures postage stamps, or as the workers create supply. When prices fall, the Currency Office creates money and puts it in circulation. And this money is demand, materialised demand. When prices rise the Currency Office destroys money, and what it destroys is demand. Thus the Currency Office controls the tone of the market, and this means that we have at last overcome economic crises and unemployment. Without our consent the price-level can neither rise or fall. Every movement up or down is a manifestation of the will of the Currency Office, for which it can be made responsible. Demand as an arbitrary act of the holders of money was bound to cause fluctuations of prices, periodic stagnation, unemployment, fraud. Free-Money makes the price-level dependent on the will of the Currency Office which uses its power, in accordance with the purpose of money, to prevent fluctuations. Confronted with the new money everyone will be forced to conclude that the traditional custom of storing up reserves of money must be abandoned, since reserve money steadily depreciates. The new money, therefore, automatically dissolves all money hoards, those of the careful householder, of the merchant and of the usurer in ambush for his prey." p. 127.

- "Under Free-Money, when sales slacken and prices decline, the explanation is no longer given that too much work has been done, that there has been overproduction. We now say that there is a shortage of money, of demand. Whereupon the National Currency Office puts more money in circulation: and since money is now simply embodied demand, this forces prices up to their proper level. We work and bring our wares to market - that is supply. The National Currency Office then considers this supply and puts a corresponding quantity of money on the market - that is demand. Demand and supply are now products of labour. There is now no trace of arbitrary action, of desires, hopes, changing prospects, speculation, left in demand. We order just the amount of demand that we require, and just this amount is created. Our production, the supply of goods, is the order for demand, and the National Currency Office executes the order." p. 134

- "And Heaven help the controller of the Currency office if he neglects to do his duty! He cannot now, like the administration of the old Banks of Issue, entrench himself behind platitudes about having to satisfy "the needs of commerce". The duties imposed on the National Currency Office are sharply defined and the weapons with which we have equipped it are powerful. The German mark, formerly a vague, indefinite thing, has now become a fixed quantity, and for this quantity the officials of the Currency Office are held responsible.

  We are no longer the sport of financiers, bankers, and adventurers; we are no longer reduced to wait in helpless resignation, until, as the phrase used to be, "the state of the market" has the creation and improved. We now control demand; for money, supply of which is in our power, is demand - a fact which cannot be too often repeated or too strongly emphasised. We can now see, grasp and measure demand - just as we can see, grasp and measure supply. **Much produce - much money; less produce - less money. That is the rule of the National Currency Office, an astonishingly simple one!**" p. 134

- "Adam Smith claimed in the *Theory of Moral Sentiments* that economic activities are primarily motivated by a desire of social recognition: 'For to what purpose is all the toil and bustle of this world? What is the end of avarice and ambition, of the pursuit of wealth, of power, and preeminence? Is it to supply the necessities of nature? The wages of the meanest labourer can supply them ... It is the vanity, not the ease, or the pleasure, which interests us.' (pp. 50-51). He went on to suggest that wealth-seeking in order to achieve greater social status may be an engine of economic growth: 'The pleasures of wealth and greatness, when considered in this complex view, strike the imagination as something grand and beautiful and noble, of which the attainment is well worth the toil and the anxiety which we are so apt to bestow upon it. And it is well that nature imposes upon us in this manner. It is this deception which rouses and keeps in continual motion the industry of mankind."' Smith (1759), pp.183-184. Here I follow the presentation in Corneo and Jeanne (2001a), p. 349.