Endogenous Weights and Multidimensional Poverty: A Cautionary Tale*

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Abstract

A strand of the burgeoning literature on counting poverty measurement computes poverty scores weighting each deprivation with weights endogenously determined by the data at hand. Notwithstanding their merits, we discuss some consequences of using endogenous weights in applied multidimensional poverty assessments. In particular, we show how a broad class of endogenous weights violates the key poverty axioms of monotonicity and subgroup consistency. We illustrate the implications of these violations for poverty assessment with the Peruvian National Household Survey ENAHO 2011.

Keywords: Multidimensional poverty, endogenous weights, measurement externalities.

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1 Introduction

It has become increasingly common to understand deprivation from a multi-dimensional perspective. Practitioners undertaking such multidimensional assessments must make several non-trivial methodological decisions, including which dimensions of deprivation to consider among the several possible and how to combine these different dimensions into one single composite index of multi-dimensional poverty. In combining these dimensions into a single index, a natural question to ask is how much weight should we assign to each of them. This paper examines the implications of using endogenous (data driven) weights on a broad set of desirable properties for multidimensional poverty indices (see Bourguignon and Chakravarty, 2003) and demonstrates their failure to satisfy key properties under endogenous weights.

In particular, we investigate analytically and empirically the consequences of using endogenous weights on the fulfillment of two important policy relevant properties that multidimensional poverty indices are expected to satisfy, namely *monotonicity* and *subgroup consistency*. Monotonicity states that if the poverty experience of an individual worsens in any dimension, then the overall poverty experience of the society to which this individual belongs, should not improve Tsui (2002). Subgroup consistency requires that changes in overall poverty in a population, should reflect the changes in poverty happening at the smaller population-subgroup level. For instance, if the population of a country is divided into two subgroups based on regions, say North and South, then if the poverty of the North increases, while the poverty of the South remains unchanged, overall poverty in the country should not decrease under fulfillment of subgroup consistency.

If monotonicity is not satisfied by the poverty index, then we might observe societal poverty falling even when the poverty of some individuals in that society may have increased, without any countervailing decrease in any other individuals' poverty. Failure to satisfy monotonicity can lead to perverse policies whereby increasing individuals deprivation in some dimensions can be deemed beneficial since it will lead to an overall decrease in multidimensional poverty. Failure of subgroup consistency, on the other hand, can lead to a situation where increase in poverty in some regions or populations subgroups, ceteris paribus, may decrease societal poverty. This in turn can lead to policies where increasing poverty in one region or one population subgroup is ignored because overall poverty has decreased. Without these key properties, it would be futile to use a poverty index to undertake any kind of comparative exercise, whether across time, regions or population groups, and thus any evaluation of anti-poverty policies would be ineffective (see Sen, 1976; Foster and Shorrocks, 1991).

This paper demonstrates that for a broad class of endogenous weights and multidimensional poverty indices based on the popular counting approach (Alkire and Foster, 2011), these two fundamental properties will be violated. Operationalisation of the counting approach entails first choosing deprivation dimensions (e.g. access to health services, quality of the dwelling, etc.) and comparing each of them against a deprivation line representing a minimally satisfactory level of that dimension. If a dimension's value is below the line then the person is deemed deprived in that dimension. The total number of dimensions the person is deprived in, is used as a threshold to determine if the person is multidimensionally poor. This originated in the sociology literature where Townsend (1979) considered a person deprived in three or more dimensions as multidimensionally poor in the context of measuring poverty in the UK. In the Alkire and Foster (2011) approach, we can vary the threshold and use different number of dimensions as the cut-off to identify who are multidimensionally poor. Then, typically, societal poverty is measured as the average weighted deprivation count faced by those who are identified as multidimensionally poor. This is broadly the path followed by around fifty countries and twenty political organisations that measure multidimensional poverty including the United Nations Development Programme's (UNDP) flagship Multidimensional Poverty Index (MPI), which is used to evaluate multidimensional poverty globally (see Alkire et al., 2015; MPPN, 2019).

For any such composite measure as the MPI, how to weight the different dimensions is a serious issue. A common approach is to use exogenous weights, which are independent of the dataset and reflect the value judgements of the society, the analyst or the policy-maker. In contrast, one can apply endogenous weights, which are determined by the dataset, to reflect the importance of the different dimensions in the composite measure of deprivation (OECD, 2008; Decanq and Lugo, 2013). Endogenous weights are broadly divided into two classes. The first relies on data reduction techniques such as Principal Components Approach (PCA), Multiple Correspondence Analysis (MCA) or Data Envelopment Approach (DEA) (see e.g. Njong and Ningaye, 2008; Asselin and Anh, 2008; Asselin, 2009; Alkire et al., 2015; Coromaldi and Drago, 2017). These methods assign the weights based on optimisation procedures applied to statistical concepts such as correlation or variance (e.g. the weights of the first principal component in PCA yield the maximum possible variance).

The second broad class of endogenous weights, which is the focus of this paper, establishes a straightforward relationship between the weight assigned to the different dimensions and the frequency of deprivation among the population in the different dimensions based on some normative judgement. For instance, if deprivation along one particular dimension becomes endemic, it may no longer serve as a distinguishing factor and hence should be weighted less in the composite index. Thus as (Deutsch and Silber, 2005, p.150) notes, "...the lower the frequency of poverty according to a given deprivation indicator, the greater the weight this indicator will receive. The idea, for example, is that if owning a refrigerator is much more common than owning a dryer, a greater weight should be given to the former indicator so that if an individual does not own a refrigerator, this rare occurrence will be taken much more into account in computing the overall degree of poverty than if some individual does not own a dryer, a case which is assumed to be more frequent." One could also argue the opposite, namely, as more people become deprived in a dimension, much of the overall deprivation could be generated from that dimension, and hence it should carry a higher weighting in the composite index. For instance, if we observe more people to be deprived in terms of health, compared to say housing, then that situation may reflect institutional shortcomings in the provision of health relative to housing and as a result, health should be given a greater weight to reflect that aspect. Examples of these frequency-driven endogenous weights are ubiquitous in the literature on multidimensional poverty measurement (e.g. see Deutsch and Silber, 2005; Njong and Ningaye, 2008; Aaberge and Brandolini, 2014; Whelan et al., 2014; Alkire et al., 2015; Cavapozzi et al., 2015; Rippin, 2016; Datt, 2017; Abdu and Delamonica, 2018).

This paper explores the problems that arise in multidimensional poverty measurement from the second broad class of endogenous weights, which includes hybrid weights where exogenous weights are used for some dimensions in combination with endogenous weights for other dimensions (e.g. see Dotter and Klasen, 2014). The use of endogenous weights, however, is not just restricted to multidimensional poverty measurement. We also find endogenous weights in other fields such as survey sampling, where both design weights and posterior weights are entirely data driven. In fact the logic of reducing the weighting of a dimension when deprivation in that dimension becomes endemic is akin to the weighting based on inverse sampling probabilities. It is well known that such weights can lead to higher standard errors and can impact inferences and regression coefficients (Kish and Frankel, 1974; Gelman, 2007; Young and Johnson, 2012; Solon et al., 2015). In a similar vein, we demonstrate in this paper that using endogenous (data driven) weights in the construction of composite indices can be problematic.

Endogenous weights, in our context, generate a *measurement externality* since they depend on the distribution of deprivations across the dimensions. Change in one person's deprivation (e.g. because she is no longer deprived in some dimension) affects the deprivation scores of many other people through its impact on the weighting vector. Our appraisal of other people's poverty is thus altered, despite the absence of any objective change in their deprivation status. By contrast, this measurement externality is nonexistent if the weights are set exogenously. This paper examines the implications of measurement externality; we derive results explaining the specific ways in which measurement externalities operate, including how monotonicity and subgroup consistency are violated due to these externalities. In addition, we illustrate the violations using a numerical example and real world data based on the 2011 Peruvian National Household Survey (ENAHO 2011).

The rest of the paper is organized as follows: Section 2 introduces the notation and discusses the basic poverty measurement framework including the important properties of monotonicity and subgroup decomposability. Although our paper is mainly based around these properties, we also discuss two other key properties of multidimensional poverty indices: the focus axiom and the transfer axiom. While the focus axiom ensures that the multidimensional poverty indices are insensitive to the experience of those who are identified as multidimensionally non-poor, the transfer axiom takes in to account the degree of inequality in the deprivations across individuals in the society. In section 3 we introduce a general class of endogenous weights as functions of deprivation frequencies. Using a numerical example we demonstrate violation of the properties monotonicity and subgroup consistency under endogenous weights. Section 4 shows how measurement externalities operate once a person's deprivation status in some dimension changes. This externality is also observed for hybrid weights, as we show. Section 5 provides the main theoretical results on measurement externality and violation of monotonicity and subgroup consistency properties under endogenous weights. In contrast, we also demonstrate analytically that under certain mild restrictions, the axioms of focus and transfer will be satisfied by multidimensional poverty indices even if the weights are endogenous. Using real-world data from Peru, section 6 presents an empirical illustration of violation of the properties of monotonicity and subgroup decomposability under a commonly used endogenous weighting rule. The final section summarises the paper with some concluding remarks.

2 Preliminaries: Counting poverty measurement

Consider a deprivation matrix $\mathbf{X_{ND}}$, with each of N rows representing an individual (or household) and each of D columns representing a dimension of deprivation. We denote any individual as n, where $n = 1, 2, ..., i, i', ...N$, and any dimension as d, where $d = 1, 2, ..., j, j', ...D$. Let $\rho_{nd} \in \{0, 1\}$ denote the deprivation of person n in indicator d in the deprivation matrix $\mathbf{X_{ND}}$. For any individual n, poverty is determined by the deprivations faced by the individual, which are given by the deprivation vector $\boldsymbol{\rho}_{\mathbf{n}}^{\mathbf{X_{ND}}} : \{\rho_{n1}, \rho_{n2}, ..., \rho_{nD}\}$. Note that, for our purpose, we assume that individuals are either fully deprived in a dimension ($\rho_{nd} = 1$) or not at all ($\rho_{nd} = 0$).

Let each dimension of $\mathbf{X_{ND}}$ be weighted, where weight in dimension d is represented as $w_d^{X_{ND}}$. Then we have a weighting vector of strictly positive entries: $\mathbf{W^{X_{ND}}} = (w_1^{X_{ND}}, w_2^{X_{ND}}, ..., w_D^{X_{ND}})$, such that: $\sum_{d=1}^{D} w_d^{X_{ND}} = 1$. As alluded before, weights can be determined either endogenously or exogenously. While exogenous weighting allocates a set value to weight each dimension (which can, for instance, remain constant across different deprivation matrices), when we use endogenous weights we take into account the distribution of the deprivation in each dimension. Thus, under endogenous weights, two different deprivation matrices $\mathbf{X_{ND}}$ and $\mathbf{X'_{ND}}$, for instance, will have different weights for the dimensions. Specifically, $\mathbf{W}^{\mathbf{X_{ND}}} = (w_1^{X_{ND}}, w_2^{X_{ND}}, ..., w_D^{X_{ND}})$ and $\mathbf{W}^{\mathbf{X'_{ND}}} = (w_1^{X'_{ND}}, w_2^{X'_{ND}}, ..., w_D^{X'_{ND}})$, where for some j and j', $w_j^{X_{ND}} \neq w_j^{X'_{ND}}$ and $w_{j'}^{X_{ND}} \neq w_{j'}^{X'_{ND}}$. We describe the weighting functions with precision later in Section 3.

