Measuring Economic Mobility in India Using Noisy Data: A Partial Identification Approach[*]

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Abstract

We examine economic mobility in India while rigorously accounting for measurement error. Such an analysis is imperative to fully understand the welfare effects of the rise in inequality that has occurred in India over the past few decades. To proceed, we extend recently developed methods on the partial identification of transition matrices and apply this methodology to newly available panel data on household consumption. We find overall mobility has been markedly low: at least 75 percent of the poor households remain poor or at-risk of being poor between 2005 and 2012. We also find Muslims, lower caste groups, and rural households are in a more disadvantageous position in terms of escaping poverty or transitioning into poverty compared to Hindus, upper caste groups, and urban households. These findings suggest inequality in India is likely to be chronic and also challenges the conventional wisdom that marginalized households are catching up on average.

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1 Introduction

There has been a phenomenal rise in economic inequality in India over the past few decades. A 2018 Oxfam study reports a significant increase in the consumption Gini index in both rural and urban areas in India from 1993-94 to 2011-12.[1] According to Global Wealth Report (GWR) 2017, between 2002 and 2012, the share of the bottom 50% of the population in the total wealth declined (increased) from 8.1% to only 4.2%, while that of top 1% increased from 15.7% to 25.7%.[2] Among the countries for which GWR gives the share of wealth held by the top 1%, only Indonesia and the US have higher shares of wealth than India. In a recent study Chancel and Piketty (2018) find that current inequality in India is at its highest level in 96 years. The authors note, "India in fact comes out as a country with one of the highest increase in top 1% income share concentration over the past thirty years" (Chancel and Piketty 2018, p. 29).

Given this dramatic rise in inequality, it is imperative to accurately measure the extent of economic mobility in India. Mobility becomes salient because the long-term welfare effects of rising inequality depend crucially on the level of economic mobility. Economic mobility (or a lack thereof) can attenuate (or accentuate) the adverse effects of inequality. Ceteris paribus, an economy with much economic mobility—one in which households move more freely throughout the income/consumption distribution—will result in a more equal distribution of lifetime incomes and consumption than an economy with low mobility. As discussed in Glewwe (2012) and Dang et al. (2014), the nexus of inequality and mobility has crucial implications for effective policy design. On the one hand, if inequality is high but mobility is low, then low socioeconomic status (SES) households will often find themselves in a poverty trap and policies should perhaps target the acquisition of assets by such households. On the other hand, if inequality and mobility are both high, but households are unable to smooth consumption during periods of low income, then policies that target the sources of consumption volatility or expand access to credit and insurance markets may be more effective.

[1] https://www.oxfamindia.org/sites/default/files/WideningGaps_IndiaInequalityReport2018.pdf.

[2] https://www.credit-suisse.com/about-us/en/reports-research/global-wealth-report.html.

Here, we seek to analyze the degree of economic mobility in India while overcoming the two data issues that have severely constrained prior attempts: (i) a lack of panel data on income or consumption, and (ii) measurement error in income or consumption data. As noted by Fields et al. (2007) among many others, studying mobility ideally requires panel data that tracks a household's income or consumption over time. However, nationally representative, household panel data for India are rarely available (Dang and Lanjouw, 2018). When available, income and consumption is known to suffer from measurement error. As noted by Vanneman and Dubey (2013, p. 441), measurement error is particularly problematic in the Indian context as "most Indian households receive income from more than one source" and this "variety of income sources and household economic strategies presents a much greater challenge for income measurement in India than is typical in rich-country data." Focusing on household consumption instead of income does not ameliorate these issues (Glewwe 1991). Vanneman and Dubey (2013, p. 443) state that, "survey measures of [consumption] expenditures have their own measurement problems (for example, respondent fatigue) and volatility (marriages, debts, and health crises can create unrepresentative spikes for some households)." Deaton and Subramaniam (1996) also note that each item of consumption "is certainly measured with some error" in household surveys, thus leading to measurement error in total expenditures as well. Meyer and Sullivan (2003, p. 1182) conclude: "In practice, survey income, expenditure and consumption are all measured with significant error."

Overcoming measurement error in income or consumption data is not trivial as previous research demonstrates that such errors are nonclassical in the sense that the errors are mean-reverting and serially correlated (Duncan and Hill 1985; Bound and Krueger 1991; Bound et al. 1994; Pischke 1995; Bound et al. 2001; Kapteyn and Ypma 2007; Gottschalk and Huynh 2010; Jäntti and Jenkins, 2015). This introduces added complications in measuring economic mobility. Pavlopoulos et al. (2012, p. 750) conclude that "ignoring [measurement error] can cause 'enormous bias' in the estimation of income/consumption dynamics."

In this paper, we overcome the dual problems of data availability and measurement error in order to examine intragenerational economic mobility in India. To do so, we extend recent work in Millimet et al. (2019) on the partial identification of transition matrices and apply our methodology to newly available panel data on household *consumption* from the Indian

Human Development Survey (IHDS). Our approach *bounds* consumption transition probabilities under different assumptions concerning misclassification errors and the underlying consumption dynamics. First, we derive sharp bounds on transition probabilities under minimal assumptions concerning the measurement error process. Second, we narrow the bounds by imposing more structure via shape restrictions, level set restrictions that relate transition probabilities across observations with different attributes (Manski 1990; Lechner 1999), and monotone instrumental variable (MIV) restrictions that assume monotonic relationships between the true consumption expenditure and certain observed covariates (Manski and Pepper 2000).

We focus on the measurement of *consumption* mobility for two reasons. First, India determines a household's official poverty status using monthly per capita consumption expenditure (see Government of India Planning Commission, 2014). As such, we are able to examine economic mobility *as it relates to poverty* or *poverty mobility*. Second, consumption is conventionally viewed as the preferred welfare indicator in developing countries because it is thought to better capture long-run welfare levels than current income (e.g., Meyer and Sullivan 2003; Carver and Grimes 2019).[3]

Our analysis yields some striking findings. First, we show that modest amounts of measurement error leads to bounds on the true mobility estimates that can be quite wide and almost uniformative in the absence of other information or restrictions. In other words, irrespective of what the estimates of tramsition probabilities based on the mismeasured data, the true mobility estimates could potentially be very different from them. This indicates that mistakenly believing mobility estimates that do not account for measurement error in data gives a false sense of certitude, and that these mobility estimates might be mislead-

[3] Meyer and Sullivan (2003, p. 1210) note:

> "Conceptual arguments as to whether income or consumption is a better measure of material well-being of the poor almost always favor consumption. For example, consumption captures permanent income, reflects the insurance value of government programs and credit markets, better accommodates illegal activity and price changes, and is more likely to reflect private and government transfers."

Furthermore, Carver and Grimmes (2019) find that a consumption-based measure outperforms (surveyed) income in predicting subjective well-being using data from New Zealand.

ing from a policy point of view. Second, the restrictions considered to address measurement error contain significant identifying power as the bounds can be severely narrowed. Third, under our most restrictive set of assumptions but allowing for misclassification errors in up to 20% of the sample, we find that the probability of being in poverty in 2012 conditional on being in poverty in 2005 is at least 28%, the probability of being in a insecure nonpoor state (i.e., monthly per capita household consumption expenditure is between the poverty line and twice the poverty line) in 2012 conditional on being in poverty in 2005 is at least 47%, and the probability of being in a secure nonpoor state (i.e., monthly per capita household consumption expenditure is at least twice the poverty line) in 2012 conditional on being in poverty in 2005 is at most 23%. Under the same set of assumptions, we also find that the probability of being in poverty in 2012 conditional on being in a insecure nonpoor state in 2005 is at least 13% and at most 15%, and the probability of being in poverty in 2012 conditional on being in a secure nonpoor state in 2005 is at least 3% and at most 6%. These figures indicate that mobility has been remarkably low in India. Finally, even upon imposition of the strongest albeit plausible set of assumptions to tighten the bounds we cannot rule out that one could be underestimating the probability of remaining poor over the seven year period and overestimating the probability of escaping poverty to a nonpoor insecure state as well as to the secure nonpoor state if one falsely believes that there is no misclassification error. This indicates that not accounting for misclassification error could actually make the mobility situation in India appear to be brighter than it actually is.

We also compare the mobility rates of various subpopulations, finding evidence of substantial heterogeneity. First, Muslims are more vulnerable to falling below poverty line over the seven year period compared to Hindus or other religious groups; they are also less likely to achieve *secure nonpoor* status, or remain *secure nonpoor*. Second, compared to Brahmin and Non-Brahmin Upper Caste groups and Other Backward Classes (OBCs), Scheduled Castes (SCs) and Scheduled Tribes (STs) are less likely to escape poverty and more likely to move into poverty. Between Brahmins and non-Brahmin Upper Castes and OBCs, OBCs are more likely to move into poverty and less likely to become *secure nonpoor* or remain *secure nonpoor*. Finally, rural households, compared to urban households, are more likely to remain in poverty. They are also less likely to escape poverty and more likely to enter into poverty than the urban households. Overall, our findings suggest inequality in India is paired with relatively low economic mobility. Our findings also challenge the conventional wisdom that marginalized households – those belonging to minority religious groups, lower castes, or living in rural regions – are catching up on average.

The rest of the paper unfolds as follows. In Section 2, we briefly review the existing studies on economic mobility pertaining to India. Section 3 presents the empirical approach. In Section 4 we describe the data. Results are presented in Section 5. The last section concludes.

2 Literature Review

Prior studies have examined economic mobility in India.[4] Early studies in this literature primarily relied on unrepresentative panel data collected for relatively small samples from rural India.[5] Moreover, none of these early studies address measurement error in income or consumption. Subsequent studies assessing income or consumption mobility in India utilize more representative data but continue to ignore measurement error (e.g., Krishna and Shariff 2011; Gautam et al. 2012; Thorat et al. 2017). Recently, a few studies analyze economic mobility in India while accounting for measurement error (Barrientos Q. et al. 2016; Pradhan and Mukherjee 2015; Azam 2016; Arunachalam and Shenoy 2017; Dang and Lanjouw 2018). However, these studies use methods that rely on strong functional form and distributional assumptions, address only certain classes of measurement error (e.g., rank preserving measurement error), or employ instrumental variable (IV) techniques where validity is often suspect and mobility across the entire distribution of income or consumption cannot be assessed.

[4] In addition to studies examining income or consumption mobility, a separate literature assesses intergenerational educational mobility (e.g., Azam and Bhatt 2015; Asher et al. 2018) and intergenerational occupational mobility (Hnatkovska et al. 2013) in India. However, given our focus on intragenerational consumption mobility, we refrain from discussing these studies further.

[5] See Fields (2007) for a brief review of this literature.

Barrientos Q. et al. (2016) use the a panel dataset of rural households collected by the National Council of Applied Economic Research (NCAER) in 1994 and 2005. The authors estimate a bivariate probit model with poverty status in 1994 and 2005 as outcomes. Poverty status in 1994 is included a covariate in the equation for poverty status in 2005 and is instrumented for using land ownership in 1994. However, measurement error in poverty status in 2005 is not addressed, nor is the fact that instrumental variables is not generally a valid solution to measurement error in binary outcomes (Black et al. 2000). Pradhan and Mukherjee (2015) use the ARIS/REDS data spanning three decades (1982-2006) to assess income mobility. The authors employ an IV strategy proposed in Glewwe (2012) to estimate the correlation between 'true' initial and final incomes. Initial income is instrumented for using the dependency ratio (i.e., the ratio of family size to the number of income earners), land ownership, land reform (a dummy that captures the effect of implementation of land reforms in the village), and rainfall shocks. In contrast to Barrientos Q. et al. (2016), Pradhan and Mukherjee (2015) find evidence of low income mobility.

The consistency of these IV studies rests on the validity of the chosen instruments. As noted by Lee et al. (2017, p. 39), "the plausibility of these instruments, as is often the case, can be debated." Specifically, one can argue that the chosen instruments may be correlated with the error term due to the nonclassical nature of the measurement error or other omitted sources of heterogeneity. Moreover, IV techniques require the specification of a particular regression model and thus estimate a single parameter to characterize mobility. Differential mobility across the full income or consumption distribution is absent.

In contrast, Azam (2016) examines economic mobility by calculating directional rank mobility (in addition to the traditional transition probabilities) following a novel approach developed in Bhattacharya and Mazumdar (2011), Mazumdar (2014), and Corak et al. (2014). This approach defines upward (downward) directional rank mobility as the probability that a household's position in the income distribution in the final period surpasses (falls below) by a given amount the household's position in the income distribution in the initial period, conditional on the household's initial position in the income distribution. The author uses longitudinal household survey data collected by the NCAER to examine rural households from 1994 to 2012, as well as data from the IHDS to examine urban households from 2005 to 2012. By focusing on ranks, rather than actual incomes, rank-preserving measurement errors – but not other types of errors – are allowed, as acknowledged in Bhattacharya and Mazumder (2011). This seems like an untenable assumption.

Arunachalam and Shenoy (2017) design a new method to detect household poverty traps and apply it to Indian data. Their method exploits a simple fact: a household just inside the threshold of a poverty trap is likely to suffer negative income growth as the trap pulls the household towards the impoverished steady state. In contrast, a household just above the threshold of a poverty trap is propelled to a higher steady state. Thus, at the threshold, the probability a household experiences negative income growth decreases. More specifically, the existence of poverty trap implies that the probability of negative income growth is a decreasing function of current household income. By contrast, if there are no poverty traps and households are converging to a single steady state, the probability of negative income growth is always rising. Using the ARIS-REDS data (1969-1999), the authors find no evidence of poverty traps. However, as noted in Arunachalam and Shenoy (2017, p. 221), "measurement error...may mask a poverty trap." To address this issue, the authors attempt to devise a consistent measure of household income across survey waves instead of using self-reported income. Nonetheless, Arunachalam and Shenoy (2017, p. 223) state that "given the complexity of a poor household's balance sheet, it is not clear what the ideal measure of income is, let alone whether our definition matches it" and that "[even] these precautions may not remove all measurement error."