When it comes to measuring societal poverty in $\mathbf{X_{ND}}$, there are two possible paths one can take (Dutta et al., 2003). One path is to first aggregate the deprivation across all the households in each dimension, yielding a dimension-specific index, and then aggregate over all the dimensions to construct the composite index. A prominent example of this in the context of poverty is the UNDP's Human Poverty Index (UNDP, 2009). Under the second path, one aggregates over dimensions for each household first, yielding a household- (or individual-) specific poverty index, and then aggregates over households (or people) to obtain a societal poverty index. The literature prefers the second path because we are interested in the individuals' poverty rather than the deprivation in the dimensions. The second approach also allows us to capture the potential prevalence of multiple deprivation concentrated among some individuals. Hence this is also the path followed by the MPI of the Human Development Report (UNDP, 2010) (which effectively replaced the Human Poverty Index). In our analysis below we take the second approach, following most major contributions to multidimensional poverty measurement including inter alia Tsui (2002); Bourguignon and Chakravarty (2003); Chakravarty and D'Ambrosio (2006); Alkire and Foster (2011).

2.1 Individual poverty

We consider a generalised version of the counting-based poverty measures proposed by Alkire and Foster (2011) where deprivation is measured through a two-step procedure, allowing us to vary the number of dimensions over which a person needs to be deprived in order to be considered multidimensionally deprived. Once we know which individuals are deprived, the measure takes a weighted aggregate of their deprivation over all the dimensions. The resulting individual poverty function thus has two components: a poverty identification function, ψ, and a poverty severity function, s (Silber and Yalonetzky, 2013).

Following Alkire and Foster (2011), a person is considered multidimensionally deprived if they are deprived in at least $k \leqslant D$ dimensions. For any individual n, let the total count of deprivations be $t_n = \sum_{d=1}^{D} \rho_{nd}$. The identification function ψ, compares t_n against a cutoff $k \in \{1, 2, .., D\}$, in order to identify the person as either poor or non-poor from a multiple-deprivation perspective. Thus,

$$\psi(t_n; k) = \mathbb{I}(t_n \geqslant k). \tag{1}$$

When $k = 1$ the poverty identification function follows a 'union' approach whereby any person with at least one deprivation is deemed poor. The 'union' approach is implicitly or explicitly adopted in practice by a swathe of the literature, especially that using endogenous weights based on data-reduction techniques (e.g. Njong and Ningaye, 2008; Asselin and Anh, 2008; Asselin, 2009; Coromaldi and Drago, 2017). On the other extreme, when $k = D$, poverty identification follows an 'intersection' approach which regards as poor only those who are deprived in all indicators. Between both extremes, several other intermediate approaches exist in a counting framework, corresponding to the other values that k can take (Alkire and Foster, 2011).

The severity component, $s(C_n) : [0, 1] \rightarrow [0, 1]$ measures the severity of the multiple-deprivation experience among poor people (Chakravarty and D'Ambrosio, 2006; Alkire and Foster, 2011; Silber and Yalonetzky, 2013), where the weighted deprivation score (or counting function) is:

$$C_n(\mathbf{X_{ND}}; \mathbf{W^{X_{ND}}}) = \sum_{d=1}^{D} w_d \rho_{nd}, \; 0 \leqslant C_n^{X_{ND}} \leqslant 1. \tag{2}$$

The severity component satisfies the following properties: $s(C_i) > s(C_j)$ whenever $C_i > C_j$, $s(0) = 0$ and $s(1) = 1$. Additionally we may also include the restriction that $s''(C_n) \geqslant 0$. Thus the severity function is monotonic in the weighted deprivation of each individual and it increases at a non-decreasing rate. Straightforward examples of $s(C_n)$ used in the literature include $s(C_n) = C_n$ (Alkire and Foster, 2011), $s(C_n) = e^{\alpha C_n} - 1$ with $\alpha > 0$ (Chakravarty and D'Ambrosio, 2006), or $s(C_n) = (C_n)^{\beta}$ with $\beta \geqslant 1$ (Datt, 2018).

Thus, for any deprivation matrix $\mathbf{X_{ND}}$, the individual poverty function for individual n, $p_n^{X_{ND}} : \{0,1\} \times [0,1] \longrightarrow [0,1]$, takes the form:

$$p_n^{X_{ND}}(t_n, C_n; k) = \psi(t_n; k)s(C_n). \tag{3}$$

It combines the identification and the severity components to yield a measure of overall deprivation at the individual level.

2.1.1 Properties of the Individual Poverty Function

In order to be useful as a poverty measure, we would demand the individual poverty function to satisfy a minimum requirement of individual monotonicity whereby an increase in an individual's deprivation along some dimension should not decrease their poverty.

We say $\mathbf{X'_{ND}}$ is obtained *by a simple increment of deprivation in dimension* j *of individual* i from $\mathbf{X_{ND}}$ if, $\rho_{ij}^{X'_{ND}} = 1, \rho_{ij}^{X_{ND}} = 0$ and $\forall (n,d) \neq (i,j)$, $\rho_{nd}^{X'_{ND}} = \rho_{nd}^{X_{ND}}$. Let $\Delta\rho_{nd} = \rho_{nd}^{X'_{ND}} - \rho_{nd}^{X_{ND}}$ and $\Delta p_n = p_n^{X'_{ND}}(t_n, C_n; k) - p_n^{X_{ND}}(t_n, C_n; k)$. Then we can state the following:

Axiom 1. *Individual monotonicity (IM): Suppose* $\mathbf{X'_{ND}}$ *is obtained from* $\mathbf{X_{ND}}$ *by a simple increment of deprivation in dimension* j *of individual* i *. Then* $\Delta p_i \geqslant 0$.

Individual monotonicity is akin to the property of monotonicity put forth by Asselin and Anh (2008), whereby the composite individual poverty function, p_n, must be monotonically increasing in the deprivation of each of the primary indicators. Here we are linking changes in the individual's deprivations to their poverty function (which depends on their counting function through equation (3)). Note that in our definition of the axiom (1) an increase in one deprivation does not necessarily translate into a strict increase in poverty. This is because, depending on the value of k, an increase in one deprivation of a non-poor person may not be enough to reclassify them as poor. Therefore, for that person, $\psi = 0$ even after the increase in that deprivation, resulting in $\Delta p_n = 0$.

2.2 Societal poverty

The societal poverty function aggregates the individual poverty experiences measured by the individual poverty function in (3). Therefore, we can represent it as $P : [0, 1]^N \longrightarrow [0, 1]$. In its most general form it could be written as:

$$P(\mathbf{X_{ND}}; \mathbf{W^{X_{ND}}}, k) = f(p_1^{X_{ND}}, p_2^{X_{ND}}, ..., p_N^{X_{ND}})$$

where $\partial P/\partial p_i \geqslant 0$ and $\partial P^2/\partial p_i \partial p_j = 0$ (assuming differentiability of P). It implies that societal poverty P should be at the least non-decreasing in its constituent parts and there is separability between them (Blackorby et al., 1978). Note that $\mathbf{W^{X_{ND}}}$ could be either exogenous or endogenous, in principle.

There are different methods of aggregation, each satisfying a set of properties. Throughout the rest of the paper, we follow a common additively decomposable societal poverty function:

$$P(\mathbf{X_{ND}}; \mathbf{W^{X_{ND}}}, k) = \frac{1}{N} \sum_{n=1}^{N} p_n^{X_{ND}} \tag{4}$$

A popular example of societal poverty function in (4) is the adjusted headcount ratio by Alkire and Foster (2011) in which $s(C_n) = C_n$. Besides being additive decomposable, the functions in (4) satisfy other key properties such as symmetry, where each individuals impact on societal poverty depends on their level of poverty and nothing else, and population invariance principle where if the population is doubled with exactly the same distribution of deprivation, societal poverty remains unchanged (see (Silber and Yalonetzky, 2013)).

2.2.1 Properties of the societal poverty function

The properties related to societal poverty functions can be broadly classified into three types: invariance axioms, dominance axioms and subgroup axioms (Foster et al., 2010, p. 497). We shall consider properties from each of these broad categories.

First, from the invariance axioms, we consider the focus axiom, which states that the changing circumstances of the non-poor should not impact our assessment of deprivation faced by the poor (as long as the non-poor do not fall into poverty themselves).

Let $Q(\mathbf{X_{ND}})$ be the set of multi-dimensionally poor people in $\mathbf{X_{ND}}$ and let $\Delta P = P(\mathbf{X'_{ND}}; \mathbf{W}^{\mathbf{X'_{ND}}}, k) - P(\mathbf{X_{ND}}; \mathbf{W}^{\mathbf{X_{ND}}}, k)$, then based on Alkire and Foster (2011) we can write the focus axiom as:

Axiom 2. *Focus (F): Suppose $\mathbf{X'_{ND}}$ is obtained from $\mathbf{X_{ND}}$ by a simple increment of deprivation in dimension j of individual i, where $i \notin Q(\mathbf{X'_{ND}})$, then $\Delta P = 0$.*

Second, from the dominance axioms we consider two key properties. First, following Bourguignon and Chakravarty (2003) and Alkire and Foster (2011), under monotonicity an increase in a person's deprivation should not decrease societal poverty P. The monotonicity axiom can be written as:

Axiom 3. *Monotonicity (M): Suppose $\mathbf{X'_{ND}}$ is obtained from $\mathbf{X_{ND}}$ by a simple increment of deprivation in dimension d of individual i, then $\Delta P \geqslant 0$.*

Additionally, we can also define another dominance axiom which captures the intuition that whenever a relatively poorer individual is made less deprived in one dimension at the expense of a relatively richer individual, then societal poverty should not increase. As an example consider a multidimensional poverty assessment over five dimensions and let health be one of them. Let individual A be deprived in one dimension but not in health and individual B be deprived in all dimensions including health. Suppose individuals are identified as multidimensionally poor if they are deprived in any dimension (i.e. a 'union' approach). Now consider a new situation where individual B is not deprived in health, but individual A is now deprived in health, with all else remaining the same. Thus individual B's total deprivation will decrease while A's will increase. In that case a transfer axiom demands that societal poverty should not increase (Bourguignon and Chakravarty, 2003). This is because, in some sense the deprivation burden after the transfer is shared more equally.

To formally express the transfer axiom we say $\widetilde{\mathbf{X}}_{\mathbf{ND}}$ is obtained from $\mathbf{X_{ND}}$ by a rank-preserving *progressive transfer* of deprivation j if for two poor individuals, $i, i' \in Q(\mathbf{X_{ND}})$, such that the weighted deprivation *score* of i is less than i''s, i.e. $C_i^{\mathbf{X_{ND}}} < C_{i'}^{\mathbf{X_{ND}}}$: $\rho_{ij} = 0$ $\rho_{i'j} = 1$; $\widetilde{\rho}_{ij} = 1$, $\widetilde{\rho}_{i'j} = 0$, $\rho_{nd} = \widetilde{\rho}_{nd}$ for all $n \neq \{i, i'\}$ and

$d \neq \{j\}$, and $C_i^{\widetilde{\mathbf{X}}_{\mathbf{ND}}} \leqslant C_{i'}^{\widetilde{\mathbf{X}}_{\mathbf{ND}}}$ (i.e. the deprivation scores do not switch ranks). Then the transfer axiom can be written as:

Axiom 4. *Transfer (T): Suppose* $\widetilde{\mathbf{X}}_{\mathbf{ND}}$ *is obtained from* $\mathbf{X}_{\mathbf{ND}}$ *by a progressive transfer, then* $\Delta P \leqslant 0$.

Finally, from the third broad category of subgroup axioms we consider the property of subgroup consistency. Following Foster and Shorrocks (1991), subgroup consistency is the requirement whereby if two societies $\mathbf{X}_{\mathbf{ND}}$ and $\mathbf{Y}_{\mathbf{ND}}$ are each composed of two subgroups, then if societal poverty of two of the subgroups across the societies are same, then the difference in poverty between $\mathbf{X}_{\mathbf{ND}}$ and $\mathbf{Y}_{\mathbf{ND}}$ should reflect the difference in poverty between the two other subgroups. In other words, the changes in the poverty of the subgroups should be reflected in the societal poverty.

For a formal definition, we first say the deprivation matrix $\mathbf{X}_{\mathbf{ND}}$ is a *subgroup decomposable matrix* if it is formed by vertical concatenation of two matrices $\mathbf{X}_{\mathbf{N_1D}}$ and $\mathbf{X}_{\mathbf{N_2D}}$ where $N = N_1 + N_2$. We represent it as $\mathbf{X}_{\mathbf{ND}} = (\mathbf{X}_{\mathbf{N_1D}} \parallel \mathbf{X}_{\mathbf{N_2D}})$. Then the axiom of Subgroup Consistency can be stated as:

Axiom 5. *Subgroup Consistency (SC): Suppose* $\mathbf{X}_{\mathbf{ND}} = (\mathbf{X}_{\mathbf{N_1D}} \parallel \mathbf{X}_{\mathbf{N_2D}})$ *and* $\mathbf{Y}_{\mathbf{ND}} = (\mathbf{Y}_{\mathbf{N_1D}} \parallel \mathbf{Y}_{\mathbf{N_2D}})$ *be two subgroup-decomposable deprivation matrices.* P *satisfies subgroup consistency if* $[P(\mathbf{X}_{\mathbf{N_1D}}; \mathbf{W}^{\mathbf{X}_{\mathbf{N_1D}}}, k) > P(\mathbf{Y}_{\mathbf{N_1D}}; \mathbf{W}^{\mathbf{Y}_{\mathbf{N_1D}}}, k) \wedge P(\mathbf{X}_{\mathbf{N_2D}}; \mathbf{W}^{\mathbf{X}_{\mathbf{N_2D}}}, k) = P(\mathbf{Y}_{\mathbf{N_2D}}; \mathbf{W}^{\mathbf{Y}_{\mathbf{N_2D}}}, k)] \Rightarrow P(\mathbf{X}_{\mathbf{ND}}; \mathbf{W}^{\mathbf{X}_{\mathbf{ND}}}, k) > P(\mathbf{Y}_{\mathbf{ND}}; \mathbf{W}^{\mathbf{Y}_{\mathbf{ND}}}, k)$.

When weights are exogenous, P in equation (4) satisfies both monotonicity and subgroup consistency (in fact the latter is implied by additive decomposability). However, as shown below, P does not satisfy either monotonicity or subgroup consistency when weights are endogenous (notwithstanding being additively decomposable).

3 A class of endogenous weights

We first introduce a general class of endogenous weights which includes some of the proposals in the literature (see Deutsch and Silber, 2005; Datt, 2017, for reviews) . For that purpose, consider a deprivation matrix $\mathbf{X_{ND}}$. We define the function of aggregate deprivation in dimension d as:

$$h_d = h(\rho_{1d}, \rho_{2d}, ..., \rho_{qd}) \tag{5}$$

such that $h(\rho_{1d}, ..., \rho_{id} = 1, ..., \rho_{qd}) - h(\rho_{1d}, ..., \rho_{id} = 0, ..., \rho_{qd}) \geqslant 0$ where q is the number of multi-dimensionally deprived people (i.e. the cardinality of the set Q). In addition, we assume that re-arranging the deprivations across individuals in dimension d, will not change the aggregate deprivation in that dimension, h_d. Therefore the aggregate deprivation is based only on those individuals who are identified as deprived. We now define the weight of dimension d as:

$$w_d = H_d(h_1, ..., h_d, ..., h_D), \tag{6}$$

where $H_d(.)$ is continuous, and $\partial H_d(h_1, ..., h_j, ..., h_D)/\partial h_j \gtreqless 0$, $\forall j$. Thus weights are based on some continuous function of the aggregate deprivation in each dimension. We allow for the possibility that if the aggregate deprivation in dimension j increases, the weight of that dimension may increase, decrease or remain unaltered. Since these are weights over dimensions, a natural constraint requires that the weights sum up to one, i.e.$\sum_{d=1}^{D} w_d = 1$.

3.1 Hybrid weights

In this paper we investigate broad cases where the weights are endogenously determined for all the dimensions. However, we could also consider situations where the weights of some dimensions are exogenously given while the rest of the dimensions bear endogenously determined weights. This type of weighting would be deemed hybrid since it includes both exogenous and endogenous weighting schemes. For instance, consider a multi-dimensional deprivation assessment over four dimensions: health, income, housing and education. Suppose the weights on health and income are determined exogenously and those on housing and education are determined endogenously. It may be reasonable to have, for instance, a certain proportion of the total weight, say θ, on health and income. Within health and income one may decide to allocate α_h and $(1 - \alpha_h)$ proportions of the total θ

weight to health and income respectively. Hence, the remainder $(1-\theta)$ proportion would be assigned to the endogenously weighted dimensions and within the endogenous dimensions the allocation of the $(1-\theta)$ proportion of the total weight will be based on endogenous weighting schemes given in (6).

Suppose S is the set of all dimensions, $S_E \subset S$ is the set of endogenously weighted dimensions with cardinality d_E, and S'_E represents the set of exogenously weighted dimensions. We can represent a hybrid weighting function for a deprivation matrix X_{ND} as $\mathbf{W^{m,X_{ND}}} = (w_1^{m,X_{ND}}, w_2^{m,X_{ND}}, ..., w_D^{m,X_{ND}})$ as:

$$\begin{aligned} w_j^m &= \theta.\alpha_j \text{ for all } j \in S'_E \\ w_j^m &= (1-\theta).H_j(h_1, ..., h_j, ..., h_{d_E}) \text{ for all } j \in S_E \subset S \end{aligned} \quad (7)$$

where H_j follows the properties in equation (6), θ is the proportion of total weight given to all the exogenous dimensions, α_d is a fixed proportion of θ going to the exogenous dimension d, such that $\sum_{d=1}^{D} w_d^m = 1$, and $d_E \geqslant 2$. Given that the weights add up to one, there needs to be a minimum of two dimensions whose weights are endogenously determined, otherwise all the weights would be exogenously determined.

3.2 Examples of endogenous weights

Equations (5), (6) and (7) characterise a broad class of endogenous weights. Specific examples of such class, which we use for our numerical example and empirical illustration, are:

$$h_d = \frac{1}{N}\sum_{n=1}^{q} \rho_{nd}. \quad (8)$$

with the endogenous weight for dimension j being:

$$w_j = H_j(h_1, ..., h_j, ..., h_D) = \frac{f(h_j)}{\sum_{d=1}^{D} f(h_d)}, \quad (9)$$

where the function $f(h_j)$ is continuous, and for $\delta > 0$, $f(h_j+\delta) - f(h_j) \gtrless 0$. One example of f with $f(h_j+\delta) - f(h_j) > 0$ is: $f(h_j) = h_j$. On the other hand, an example of f with $f(h_j + \delta) - f(h_j) \leqslant 0$ is: $f(h_j) = -\ln[h(_j)]$ (see Deutsch and Silber, 2005, p. 150). Another possibility is: $f(h_j) = 1 - h_j$. We use this functional form along with equations (8) and (9) for our numerical and empirical expositions. In the original formulations of Cerioli and Zani (1990) and Cheli and Lemmi (1995), the function h maps from a population average of fuzzy membership functions related to dimension j; but, as Deutsch and Silber (2005) explain, these averages boil down to h_d in equation (8) whenever the variable is dichotomic.

This can be easily extended to accommodate the general class of hybrid weights given by (7). As before, let S be the set of all dimensions and $S_E \subset S$ be the set of endogenous dimensions with cardinality d_E and S'_E be the set of exogenous dimensions. Therefore the number of exogenous dimensions is $D - d_E$. Suppose the proportion of total weight given to exogenous variables is $\theta = 0.5$. Furthermore, assume that the weight is equally divided among the exogenous dimensions. Then an example of a hybrid weighting function would be given by:

$$\begin{aligned} w_j^m &= \frac{0.5}{d_E} \, \forall \, j \in S'_E, \\ w_j^m &= \frac{0.5 f(h_j)}{\sum\limits_{d \in S_E} f(h_d)} \forall \, j \in S_E \subset S \end{aligned} \tag{10}$$

3.3 Numerical example

To demonstrate the violation of the key properties of monotonicity and subgroup consistency by a multidimensional counting poverty index under endogenous weights, we consider a simple deprivation matrix with ten individuals (in rows) and four dimensions (in columns) in Table 1:

Table 1: Deprivation Matrix showing the Initial Distribution of Deprivation

Individuals	Dimensions				Deprivation Count
	D1	D2	D3	D4	
1	0	0	0	1	1
2	1	0	1	0	2
3	0	0	1	1	2
4	0	0	1	1	2
5	1	0	1	1	3
6	0	1	1	1	3
7	0	1	1	1	3
8	1	1	1	1	4
9	0	1	1	1	3
10	0	0	1	1	2

As a first step, we identify the individuals who are multidimensionally poor using equation (1) which depends on individuals' total number of deprived dimensions. We use cutoff values $k = 1, 2, 3, 4$. Since k determines the number of poor people (q), there will be a different vector of endogenous weights for each k (as per equation 1). For this numerical example, we calculate individual poverty based on (i) $p_n = \psi(t_n; k)C_n$; and (ii) $p_n = \psi(t_n; k)C_n^2$. The societal poverty P is an average of individual poverty functions as in (4). We use endogenous weights of functional form given by equation (9), with $f(h_j) = 1 - h_j$ for our computations.