Finally, Dang and Lanjouw (2018), using three cross-sectional rounds of data from the National Sample Survet (NSS), compute rates of economic mobility using a synthetic panel approach developed in Dang et al. (2014). The authors posit a static model of consumption using only covariates that are collected in one survey round but whose values can be inferred for the other round (e.g., time invariant variables). The model estimates, along with various assumptions concerning how unobserved determinants of consumption are correlated over time, are used to estimate a poverty transition matrix. The synthetic panel approach implicitly addresses measurement error through the imputation process as missing data can be considered an extreme form of measurement error. However, measurement error in observed consumption used to estimate the static model and compute the poverty transition matrix is not addressed.

Our approach, relative to the approaches used in the existing literature, offers several distinct advantages. First, our approach is based on assumptions that are transparent, easily understood by policymakers, and easy to impose or not impose depending on one's beliefs. Second, the approach is suitable to address a wide class of measurement errors compared to existing methods. Third, given that the approach focuses on estimation of transition matrices, it allows us to examine mobility over the entire distribution of income and consumption. Fourth, the bounds require only data from two points in time and no auxiliary sources of information. Finally, our approach is easy to implement (through the creation of a generic Stata command).[6]

3 Empirical Framework

3.1 Setup

The setup is identical to previous work in Millimet et al. (2019). Thus, we provide only a brief overview, and focus on the extensions in Section 3.2. We relegate the formal derivations to Appendix A.

To begin, let y^*_{it}, denote the true consumption for household i, $i = 1, ..., N$, in period t, $t = 0, 1$. Define the true $K \times K$ transition matrix as $P^*_{0,1}$, given by

$$P^*_{0,1} = \begin{bmatrix} p^*_{11} & \cdots & \cdots & p^*_{1K} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ p^*_{K1} & \cdots & \cdots & p^*_{KK} \end{bmatrix}. \tag{1}$$

[6] Available at http://faculty.smu.edu/millimet/code.html.

Elements of this matrix have the following form

$$
\begin{aligned}
p^*_{kl} &= \frac{\Pr(\zeta^0_{k-1} \leq y^*_0 < \zeta^0_k,\ \zeta^1_{l-1} \leq y^*_1 < \zeta^1_l)}{\Pr(\zeta^0_{k-1} \leq y^*_0 < \zeta^0_k)} \\
&= \frac{\Pr(y^*_0 \in k,\ y^*_1 \in l)}{\Pr(y^*_0 \in k)} \qquad k, l = 1, ..., K,
\end{aligned} \tag{2}
$$

where the ζs are cutoff points between the K partitions such that $0 = \zeta^t_0 < \zeta^t_1 < \zeta^t_2 < \cdots < \zeta^t_{K-1} < \zeta^t_K < \infty$, $t = 0, 1$. Thus, p^*_{kl} is a conditional probability. A complete lack of mobility implies p^*_{kl} equals unity if $k = l$ and zero otherwise. Finally, we can define *conditional* transition matrices, conditioned upon $X = x$, where X denotes a vector of observed attributes. Denote the conditional transition matrix as $P^*_{0,1}(x)$, with elements given by

$$
\begin{aligned}
p^*_{kl}(x) &= \frac{\Pr(\zeta^0_{k-1} \leq y^*_0 < \zeta^0_k,\ \zeta^1_{l-1} \leq y^*_1 < \zeta^1_l | X = x)}{\Pr(\zeta^0_{k-1} \leq y^*_0 < \zeta^0_k | X = x)} \\
&= \frac{\Pr(y^*_0 \in k,\ y^*_1 \in l | X = x)}{\Pr(y^*_0 \in k | X = x)} \qquad k, l = 1, ..., K.
\end{aligned} \tag{3}
$$

Implicit in this definition is the assumption that X includes only time invariant attributes. Moreover, while the probabilities are conditional on X, the cutoff points ζ are not. Thus, we are capturing movements within the *overall* distribution among those with $X = x$.

As discussed in more detail in Section 4, in this study we set $K = 3$. The outcome, y^*, denotes household consumption expenditure *relative to* the official poverty line. In each period t, the partitions are set as $\zeta^t_0 = 0$, $\zeta^t_1 = 1$, $\zeta^t_2 = 2$, and $\zeta^t_3 \to \infty$, $t = 0, 1$. Thus, partition one includes households categorized as officially poor in period t. Partition 2 includes households between 100% and 200% of the official poverty line in period t. Partition 3 includes all households with consumption exceeding 200% of the official poverty line in period t. In the terminology of Millimet et al. (2019), $P^*_{0,1}$ and $P^*_{0,1}(x)$ utilize unequal-sized partitions in our application since the sample is not equally split among the partitions. This means that mobility is not zero-sum; for example, a household may move up to a new partition without another household having to move down.

Our objective is to learn about the elements of $P^*_{0,1}$ or $P^*_{0,1}(x)$. With a random sample

$\{y^*_{it}, x_i\}$, the transition probabilities are nonparametrically identified; consistent estimates are given by the empirical transition probabilities. However, as stated previously, ample evidence indicates that consumption is measured with error. Let y_{it} denote the observed consumption for household i in period t. With data $\{y_{it}, x_i\}$, the empirical transition probabilities are inconsistent for p^*_{kl} and $p^*_{kl}(x)$. Rather than invoking overly strong, and likely implausible assumptions, to point identify the transition probabilities, our goal is to bound the probabilities given in (2) and (3).

To proceed, we characterize the relationships between the true partitions of $\{y^*_{it}\}_{t=0}^1$ and the observed partitions of $\{y_{it}\}_{t=0}^1$ using the following joint probabilities:

$$\theta_{(k,l)}^{(k',l')} = \Pr(y_0 \in k',\ y_1 \in l',\ y^*_0 \in k,\ y^*_1 \in l). \tag{4}$$

While conditional misclassification probabilities are more intuitive, these joint probabilities are easier to work with (e.g., Kreider et al. 2012).

In (4) the subscript (k, l) indexes the true partitions in periods 0 and 1 and the superscript (k', l') indicates the observed partitions. With this notation, we can now rewrite the elements of $P^*_{0,1}$ as

$$\begin{aligned} p^*_{kl} &= \frac{\Pr(y^*_0 \in k, y^*_1 \in l)}{\Pr(y^*_0 \in k)} \\ &= \frac{\Pr(y_0 \in k, y_1 \in l) + \sum\limits_{\substack{k',l'=1,2,\ldots,K \\ (k',l')\neq(k,l)}} \theta_{(k,l)}^{(k',l')} - \sum\limits_{\substack{k',l'=1,2,\ldots,K \\ (k',l')\neq(k,l)}} \theta_{(k',l')}^{(k,l)}}{\Pr(y_0 \in k) + \sum\limits_{\substack{k',l',\tilde{l}=1,2,\ldots,K \\ k'\neq k}} \theta_{(k,\tilde{l})}^{(k',l')} - \sum\limits_{\substack{k',l',\tilde{l}=1,2,\ldots,K \\ k'\neq k}} \theta_{(k',\tilde{l})}^{(k,l')}} \\ &\equiv \frac{r_{kl} + Q_{1,kl} - Q_{2,kl}}{p_k + Q_{3,k} - Q_{4,k}}. \end{aligned} \tag{5}$$

$Q_{1,kl}$ measures the proportion of false *negatives* associated with partition kl (i.e., the probability of being misclassified conditional on kl being the true partition). $Q_{2,kl}$ measures the proportion of false *positives* associated with partition kl (i.e., the probability of being misclassified conditional on kl being the observed partition). Similarly, $Q_{3,k}$ and $Q_{4,k}$ measure the proportion of false *negatives* and *positives* associated with partition k, respectively.

The data identify r_{kl} and p_k (and, hence, $p_{kl} \equiv r_{kl}/p_k$), but not the misclassification parameters, θ. One can compute sharp bounds by searching across the unknown misclassification parameters. However, absent further restrictions, obtaining informative bounds on the transition probabilities is not possible. In the Section 3.2, we introduce assumptions on the θs to potentially yield informative bounds. Some of these assumptions are considered in Millimet et al. (2019), while some are new. Section 3.3 considers restrictions on the underlying mobility process and are identical to those considered in Millimet et al. (2019).

3.2 Misclassification

3.2.1 Assumptions

Allowing for measurement error, we obtain bounds on the elements of $P^*_{0,1}$, given in (5).[7] We begin by considering the following misclassification assumptions from Millimet et al. (2019).

Assumption 1 (Classification-Preserving Measurement Error)**.** *Misreporting does not alter an observation's partition in the consumption distribution in either period. Formally,* $\sum_{k,l} \theta^{kl}_{kl} = 1$ *or, equivalently,*

$$\sum_{\substack{k,k',l,l'=1,2,\ldots,K \\ (k',l')\neq(k,l)}} \theta^{(k',l')}_{(k,l)} = 0.$$

Assumption 2 (Maximum Misclassification Rate)**.**

(i) (Arbitrary Misclassification) *The total misclassification rate in the data is bounded from above by* $Q \in (0,1)$*. Formally,* $1 - \sum_{k,l} \theta^{00}_{kl} \leq Q$ *or, equivalently,*

$$\sum_{\substack{k,k',l,l'=1,2,\ldots,K \\ (k',l')\neq(k,l)}} \theta^{(k',l')}_{(k,l)} \leq Q.$$

(ii) (Uniform Misclassification) *The total misclassification rate in the data is bounded from*

[7] In the interest of brevity, we focus attention from here primarily on the unconditional transition matrix. We return to the conditional transition matrix in Section 3.3.

above by $Q \in (0,1)$ and is uniformly distributed across partitions. Formally,

$$\begin{aligned}
\sum_{\substack{k,k',l,l'=1,2,\ldots,K \\ (k',l')\neq(k,l)}} \theta_{(k,l)}^{(k',l')} &\leq Q \\
\sum_{\substack{k',l,l'=1,2,\ldots,K \\ (k',l')\neq(k,l)}} \theta_{(k,l)}^{(k',l')} &\leq \frac{Q}{K} \;\; \forall k \\
\sum_{\substack{k',l,l'=1,2,\ldots,K \\ (k',l')\neq(k,l)}} \theta_{(k,l)}^{(k',l')} &\leq \frac{Q}{K} \;\; \forall l.
\end{aligned}$$

Assumption 3 (Uni-Directional Misclassification)**.** *Misclassification occurs strictly in the upward direction. Formally,*

$$\begin{aligned}
\theta_{(k,l)}^{(k',l')} &= 0 \;\forall k' < k \\
\theta_{(k,l)}^{(k',l')} &= 0 \;\forall l' < l.
\end{aligned}$$

Assumption 1 is quite strong, but is simply used as a benchmark. Under this assumption, measurement error is allowed as long as it does not cause observations to be classified into incorrect partitions. Assumption 2 places restrictions on the total amount of misclassification allowed in the data. As discussed in Millimet et al. (2019), the amount of misclassification is unknown and dependent on the choice of K. We consider sensitivity to the choice of Q in the application. Under Assumption 2, the number of θs defined by (4) is $K^2(K^2-1)$; which, in our case, equals 72. Assumptions 1 and 2 limit the sum of these parameters, but not the unique number of parameters. In contrast, Assumption 3 rules out the possibility of false positives (negatives) occurring in the worst (best) partition. It reduces the number of misclassification parameters in our case from 72 to 27. Note, Assumption 3 is consistent with mean-reverting measurement error as long as the negative measurement errors for observations with high consumption are not sufficient to lead to misclassification.

Finally, we consider the following two additional assumptions not considered in Millimet et al. (2019). It may be imposed in combination with any of the preceding assumptions.

Assumption 4 (Temporal Independence)**.** *Misclassification probabilities are independent* *across time periods. Formally,* $\theta_{(k,l)}^{(k',l')}$ *simplifies to*

$$\alpha_k^{k'} \bullet \beta_l^{l'},$$

where $\alpha_k^{k'}$ $(\beta_l^{l'})$ *is the probability of being observed in partition* k' (l') *in the initial (terminal) period when the true partition is* k (l)*.*

Assumption 5 (Temporal Invariance)**.** *Misclassification probabilities are independent across time periods as well as temporally invariant. Formally,* $\theta_{(k,l)}^{(k',l')}$ *simplifies to*

$$\alpha_k^{k'} \bullet \alpha_l^{l'}.$$

In other words, $\alpha_k^{k'} = \beta_k^{k'} \ \ \forall k$*.*

Assumption 4 restricts misreporting behavior such that the decision to misreport is independent across periods. This rules out a household's consumption history affecting its propensity to misreport its current consumption. It reduces the number of misclassification parameters to $2K(K-1)$; or, in our case, to 12. Combining Assumptions 4 and 3 further reduces the number of parameters in our case to six. Assumption 5 further restricts the probability of misreporting in particular directions to be constant over the sample period. This assumes that data accuracy and other sources of measurement error such as stigma do not change across time periods. This restriction further reduces the number of parameters to $K(K-1)$; or, in our case, to six. Lastly, combining Assumptions 5 and 3 reduces the number of parameters in our case to three.

3.2.2 Bounds

Under Assumption 1 consistent estimates are given by the empirical transition probabilities (Proposition 1 in Millimet et al. (2019)):

$$\widehat{p}_{kl} = \frac{\sum_i \mathrm{I}(y_{0i} \in k, y_{1i} \in l)}{\sum_i \mathrm{I}(y_{0i} \in k)}.$$

Absent this assumption, the transition probabilities are no longer nonparametrically identified. The bounds under various combinations of Assumptions 2 – 5 are detailed in Appendix A.

3.3 Mobility

3.3.1 Assumptions

The preceding section provides bounds on the transition probabilities considering only restrictions on the misclassification process. Here, we introduce restrictions on the mobility process that may further serve to tighten the bounds. The restrictions may be imposed alone or in combination.