3.3.1 Violation of Monotonicity

To show the violation of monotonicity, we compute P on the initial deprivation matrix in Table 1 and a changed deprivation matrix derived from Table 1 by an increase in deprivation of individual 2 in dimension D4. Although the total number deprivations in society now has increased, we show that the under endogenous weights P will decrease for several measurement choices (e.g. of k) thus violating monotonicity. In addition, we demonstrate that this violation also holds under hybrid weights, attaching exogenous weights to dimensions D2 and D3, and endogenous weights to dimension D1 and D4. Dimensions D2 and D3 has 35 and 15 percent of the overall weight respectively, giving the exogenous dimensions a share of 50 percent of the total weight. The endogenous weights share jointly 50 percent of the overall weight. For the hybrid weights we follow (10).

Table 2 shows the computed results for (a) the head-count ratio (proportion of multidimensionally poor in the population (i.e. $P_1 = \frac{1}{N}\sum_{n=1}^{N}\psi(t_n;k)$)), (b) societal poverty when severity functions are linear (i.e. $P_2 = \frac{1}{N}\sum_{n=1}^{N}\psi(t_n;k)C_n$), (c) societal poverty when severity functions are squared (i.e. $P_3 = \frac{1}{N}\sum_{n=1}^{N}\psi(t_n;k)C_n^2$), and (d) societal poverty under linear severity function but with hybrid weights as described above (P_4). The first four columns of results relate to the deprivation matrix in Table 1, whereas the last four columns correspond to the deprivation matrix when deprivation of individual 2 increases. In both cases, the rows show results for different choices of the k cutoff.

Table 2: Societal poverty and Monotonicity

	Initial deprivation matrix				Changed deprivation matrix: Simple increment			
k	P_1	P_2	P_3	P_4	P_1	P_2	P_3	P_4
1	1.00	0.420	0.256	0.463	1.00	0.386	0.242	0.425
2	0.90	0.438	0.270	0.481	0.90	0.420	0.263	0.463
3	0.50	0.375	0.290	0.373	0.60	0.443	0.335	0.435
4	0.10	0.100	0.100	0.100	0.10	0.100	0.100	0.100

Note in Table 2 the differences in the values of societal poverty indices between the two deprivation matrices. For values of $k = 1, 2$ societal poverty decreases even though the only change was an increase in one person's deprivation. This is true not only when the severity function is linear, but also when it is squared. We also find that this is the case for hybrid weights. Hence, the monotonicity axiom is clearly violated. The intuition is that, after the change undergone by individual

2, the decreased weight of dimension D4 reduces the deprivation score of all the individuals who are deprived in that dimension. Since most people are deprived in dimension D4, the impact is big enough to trigger a reduction in societal poverty.

However for $k = 3$, the headcount (P_1) is 0.6 instead of 0.5 because individual 2 now is deprived in three dimensions rather than two. The poverty of individual 2 is now part of the $k = 3$ case, which was not the case in the original deprivation matrix. This new identification of individual 2 as poor (when $k = 3$) dominates any decrease in the other individuals' poverty. Hence, societal poverty increases for $k = 3$. For $k = 4$ societal poverty remains unaltered because the proportion of individuals deprived in every dimension remains unaltered (even after becoming deprived in dimension D4, individual 2 is not identified as poor when $k = 4$). In fact, only one individual (8) is poor in all dimensions in both the matrices. We analytically show later that when a person is deprived in all dimensions, then the changes in weights do not impact their deprivation score. Hence, weights become irrelevant in counting poverty indices when the poor are identified using an intersection approach (deeming someone poor only if they are deprived in every dimension).

3.3.2 Violation of Subgroup Consistency

We divide our initial population into two subgroups. Subgroup I consists of individuals 1, 2, 3 and 8 from Table 1, while Subgroup II comprises the remaining seven individuals. Subgroup consistency implies that if, say, poverty in a subgroup changes with all else remaining unchanged, then societal poverty should reflect that change. To that end, in this example, we decrease the poverty in subgroup 1, while keeping everything else unchanged. Under subgroup consistency this would mean that societal poverty should decrease too.

For Subgroup I, Subgroup II and the whole population we compute P_1, P_2, P_3 and P_4 under different k cut-off values: Table 3a shows the results for the original deprivation matrix as given in Table 1.

Table 3a: Subgroup Consistency: Societal Poverty under Initial Deprivation Matrix

	Subgroup I				Subgroup II				Total Population			
k	P_1	P_2	P_3	P_4	P_1	P_2	P_3	P_4	P_1	P_2	P_3	P_4
1	1.00	0.464	0.321	0.492	1.00	0.292	0.135	0.408	1.00	0.420	0.256	0.463
2	0.75	0.438	0.320	0.450	1.00	0.292	0.135	0.408	0.90	0.438	0.270	0.481
3	0.25	0.250	0.250	0.250	0.67	0.417	0.264	0.430	0.50	0.375	0.290	0.373
4	0.25	0.250	0.250	0.250	0.00	0.000	0.000	0.000	0.10	0.100	0.100	0.100

Table 3b shows the computation for societal poverty when we decrease the deprivation of individuals 3 and 8 in dimension D4 and D3 respectively. Note that change in deprivation is happening both in dimensions with exogenous and endogenous weights. This should only impact the poverty values for Subgroup I and the overall population.

Table 3b: Subgroup Consistency: Societal Poverty under the Changed Deprivation Matrix

	Subgroup I				Subgroup II				Total Population			
k	P_1	P_2	P_3	P_4	P_1	P_2	P_3	P_4	P_1	P_2	P_3	P_4
1	1.00	0.417	0.225	0.413	1.00	0.292	0.135	0.408	1.00	0.453	0.265	0.466
2	0.50	0.295	0.184	0.300	1.00	0.292	0.135	0.408	0.80	0.458	0.285	0.455
3	0.25	0.173	0.120	0.213	0.67	0.417	0.264	0.430	0.50	0.356	0.254	0.358
4	0.00	0.000	0.000	0.000	0.00	0.000	0.000	0.000	0.00	0.000	0.000	0.000

Comparing Tables 3a and 3b we can see that for $k = 1$ and $k = 2$, societal poverty in Subgroup I has decreased. This holds true for all the three measures of poverty, P_2, P_3 and P_4. As expected, Subgroup II's poverty remains unchanged because none of the individuals whose deprivation status changed is in that group. Yet when it comes to the whole population, instead of a decrease in societal poverty, for $k = 1$, we see an increase in societal poverty for all the poverty measures including the hybrid weights, and for $k = 2$, for P_2 and P_3 only. Thus the axiom of subgroup consistency is violated.

This violation is occurring because, when the total population is considered, the change in the weights triggers poverty increases among some individuals in Subgroup II. For $k = 3$, we can see that societal poverty is now reduced in line with what the axiom of subgroup consistency would suggest. As the number of multidimensionally poor is reduced (due to increase in the cutoffs k) the decrease in poverty of individual 8 now dominates any possible increase in individual poverty. For $k = 4$, there is no societal deprivation in Table 3b because there are no longer individuals deprived in all dimensions in the new deprivation matrix, as individual 8, the only one deprived in all dimensions (see Table 1), ceases to be deprived in one dimension.

4 Endogenous weights and measurement externalities

Why do we observe this violation of basic properties when using endogenous weights? In this section, we investigate this issue in greater depth. Our focus will be on the weighted deprivation score (or counting function) C_n, because that is where the endogenous weights come in to play. We establish that the change in one person's deprivation status in one dimension changes the counting functions for everyone else too. Thus, there are clear measurement externalities among individuals, which as will be shown later, lead to situations where fundamental properties of poverty functions are violated.

Consider a deprivation matrix $\mathbf{X_{ND}}$, where individual i is considered deprived. Let $\mathbf{X'_{ND}}$ be obtained from $\mathbf{X_{ND}}$ by *a simple increment of deprivation in dimension j of individual i* in $\mathbf{X_{ND}}$. Then the change in the counting function of any individual $n \neq i$, who is also identified as deprived, is:

$$\Delta C_n = C_n^{X'_{ND}} - C_n^{X_{ND}} = \rho_{nj}\Delta w_j + \sum_{\substack{d=1\\d\neq j}}^{D} \rho_{nd}\Delta w_d, \tag{11}$$

where $\Delta C_n = C_n^{X'_{ND}} - C_n^{X_{ND}}$; and $\forall d \in \{1, 2, ..., D\}$, $\Delta w_d = w_d^{X'_{ND}} - w_d^{X_{ND}}$. For simplicity of notation we denote $\rho_{nj}^{X_{ND}} = \rho_{nj}$, $\rho_{nd}^{X'_{ND}} = \rho'_{nd}$.

For the ith individual who became deprived in the jth dimension, we know that $\rho'_{ij}w'_j - \rho_{ij}w_j = w'_j$. Thus, ΔC_i due to a change in the status of person i with respect to dimension j, is given by:

$$\Delta C_i = w'_j + \sum_{\substack{d=1\\d\neq j}}^{D} \rho_{id}\Delta w_d. \tag{12}$$

As long as person i is also deprived in some other dimension, the changes in the other weights produced by the change in i's status regarding j (i.e. $\forall d \neq j$, Δw_d) also affect the total change in C_i. These same changes in weights led by the change in deprivation status of person i in dimension j produce, in turn, changes in the counting function of every other person.

Note that in (11), the change in the counting function will depend on how the endogenous weights change (the signs of Δw_d $\forall d$), and thus will depend on the weighting rule. Hence, *a priori*, the change in any person's counting function (which in turn affects her individual poverty measure, p_n) is ambiguous. Proposition 1 captures how changes in ρ_{ij} can impact weights in each dimension and, through that channel, the counting function of everybody besides person i:

Proposition 1. *Suppose $\mathbf{X'_{ND}}$ is obtained from $\mathbf{X_{ND}}$ by a simple increment of deprivation of deprivation in dimension j for individual i. For all $n \neq i$:*

(i) if $\forall d,\ \rho_{nd} = 0,$ *or* $\forall d,\ \rho_{nd} = 1,$ *then* $\Delta C_n = 0,$

(ii) if $0 < \sum_{d=1}^{D} \rho_{nd} < D,$ *then* $\begin{cases} \Delta C_n \lesseqqgtr 0 \Longleftrightarrow \Delta w_j \lesseqqgtr 0 & \textit{if } \rho_{nj} = 1 \\ \Delta C_n \gtreqqless 0 \Longleftrightarrow \Delta w_j \lesseqqgtr 0 & \textit{if } \rho_{nj} = 0 \end{cases}.$

Proof: Let $\forall d \in \{1, 2, ..., D\}$, $\Delta w_d = \{H_d(h_1, ..., h_j + \delta, ..., h_D) - H_d(h_1, ..., h_j, ..., h_D)\}$, where $\delta \equiv h_j(\rho_{1j}, ..., \rho_{ij} = 1, ..., \rho_{Nj}) - h_j(\rho_{1j}, ..., \rho_{ij} = 0, ..., \rho_{Nj})$.