First, we consider shape restrictions which place inequality constraints on the population transition probabilities. Specifically, we assume that large transitions are less likely than smaller ones.

Assumption 6 (Shape Restrictions)**.** *The transition probabilities are weakly decreasing in the size of the transition. Formally, p^*_{kl} is weakly decreasing in $|k-l|$, the absolute difference between k and l.*

This assumption implies that within each row or each column of the transition matrix, the diagonal element (i.e., the conditional staying probability) is the largest. The remaining elements decline weakly monotonically moving away from the diagonal element. This assumption, which may be plausible if large jumps in consumption are less common than small ones.

Second, we consider level set restrictions which place equality constraints on population transition probabilities across observations with different observed attributes (Manski 1990; Lechner 1999).

Assumption 7 (Level Set Restrictions)**.** *The conditional transition probabilities, given in (3), are constant across a range of conditioning values. Formally, $p^*_{kl}(x)$ is constant for all $x \in \mathcal{A}_x \subset \mathcal{R}_m$, where x is an m-dimensional vector.*

For instance, if x denotes the age of the household head in years, one might wish to assume that $p^*_{kl}(z)$ is constant for all z within a fixed window around x.

From (3) and (5), we have

$$
\begin{aligned}
p^*_{kl}(x) &= \frac{\Pr(y_0 \in k, y_1 \in l | X = x) + \sum\limits_{\substack{k',l'=1,2,\ldots,K \\ (k',l')\neq(k,l)}} \theta^{(k',l')}_{(k,l)}(x) - \sum\limits_{\substack{k',l'=1,2,\ldots,K \\ (k',l')\neq(k,l)}} \theta^{(k,l)}_{(k',l')}(x)}{\Pr(y_0 \in k | X = x) + \sum\limits_{\substack{k',l',\tilde{l}=1,2,\ldots,K \\ k'\neq k}} \theta^{(k',l')}_{(k,\tilde{l})}(x) - \sum\limits_{\substack{k',l',\tilde{l}=1,2,\ldots,K \\ k'\neq k}} \theta^{(k,l')}_{(k',\tilde{l})}(x)} \\
&\equiv \frac{r_{kl}(x) + Q_{1,kl}(x) - Q_{2,kl}(x)}{p_k(x) + Q_{3,k}(x) - Q_{4,k}(x)} \qquad (6)
\end{aligned}
$$

where now $Q_{j,\cdot}(x)$, $j = 1, \ldots, 4$, represent the proportions of false positives and negatives conditional on x. As such, we also consider the following assumption regarding the conditional misclassification probabilities.

Assumption 8 (Independence). *Misclassification rates are independent of the observed attributes of observations, x. Formally,*

$$\theta^{(k',l')}_{(k,l)}(x) = \theta^{(k',l')}_{(k,l)}, \ \forall k, k', l, l', x.$$

The plausibility of Assumption 8 depends on one's conjectures concerning the measurement error process. However, two points are important to bear in mind. First, the misclassification probabilities, $\theta^{(k'-k,l'-l)}_{(k,l)}$, are specific to a pair of true and observed partitions. As a result, even if misclassification is more likely at certain parts of the consumption distribution and x is correlated with consumption, this does not necessarily invalidate Assumption 8. Second, Assumption 8 does not imply that misclassification rates are independent of all individual attributes, only those included in the variables used to define the level set restrictions.

Finally, we consider monotonicity restrictions which place inequality constraints on population transition probabilities across observations with different observed attributes (Manski and Pepper 2000; Chetverikov et al. 2018).

Assumption 9 (Monotonicity). *The conditional probability of upward mobility is weakly increasing in a vector of attributes, u, and the conditional probability of downward mobility* *is weakly decreasing in the same vector of attributes. Formally, if* $u_2 \geq u_1$*, then*

$$\begin{aligned}
p_{11}^*(u_1) &\geq p_{11}^*(u_2) \\
p_{KK}^*(u_1) &\leq p_{KK}^*(u_2) \\
p_{kl}^*(u_1) &\leq p_{kl}^*(u_2) \;\; \forall l > k \\
p_{kl}^*(u_1) &\geq p_{kl}^*(u_2) \;\; \forall l < k.
\end{aligned}$$

For instance, if u denotes the education of an individual, one might wish to assume that the probability of upward (downward) mobility is no lower (higher) for individuals with more education. Note, the monotonicity assumption provides no information on the conditional staying probabilities, $p_{kk}^*(u)$, for $k = 2, ..., K-1$.

3.3.2 Bounds

The bounds under various combinations of Assumptions 2 – 9 are relegated to Appendix A. However, as discussed in Millimet et al. (2019), estimates of the bounds suffer from finite sample bias as they rely on infima and suprema. To circumvent this issue, we follow this previous work and utilize a bootstrap bias correction, based on subsampling with replicate samples of size $N/2$. To obtain confidence intervals, we utilize subsampling along with the Imbens-Manski (2004) correction to obtain 90% confidence intervals (CIs).[8] As with the bias correction, we set the size of the replicate samples to $N/2$.

4 Data

4.1 Indian Human Development Survey

The data come from the IHDS. IHDS is a nationally representative multi-topic panel household survey conducted by NCAER in New Delhi and University of Maryland (Desai et al. 2010; Desai et al. 2015). It was designed to complement existing Indian household surveys

[8] The literature on inference in partially identified models is expanding rapidly. However, as discussed in Millimet et al. (2019), the Imbens-Manski (2004) approach is preferable in the current context.

by bringing together a wide range of socio-economic topics in a single survey. The sample was drawn using stratified random sampling.

The first wave of the survey was conducted in 2004-05 and covered 41,554 households in 1,503 villages and 971 urban neighborhoods across India.[9] The second wave was conducted in 2011-12 and covered 42,152 households, 40,018 of whom were interviewed in the first wave.[10] Both waves of the IHDS are based on interviews with a knowledgeable informant from the household. The interviews covered health, education, employment, economic status, marriage, fertility, gender relations, and social capital. The survey instruments were translated into 13 Indian languages and were administered by local interviewers. Both waves are now publicly available through the Inter-university Consortium for Political and Social Research (ICPSR).

The IHDS is well-suited to our inquiry for the following reasons. First, the IHDS is the most recent household panel survey conducted in India. Second, it is a nationwide panel that follows rural and urban households. Moreover, the sample size is fairly large compared to other panel surveys conducted in India. The closest alternative is the ARIS/REDS panel study of Indian households. Although the ARIS/REDS data, collected in four rounds between 1971 and 2006, covers a longer time period than the IHDS, it surveys only rural households. Furthermore, the sample size of ARIS/REDS is much smaller; the first round of the ARIS/REDS covered 4,527 households across 259 villages and the latest round covered 9,500 households.[11]

[9] The survey covered all the states and union territories of India except Andaman and Nicobar, and Lakshadweep. These two account for less than 0.05 percent of India's population.

[10] However, this includes 'split-households,' which refers to those households who had split from a single unit to multiple units between 2005 and 2012. Specifically, of the 40,018 households, the number of households who were living as a single unit in both the first and second survey rounds was 30,462, while the rest of were 'split households'. For the 'split households', although their second period characteristics (e.g., household income) differs from one another, their first period characteristics are the same (since they were living as a single entity in the first period).

[11] Other alternatives are a small sample from six ICRISAT villages beginning in the mid-1970s (Naschold 2012; Dercon et al. 2013) and the long-term study of the village of Palanpur since the 1950s (Himanshu and Stern 2011). However, neither is as large or as representative as the IHDS.

4.2 Analytic Sample

We examine economic mobility using per capita monthly consumption expenditure. It is derived from total annual household consumption expenditure. In the 2004-05 wave, this is aggregated from information on forty-seven different consumption categories. In the 2011-12 wave, this is aggregated from information on fifty-two different consumption categories.

Our outcome variable is based on the poverty ratio (POVRATIO) of the household, defined as the ratio of a household's per capita monthly household consumption expenditure to the corresponding poverty line. The poverty line is based on the official Indian poverty line for per capita monthly consumption as recommended by the Suresh Tendulkar committee in 2012. The poverty line is state-, year-, and urban-/rural-specific (see Table B1 in Appendix B).[12] As noted by the former Deputy Chairman of the Planning Commission of India, Montek Singh Ahluwalia, the Tendulkar poverty line is to be used as a relevant reference point "to see how development is helping to take more and more individuals above a fixed line over time and across states."[13] According to a recent report in *The Hindu* (18 March 2016), the NITI Ayog, the policy think tank of the Government of India that replaced the Planning Commission in 2015, also favors the use of the Tendulkar poverty line for tracking progress in combating extreme poverty.[14]

Using POVRATIO, we partition the households into three parts. The first partition consists of households whose POVRATIO is less than one; these are the households who are officially classified as poor. The second partition consists of households whose POVRATIO is at least one but does not exceed 2; we refer to these households are *insecure nonpoor* since these households are at-risk of becoming impoverished.[15] The third partition consists of the households whose POVRATIO is at least 2; we will refer to these as *secure nonpoor*. Using these three partitions, we estimate 3×3 consumption transition matrices for the full sample

[12] See Government of India Planning Commission (2009) report for details about the Suresh Tendulkar poverty line.

[13] https://www.thehindu.com/news/national/tendulkar-poverty-line-will-remain-reference-point/article2509910.ece.

[14] https://www.thehindu.com/business/Industry/niti-aayog-task-force-backs-tendulkar-poverty-line/article8371390.ece.

[15] The term insecure nonpoor has been used in an USAID report to describe people living at a level less than twice the poverty line in Uganda (see https://www.un.org/development/desa/dspd/wp-content/uploads/sites/22/2018/03/Uganda-Case-Study.pdf)

as well as for different subsamples.

When imposing level set restrictions, we use age of the household head in the initial period. Specifically, we group households into the age bins – less than 35, 35-44, 45-54, 55-64, 65 and above – and impose the restriction that mobility is constant across adjacent bins. For example, we tighten the bounds for households where the head is, say, 35-44 by assuming that mobility is constant across households where the head is less than 35 years of age and where the head's age is between 45 and 54 years. When imposing the monotonicity restrictions, we use the education level of the household head in the initial period. Here, households are grouped into four bins based on years of completed schooling: zero, 1-5, 6-10, and 11-15.

Our sample consists of 38,737 households from across India: these are the households who were interviewed in both the waves of the survey, who have no missing (or invalid) information on consumption, income, and other demographic characteristics of the household head (e.g., education, age, gender, caste, religion), annual total household income and annual total consumption expenditure are non-negative, and age of the household head in the first round is at least 18. Summary statistics are presented in Table 1.

5 Results

5.1 Full Sample Analysis

Results for the 3×3 transition matrix based on the full sample are presented in Tables 2-6.[16] In all tables, for both the time periods, the partitions consisting of poor households, insecure nonpoor households, and secure nonpoor households are represented by the numbers 1, 2, and 3 respectively. Overall, between 2005 and 2012, the observed poverty rate declined from 33.5% to 15.4%, the proportion of insecure nonpoor households rose slightly from 42.7% to 43.9%, and the proportion of secure nonpoor households increased from 23.7% to more than 40% (see Table 1). Turning to mobility, Table 2 presents our baseline results under the strong assumption of Classification-Preserving Measurement Error (which is equivalent to the as-

[16] For brevity, we do not report bounds based on all possible combinations of restrictions. Unreported results are available upon request.

sumption that there is no misclassification). The probability of a household remaining in poverty across the initial and terminal period is 0.279, the probability of remaining insecure nonpoor is 0.472, and the probability of remaining secure nonpoor is 0.670. Furthermore, we find that the probabilities of observing larger transitions in the consumption distribution are less likely than smaller movements. For example, the probability of moving from impoverished to insecure nonpoor is 0.510; the probability of moving from impoverished to secure nonpoor is 0.211. Similarly, the probability of moving from the secure nonpoor state to insecure nonpoor is 0.281; the probability of moving from the secure nonpoor state to impoverished is 0.048.

Misclassification Assumptions **Table 3 allows for a wider class of misclassification errors. For our baseline analysis, we assume the maximum misclassification rate as 20% ($Q = 0.20$). This choice is primarily guided by the findings of Millimet et al. (2019). Specifically, Millimet et al. (2019) using data from the US and employing a simulation-based approach to quantify Q show that the misclassification rate is roughly 20% when the data is discretized into three partitions. Since it is unlikely that the extent of measurement error is lower in a developing country like India compared to the US, assuming $Q = 20\%$ for our baseline analysis seems reasonable. However, since there is a possibility that Q could actually be higher than 20%, in Appendix C, we present bounds for additional values of Q ranging upto 40%.**[17]

Panels I and II in Table 3 restrict misclassification errors to be arbitrary (Assumption 2(i)) and uniform (Assumption 2(ii)), respectively. Panels III and IV further restrict the misclassification errors to be only in the upward direction (Assumption 3). Thus, Column A presents results under Assumptions 2(i) and 2(ii) with and without Assumption 3. Column B adds the assumption of temporal independence (Assumption 4) to the preceding assumptions,

[17] It would have been ideal to carry out an exercise similar to Millimet et al. (2019) to quantify Q in context of India. However, in the present paper we could not do that. This is because Millimet et al.'s (2019) approach relies heavily on the paramter values of the measurement error process which they obtain from the previous literature on measurement error in US income and consumption data. For India (and almost all other developing countries) such information is absent due to lack of previous research aiming to quantify measurement error in consumption data for India.

while Column C adds the assumption of temporal invariance (Assumption 5).

In Panel IA the bounds are nearly uninformative. This means a relatively moderate amount of arbitrary misclassification, in the absence of other information, results in an inability to say much more about mobility over this time period. This indicates that mistakenly believing mobility estimates that do not account for measurement error gives a false sense of certitude. For example, while based on the assumption of no misclassification error (or rank preserving misclassification error), one may be tempted to believe that probability of a household remaining in poverty across the initial and terminal period is 28%, in reality the true estimate of the probability of a household remaining in poverty across the initial and terminal period could be anything between zero and 88%. Similarly, while based on the assumption of no misclassification error (or rank preserving misclassification error), one may be tempted to believe that probability of a household moving out of poverty to a nonpoor secure state initial and terminal period is 21%, in reality the true estimate could anything between zero and 81%. In order to get back some of this certitude and for being able to something meaningful about the mobility rates while accounting for misclassification error, we go on to add misclassification assumptions and mobility restrictions

We start by adding the assumption of temporal independence. As evident from the results reported in Panel IIB, with the added assumption of temporal independence, the bounds narrow significantly. For example, the probability of remaining impoverished is at least 8.0%, and of transitioning from impoverished to insecure (secure) nonpoor is at least 31.1% (1.2%).[18] For households initially insecure nonpoor, their chances of remaining insecure nonpoor is at least 31.6%, and of transitioning from insecure nonpoor to secure nonpoor (impoverished) is at least (at most) 25.7% (27.1%). Adding the stronger assumption of temporal invariance in Panel IC, the bounds are further tightened. For example, for the poor, the bounds on probability remaining in poverty over the sample period narrows from $[0.080, 0.478]$ in Panel 1B to $[0.107, 0.366]$. While the assumptions of temporal independence and temporal invariance have a lot of identifying power, these assumptions rule out behaviors such as households impoverished in the initial period over-reporting their consumption in

[18] Throughout the discussion of the results, unless otherwise noted, we focus on the point estimates for simplicity. The confidence intervals are generally not much wider than the point estimates of the bounds.

both the periods to avoid any stigma, or misclassification probabilities changing over time due to changes in stigma associated with poverty. Nonetheless, since the waves are seven years apart, these assumptions may be plausible.

In Panel II misclassification errors are assumed to be uniformly distributed across the three partitions. This is a strong assumption as mean-reverting measurement error might imply greater misclassification at the tails of the distribution. With this in mind, the bounds are considerably narrowed relative to their counterparts in Panel I. In Panel IIA, the conditional staying probability for impoverished households is at least 8.0% and at most 59.7%. The conditional staying probability for insecure (secure) nonpoor households is at least 31.6% (39.0%). The probability of transitioning from impoverished to insecure (secure) nonpoor is at least 31.1% (1.2%). Conversely, the probability of transitioning from insecure (secure) nonpoor to impoverished is at most 32.2% (45.6%). However, we are not able to rule out the possibility (at the 90% confidence level) that no households transition into poverty over the sample period from insecure nonpoor and secure nonpoor states. As in Panel I, the bounds in Panels IIB and IIC are considerably more narrow. In Panel IIB, for example, the bounds on probability of remaining impoverished in 2005 and 2012 narrow from $[0.080, 0.597]$ to $[0.146, 0.345]$. Similarly, the bounds on probability of transitioning from impoverished to insecure (secure) nonpoor state narrow from $[0.311, 0.886]$ ($[0.012, 0.512]$) to $[0.377, 0.627]$ ($[0.078, 0.278]$). In Panel IIC, the bounds narrow further. Now, the bounds on probability of remaining impoverished are $[0.209, 0.329]$. Similarly, the bounds on probability of transitioning from impoverished to insecure (secure) nonpoor narrows to $[0.444, 0.601]$ ($[0.161, 0.261]$).

Panels III and IV add Assumption 3. This assumption is consistent with mean-reverting measurement error as long as downward measurement errors among households truly in the upper part of the distribution are not sufficient to cause them to be misclassified into lower partitions, which seems reasonable. This assumption has no identifying power on the downward transition probabilities. However, it does tighten the remaining bounds. In Panel IIIA, bounds on the probability of remaining impoverished are $[0.174, 0.876]$. In Panel IVA the bounds on the probability of remaining impoverished are narrowed further to $[0.232, 0.478]$. Thus, the probability of escaping poverty is at least 17.4% and 23.2%, respectively. Furthermore, in Panel IIIA, the probability of transitioning from impoverished to insecure (secure) nonpoor is at most 82.6% (50.6%); 70.9% (34.2%) in Panel IVA. These bounds, as before, significantly narrow in Columns B and C. For example, in Panel IIIB (Panel IIIC), the bounds on the probability of remaining impoverished are $[0.279, 0.478]$ ($[0.279, 0.366]$). In Panel IVB (Panel IVC), the corresponding bounds are $[0.279, 0.345]$ ($[0.279, 0.329]$). These bounds are very narrow.

While the assumptions of uniform and uni-directional misclassification certainly tighten the bounds, Table 3 highlights the limited information that can be learned under Assumptions 2 and 3 alone, allowing for a 20% misclassification rate. Thus, even relatively modest amounts of misclassification add considerable uncertainty to estimates of poverty mobility. That said, one still learns that even if misclassification is not temporally independent or temporally invariant, the seven-year poverty persistence rate is at least 17.4% under Assumptions 2(i) and 3 and is at least 23.2% under Assumptions 2(ii) and 3. Moreover, at most 50.6% (34.2%) transition from impoverished to secure nonpoor over the seven-year sample period under Assumptions 2(i) (2(ii)) and 3. Adding Assumptions 4 or 5 certainly adds to identification.

Level Set Restrictions Table 4 imposes different combinations of assumptions along with Assumptions 7 and 8. The level set restrictions are based on the age of the household head in the initial period. The level set restriction (Assumption 7) seem reasonable in that households with similarly aged heads may face the same mobility rates. The independence assumption (Assumption 8) may be more problematic as it assumes that households with different aged heads have similar patterns of mis-reporting. This might be violated if different cohorts experience different levels of stigma associated with economic well-being. However, absent Assumption 7 without Assumption 8 has no identifying power. With that in mind, our primary result is that the level set and independence restrictions considered here have a modest amount of identifying power relative to the corresponding bounds in Table 3.

For example, bounds for conditional staying probability for impoverished (insecure nonpoor) under arbitrary misclassification in Panel IA of Table 3 are $[0.000, 0.876]$ ($[0.004, 0.940]$) and are $[0.000, 0.851]$ ($[0.030, 0.929]$) in Panel IA in Table 4. Bounds on the probability of transitioning from secure nonpoor to impoverished under uniform, uni-directional, and temporally independent (invariant) misclassification in Panel IVB (Panel IVC) are $[0.000, 0.142]$ ($[0.000, 0.119]$). This is tightened to $[0.000, 0.131]$ ($[0.026, 0.086]$) in Panel IIIC in Table 4. In this final case, this means that under the Assumptions 2(ii), 3, 5, 7, and 8, we reject (at the 90% confidence level) the hypothesis that no households transition secure to nonpoor to impoverished over the seven-year sample period. The majority of the remaining bounds in Table 4 are similarly tightened relative to their counterparts in Table 3.

Shape Restrictions Table 5 adds the shape restriction (Assumption 6) to the previous set of assumptions considered in Table 4. The shape restriction imposes the belief that households are more likely to experience smaller transitions in the consumption distribution than larger transitions. This assumptions seems reasonable given the typical assumed pattern of consumption dynamics. Compared to Table 4, we find that adding the shape restriction modestly narrows some, but not all, of the bounds. In particular, the bounds in Panels IIC, IIIB, and IIIC are not narrowed at all, while the bounds in Panels IA, IB, and IIB are only minorly affected. Thus, under the strongest set of assumptions considered to this point, shape restrictions have limited or no additional identifying power.

The largest impact is on the bounds in Panels IIA and IIIA. For example, under uniform and independent misclassification errors in Panel IIA, the bounds on the conditional staying probability for being impoverished are $[0.318, 0.574]$ versus $[0.107, 0.574]$ in Panel IIA of Table 4. When adding the uni-directional assumption in Panel IIIA, the bounds on the conditional staying probability for being impoverished are $[0.318, 0.463]$ versus $[0.250, 0.463]$ in Panel IIIA of Table 4.

Monotonicity Restrictions Table 6 adds the monotonicity restriction (Assumption 9) to the previous set of assumptions considered in Table 5. The monotonicity restriction requires upward mobility to be weakly increasing in the household head's education level. Given the returns to human capital, this seems to be a reasonable assumption. The monotonicity assumption has some additional identifying power across all panels.

First, under arbitrary and independent misclassification errors in Panel IA, the bounds now exclude the lower endpoint of zero in some instances. Moreover, adding the assumption of temporally invariant errors in Panel IC, the probability of remaining impoverished increases from at least 16.1% to at least 23.6% and falls from at most 35.7% to at most 34.2%. Second, under our strongest set of assumptions (i.e., uniform, independent, unidirectional and temporally invariant misclassification) in Panel IIIC, bounds on the conditional staying probability of remaining impoverished narrow from $[0.281, 0.327]$ without monotonicity in Table 5 to $[0.281, 0.313]$. Similarly, monotonicity tightens the bounds on the probability of transitioning from being impoverished to insecure (secure) nonpoor; from $[0.446, 0.542]$ ($[0.177, 0.227]$) without monotonicity in Table 5 to $[0.468, 0.506]$ ($[0.187, 0.227]$) with monotonicity in Table 6. Finally, monotonicity tightens the bounds on the probability of entering poverty from insecure nonpoor state over the seven-year period from $[0.124, 0.145]$ to $[0.126, 0.145]$.

To sum up, over the seven-year sample period, under the strongest set of assumptions, between 28% to 31% impoverished households remain so, between 47% and 51% escape poverty but remain at-risk of poverty, and between 19% and 23% impoverished households exceed twice the poverty line. If we relax the assumption of temporal independence and invariance, but continue to maintain the other assumptions, our findings we find that between 32% and 44% impoverished households remain so, between 37% and 45% households escape poverty but remain at-risk, and between 13% and 31% households exceed the poverty line. Under the assumption of no misclassification error, these corresponding figures are 28%, 51% and 21%. These results have two implications. First, mobility out of poverty over the seven year period is remarkably low in India. Second, if we mistakenly believe that there is no misclassification error, we might be underestimating the probability of the poor remaining poor, and over estimating the probability of the poor escaping poverty and becoming insecure nonpoor or secure nonpoor. In other words, not accounting for measurement error could actually make the mobility situation in India appear brighter than it actually is.

Sensitivity to Q Perhaps the biggest unknown in constructing the bounds, in the Indian context, is the appropriate choice of Q. There is not available administrative data that one can draw upon, or other source of institutional knowledge to know the extent of misclassification. However, the advantage of the partial identification approach is that we can investigate the sensitivity of the bounds to the choice of Q. To this end, we re-estimate the bounds for several values of Q ranging from 0 to 0.40. For the sake of computational time, we focus on the point estimates of the bounds, not the confidence regions. Select results are presented in Figures C1-C5 in Appendix C.

There are four primary takeaways. First, with small Q, the bounds on transition probabilities are informative under arbitrary and uniform misclassification alone. For example, under arbitrary misclassification with $Q = 0.05$, the conditional staying probability of being impoverished is at least 13% and no more than 43%. At least 36% and no more than 66% transition from impoverished to insecure nonpoor, while at least 6% but no more than 36% transition to secure nonpoor. Under uniform misclassification with $Q = 0.10$, the conditional staying probability of being impoverished is at least 18% and no more than 42%. These bounds get reasonably narrowed if we impose additional shape, level set, and/or monotonicity restrictions.

Second, for misclassification error rates higher than $Q = 0.20$, not much can be learned from the estimated bounds under arbitrary or uniform misclassification errors alone. However, if we impose the shape restriction, the bounds under uniform misclassification on at least some of the transition probabilities become reasonably narrow and informative. For example, under uniform misclassification with $Q = 0.30$ and the shape restriction, the conditional staying probability of being impoverished is at least 21%; in contrast, without shape restriction the minimum conditional staying probability is zero. On the other hand, at least 21% of the impoverished will transition to insecure nonpoor and at the most 58% will transition to secure nonpoor. As such, the shape restriction has some identifying power even when 30% households are misclassified. Instead of shape restrictions, if we impose the level set and/or monotonicity restriction and assume that misclassification is independent and uniform, the bounds are not very informative for $Q \geq 0.30$.

Third, with high Q, the restriction of temporal independence or temporal invariance combined with the shape restriction and assumption of uni-directional misclassification have significant identifying power. For example, under arbitrary, uni-directional, and temporally independent misclassification with $Q = 0.30$ and the shape restriction, bounds on the conditional staying probability of being impoverished are $[0.210, 0.580]$ and the bounds on the probability of transitioning from being impoverished to secure nonpoor are $[0.000, 0.270]$; under arbitrary misclassification with only the shape restriction, both of these bounds include the entire unit interval. Instead of temporal independence, if we invoke the assumption of temporal invariance, these bounds become remarkably tighter. Specifically, under arbitrary, uni-directional, and temporally independent misclassification with $Q = 0.30$ and the shape restrictions, bounds on the conditional staying probability of being impoverished are $[0.360, 0.390]$ and the bounds on the probability of transitioning from being impoverished to secure nonpoor are $[0.210, 0.250]$. Not only do these assumptions yields a lot information when examining transitions out of poverty, but they also have substantial identifying power when examining transitions into poverty. For example, under arbitrary, uni-directional, and temporally independent (invariant) misclassification with $Q = 0.30$ and shape restrictions, now we learn that at most 35% (23%) of the secure nonpoor become impoverished over the seven-year sample period.

In sum, we find that, despite allowing nearly 1 in 3 households to be misclassified, at least 4 out of 10 impoverished households will either remain impoverished or insecure nonpoor over the sample period under transparent and reasonably plausible assumptions (uniform misclassification with shape restrictions). If we are willing to impose stronger restrictions (arbitrary, uni-directional, and temporally invariant misclassification with shape restrictions), the number of impoverished households remaining impoverished or insecure nonpoor is at least 7 out of 10.

5.2 Heterogeneity Analysis

Finally, we analyze heterogeneity in economic mobility across different population subgroups. In Table 7, we report the results for the 3×3 transition matrix by religion. Table 8 contains the results by caste. Table 9 presents the results by geographic area (urban or rural). For brevity, for all the cases, we present results only based on a select combination of misclassification assumptions in combination with level set, shape, and monotonicity restrictions.