Case (i): Suppose individual n is not deprived in any dimension. In that case, $\forall d$, $\rho_{nd} = 0$. Thus from (11), we know that $\Delta C_n = 0$. Now suppose individual n is deprived in all D dimensions. Since $\sum_{d=1}^{D} w_d = 1$, we can deduce that:

$$\sum_{d=1}^{D} \Delta w_j = 0 \tag{13}$$

Thus,

$$\Delta w_j = -\sum_{\substack{d=1 \\ d\neq j}}^{D} \Delta w_d. \tag{14}$$

Hence, from (11), $\Delta C_n = 0$.

Case (ii): Suppose for n, $\rho_{nj} = 1$ and $\exists d \neq j$ such that $\rho_{nd} = 0$. Then, from (14), we can infer:

$$|\Delta w_j| > \left| \sum_{\substack{d=1 \\ d\neq j}}^{D} \rho_{nd} \Delta w_d \right|,$$

since the right-hand side of the inequality aggregates over only those dimensions in which individual n is deprived, except j. Thus:

$$\Delta C_n \gtreqless 0 \Longleftrightarrow \Delta w_j = \{H_j(h_1, ..., h_j + \delta, ..., h_D) - H_j(h_1, ..., h_j, ..., h_D)\} \gtreqless 0.$$

On the other hand if, for n, $\rho_{nj} = 0$; then from (11) we get:

$$\Delta C_n = \sum_{\substack{d=1 \\ d\neq j}}^{D} \rho_{nd} \Delta w_d.$$

Then:

$$\sum_{\substack{d=1 \\ d\neq j}}^{D} \rho_{nd} \Delta w_d \gtreqless 0 \text{if } \Delta w_j \lesseqgtr 0.$$

Thus: $\Delta C_n \gtreqless 0 \Longleftrightarrow \Delta w_j = \{H_j(h_1, ..., h_j + \delta, ..., h_D) - H_j(h_1, ..., h_j, ..., h_D)\} \lesseqgtr 0$. ■

Proposition 1 reveals that the effect of rendering individual i deprived in dimension j on the counting function of other individuals, $n \neq i$, only depends on the direction of change in the weight of dimension j, in combination with the deprivation status of n in dimension j. Note that dimension j is the only one where the number of deprived people changes. It plays a central role in understanding the changes in the counting function and, as result, changes in individual poverty levels. *If individual n is deprived in j, then an increase (respectively decrease) in the weight of j leads to an increase (respectively decrease) in n's counting function. Otherwise, if n is not deprived in j then an increase (respectively decrease) in the weight of j reduces (respectively increases) n's counting function.*

Will this result hold for hybrid weights too? So long as the dimension in which the changes occur is endogenous, we will see a similar impact as before on the counting function. Consider a deprivation matrix X_{ND} with an endogenous weighting vector $\mathbf{W^m} = (w_1^m, w_2^m, ..., w_D^m)$, where dimension j is endogenously determined. Now suppose we get X'_{ND} from X_{ND}, through a simple increment of deprivation in dimension j for person i. Then we can write (11) and (12) as:

$$\Delta C_n = \rho_{nj}\Delta w_j^m + \sum_{\substack{d=1\\d\neq j}}^{d_E} \rho_{nd}\Delta w_d^m \tag{15}$$

$$\Delta C_i = w_j'^m + \sum_{\substack{d=1\\d\neq j}}^{d_E} \rho_{id}\Delta w_d^m. \tag{16}$$

where $d \in S_E \subset S$ and Δw_d^m is based on (7). Note that for the dimensions whose weights are exogenous: $\Delta w_d^m = 0$. Thus we can write the following proposition:

Proposition 2. *Suppose $\mathbf{X'_{ND}}$ be obtained from $\mathbf{X_{ND}}$ by a simple increment of deprivation in dimension j of individual i. Suppose we have a hybrid weighting vector $\mathbf{W^{m,X_{ND}}} = (w_1^{m,X_{ND}}, w_2^{m,X_{ND}}, ..., w_D^{m,X_{ND}})$, where dimension $j \in S_E$, the set of endogenously determined dimensions. For all $n \neq i$,*

(i) if $\forall d,\ \rho_{nd} = 0,$ *or* $\forall d,\ \rho_{nd} = 1,$ *then* $\Delta C_n = 0,$

(ii) if $0 < \sum_{d=1}^{D} \rho_{nd} < D,$ *and* $\forall d \in S_E,\ \rho_{nd} = 1$ *then* $\Delta C_n = 0,$

(iii) if $0 < \sum_{d=1}^{D} \rho_{nd} < D,$ *and* $\exists d \in S_E,\ \rho_{nd} = 0$ *then* $\begin{cases} \Delta C_n \lesseqqgtr 0 \Longleftrightarrow \Delta w_j^m \lesseqqgtr 0 & \textit{if } \rho_{nj} = 1 \\ \Delta C_n \gtreqqless 0 \Longleftrightarrow \Delta w_j^m \lesseqqgtr 0 & \textit{if } \rho_{nj} = 0 \end{cases}$

Proof: The part (i) of the proof is similar to Proposition 1. For part (ii),

$\sum_{d=1}^{D} w_d^m = 1$, implies:

$$\sum_{d=1}^{D} \Delta w_j^m = 0 \tag{17}$$

Given that exogenous weights do not change, from (17) one can deduce that:

$$\Delta w_j = -\sum_{\substack{d=1 \\ d \neq j}}^{d_E} \Delta w_d \tag{18}$$

Note that (18) holds over S_E. Thus if $\forall d \in S_E,\ \rho_{nd} = 1$, then from (15) we can immediately deduce that $\Delta C_n = 0$. For part (iii), using similar logic as in Proposition 1 and Δw_j^m, if $\exists d \in S_E,\ \rho_{nd} = 0$, then ΔC_n will be dependent on Δw_j^m. ■

Proposition 2 demonstrates that so long as the change in the level of deprivation of individual i takes place in an endogenous dimension, then measurement externalities will spill over to other individuals, particularly those who are not deprived in all the dimensions whose weights are determined endogenously. This means that the impact of a change in a person's deprivation in a certain dimension on the overall deprivation of others is ambiguous. Hence, societal poverty may increase, decrease or remain the same. Thus, for hybrid weights the broad thrust of our results based on endogenous weights will carry through.

In fact, this problem of measurement externalities is also present in the measurement of monetary poverty with so-called strongly relative poverty lines where the poverty line is usually set as a proportion of the mean or the median of the income distribution (Foster et al., 2013; Ravallion, 2016). A typical example would be for the poverty line to be set at 60 percent of the median income as is done in the UK. If the relative poverty line is based on the mean, then a change in any person's income will generate changes in everyone else's individual poverty function by way of changes in the poverty line itself. At a more practical level, Ravallion (2016, p. 210) reports some empirical cases in Ireland and New Zealand where relative poverty measures moved in exact opposite direction to their counterparts based on absolute poverty lines (i.e. exogenously determined).

The arguments presented in this paper are independent of the externality issues involved in relative poverty lines of monetary poverty measures. For us the 'deprivation line' for each dimension, which determines whether a person is deprived in that specific dimension or not, is fixed and insensitive to the distribution of deprivations. In fact, we take the deprivation of individuals in the different dimensions as a primitive for our analysis. Thus we do not have any externality issues emanating from changing the deprivation lines.

5 Endogenous weights and societal poverty

In this section we discuss how endogenous weights impact on the fulfillment of important properties of multidimensional poverty. The discuss specific properties that taken together, come from a broad set of axioms, such as invariance axioms, dominance axioms and subgroup axioms. We demonstrate that while multidimensional poverty based on endogenous weights under certain conditions satisfies the focus axiom and the transfer axiom, it will invariably violate axioms of monotonicity and subgroup consistency.

5.1 Focus

The Focus axiom effectively states that any changes in the deprivations of the non-poor should not alter societal poverty assessments (as long as the non-poor do not fall into poverty). Hence the 'focus' remains on the poor. In this section, we show that under endogenous weights, given the identification function in equation 1 and the class of endogenous weights based on equations 5 and 6, changes in the non-poor's deprivation does not change overall poverty. We demonstrate that by showing that an increase the deprivation of the non-poor does not lead to a different poverty level.

Consider a deprivation matrix $\mathbf{X_{ND}}$ such that someone is identified as poor if they are deprived in at least k dimensions, i.e. for any individual n, $\psi(t_n; k) = \mathbb{I}(t_n \geqslant k)$. Thus $p_n^{X_{ND}} = 0$ for all n such that $t_n < k$. Note that it is still possible for non-poor individuals to be deprived in more than one dimension. The societal poverty from (3) can be written as:

$$P(\mathbf{X_{ND}}; \mathbf{W}^{\mathbf{X_{ND}}}, k) = \frac{1}{N}\sum_{n=1}^{q} p_n^{X_{ND}}, \tag{19}$$

where q is the number of poor people in the society. Now suppose $\mathbf{X'_{ND}}$ was derived from $\mathbf{X_{ND}}$ by a simple increment of deprivation for some non-poor individual $i \notin Q$. Clearly, if $i \notin Q(\mathbf{X'_{ND}})$, i.e. $t_i < t'_i < k$, then the number of poor people remains the same, which implies that the weights based on (5) and (6) do not change either. Hence, $P(\mathbf{X'_{ND}}; \mathbf{W^{X_{ND}}}, k) = P(\mathbf{X_{ND}}; \mathbf{W^{X_{ND}}}, k)$. Thus the Focus axiom is satisfied.

However, this result whereby the focus axiom is satisfied works because: (i) our identification function is based purely on the total number of deprived dimensions, which essentially weights them exogenously; and (ii) our broad class of endogenous weights does not take into account the non-poor's deprivation status in any dimension. Modifying any of these functional assumptions would immediately trigger a violation of the focus axiom. For example, if equation 5 were replaced with $h_d = h(\rho_{1d}, \rho_{2d}, ..., \rho_{Nd}$ (notice N replacing q at the end), then changes in the deprivation status of the non-poor in any dimension would alter the endogenous weights with concomitant measurement externalities and changes in societal poverty level.

Likewise,changing the identification function ψ by replacing t_i with an endogenously weighted sum of deprivations (essentially the deprivation score C_n) applying to the whole matrix $\mathbf{X_{ND}}$ would also lead to a violation of the Focus axiom, *unless we adopted the union approach to the identification of the poor by setting* $k = 0$. In other words, unless we allowed anybody in society to be potentially poor as long as they are deprived in at least one dimension.

5.2 Monotonicity

One of the main implications of Proposition 1 for societal poverty indices based on endogenous weights (at least those of the form (6)) is that they can violate the desirable axiom of monotonicity (axiom 3). This violation implies, inter alia, that when poor individuals in a society become less deprived, societal poverty may increase. In order to understand how this situation comes about, it is important to derive the impact produced by this change in the deprivation status of person i on the societal poverty index.