Religion India contains multiple religious groups; namely, Hindus and various minority groups including Muslims, Christian, Sikh, Buddhist, Jain, Tribal, and others. We divide the minority groups into two groups, Muslims and others. Table 7 presents the results.

Under arbitrary and independent misclassification with level set, shape, and monotonicity restrictions in Panel I, the bounds are wide but informative. For example, the conditional staying probability of being impoverished is at least 3.8% for Hindus, 10.0% for Muslims, and 5.2% for others. While this suggests a disadvantage for Muslims, the conditional staying probability of being impoverished is at most 67.4% for Muslims, while it is at most 74.5% and 76.0% for Hindus and others, respectively. Moreover, the probability of transitioning from impoverished to insecure nonpoor is at least 19.5% for Muslims, while it is at least only 7.8% and 11.5% for Hindus and others, respectively. On the other hand, the conditional staying probability of being secure nonpoor may be as low as 3.9% for Muslims, whereas it is at least 10.2% and 17.7% for Hindus and others, respectively.

Under the assumption of uniform and independent misclassification in Panel II, the bounds for all subsamples are narrowed considerably. For example, bounds on the conditional staying probability of being impoverished are narrowed from $[0.038, 0.745]$ to $[0.312, 0.515]$ for Hindus, from $[0.100, 0.674]$ to $[0.394, 0.441]$ for Muslims, and from $[0.052, 0.760]$ to $[0.267, 0.445]$ for others; bounds on the probability of transitioning from insecure nonpoor to impoverished narrow from $[0.000, 0.549]$ to $[0.006, 0.285]$ for Hindus, from $[0.000, 0.548]$ to $[0.017, 0.299]$ for Muslims, and from $[0.000, 0.559]$ to $[0.033, 0.209]$ for others. Again, we find that Muslims have the highest minimum, but the smallest maximum, conditional staying probability of being impoverished over the sample period.

Adding the assumption of uni-directional misclassification (results not shown) modestly narrows the bounds. However, if we also add the assumption of temporal independence or temporal invariance (Panels III and IV), the bounds narrow significantly. Specifically, under the assumptions of uniform, independent, uni-directional, and temporally invariant misclassification in Panel IV, the following findings stand out. First, Muslims have the lowest conditional staying probability of remaining impoverished and highest probability of transitioning from impoverished to insecure nonpoor as the bounds for Muslims do not overlap the bounds for Hindus and others. The bounds on the probability of staying impoverished and of transitioning from impoverished to insecure nonpoor are $[0.268, 0.279]$ and $[0.524, 0.560]$ for Muslims, $[0.288, 0.315]$ and $[0.463, 0.520]$ for Hindus, and $[0.304, 0.316]$ and $[0.451, 0.480]$ for others. However, in terms of transitioning from impoverished to secure nonpoor, Muslims are not in an unambiguously better position than Hindus and others; in fact, others are strictly better off than Muslims in that the bounds for this group do not intersect the bounds for Muslims.

Second, the minimum probability of transitioning from either insecure (secure) nonpoor to impoverished is highest for the Muslims; the probability is at least 13.5% (6.6%) for Muslims, 12.1% (2.7%) for Hindus, and 11.8% (1.9%) for others. Third, insecure nonpoor Muslims have the lowest probability of transitioning to secure nonpoor. Specifically, bounds on the probability of transitioning from insecure nonpoor to secure nonpoor are $[0.338, 0.341]$ for Muslims, $[0.380, 0.414]$ for Hindus, and $[0.468, 0.494]$ for others. Thus, there is a strict ranking among the three groups. Finally, there is also a strict ranking among the groups in terms of the conditional staying probability of being secure nonpoor and the probability of transitioning from secure nonpoor to insecure nonpoor. Bounds on the conditional staying probability of being secure nonpoor are $[0.594, 0.599]$ for Muslims, $[0.665, 0.666]$ for Hindus, and $[0.746, 0.778]$ for others. Bounds on the probability of transitioning from secure nonpoor to insecure poor are $[0.325, 0.335]$ for Muslims, $[0.259, 0.307]$ for Hindus, and $[0.192, 0.235]$ for others.

In sum, Muslims seem to be doing better than Hindus or other religious groups in terms of escaping poverty. However, in terms of transitioning to secure nonpoor or remaining secure nonpoor, Muslims are at a disadvantage compared to Hindus and other religious groups.

Caste Next, we explore heterogeneity in economic mobility across castes; namely, Brahmins and non-Brahmin Upper Castes (UCs), Scheduled Castes and Scheduled Tribes (SCs/STs), and Other Backward Classes (OBCs). The results are shown in Table 8.

Under arbitrary and independent misclassification with level set, shape, and monotonicity restrictions in Panel I, the bounds are modestly informative. For example, the conditional staying probability of being impoverished is at least 1.1% for UCs, 8.4% for SCs/STs, and 3.8% for OBCs. While suggestive of a disadvantage for SCs/STs, the conditional staying probability of being impoverished is at most 72.9% for SCs/STs, while it is at most 77.8% for UCs. Moreover, the probability of transitioning from impoverished to insecure nonpoor is at least 11.1% for SCs/STs, while it may be as low as 5.2% and 7.9% for UCs and OBCs, respectively. On the other hand, the conditional staying probability of being secure nonpoor may be as low as 2.5% for SCs/STs, whereas it is at least 24.9% and 6.2% for UCs and OBCs, respectively.

Under the assumption of uniform and independent misclassification in Panel II, the bounds for all subsamples are significantly narrowed. For example, bounds on the conditional staying probability of being impoverished are narrowed from $[0.011, 0.778]$ to $[0.231, 0.493]$ for UCs, from $[0.084, 0.729]$ to $[0.355, 0.542]$ for SCs/STs, and from $[0.038, 0.704]$ to $[0.330, 0.487]$ for OBCs; bounds on the probability of transitioning from insecure nonpoor to impoverished narrow from $[0.000, 0.499]$ to $[0.002, 0.212]$ for UCs, from $[0.000, 0.636]$ to $[0.044, 0.367]$ for SCs/STs, and from $[0.000, 0.532]$ to $[0.016, 0.269]$ for OBCs. This is suggestive of SCs/STs being at a disadvantage relative to other castes, but it is inconclusive given the overlap in the bounds.

Adding the assumption of uni-directional misclassification (results not shown) modestly narrows the bounds. However, combining this with the assumption of temporal independence or temporal invariance (Panels III and IV) significantly narrows the bounds. In particular, under the assumptions of uniform, independent, uni-directional, and temporally invariant misclassification in Panel IV, the following conclusions emerge. First, SCs/STs have the highest conditional staying probability of remaining impoverished and lowest probability of transitioning from impoverished to insecure or secure nonpoor as the bounds for SCs/STs do not overlap the bounds for the other castes. Among the remaining groups, there is some evidence in favor of UCs being in a more advantageous position than OBCs. Specifically, bounds on the probability of staying impoverished and of transitioning from impoverished to insecure (secure) nonpoor are $[0.368, 0.370]$ and $[0.456, 0.491]$ ($[0.141, 0.174]$) for SCs/STs, $[0.244, 0.252]$ and $[0.506, 0.530]$ ($[0.242, 0.266]$) for UCs, and $[0.249, 0.283]$ and $[0.496, 0.547]$ ($[0.204, 0.235]$) for OBCs.

Second, bounds on the probability of staying secure nonpoor and of transitioning from secure nonpoor to insecure poor (impoverished) are $[0.459, 0.508]$ and $[0.329, 0.383]$ ($[0.158, 0.180]$)

for SCs/STs, $[0.751, 0.762]$ and $[0.216, 0.227]$ ($[0.017, 0.032]$) for UCs, and $[0.613, 0.633]$ and $[0.315, 0.339]$ ($[0.040, 0.072]$) for OBCs. Again, there is a clear ranking with UCs being in the most secure economic position, followed by OBCs and then SCs/STs.

Finally, insecure nonpoor SCs/STs have the lowest probability of transitioning to secure nonpoor. Specifically, bounds on the probability of transitioning from insecure nonpoor to secure nonpoor are $[0.299, 0.308]$ for SCs/STs, $[0.460, 0.461]$ for UCs, and $[0.382, 0.411]$ for OBCs. Thus, there continues to be a strict ranking among the three groups.

In sum, SCs/STs have the lowest probability of escaping poverty and highest probability of entering poverty over the sample period. Comparing UCs and OBCs, OBCs have a higher probability of becoming impoverished and lower probability of becoming or remaining secure nonpoor. However, in terms of remaining impoverished or transitioning just above the poverty line, it is not clear – even under our most stringent assumptions – whether OBCs are worse off than UCs.

Geographic Location Lastly, we explore heterogeneity in economic mobility across households living in rural and urban areas. Table 9 displays the results.

As in the previous cases, under arbitrary and independent misclassification with level set, shape, and monotonicity restrictions in Panel I, the bounds are at best modestly informative. For example, the conditional staying probability of being impoverished is at least 2.4% and at most 70.5% for urban households; the corresponding probabilities are 5.0% and 74.7% for rural households. In addition, the probability of transitioning from impoverished to insecure nonpoor is at least 9.2% for urban households, yet at least only 8.6% for rural households. Moreover, the conditional staying probability of being secure nonpoor may be as low as 4.9% for rural households, whereas it is at least 22.2% for urban households. The probability of transitioning from secure nonpoor to insecure nonpoor (impoverished) is at most 74.7% (45.7%) for urban households, whereas it is at most 90.5% (57.1%) for rural households.

Under the assumption of uniform and independent misclassification in Panel II, the bounds for all subsamples are significantly narrowed. For example, bounds on the conditional staying probability of being impoverished are narrowed from $[0.024, 0.705]$ to $[0.265, 0.477]$ for urban households and from $[0.050, 0.747]$ to $[0.331, 0.524]$ for rural households; bounds on the probability of transitioning from insecure nonpoor to impoverished narrow from $[0.000, 0.499]$ to $[0.004, 0.178]$ for urban households and from $[0.000, 0.578]$ to $[0.011, 0.321]$ for rural households. This is suggestive of a distinct advantage for urban households, but it is inconclusive given the overlap in the bounds.

Adding the assumption of uni-directional misclassification (results not shown) continues to narrow the bounds only modestly. However, combining this with the assumption of temporal independence or temporal invariance (Panels III and IV) significantly narrows the bounds. Specifically, under the assumptions of uniform, independent, uni-directional, and temporally invariant misclassification in Panel IV, we document the following. First, rural households have a higher conditional staying probability of remaining impoverished and lower probability of transitioning from impoverished to insecure or secure nonpoor as the bounds do not overlap across the two groups. Specifically, bounds on the probability of staying impoverished and of transitioning from impoverished to insecure (secure) nonpoor are $[0.228, 0.261]$ and $[0.514, 0.531]$ ($[0.225, 0.226]$) for urban households; the corresponding bounds are $[0.312, 0.325]$ and $[0.463, 0.472]$ ($[0.203, 0.216]$) for rural households.

Second, bounds on the probability of staying secure nonpoor and of transitioning from secure nonpoor to insecure poor (impoverished) are $[0.762, 0.774]$ and $[0.210, 0.210]$ ($[0.016, 0.028]$) for urban households; the corresponding bounds for rural households are $[0.580, 0.586]$ and $[0.290, 0.384]$ ($[0.030, 0.130]$) for rural households. Again, there is a clear ranking as the bounds do not overlap. Finally, insecure nonpoor rural households have a lower probability of transitioning to secure nonpoor. Specifically, bounds on the probability of transitioning from insecure nonpoor to secure nonpoor are $[0.355, 0.383]$ for rural households, but are $[0.449, 0.464]$ for urban households.

In sum, our findings indicate that, compared to urban households, rural households have greater probability of remaining impoverished over the sample period. Rural households also have a lower probability of escaping poverty and higher probability of becoming impoverished than their urban counterparts.

6 Conclusion

In this paper, we provide bounds on the extent of economic mobility in India over the period 2005 to 2012 using IHDS panel data on household consumption while rigorously accounting for measurement error in a transparent manner. Methodologically, we extend recent work in Millimet et al. (2019) on the partial identification of transition matrices by considering the identifying power of additional assumptions on the misclassification process: temporal independence and temporal invariance. In the application, we reveal how little can be learned about poverty dynamics under relatively small amounts of misclassification absent additional information. We then show the identifying power the accompanies additional information via assumptions on the nature of the misclassification as well as restrictions on consumption dynamics.

We find that, under reasonable assumptions, for the population as a whole, mobility in India is remarkably low: allowing for misclassification errors in up to 20% of the sample, at least 3 in 10 poor households remain poor between 2005 and 2012, at least 4 in 10 households manage to escape poverty but remain in at-risk, and at most 3 in 10 poor households manage to attain the status of secure nonpoor. Further, we show that if we mistakenly believe that there is no misclassification error, we might be underestimating the probability of the poor remaining poor, and over estimating the probability of the poor escaping poverty and becoming insecure nonpoor. Under stronger assumptions, we also find clear rankings among different population subgroups in terms of economic status. Among religious groups, Muslims are at a disadvantage compared to Hindus or other religious groups. Among castes, SCs/STs are the worst off, followed by OBCs and then UCs. Finally, rural households are at a distinct disadvantage relative to urban households.

Our results for the population as a whole suggest that inequality in India can be characterized as chronic as households belonging to the lower rungs of the economic ladder are likely to find themselves caught in a poverty trap. As a result, our findings suggest that poverty reduction efforts should focus on ways to improve the permanent economic status of households, possibly through acquisition of assets and capabilities, rather than on ways to deal with temporary volatility. Our findings also challenge the conventional wisdom that marginalized groups in India – households belonging to minority communities, lower castes, or living in rural regions – are catching up on average. This casts doubt about the efficacy of existing affirmative action and social programs in improving the economic status of marginalized groups in India.