For any $\mathbf{X_{ND}}$, let $\Pr[\rho_{nj} = 1 | n \neq i] \equiv \frac{1}{N-1} \sum_{n=1, n\neq i}^{N} \mathbb{I}(\rho_{nj} = 1 | n \neq i)$ (and similar definition for $\Pr[\rho_{nj} = 0 | n \neq i]$). Suppose $\mathbf{X'_{ND}}$ is obtained from $\mathbf{X_{ND}}$ *by increasing* *deprivation in dimension j of individual i.* Then:

$$\begin{aligned}\Delta P = \;& \tfrac{1}{N}\Delta p_i \\ & + \tfrac{N-1}{N}\Pr[\rho_{nj}=1|n\neq i]\tfrac{1}{(N-1)\Pr[\rho_{nj}=1|n\neq i]}\textstyle\sum_{n=1,n\neq i}^{N}\mathbb{I}(\rho_{nj}=1|n\neq i)\Delta p_n \\ & + \tfrac{N-1}{N}\Pr[\rho_{nj}=0|n\neq i]\tfrac{1}{(N-1)\Pr[\rho_{nj}=0|n\neq i]}\textstyle\sum_{n=1,n\neq i}^{N}\mathbb{I}(\rho_{nj}=0|n\neq i)\Delta p_n\end{aligned} \tag{20}$$

where:

$$\Delta p_n = \psi(t_n;k)s(C_n^{X_{ND}} + \Delta C_n) - \psi(t_n;k)s(C_n^{X_{ND}}). \tag{21}$$

That is, the change in societal poverty, ΔP, depends on (i) the change in person i's individual poverty (Δp_i), (ii) the total change in deprivation of other individuals deprived in j (captured in (20) as the average change in the poverty of other people deprived in j $\left(\frac{1}{(N-1)\Pr[\rho_{nj}=1|n\neq i]}\sum_{n=1,n\neq i}^{N}\mathbb{I}(\rho_{nj}=1|n\neq i)\Delta p_n\right)$ multiplied by the proportion of people, other than i, deprived in j ($\Pr[\rho_{nj}=1|n\neq i]$)), and (iii) the total change in deprivation of other individuals who are not deprived in j (shown in (20) as the average change in the poverty of other people not deprived in j $\left(\frac{1}{(N-1)\Pr[\rho_{nj}=0|n\neq i]}\sum_{n=1,n\neq i}^{N}\mathbb{I}(\rho_{nj}=0|n\neq i)\Delta p_n\right)$ multiplied by the proportion of people, other than i, not deprived in j ($\Pr[\rho_{nj}=0|n\neq i]$)).

In the following discussion we show how the three components highlighted above react to an increase in one person's deprivation. First we show that an increase in deprivation in any one dimension for any individual increases their poverty. In others words we show that individual monotonicity (axiom 1) is satisfied.

A helpful corollary stems from (7) and the definition of individual poverty (3):

Corollary 1. *Let $\mathbf{X}'_{\mathbf{ND}}$ be obtained from $\mathbf{X}_{\mathbf{ND}}$ by a simple increment of deprivation in dimension j of individual i. Then individual i's poverty function does not decrease, that is $\Delta p_i \geqslant 0$.*

Proof: First we prove that $\Delta\rho_{ij} > 0$ leads to $\Delta C_i > 0$. From equation (12) we can get:

$$\Delta C_i = w'_j + \sum_{\substack{d=1 \\ d\neq j}}^{D} \rho_{id} \Delta w_d, \tag{22}$$

where w'_j is the weight of dimension j in $\mathbf{X'_{ND}}$. Since $\sum_{d=1}^{D} \Delta w_d = 0$, then $\Delta w_j \gtrless 0$ implies $\sum_{d=1,d\neq j}^{D} \Delta w_d \lesseqgtr 0$. Thus:

$$|\Delta w_j| = \left| \sum_{\substack{d=1 \\ d\neq j}}^{D} \Delta w_d \right| \geqslant \left| \sum_{\substack{d=1 \\ d\neq j}}^{D} \rho_{id} \Delta w_d \right|. \tag{23}$$

Suppose, $\Delta w_j > 0$. Thus from (23) $|w'_j| > |\sum_{d=1,d\neq j}^{D} \rho_{id} \Delta w_d|$ which from (22) implies $\Delta C_i > 0$. Likewise if $\Delta w_j < 0$, we know from (23) $\sum_{d=1,d\neq j}^{D} \Delta w_d > 0$. Given $w'_j > 0$ we can deduce from (22) that $\Delta C_i > 0$.

Let t_i be the total number of dimensions in which individual i is deprived and k is the cut-off for the number of dimensions one has to be deprived to be identified as multidimensionally poor. Then if $t_i \geqslant k$, given $\Delta C_i \geqslant 0$ and the definition of p_n, we can infer that $\Delta p_n \geqslant 0$. Likewise, if $t_i < k$ and $t'_i \geqslant k$ given $\Delta C_i > 0$, then again $\Delta p_n > 0$. Otherwise $\Delta p_n = 0$. ■

An increase in a person's deprivation does not decrease their individual poverty function, therefore the latter satisfies individual monotonicity (axiom 1). Thus, the main problem with counting poverty functions relying on endogenous weights lies elsewhere with the presence of *measurement externalities*.

Next we investigate how the poverty of other individuals change as a result of the change in i's deprivation. Two helpful corollaries stem from (11) combined with Proposition 1 and the definition of individual poverty (3):

Corollary 2. *Let $\mathbf{X'_{ND}}$ be obtained from $\mathbf{X_{ND}}$ by a simple increment of deprivation in dimension j of individual i. Suppose $\Delta w_j > 0$. For any individual $n \neq i$:*

$$\begin{array}{ll} \Delta p_n \geqslant 0 \iff \Delta w_j > |\sum_{d=1,d\neq j}^{D} \rho_{nd} \Delta w_d| & \textit{if } \rho_{nj} = 1 \\ \Delta p_n \leqslant 0 \iff \sum_{d=1,d\neq j}^{D} \rho_{nd} \Delta w_d < 0 & \textit{if } \rho_{nj} = 0 \end{array}.$$

When $\Delta w_j > 0$, from (23) we know that $\sum_{d=1,d\neq j}^{D} \rho_{nd} \Delta w_d < 0$. Thus, $\Delta C_n > 0$. Hence if either $t_n \geqslant k$, or $[t_n < k$ and $t'_n \geqslant k]$, then $\Delta p_n > 0$. Otherwise, $\Delta p_n = 0$.

Using a similar logic, we get the following result when $\Delta w_j < 0$:

Corollary 3. *Let $\mathbf{X}'_{\mathbf{ND}}$ be obtained from $\mathbf{X}_{\mathbf{ND}}$ by a simple increment of deprivation in dimension j of individual i. Suppose $\Delta w_j < 0$. For any individual $n \neq i$:*

$$\begin{array}{ll} \Delta p_n \leqslant 0 \Longleftrightarrow |\Delta w_j| > \left|\sum_{d=1,d\neq j}^{D} \rho_{nd}\Delta w_d\right| & if \rho_{nj} = 1 \\ \Delta p_n \geqslant 0 \Longleftrightarrow \sum_{d=1,d\neq j}^{D} \rho_{nd}\Delta w_d > 0 & if\ \rho_{nj} = 0 \end{array}.$$

Corollary 2 and Corollary 3 demonstrate that, with endogenous weights, $\Delta\rho_{ij} \neq 0$ is bound to produce changes in the poverty of other individuals, $\Delta p_n \neq 0$ where $n \neq i$, which will differ based on their deprivation status regarding j. Therefore, the aforementioned average changes (among those deprived in j and among those not deprived in j) will bear *opposite signs*. Hence, a priori, expression (20) may be positive, negative, or even nil. Thus we can deduce the following result:

Proposition 3. *Let $\mathbf{X}'_{\mathbf{ND}}$ be obtained from $\mathbf{X}_{\mathbf{ND}}$ by a simple increment of deprivation in dimension j of individual i. Then, $\Delta P \gtreqless 0$, thereby violating monotonicity (axiom 3).*

This is a general result, not relying on any particular functional form of the weighting function, or any particular parameters or data. It demonstrates that the change in societal poverty, ΔP, resulting from a change in deprivation in any one dimension experienced by any one poor individual would be ambiguous, thereby violating monotonicity (axiom 3). Therefore, under endogenous weights it is quite possible that if the deprivation of an individual increases in some dimension, overall poverty will decline. Note that this result also holds for any hybrid weighting rule where endogenous weights have been used alongside exogenous weights.

From (21) we know the magnitude of change depends on k; therefore the same change in the deprivation status of i (regarding j) may generate different values and signs for ΔP, depending on the choice of k. Likewise, the specific functional forms chosen for the weights and the severity function, s, also influence the total effect. This was evident in the numerical example in the previous section.

Finally, note that, by contrast, with exogenous weights the score of everybody, except i, remains unaltered: $\Delta C_n = 0,\ \forall n \neq i$. Consequently: $\Delta p_n = 0,\ \forall n \neq i$. Hence, finally, $\Delta P = \frac{1}{N}\Delta p_i$. That is, with exogenous weights, societal poverty changes coherently with the change in person i's individual poverty, as the latter does not affect the poverty measurement of anybody else. Hence monotonicity is fulfilled.

5.3 Transfer

The transfer axiom is also a part of the broad group of dominance properties related to poverty indices (Foster et al., 2010). The basic intuition of our transfer axiom is that if a poor person experiences a new deprivation in a certain dimension, whilst another poor person suffering from a higher deprivation score ceases to be deprived in that same dimension, then we would consider that overall poverty should not increase (as long as the two deprivation scores involved do not switch ranks). Here we demonstrate that for the endogenous weights the transfer property holds as long as the severity function s is convex.

Consider two poor individuals $i, i' \in Q(\mathbf{X_{ND}})$, such that total deprivation score of i is less than n, i.e. $C_i < C_n$. Moreover, we obtain $\mathbf{X'_{ND}}$ from $\mathbf{X_{ND}}$ the following way: $\rho_{ij} = 0 \ \rho_{i'j} = 1;\ \rho'_{ij} = 1,\ \rho'_{i'j} = 0$, and $\rho_{nd} = \rho'_{nd}$ for all $n \neq \{i, i'\}$ and $d \neq \{j\}$. Imagine also that $C'_i \leqslant C'_n$.

Now recall function h_d (equation 5). If h_d is a symmetric function (i.e. a permutation of the deprivation statuses ρ_{id} will not affect h_d) then it should be the case that: $h_j(\rho_{1j}, ..., \rho_{ij} = 0, \rho_{nj} = 1, ..., \rho_{qj}) = h_j(\rho_{1j}, ..., \rho_{ij} = 1, \rho_{nj} = 0, ..., \rho_{qj})$. Therefore, since all h_d remain unchanged, the endogenous weights remain unaltered in turn, i.e. $\Delta w_d = 0 \ \forall d \in \{1, ..., D\}$. Hence the change in poverty due to the transfer is the following:

$$\begin{aligned} N\Delta P = \Delta p_i + \Delta p_n = & [\psi(t_i + 1; k)s(C_i^{X_{ND}} + w_j) - \psi(t_i; k)s(C_i^{X_{ND}})] \\ & + [\psi(t_n - 1; k)s(C_n^{X_{ND}} - w_j) - \psi(t_n; k)s(C_n^{X_{ND}})] \end{aligned} \tag{24}$$

Since i and n remain poor after the transfer, equation 24 boils down to:

$$N\Delta P = [s(C_i^{X_{ND}} + w_j) - s(C_i^{X_{ND}})] + [s(C_n^{X_{ND}} - w_j) - s(C_n^{X_{ND}})] \tag{25}$$

Now, it is clear from equation 25 that the convexity of s guarantees that $\Delta P \leqslant 0$

in the event of a transfer (since $|s(C_n^{X_{ND}} - w_j) - s(C_n^{X_{ND}})| \geqslant |s(C_i^{X_{ND}} + w_j) - s(C_i^{X_{ND}})|$).