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Table 1. Summary Statistics

	2005 (Wave 1)		2012 (Wave 2)	
	Mean	SD	Mean	SD
Household Consumption				
Total Consumption (in Rs.)	54,493.00	52,225.89	115,364.00	119,271.50
Per Capita Consumption (in Rs.)	863.08	896.88	2200.60	2584.49
Poverty Status				
Poor (POVRATIO < 1)	0.3347	0.4719	0.1541	0.3611
Insecure Nonpoor (POVRATIO >= 1, < 2)	0.4274	0.4947	0.4393	0.4963
Secure Nonpoor (POVRATIO >= 2)	0.2379	0.4258	0.4066	0.4912
Household Size	5.85	3.02	4.87	2.33
Age Group (Household Head)				
<=34	0.1520	0.3590		
35-44	0.2536	0.4351		
45-54	0.2647	0.4412		
55-64	0.1897	0.3921		
>=65	0.1400	0.3470		
Education (Household Head) : Years of Schooling				
0 (Illiterate)	0.3636	0.4810		
1-5 years	0.2074	0.4054		
6-10 years	0.3094	0.4622		
11-15 years	0.1196	0.3245		
Caste (Household Head)				
Brahmin and Others	0.3008	0.4586		
OBC	0.3996	0.4898		
SC/ST	0.2996	0.4581		
Religion				
Hindu	0.8107	0.3917		
Muslims	0.1155	0.3196		
Other Religions	0.0738	0.2615		
Percentage of Males (Household Head)	0.9054	0.2927		
Percentage of Urban Residents (Household Head)	0.2994	0.4580		
Number of Observations	38,737		38,737	

Notes: POVRATIO is defined as the ratio of household per capita monthly household consumption expenditure to the poverty line per capita monthly consumption expenditure. In our analysis we use information pertaining to education level, age, caste, religion and the living region of household heads only from the first wave; hence we report summary statistics of these variables only for the first wave.

Table 2. Full Sample Transition Matrices: Classification-Perserving Misclassification

	1	2	3
1	[0.279, 0.279]	[0.510, 0.510]	[0.211, 0.211]
	(0.272, 0.285)	(0.504, 0.516)	(0.206, 0.217)
2	[0.115, 0.115]	[0.472, 0.472]	[0.413, 0.413]
	(0.111, 0.120)	(0.466, 0.472)	(0.407, 0.419)
3	[0.048, 0.048]	[0.281, 0.281]	[0.670, 0.670]
	(0.045, 0.052)	(0.273, 0.289)	(0.663, 0.678)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Point estimates for bounds provided in brackets obtained using 100 subsamples of size N/2 for bias correction. 90% Imbens-Manski confidence intervals for the bounds provided in parentheses obtained using 250 subsamples of size N/2. See text for further details.

Table 3. Full Sample Transition Matrices: Misclassification Assumptions

	A. Without Temporal Independence/Invariance				B. With Temporal Independence				C. With Temporal Invariance		
I. Arbitrary Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.000, 0.876]	[0.000, 1.000]	[0.000, 0.809]	**1**	[0.080, 0.478]	[0.311, 0.773]	[0.012, 0.410]	**1**	[0.107, 0.366]	[0.395, 0.734]	[0.111, 0.311]
	(0.000, 0.883)	(0.000, 1.000)	(0.000, 0.815)		(0.074, 0.483)	(0.306, 0.786)	(0.007, 0.415)		(0.100, 0.370)	(0.391, 0.742)	(0.107, 0.316)
2	[0.000, 0.583]	[0.004, 0.940]	[0.000, 0.881]	**2**	[0.000, 0.271]	[0.316, 0.628]	[0.257, 0.569]	**2**	[0.037, 0.194]	[0.408, 0.507]	[0.334, 0.543]
	(0.000, 0.589)	(0.000, 0.946)	(0.000, 0.887)		(0.000, 0.275)	(0.311, 0.633)	(0.252, 0.573)		(0.034, 0.197)	(0.402, 0.511)	(0.330, 0.549)
3	[0.000, 0.889]	[0.000, 1.000]	[0.000, 1.000]	**3**	[0.000, 0.329]	[0.001, 0.562]	[0.390, 0.999]	**3**	[0.000, 0.189]	[0.140, 0.334]	[0.666, 0.670]
	(0.000, 0.899)	(0.000, 1.000)	(0.000, 1.000)		(0.000, 0.333)	(0.000, 0.568)	(0.384, 1.000)		(0.000, 0.192)	(0.134, 0.343)	(0.656, 0.678)
II. Uniform Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.080, 0.597]	[0.311, 0.886]	[0.012, 0.512]	**1**	[0.146, 0.345]	[0.377, 0.627]	[0.012, 0.257]	**1**	[0.209, 0.329]	[0.444, 0.601]	[0.161, 0.261]
	(0.074, 0.604)	(0.306, 0.893)	(0.007, 0.519)		(0.141, 0.350)	(0.372, 0.635)	(0.007, 0.260)		(0.203, 0.333)	(0.440, 0.607)	(0.157, 0.266)
2	[0.000, 0.322]	[0.316, 0.743]	[0.257, 0.674]	**2**	[0.011, 0.167]	[0.368, 0.534]	[0.309, 0.465]	**2**	[0.076, 0.155]	[0.445, 0.492]	[0.370, 0.469]
	(0.000, 0.326)	(0.311, 0.748)	(0.252, 0.680)		(0.008, 0.171)	(0.363, 0.541)	(0.304, 0.469)		(0.073, 0.158)	(0.440, 0.496)	(0.366, 0.474)
3	[0.000, 0.456]	[0.001, 0.610]	[0.390, 0.999]	**3**	[0.000, 0.142]	[0.095, 0.375]	[0.484, 0.905]	**3**	[0.000, 0.119]	[0.211, 0.331]	[0.669, 0.670]
	(0.000, 0.464)	(0.000, 0.616)	(0.384, 1.000)		(0.000, 0.145)	(0.088, 0.381)	(0.478, 0.912)		(0.000, 0.121)	(0.204, 0.337)	(0.662, 0.678)
III. Arbitrary, Uni-Directional Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.174, 0.876]	[0.000, 0.826]	[0.000, 0.506]	**1**	[0.279, 0.478]	[0.311, 0.709]	[0.012, 0.257]	**1**	[0.279, 0.366]	[0.395, 0.610]	[0.111, 0.239]
	(0.174, 0.883)	(0.000, 0.829)	(0.000, 0.509)		(0.274, 0.483)	(0.306, 0.715)	(0.007, 0.260)		(0.274, 0.370)	(0.391, 0.615)	(0.107, 0.243)
2	[0.000, 0.583]	[0.004, 0.940]	[0.000, 0.776]	**2**	[0.000, 0.271]	[0.316, 0.628]	[0.257, 0.483]	**2**	[0.115, 0.194]	[0.408, 0.507]	[0.334, 0.437]
	(0.000, 0.589)	(0.000, 0.946)	(0.000, 0.786)		(0.000, 0.275)	(0.311, 0.633)	(0.252, 0.491)		(0.112, 0.197)	(0.402, 0.511)	(0.330, 0.443)
3	[0.000, 0.889]	[0.000, 1.000]	[0.000, 1.000]	**3**	[0.000, 0.329]	[0.001, 0.562]	[0.390, 0.999]	**3**	[0.000, 0.189]	[0.140, 0.334]	[0.666, 0.670]
	(0.000, 0.899)	(0.000, 1.000)	(0.000, 1.000]		(0.000, 0.333)	(0.000, 0.568)	(0.384, 1.000)		(0.000, 0.192)	(0.134, 0.343)	(0.656, 0.678)
IV. Uniform, Uni-Directional Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.232, 0.478]	[0.311, 0.709]	[0.012, 0.342]	**1**	[0.279, 0.345]	[0.444, 0.576]	[0.078, 0.231]	**1**	[0.279, 0.329]	[0.444, 0.560]	[0.161, 0.227]
	(0.228, 0.483)	(0.306, 0.715)	(0.007, 0.346)		(0.274, 0.350)	(0.439, 0.581)	(0.074, 0.235)		(0.274, 0.333)	(0.440, 0.565)	(0.157, 0.231)
2	[0.000, 0.322]	[0.316, 0.743]	[0.257, 0.492]	**2**	[0.075, 0.167]	[0.405, 0.524]	[0.309, 0.427]	**2**	[0.115, 0.155]	[0.445, 0.492]	[0.370, 0.423]
	(0.000, 0.326)	(0.311, 0.748)	(0.252, 0.496)		(0.071, 0.171)	(0.400, 0.528)	(0.304, 0.433)		(0.112, 0.158)	(0.440, 0.496)	(0.366, 0.428)
3	[0.000, 0.456]	[0.001, 0.610]	[0.390, 0.931]	**3**	[0.000, 0.142]	[0.188, 0.375]	[0.484, 0.812]	**3**	[0.000, 0.119]	[0.211, 0.331]	[0.669, 0.670]
	(0.000, 0.464)	{0.000, 0.616)	(0.384, 0.941)		(0.000, 0.145)	(0.181, 0.381)	(0.478, 0.819)		(0.000, 0.121)	(0.204, 0.337)	(0.662, 0.678)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Point estimates for bounds provided in brackets obtained using 100 subsamples of size N/2 for bias correction. 90% Imbens-Manski confidence intervals for the bounds provided in parentheses obtained using 250 subsamples of size N/2. For all cases $Q = 0.20$. See text for further details.

Table 4. Full Sample Transition Matrices: Level Set Restrictions

	A. Without Temporal Independence/Invariance				B. With Temporal Independence				C. With Temporal Invariance		
I. Arbitrary, Independent Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.000,0.851]	[0.000,1.000]	[0.000,0.769]	**1**	[0.107,0.463]	[0.318,0.757]	[0.025,0.386]	**1**	[0.143,0.357]	[0.397,0.725]	[0.121,0.291]
	(0.000,0.861)	(0.000,1.000)	(0.000,0.777)		(0.099,0.470)	(0.311,0.774)	(0.018,0.392)		(0.133,0.362)	(0.392,0.734)	(0.115,0.297)
2	[0.000,0.573]	[0.030,0.929]	[0.000,0.834]	**2**	[0.000,0.258]	[0.338,0.614]	[0.287,0.527]	**2**	[0.057,0.180]	[0.435,0.495]	[0.360,0.489]
	(0.000,0.579)	(0.021,0.937)	(0.000,0.844)		(0.000,0.262)	(0.330,0.621)	(0.279,0.535)		(0.052,0.184)	(0.425,0.500)	(0.355,0.500)
3	[0.000,0.824]	[0.000,1.000]	[0.000,1.000]	**3**	[0.000,0.304]	[0.034,0.526]	[0.429,0.966]	**3**	[0.019,0.165]	[0.169,0.311]	[0.666,0.670]
	(0.000,0.836)	(0.000,1.000)	(0.000,1.000)		(0.000,0.309)	[0.034,0.526]	(0.421,0.976)		(0.003,0.179)	(0.157,0.317)	(0.656,0.678)
II. Uniform, Independent Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.107,0.574]	[0.318,0.868]	[0.025,0.478]	**1**	[0.171,0.332]	[0.410,0.623]	[0.089,0.258]	**1**	[0.236,0.316]	[0.445,0.590]	[0.175,0.242]
	(0.099,0.584)	(0.311,0.879)	(0.018,0.486)		(0.164,0.338)	(0.402,0.635)	(0.083,0.264)		(0.227,0.321)	(0.440,0.599)	(0.169,0.248)
2	[0.000,0.306]	[0.338,0.713]	[0.287,0.623]	**2**	[0.063,0.153]	[0.423,0.513]	[0.339,0.424]	**2**	[0.096,0.140]	[0.450,0.486]	[0.380,0.454]
	(0.000,0.311)	(0.330,0.721)	(0.279,0.633)		(0.052,0.157)	(0.413,0.522)	(0.332,0.433)		(0.090,0.144)	(0.440,0.492)	(0.371,0.469)
3	[0.000,0.412]	[0.034,0.571]	[0.429,0.966]	**3**	[0.000,0.131]	[0.119,0.355]	[0.514,0.881]	**3**	[0.026,0.086]	[0.245,0.304]	[0.669,0.670]
	(0.000,0.420)	(0.024,0.579)	(0.421,0.976)		(0.000,0.135)	(0.108,0.363)	(0.506,0.892)		(0.016,0.104)	(0.233,0.311)	(0.662,0.678)
III. Uniform, Independent, Uni-directional Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.250,0.463]	[0.318,0.703]	[0.025,0.324]	**1**	[0.279,0.345]	[0.444,0.576]	[0.086,0.217]	**1**	[0.281,0.327]	[0.446,0.542]	[0.177,0.227]
	(0.244,0.470)	(0.311,0.710)	(0.018,0.328)		(0.274,0.350)	(0.439,0.581)	(0.079,0.221)		(0.274,0.332)	(0.440,0.551)	(0.170,0.231)
2	[0.000,0.306]	[0.338,0.713]	[0.287,0.457]	**2**	[0.117,0.153]	[0.473,0.509]	[0.338,0.374]	**2**	[0.124,0.145]	[0.450,0.486]	[0.379,0.423]
	(0.000,0.311)	(0.330,0.721)	(0.279,0.464)		(0.104,0.157)	(0.457,0.516)	(0.331,0.390)		(0.117,0.150)	(0.440,0.492)	(0.373,0.428)
3	[0.000,0.412]	[0.034,0.571]	[0.429,0.889]	**3**	[0.000,0.131]	[0.220,0.355]	[0.514,0.780]	**3**	[0.026,0.086]	[0.245,0.304]	[0.669,0.670]
	(0.000,0.420)	(0.024,0.579)	(0.421,0.905)		(0.000,0.135)	(0.208,0.363)	(0.506,0.792)		(0.016,0.104)	(0.233,0.311)	(0.662,0.678)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Point estimates for bounds provided in brackets obtained using 100 subsamples of size N/2 for bias correction. 90% Imbens-Manski confidence intervals for the bounds provided in parentheses obtained using 250 subsamples of size N/2. For all cases Q = 0.20. See text for further details.