Finally, note that if s is not convex, the transfer axiom is violated (as it is apparent from equation 25), because the convexity of s is a requirement for the fulfilment of the transfer axiom. In other words, it is a necessary condition irrespective of how the weights are determined (whether endogenously or not).

5.4 Subgroup consistency

Another key implication of Proposition 1, is that societal poverty indices based on endogenous weights (those of the form (6) can also violate the desirable property of subgroup consistency (axiom 5). In other words, we may find that poverty of a subgroup of the population had declined, all else subgroups remaining unchanged, yet poverty of the whole society has increased. However, this does not mean that subgroup consistency is only relevant when the poverty of one subpopulation changes and all other subpopulations' remaining unchanged. Through repeated application of the property of subgroup consistency we can compare situations when the poverty of one or more of the subpopulations changes (Foster and Szekely, 2008). Violation of subgroup consistency, on the other hand, will essentially imply that societal poverty may increase even if the poverty of all the subpopulations have reduced. Thus this is a powerful axiom which ensures that changes in the poverty of the total population is consistent with the changes happening at the subpopulation level. We claim the following:

Proposition 4. *Suppose for any deprivation matrix* $\mathbf{X_{ND}}$*, the societal poverty measure is given by an additively decomposable poverty function* $P(\mathbf{X_{ND}}; \mathbf{W}, k)$*, where* $\mathbf{W}$ *represents the class of endogenous weights in (6). Then* $P(\mathbf{X_{ND}}; \mathbf{W}, k)$ *fails to satisfy the subgroup consistency (axiom 5).*

Proof: Consider a deprivation matrix decomposed by subgroups $X_{ND} = (X_{N_1D} \parallel X_{N_2D})$ where $N = N_1 + N_2, \forall n \in X_{N_1D}$, $\rho_{nj} = 1$ and $\forall n \in X_{N_2D}, \rho_{nj} = 0$. Suppose $X'_{ND} = (X_{N_1D} \parallel X'_{N_2D})$, where X'_{N_2D} is obtained from X_{N_2D} by increasing deprivation of person i in dimension j, i.e. $\Delta\rho_{ij} = 1$, $i \in X_{N_2D}$. Suppose for X'_{N_2D}: $\Delta w_j = w_j(\rho_{ij} = 1) - w_j(\rho_{ij} = 0) < 0$.

To be subgroup consistent it must be the case that $\Delta P^{X'_{ND}-X_{ND}} \lesseqqgtr 0$ if and only if $\Delta P^{X'_{N_2D}-X_{N_2D}} \lesseqqgtr 0$. Applying (20) we get:

$$\Delta P^{X'_{N_2D}-X_{N_2D}} = \frac{1}{N_2}\Delta p_i^{X'_{N_2D}-X_{N_2D}} + \frac{1}{N_2}\sum_{n\neq i}^{N_2}\mathbb{I}(\rho_{nm}=0)\Delta p_n^{X'_{N_2D}-X_{N_2D}}. \tag{26}$$

In (26), $\Delta p_i^{X'_{N_2D}-X_{N_2D}} \geqslant 0$ from Corollary (1). Also $\Delta p_n^{X'_{N_2D}-X_{N_2D}} \geqslant 0 \ \forall n \neq i$ due to Corollary 3. Therefore, $\Delta P^{X'_{N_2D}-X_{N_2D}} \geqslant 0$. Now:

$$\Delta P^{X'_{ND}-X_{ND}} = \frac{1}{N}\Delta p_i^{X'_{ND}-X_{ND}} + \frac{1}{N}[\sum_{n\neq i}^{N}\mathbb{I}(\rho_{nm}=0)\Delta p_n^{X'_{ND}-X_{ND}} + \sum_{n\neq i}^{N}\mathbb{I}(\rho_{nm}=1)\Delta p_n^{X'_{ND}-X_{ND}}] \tag{27}$$

Again, in (27) $\Delta p_i^{X'_{ND}-X_{ND}} \geqslant 0$. Likewise, $\sum_{n\neq i}^{N}\mathbb{I}(\rho_{nm}=0)\Delta p_n^{X'_{ND}-X_{ND}} \geqslant 0$. However, from Corollary 3, $\sum_{n\neq i}^{N}\mathbb{I}(\rho_{nm}=1)\Delta p_n^{X'_{ND}-X_{ND}} \leqslant 0$. Therefore, $\Delta P^{X'_{ND}-X_{ND}} \lesseqqgtr 0$, unlike $\Delta P^{X'_{N_2D}-X_{N_2D}} \geqslant 0$. In fact, with $N_1 \to \infty$, we can obtain $\Delta P^{X'_{ND}-X_{ND}} \leqslant 0$. ■

Proposition 4 thus demonstrates, in general terms, that endogenous weights will lead to the violation of subgroup consistency (axiom 5). Note that the class of endogenous weights used is very general. As before, this result will also apply to any hybrid weights which includes both endogenous and exogenous weights, so long as the changes in deprivation status are happening in dimensions where the weights are endogenous. In our numerical example, in Section 3.3, where we have shown the violation of subgroup consistency for hybrid weights, the changes in deprivation happened also in dimensions with endogenous weights too. If the changes happen solely in the dimensions with exogenous weights, subgroup consistency will not be violated, as weights would remain the same for all the dimensions.

6 Empirical illustration

We now illustrate the aforementioned measurement externalities in counting poverty indices with endogenous weights, and how they lead to violations of monotonicity and subgroup consistency, in real world data rather than a constructed example, by assessing the impact of a change in one household's deprivation status in one single variable on regional poverty rates in Peru, using the Peruvian National Household Survey 2011 (ENAHO 2011). What is important to keep in mind here is not the magnitude of the change, since it will invariably be small given that the changes happen in a few households in a large data set, but the direction of the change.

For all illustrations, we use seven deprivation indicators covering dwelling infrastructure quality and access to basic services (electricity, telephone, water, and sewerage). The specific indicators, together with their respective deprivation lines, are the following:

- Quality of dwelling floor: household deprived if earth floor or any other lower quality material (not deprived if floor made of parquet, polished wood, tiles, wood, concrete, 'azulejo' tiles).
- Quality of dwelling walls: household deprived if walls made of 'quincha', fiber, or other inferior material (not deprived if walls made of brick, concrete, mud brick, wood, mud and stone, rammed earth).
- Quality of dwelling roof: household deprived if roof made of canes, straw, fiber, palm leaves, or other inferior material (not deprived if roof made of concrete, wood, tiles, sheet metal).
- Access to electricity: household deprived if dwelling lacks electricity service.
- Access to telephone: household deprived if dwelling lacks either land-line or mobile telephone.
- Access to water: household deprived if dwelling lacks access to public network (in urban areas), or deprived if lacking access to public network, water reservoir, or water truck (in rural areas).
- Access to sewerage: household deprived if dwelling lacks access to public network (in urban areas), or deprived if lacking access to public network, or sceptic tank (in rural areas).

All computations are performed for the following geographic regions (sample sizes in parentheses): Northern Coast (3,276), Central Coast (1,951), Southern Coast (1,414), Northern Highlands (1,542), Central Highlands (4,716), Southern Highlands (3,564), Rainforest (5,081), and Metropolitan Lima (2,776).

For the construction of endogenous weights we used the formula given by (9), with $f(h_j) = 1 - h_j$. Weights were computed separately for each geographic domain and depend on the poverty cut-off k since the latter determines the number of poor people, q, which in turn affects the computation of h_j (see 5). As with our numerical example, we compute the individual poverty function $p_n = \psi C_n$. The ensuing societal poverty measure is also known as the *adjusted headcount ratio* (Alkire and Foster, 2011) and will be denoted by P_2. Secondly, we also consider the individual poverty function $p_n = \psi C_n^2$ proposed by Silber and Yalonetzky (2013). Unlike the previous individual poverty index, this functional form features a severity function that is sensitive to the distribution of deprivation scores among the poor (i.e. fulfills the transfer axiom). The ensuing societal poverty index will be denoted by P_3.

Table 4 shows the baseline regional estimates of societal poverty as measured by the adjusted headcount ratio (P_2) with the values for k reported on the left-most column. Among some interesting results, note that the relative ranks of the regions based on societal poverty are fairly robust to different choices of k. Only when $k = 7$ (the intersection approach), usually involving very few people with the most severe poverty experience, some regional ranks flip about (e.g. in the comparison between the Central and the Southern Highlands, or between the Central and Southern Coast). For most values of k the regions turn up ordered in terms of societal poverty the following way (from lowest to highest values): Lima, Southern Coast, Central Coast, Northern Coast, Central Highlands, Southern Highlands, Northern Highlands, Rainforest. In a nutshell, the coastal regions are the least poor, followed by the Highlands, and then the Rainforest exhibits the highest societal poverty levels.

6.1 Violation of Monotonicity

In order to show how poverty measures based on endogenous weights violate the axiom of Monotonicity we performed the following computations using counterfactual deprivation matrices: In each geographical domain we chose a household that was deprived only in two deprivations, including flooring. To all those households

Table 4: Baseline actual estimates of the adjusted headcount ratio with endogenous weights by region

k	Northern Coast	Central Coast	Southern Coast	Northern Highlands	Central Highlands	Southern Highlands	Rainforest	Lima
1	0.174	0.143	0.099	0.242	0.199	0.211	0.291	0.065
2	0.134	0.116	0.076	0.213	0.165	0.176	0.259	0.039
3	0.087	0.076	0.053	0.162	0.118	0.131	0.217	0.022
4	0.047	0.049	0.030	0.090	0.068	0.084	0.169	0.009
5	0.018	0.023	0.017	0.032	0.029	0.039	0.117	0.003
6	0.003	0.011	0.006	0.014	0.006	0.003	0.072	0.000
7	0.000	0.001	0.002	0.002	0.001	0.000	0.037	0.000

(one per region) we *reduced* the deprivation in quality of dwelling floor (i.e. we "granted" them a good-quality floor and reduced their number of deprivations to one). Since $f(h_j) = 1 - h_j$, the removal of these deprivations increases their weight while decreasing the weights of the other dimensions.

As indicated by Proposition 1, and Corollary 3, all households which were deprived in floor quality, experience an increase in their counting function, while the opposite occurs to household that were not deprived in floor quality. The total impact on societal poverty depends on the proportions of household deprived, and not deprived, in floor quality, as well as on the average change in their respective individual poverty functions as a consequence of the simulation (following expressions (20) and (21)). Additionally, the change in the individual poverty of the household whose deprivation was relieved by the simulated relief also contributes to the total change in each region's societal poverty.

Table 5 summarizes the changes in societal poverty measured by P_2 and P_3 produced by the re-computation involving the removal of floor deprivations. The magnitude of changes is small, as should be expected given that only one household per geographic domain was directly affected by the simulation, and that each region has a relatively large sample size (1,414 valid observations in the case of the region with the smallest sample size). Moreover, we only report poverty differences for $k = 1, 2$ because the households affected by the removal of deprivation are not identified as poor in the original deprivation matrix whenever $k > 2$, hence the differences in regional societal poverty are nil for these poverty identification criteria.

As predicted by the theoretical results in the preceding sections, the differences in regional societal poverty are not always negative even though in all cases one person is free from one deprivation, *ceteris paribus*. For instance, in the case of P_2 societal poverty actually increases in the Northern Highlands. This is a clear violation of monotonicity, further corroborated by increases in P_3 for 5 out of 16 region-cutoff combinations (bottom of Table 5) despite reductions in the number of deprivations.

Table 5: Monotonicity: Difference in societal poverty from baseline data in each region

	P_2			P_3		
Regions	k=1	k=2	k=3,..,k=7	k=1	k=2	k=3,..,k=7
Northern Coast	-0.026	-0.077	0.000	-0.008	-0.018	0.000
Central Coast	-0.071	-0.120	0.000	-0.022	-0.023	0.000
Southern Coast	-0.092	-0.200	0.000	-0.038	-0.053	0.000
Norther Highlands	0.038	-0.051	0.000	0.030	0.031	0.000
Central Highlands	-0.016	-0.030	0.000	0.003	0.003	0.000
Southern Highlands	-0.091	-0.048	0.000	0.002	-0.001	0.000
Rain Forest	-0.024	-0.042	0.000	-0.008	-0.007	0.000
Lima	-0.050	-0.110	0.000	-0.021	-0.031	0.000

Note: The values in the table are all multiples of 10^{-3}

Finally, three observations are worth bearing in mind regarding this computation with counterfactual deprivation matrices. Firstly, it changes the counting functions of many households in different directions, thereby affecting their ranking within society in terms of p_n, with potential implications for social program targeting that prioritises households with the lowest poverty indices.

Secondly, in the preceding empirical illustration we found 6 instances of societal poverty increases despite reductions in deprivation out of 32 combinations (2 values of k times 8 regions times 2 poverty measures). Similar manifestations of monotonicity violations may be more or less ubiquitous depending on the dataset and alternative methodological choices including societal poverty functions and selection of deprivation indicators. The key message is that monotonicity is not guaranteed when using counting poverty measures with the broad classes of endogenous weights considered in the theoretical sections.

Thirdly, had this computation been performed with exogenous weights, the counting functions of households not directly affected would have remained unchanged, so that: $\Delta P = \frac{1}{N}\Delta p_i$ where i is the person whose deprivation in sewerage we relieved by simulation. In our particular case, we would get $\Delta p_i = -w_e$ (where w_e is the exogenous weight attached to floor quality) for $t_i \geqslant k$, and $\Delta p_i = 0$ for $k > t_i$. Clearly, with exogenous weights household rankings would change significantly less, and societal poverty would decrease in every geographic domain for $t_i \geqslant k$. For other values of k societal poverty would not be affected.

6.2 Violation of Subgroup consistency

In order to show how poverty measures based on endogenous weights violate the axiom of subgroup consistency we considered only three regions, Southern coast, Southern highlands and Lima with Lima as Subgroup I and Southern coast and Southern highland as Subgroup II. As before (with the numerical example in Section 3.3) we compute estimates for the head count ratio (P_1), and the societal poverty when severity functions are linear (P_2) and societal poverty when severity functions are squared (P_3) for Subgroup I, Subgroup II and the total population which takes all three regions in to account. Table 6a shows the societal poverty based on the data, before any changes are made.

Table 6a: Subgroup Consistency: Societal Poverty under Baseline Data

	Subgroup I Lima			Subgroup II Southern Coast Southern Highlands			Pooled Population		
k	P_1	P_2	P_3	P_1	P_2	P_3	P_1	P_2	P_3
1	0.27	0.071	0.024	0.60	0.210	0.099	0.48	0.165	0.076
2	0.14	0.053	0.022	0.44	0.193	0.099	0.33	0.146	0.075
3	0.06	0.031	0.016	0.29	0.159	0.093	0.21	0.115	0.068
4	0.02	0.014	0.009	0.17	0.113	0.077	0.12	0.079	0.053
5	0.01	0.005	0.004	0.09	0.068	0.052	0.06	0.046	0.035
6	0.00	0.001	0.001	0.03	0.027	0.023	0.02	0.018	0.015
7	0.00	0.000	0.000	0.00	0.001	0.001	0.00	0.001	0.001

Note that for subgroup I there is no household which is deprived in all dimensions. In order to check for subgroup consistency we change the deprivation of some households in Lima. We reduce deprivation in the quality of the flooring of the house for three households and increased deprivation in terms the quality of the wall of the house for two households. We leave Subgroup II unchanged. Table 6b shows the difference between the baseline distribution and the distribution after the changes in deprivation for societal poverty measures in P_2 and P_3.

Table 6b: Subgroup Consistency: Societal poverty when poverty in Subgroup I has decreased

	Subgroup I Lima		Subgroup II Southern Coast Southern Highlands		Pooled Population	
k	P_2	P_3	P_2	P_3	P_2	P_3
1	-0.326	-0.303	0.000	0.000	0.068	0.126
2	-0.324	-0.298	0.000	0.000	0.045	0.104
3	-0.443	-0.424	0.000	0.000	-0.034	0.022
4	-0.484	-0.473	0.000	0.000	-0.097	-0.055
5	-4.628	-2.861	0.000	0.000	-1.607	-0.948
6	3.088	2.646	0.000	0.000	1.114	0.961
7	0.000	0.000	0.000	0.000	0.000	0.000

Note: The values in the table are all multiples of 10^{-4}

Negative values mean that societal poverty has decreased compared to the baseline, whereas positive values show an increase vis-a-vis the baseline. From Table 6b we observe societal poverty for subgroup I has decreased for both P_2 and P_3 and for $k = 1, 2, 3, 4$ and 5. Subgroup II obviously remains unchanged. In accordance with the axiom of subgroup consistency (5, as defined in section 2.2.1), we would expect societal poverty for the pooled population to decrease as well, when measured with the same indices and choices of k. However from Table 6b it is clear that for the pooled population, societal poverty for $k = 1, 2$ has increased for P_2; for P_3 it has increased for $k = 1, 2,$ and 3. Thus for several values of the cutoff k and different measures of societal poverty, subgroup consistency is violated.

As with the illustration discussed in the previous subsection, this violation may severe the relationship between trends in national poverty and geographic targeting (whereby anti-poverty programs are prioritised for people living in the most deprived regions; Bigman and Fofack (2000)) if both the national trends and the geographic targeting were operationalised with the same multidimensional poverty measures and these, in turn, were based on endogenous weights. In such circumstances, it could happen that a well-implemented anti-poverty program succeeds in reducing poverty in a country's most deprived region while seemingly failing to induce an amelioration at the national level. Meanwhile, the problem would be absent if weights of the dimensions were exogenous and thus common across the different deprivation matrices.

7 Conclusions

The use of endogenous weights in multidimensional poverty measurement has enjoyed some popularity, yet the implications of letting weights depend on the dataset have not been studied in depth, above and beyond some reflections and sensible warnings (e.g. Alkire et al., 2015). In this paper we focused on a broad class of endogenous weights, inspired by the examples compiled in Deutsch and Silber (2005). Interestingly the main idea of using frequency weights to generate endogenous weights, is similar to the rank-ordering weights proposed by Sen (1976). The idea also stems from applying the frequency-based weights of the fuzzy-set literature (e.g. as summarised in Deutsch and Silber, 2005) to multidimensional poverty measurement with binary variables. Similar to the Sen (1976) measure, we also find that endogenous weighting leads to violations of monotonicity and subgroup consistency in the context of counting poverty measurement.

Firstly, we precisely explained how genuine changes in the deprivation status of a household (or individual) generate measurement externalities in the form of changes in the counting function of many other households (or people), despite the absence of any changes in the latter's deprivation profiles. These transformations operate through the effect of the original changes in deprivation status on the weights.

Secondly, we found that, depending on the composition of the different contradictory effects on a population's societal poverty measure, new deprivations in one household can increase, decrease, or leave unchanged, the value of societal poverty. That is, societal poverty indices based on endogenous weights violate monotonicity and subgroup consistency. Even though this analysis is yet to be extended to cover endogenous weights based on Multiple Correspondence Analysis (MCA) formally, it is very likely that indices based on these weights suffer from the same problem. As explained by Asselin and Anh (2008) and Asselin (2009, p. 29-30), MCA weights are an inverse function of their respective category's frequency and a direct function of the covariance between that category and the normalised score on a particular factorial axis. In fact, Asselin and Anh (2008) acknowledge this potential problem of inconsistency empirically, and suggest some methods to overcome it (involving dropping indicators or using more than one eigenvector). However, as mentioned, the formal analysis performed in this paper has not yet been implemented for weights based on this technique.

By contrast, societal poverty indices based on exogenous weights satisfy monotonicity and subgroup consistency, so that if one household's poverty experience worsens, then societal poverty will increase, *ceteris paribus*. Likewise, with exogenous weights, the change in the poverty experience of one household produces fewer changes in households' ranks. In fact, relative comparisons between the other households are left unaffected; whereas with endogenous weights many other comparisons of individual poverty are affected, even though deprivation profiles remain intact. Of course, resorting to exogenous weights involves tricky, even potentially arbitrary choices. Best-practice suggestions for choosing exogenous weights are in their infancy, but emerging. For instance, Esposito and Chiappero-Martinetti (2019) monitor multidimensional poverty in the Dominican Republic using exogenous weights generated from a field experiment (independent from their main dataset).

One way of reflecting the relative occurrence of each deprivation in the weights, while satisfying monotonicity and subgroup consistency, could be to compute endogenous weights with one particular dataset and then leave them fixed toward future comparisons. This is precisely what Asselin and Anh (2008) do in their application to poverty comparisons in Vietnam with weights derived from MCA. However this option would not really simplify the complexity of the decision regarding weight selection, since one would still need to decide on the dataset to use in poverty comparisons (e.g. should one use a particular dataset or pool datasets?). Moreover, as pointed out by Alkire et al. (2015, p. 99), if datasets are pooled to compute weights based on data reduction techniques (e.g. MCA, principal component analysis, factor analysis, etc), there is no guarantee that a poverty comparison will be robust to sample updating, e.g. adding new time periods and including the new datasets in a recalculation of weights. Clearly, the latter decisions are hardly less arbitrary than choosing a vector of exogenous weights.

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