Table 5. Full Sample Transition Matrices: Level Set + Shape Restrictions

	A. Without Temporal Independence/Invariance				B. With Temporal Independence				C. With Temporal Invariance		
I. Arbitrary, Independent Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.000,0.851]	[0.000,0.851]	[0.000,0.769]	**1**	[0.318,0.463]	[0.318,0.463]	[0.074,0.357]	**1**	[0.161,0.357]	[0.398,0.570]	[0.150,0.268]
	(0.000,0.861)	(0.000,0.861)	(0.000,0.777)		(0.311,0.470)	(0.311,0.470)	(0.060,0.368)		(0.133,0.362)	(0.392,0.642)	(0.127,0.283)
2	[0.000,0.573]	[0.030,0.929]	[0.000,0.834]	**2**	[0.000,0.258]	[0.338,0.614]	[0.287,0.527]	**2**	[0.057,0.180]	[0.435,0.495]	[0.360,0.485]
	(0.000,0.579)	(0.021,0.937)	(0.000,0.844)		(0.000,0.262)	(0.330,0.621)	(0.279,0.535)		(0.052,0.184)	(0.425,0.500)	(0.355,0.492)
3	[0.000,0.573]	[0.000,0.929]	[0.000,1.000]	**3**	[0.000,0.258]	[0.034,0.526]	[0.429,0.966]	**3**	[0.019,0.158]	[0.176,0.311]	[0.666,0.670]
	(0.000,0.579)	(0.000,0.937)	(0.000,1.000)		(0.000,0.262)	(0.024,0.534)	(0.421,0.976)		(0.003,0.177)	(0.164,0.317)	(0.656,0.678)
II. Uniform, Independent Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.318,0.574]	[0.318,0.574]	[0.025,0.365]	**1**	[0.219,0.332]	[0.410,0.523]	[0.145,0.258]	**1**	[0.236,0.316]	[0.445,0.590]	[0.175,0.242]
	(0.311,0.584)	(0.311,0.584)	(0.018,0.379)		(0.182,0.338)	(0.402,0.571)	(0.103,0.264)		(0.227,0.321)	(0.440,0.599)	(0.169,0.248)
2	[0.000,0.306]	[0.338,0.713]	[0.287,0.623]	**2**	[0.063,0.153]	[0.423,0.513]	[0.339,0.424]	**2**	[0.096,0.140]	[0.450,0.486]	[0.380,0.446]
	(0.000,0.311)	(0.330,0.721)	(0.279,0.633)		(0.052,0.157)	(0.413,0.522)	(0.332,0.433)		(0.090,0.144)	(0.441,0.492)	(0.371,0.457)
3	[0.000,0.306]	[0.034,0.571]	[0.429,0.966]	**3**	[0.000,0.131]	[0.119,0.355]	[0.514,0.881]	**3**	[0.026,0.086]	[0.245,0.304]	[0.669,0.670]
	(0.000,0.311)	(0.024,0.579)	(0.421,0.976)		(0.000,0.135)	(0.108,0.363)	(0.506,0.892)		(0.016,0.104)	(0.233,0.311)	(0.662,0.678)
III. Uniform, Independent, Uni-directional Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.318,0.463]	[0.318,0.463]	[0.074,0.324]	**1**	[0.279,0.345]	[0.444,0.576]	[0.086,0.217]	**1**	[0.281,0.327]	[0.446,0.542]	[0.177,0.227]
	(0.311,0.470)	(0.311,0.470)	(0.060,0.328)		(0.274,0.350)	(0.439,0.581)	(0.079,0.221)		(0.274,0.332)	(0.440,0.551)	(0.170,0.231)
2	[0.000,0.306]	[0.338,0.713]	[0.287,0.457]	**2**	[0.117,0.153]	[0.473,0.509]	[0.338,0.374]	**2**	[0.124,0.145]	[0.450,0.486]	[0.379,0.423]
	(0.000,0.311)	(0.330,0.721)	(0.279,0.464)		(0.104,0.157)	(0.457,0.516)	(0.331,0.390)		(0.117,0.150)	(0.441,0.492)	(0.373,0.428)
3	[0.000,0.306]	[0.034,0.571]	[0.429,0.889]	**3**	[0.000,0.131]	[0.220,0.355]	[0.514,0.780]	**3**	[0.026,0.086]	[0.245,0.304]	[0.669,0.670]
	(0.000,0.311)	(0.024,0.579)	(0.421,0.905)		(0.000,0.135)	(0.208,0.363)	(0.506,0.792)		(0.016,0.104)	(0.233,0.311)	(0.662,0.678)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Point estimates for bounds provided in brackets obtained using 100 subsamples of size N/2 for bias correction. 90% Imbens-Manski confidence intervals for the bounds provided in parentheses obtained using 250 subsamples of size N/2. For all cases $Q = 0.20$. See text for further details.

Table 6. Full Sample Transition Matrices: Level Set + Shape + Monotonicity Restrictions

	A. Without Temporal Independence/Invariance				B. With Temporal Independence				C. With Temporal Invariance		
I. Arbitrary, Independent Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.043,0.735]	[0.088,0.790]	[0.000,0.707]	**1**	[0.318,0.435]	[0.369,0.448]	[0.131,0.314]	**1**	[0.236,0.342]	[0.426,0.570]	[0.170,0.268]
	(0.038,0.747)	(0.077,0.799)	(0.000,0.713)		(0.311,0.443)	(0.358,0.457)	(0.113,0.331)		(0.210,0.348)	(0.418,0.635)	(0.156,0.276)
2	[0.000,0.555]	[0.070,0.913]	[0.037,0.498]	**2**	[0.059,0.247]	[0.405,0.610]	[0.287,0.366]	**2**	[0.084,0.180]	[0.435,0.495]	[0.360,0.455]
	(0.000,0.561)	(0.062,0.921)	(0.031,0.508)		(0.038,0.253)	(0.392,0.617)	(0.279,0.393)		(0.073,0.184)	(0.426,0.500)	(0.355,0.466)
3	[0.000,0.538]	[0.000,0.854]	[0.099,1.000]	**3**	[0.000,0.246]	[0.079,0.526]	[0.429,0.921]	**3**	[0.021,0.116]	[0.218,0.311]	[0.667,0.669]
	(0.000,0.544)	(0.000,0.861)	(0.095,1.000)		(0.000,0.251)	(0.067,0.534)	(0.421,0.933)		(0.003,0.137)	(0.207,0.317)	(0.656,0.677)
II. Uniform, Independent Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.318,0.515]	[0.369,0.534]	[0.052,0.314]	**1**	[0.301,0.311]	[0.444,0.523]	[0.248,0.254]	**1**	[0.236,0.303]	[0.469,0.567]	[0.188,0.240]
	(0.311,0.527)	(0.358,0.547)	(0.042,0.331)		(0.182,0.318)	(0.433,0.571)	(0.103,0.264)		(0.227,0.309)	(0.461,0.593)	(0.180,0.247)
2	[0.009,0.290]	[0.380,0.702]	[0.287,0.427]	**2**	[0.063,0.153]	[0.423,0.513]	[0.339,0.424]	**2**	[0.107,0.140]	[0.458,0.482]	[0.382,0.433]
	(0.004,0.298)	(0.372,0.710)	(0.279,0.438)		(0.052,0.157)	(0.415,0.522)	(0.332,0.433)		(0.100,0.144)	(0.452,0.488)	(0.376,0.442)
3	[0.000,0.290]	[0.079,0.571]	[0.429,0.921]	**3**	[0.000,0.131]	[0.148,0.355]	[0.514,0.852]	**3**	[0.034,0.056]	[0.275,0.298]	[0.669,0.669]
	(0.000,0.297)	(0.067,0.579)	(0.421,0.933)		(0.000,0.135)	(0.136,0.363)	(0.506,0.864)		(0.025,0.074)	(0.264,0.311)	(0.662,0.677)
III. Uniform, Independent, Uni-directional Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.318,0.435]	[0.369,0.453]	[0.131,0.314]	**1**	[0.313,0.329]	[0.471,0.564]	[0.123,0.216]	**1**	[0.281,0.313]	[0.468,0.506]	[0.187,0.227]
	(0.311,0.443)	(0.358,0.462)	(0.113,0.328)		(0.286,0.336)	(0.461,0.581)	(0.115,0.221)		(0.274,0.319)	(0.459,0.546)	(0.180,0.231)
2	[0.011,0.290]	[0.380,0.702]	[0.287,0.410]	**2**	[0.117,0.153]	[0.473,0.509]	[0.338,0.374]	**2**	[0.126,0.145]	[0.458,0.482]	[0.379,0.404]
	(0.005,0.298)	(0.372,0.709)	(0.279,0.418)		(0.104,0.157)	(0.457,0.516)	(0.331,0.390)		(0.121,0.150)	(0.452,0.488)	(0.373,0.411)
3	[0.000,0.290]	[0.079,0.571]	[0.429,0.857]	**3**	[0.000,0.131]	[0.239,0.355]	[0.514,0.761]	**3**	[0.032,0.057]	[0.275,0.299]	[0.669,0.669]
	(0.000,0.297)	(0.067,0.579)	(0.421,0.876)		(0.000,0.135)	(0.227,0.363)	(0.506,0.773)		(0.024,0.076)	(0.262,0.311)	(0.662,0.677)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Point estimates for bounds provided in brackets obtained using 100 subsamples of size N/2 for bias correction. 90% Imbens-Manski confidence intervals for the bounds provided in parentheses obtained using 250 subsamples of size N/2. For all cases Q = 0.20. See text for further details.

Table 7. Subsample Transition Matrices (Religion): Level Set + Shape + Monotonicity Restrictions

	A. Hindu				B. Muslim				C. Others		
I. Arbitrary, Independent Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.038,0.745]	[0.078,0.794]	[0.000,0.710]	**1**	[0.100,0.674]	[0.195,0.745]	[0.000,0.665]	**1**	[0.052,0.760]	[0.115,0.802]	[0.000,0.718]
	(0.032,0.757)	(0.065,0.804)	(0.000,0.717)		(0.084,0.696)	(0.168,0.770)	(0.000,0.685)		(0.033,0.801)	(0.068,0.837)	(0.000,0.750)
2	[0.000,0.549]	[0.070,0.913]	[0.039,0.493]	**2**	[0.000,0.548]	[0.115,0.939]	[0.016,0.430]	**2**	[0.000,0.559]	[0.111,0.828]	[0.066,0.548]
	(0.000,0.557)	(0.060,0.922)	(0.032,0.504)		(0.000,0.562)	(0.092,0.961)	(0.003,0.477)		(0.000,0.583)	(0.084,0.850)	(0.046,0.592)
3	[0.000,0.531]	[0.000,0.853]	[0.102,1.000]	**3**	[0.000,0.546]	[0.000,0.938]	[0.039,1.000]	**3**	[0.000,0.512]	[0.000,0.767]	[0.177,1.000]
	(0.000,0.538)	(0.000,0.862)	(0.096,1.000)		(0.000,0.560)	(0.000,0.961)	(0.026,1.000)		(0.000,0.531)	(0.000,0.787)	(0.160,1.000)
II. Uniform, Independent Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.312,0.515]	[0.357,0.537]	[0.043,0.331]	**1**	[0.394,0.441]	[0.438,0.458]	[0.122,0.169]	**1**	[0.267,0.445]	[0.418,0.539]	[0.137,0.315]
	(0.303,0.529)	(0.347,0.552)	(0.035,0.350)		(0.372,0.462)	(0.417,0.501)	(0.099,0.211)		(0.232,0.524)	(0.382,0.587)	(0.094,0.385)
2	[0.006,0.285]	[0.387,0.701]	[0.292,0.421]	**2**	[0.017,0.299]	[0.404,0.730]	[0.231,0.379]	**2**	[0.033,0.209]	[0.422,0.597]	[0.348,0.370]
	(0.001,0.296)	(0.377,0.711)	(0.284,0.438)		(0.003,0.315)	(0.384,0.752)	(0.213,0.411)		(0.000,0.234)	(0.395,0.622)	(0.323,0.413)
3	[0.000,0.285]	[0.086,0.567]	[0.433,0.914]	**3**	[0.000,0.299]	[0.058,0.698]	[0.283,0.942]	**3**	[0.000,0.208]	[0.091,0.455]	[0.545,0.909]
	(0.000,0.293)	(0.071,0.578)	(0.422,0.929)		(0.000,0.313)	(0.025,0.722)	(0.250,0.975)		(0.000,0.233)	(0.053,0.482)	(0.518,0.947)
III. Uniform, Independent, Uni-directional, Temporally Independent Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.309,0.330]	[0.464,0.571]	[0.120,0.217]	**1**	[0.274,0.275]	[0.537,0.596]	[0.130,0.188]	**1**	[0.302,0.331]	[0.463,0.490]	[0.181,0.206]
	(0.281,0.338)	(0.453,0.577)	(0.111,0.223)		(0.253,0.295)	(0.512,0.631)	(0.115,0.213)		(0.272,0.356)	(0.420,0.554)	(0.140,0.232)
2	[0.121,0.149]	[0.478,0.506]	[0.345,0.373]	**2**	[0.131,0.164]	[0.509,0.542]	[0.302,0.327]	**2**	[0.141,0.147]	[0.397,0.419]	[0.442,0.456]
	(0.106,0.154)	(0.459,0.514)	(0.335,0.392)		(0.101,0.176)	(0.488,0.559)	(0.283,0.349)		(0.105,0.161)	(0.365,0.440)	(0.418,0.487)
3	[0.000,0.131]	[0.240,0.354]	[0.515,0.755]	**3**	[0.000,0.159]	[0.321,0.443]	[0.436,0.579]	**3**	[0.000,0.111]	[0.202,0.267]	[0.622,0.769]
	(0.000,0.135)	(0.226,0.363)	(0.505,0.771)		(0.000,0.169)	(0.276,0.470)	(0.398,0.635)		(0.000,0.119)	(0.169,0.288)	(0.593,0.814)
IV. Uniform, Independent, Uni-directional, Temporally Invariant Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.288,0.315]	[0.463,0.520]	[0.191,0.228]	**1**	[0.268,0.279]	[0.524,0.560]	[0.171,0.198]	**1**	[0.304,0.316]	[0.451,0.480]	[0.216,0.245]
	(0.281,0.321)	(0.453,0.536)	(0.184,0.232)		(0.251,0.293)	(0.504,0.590)	(0.158,0.210)		(0.279,0.335)	(0.423,0.527)	(0.194,0.269)
2	[0.121,0.149]	[0.454,0.483]	[0.380,0.414]	**2**	[0.135,0.151]	[0.508,0.523]	[0.338,0.341]	**2**	[0.118,0.150]	[0.374,0.414]	[0.468,0.494]
	(0.117,0.154)	(0.444,0.489)	(0.373,0.421)		(0.123,0.162)	(0.480,0.539)	(0.320,0.361)		(0.104,0.164)	(0.344,0.440)	(0.447,0.519)
3	[0.027,0.077]	[0.259,0.307]	[0.665,0.666]	**3**	[0.066,0.081]	[0.325,0.335]	[0.594,0.599]	**3**	[0.019,0.062]	[0.192,0.235]	[0.746,0.778]
	(0.016,0.099)	(0.246,0.315)	(0.655,0.674)		(0.016,0.136)	(0.296,0.372)	(0.563,0.620)		(0.000,0.100)	(0.164,0.267)	(0.731,0.799)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Point estimates for bounds provided in brackets obtained using 100 subsamples of size N/2 for bias correction. 90% Imbens-Manski confidence intervals for the bounds provided in parentheses obtained using 250 subsamples of size N/2. For all cases $Q = 0.20$. See text for further details.

Table 8. Subsample Transition Matrices (Caste): Level Set + Shape + Monotonicity Restrictions

	A. Brahmin/Other Upper Castes				B. SC/ST				C. OBC		
I. Arbitrary, Independent Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.011,0.778]	[0.052,0.843]	[0.000,0.824]	**1**	[0.084,0.729]	[0.111,0.744]	[0.000,0.561]	**1**	[0.038,0.704]	[0.079,0.774]	[0.000,0.714]
	(0.005,0.812)	(0.029,0.857)	(0.000,0.837)		(0.068,0.744)	(0.100,0.770)	(0.000,0.570)		(0.027,0.722)	(0.061,0.787)	(0.000,0.724)
2	[0.000,0.499]	[0.106,0.867]	[0.061,0.518]	**2**	[0.000,0.636]	[0.058,0.956]	[0.006,0.538]	**2**	[0.000,0.532]	[0.094,0.891]	[0.042,0.469]
	(0.000,0.511)	(0.092,0.879)	(0.049,0.527)		(0.000,0.647)	(0.047,0.966)	(0.000,0.565)		(0.000,0.542)	(0.082,0.902)	(0.033,0.483)
3	[0.000,0.449]	[0.000,0.724]	[0.249,1.000]	**3**	[0.000,0.631]	[0.000,0.949]	[0.025,1.000]	**3**	[0.000,0.527]	[0.000,0.861]	[0.062,1.000]
	(0.000,0.458)	(0.000,0.736)	(0.236,1.000)		(0.000,0.642)	(0.000,0.959)	(0.020,1.000)		(0.000,0.536)	(0.000,0.873)	(0.053,1.000)
II. Uniform, Independent Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.231,0.493]	[0.384,0.532]	[0.050,0.385]	**1**	[0.355,0.542]	[0.368,0.542]	[0.040,0.277]	**1**	[0.330,0.487]	[0.381,0.517]	[0.074,0.290]
	(0.213,0.521)	(0.359,0.555)	(0.026,0.429)		(0.341,0.556)	(0.354,0.573)	(0.030,0.306)		(0.321,0.504)	(0.367,0.531)	(0.060,0.312)
2	[0.002,0.212]	[0.420,0.626]	[0.360,0.424]	**2**	[0.044,0.367]	[0.382,0.772]	[0.184,0.452]	**2**	[0.016,0.269]	[0.407,0.679]	[0.304,0.387]
	(0.000,0.227)	(0.407,0.636)	(0.350,0.469)		(0.030,0.379)	(0.369,0.791)	(0.171,0.484)		(0.005,0.281)	(0.393,0.692)	(0.291,0.431)
3	[0.000,0.210]	[0.072,0.415]	[0.585,0.928]	**3**	[0.000,0.368]	[0.061,0.758]	[0.160,0.939]	**3**	[0.000,0.269]	[0.082,0.628]	[0.368,0.918]
	(0.000,0.224)	(0.057,0.426)	(0.574,0.943)		(0.000,0.379)	(0.034,0.775)	(0.150,0.966)		(0.000,0.282)	(0.061,0.641)	(0.351,0.939)
III. Uniform, Independent, Uni-directional, Temporally Independent Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.224,0.275]	[0.505,0.630]	[0.140,0.271]	**1**	[0.374,0.387]	[0.472,0.523]	[0.104,0.142]	**1**	[0.276,0.290]	[0.500,0.589]	[0.125,0.223]
	(0.210,0.297)	(0.484,0.643)	(0.098,0.298)		(0.347,0.395)	(0.457,0.545)	(0.092,0.156)		(0.239,0.300)	(0.488,0.601)	(0.111,0.242)
2	[0.077,0.119]	[0.412,0.458]	[0.422,0.470]	**2**	[0.186,0.194]	[0.504,0.541]	[0.265,0.302]	**2**	[0.114,0.139]	[0.480,0.505]	[0.355,0.380]
	(0.066,0.125)	(0.399,0.469)	(0.407,0.487)		(0.172,0.205)	(0.478,0.553)	(0.251,0.312)		(0.096,0.145)	(0.455,0.516)	(0.344,0.396)
3	[0.000,0.083]	[0.215,0.262]	[0.655,0.785]	**3**	[0.000,0.199]	[0.361,0.513]	[0.332,0.632]	**3**	[0.000,0.133]	[0.286,0.397]	[0.470,0.714]
	(0.000,0.087)	(0.201,0.273)	(0.643,0.799)		(0.000,0.220)	(0.310,0.537)	(0.291,0.690)		(0.000,0.139)	(0.264,0.412)	(0.454,0.736)
IV. Uniform, Independent, Uni-directional, Temporally Invariant Misclassification											
	1	**2**	**3**		**1**	**2**	**3**		**1**	**2**	**3**
1	[0.244,0.252]	[0.506,0.530]	[0.242,0.266]	**1**	[0.368,0.370]	[0.456,0.491]	[0.141,0.174]	**1**	[0.249,0.283]	[0.496,0.547]	[0.204,0.235]
	(0.217,0.283)	(0.487,0.561)	(0.227,0.289)		(0.344,0.378)	(0.442,0.522)	(0.134,0.181)		(0.240,0.290)	(0.485,0.565)	(0.195,0.246)
2	[0.100,0.105]	[0.433,0.439]	[0.460,0.461]	**2**	[0.184,0.202]	[0.499,0.508]	[0.299,0.308]	**2**	[0.112,0.144]	[0.458,0.491]	[0.382,0.411]
	(0.088,0.111)	(0.416,0.448)	(0.450,0.479)		(0.171,0.215)	(0.489,0.517)	(0.290,0.319)		(0.107,0.150)	(0.450,0.499)	(0.374,0.421)
3	[0.017,0.032]	[0.216,0.227]	[0.751,0.762]	**3**	[0.158,0.180]	[0.329,0.383]	[0.459,0.508]	**3**	[0.040,0.072]	[0.315,0.339]	[0.613,0.633]
	(0.011,0.053)	(0.203,0.239)	(0.744,0.772)		(0.128,0.191)	(0.306,0.437)	(0.435,0.526)		(0.028,0.103)	(0.296,0.356)	(0.601,0.641)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Point estimates for bounds provided in brackets obtained using 100 subsamples of size N/2 for bias correction. 90% Imbens-Manski confidence intervals for the bounds provided in parentheses obtained using 250 subsamples of size N/2. For all cases Q = 0.20. See text for further details.

Table 9. Subsample Transition Matrices (Rural/Urban): Level Set + Shape + Monotonicity Restrictions

	A. Urban				B. Rural		
I. Arbitrary, Independent Misclassification							
	1	**2**	**3**		**1**	**2**	**3**
1	[0.024,0.705]	[0.092,0.791]	[0.000,0.768]	**1**	[0.050,0.747]	[0.086,0.797]	[0.000,0.686]
	(0.016,0.733)	(0.069,0.805)	(0.000,0.782)		(0.044,0.758)	(0.074,0.810)	(0.000,0.693)
2	[0.000,0.499]	[0.125,0.867]	[0.087,0.500]	**2**	[0.000,0.578]	[0.055,0.929]	[0.016,0.499]
	(0.000,0.510)	(0.111,0.882)	(0.071,0.512)		(0.000,0.586)	(0.046,0.938)	(0.011,0.512)
3	[0.000,0.457]	[0.000,0.747]	[0.222,1.000]	**3**	[0.000,0.571]	[0.000,0.905]	[0.049,1.000]
	(0.000,0.466)	(0.000,0.759)	(0.206,1.000)		(0.000,0.579)	(0.000,0.914)	(0.044,1.000)
II. Uniform, Independent Misclassification							
	1	**2**	**3**		**1**	**2**	**3**
1	[0.265,0.477]	[0.403,0.512]	[0.079,0.332]	**1**	[0.331,0.524]	[0.362,0.548]	[0.042,0.308]
	(0.248,0.500)	(0.383,0.535)	(0.055,0.369)		(0.322,0.536)	(0.354,0.562)	(0.034,0.324)
2	[0.004,0.178]	[0.449,0.626]	[0.361,0.373]	**2**	[0.011,0.321]	[0.362,0.727]	[0.253,0.435]
	(0.000,0.200)	(0.428,0.642)	(0.346,0.421)		(0.005,0.329)	(0.354,0.736)	(0.244,0.448)
3	[0.000,0.175]	[0.050,0.396]	[0.604,0.950]	**3**	[0.000,0.318]	[0.096,0.681]	[0.303,0.904]
	(0.000,0.196)	(0.033,0.409)	(0.591,0.967)		(0.000,0.325)	(0.078,0.690)	(0.291,0.922)
III. Uniform, Independent, Uni-directional, Temporally Independent Misclassification							
	1	**2**	**3**		**1**	**2**	**3**
1	[0.263,0.264]	[0.522,0.587]	[0.150,0.216]	**1**	[0.293,0.340]	[0.464,0.557]	[0.107,0.205]
	(0.241,0.278)	(0.503,0.624)	(0.135,0.250)		(0.288,0.347)	(0.456,0.571)	(0.099,0.211)
2	[0.064,0.115]	[0.438,0.470]	[0.415,0.466]	**2**	[0.140,0.168]	[0.479,0.518]	[0.314,0.352]
	(0.045,0.121)	(0.425,0.482)	(0.398,0.476)		(0.128,0.174)	(0.464,0.526)	(0.299,0.366)
3	[0.000,0.084]	[0.200,0.252]	[0.664,0.800]	**3**	[0.000,0.165]	[0.266,0.429]	[0.406,0.734]
	(0.000,0.088)	(0.180,0.264)	(0.652,0.820)		(0.000,0.171)	(0.247,0.440)	(0.394,0.753)
IV. Uniform, Independent, Uni-directional, Temporally Invariant Misclassification							
	1	**2**	**3**		**1**	**2**	**3**
1	[0.228,0.261]	[0.514,0.531]	[0.225,0.226]	**1**	[0.312,0.325]	[0.463,0.472]	[0.203,0.216]
	(0.220,0.288)	(0.496,0.566)	(0.212,0.239)		(0.289,0.331)	(0.455,0.547)	(0.164,0.222)
2	[0.086,0.106]	[0.439,0.452]	[0.449,0.464]	**2**	[0.140,0.164]	[0.469,0.493]	[0.355,0.383]
	(0.080,0.112)	(0.427,0.462)	(0.438,0.471)		(0.135,0.169)	(0.462,0.498)	(0.347,0.390)
3	[0.016,0.028]	[0.210,0.210]	[0.762,0.774]	**3**	[0.030,0.130]	[0.290,0.384]	[0.580,0.586]
	(0.006,0.052)	(0.194,0.223)	(0.754,0.800)		(0.000,0.146)	(0.274,0.402)	(0.569,0.593)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Point estimates for bounds provided in brackets obtained using 100 subsamples of size N/2 for bias correction. 90% Imbens-Manski confidence intervals for the bounds provided in parentheses obtained using 250 subsamples of size N/2. For all cases Q = 0.20. See text for further details.

Supplemental Appendix

Measuring Economic Mobility in India Using Noisy Data:
A Partial Identification Approach

A Derivation of Bounds

A.1 Misclassification Assumptions

- Baseline case: Assumption 2(i), 2(ii)

$$\theta_{kl}^{k'l'} = \Pr(y_o \in k', y_1 \in l', y_o^* \in k, y_1^* \in l)$$

  - 72 elements
    - General: # elements $= K^2(K^2 - 1)$
  - Under Assumption 2(i)

$$\sum \theta_{kl}^{k'l'} \leq Q$$

  - Under Assumption 2(ii)

$$\begin{aligned}
\sum \theta_{11}^{k'l'} + \sum \theta_{12}^{k'l'} + \sum \theta_{13}^{k'l'} &\leq Q/3 = Q/K \qquad \text{(generally)} \\
\sum \theta_{21}^{k'l'} + \sum \theta_{22}^{k'l'} + \sum \theta_{23}^{k'l'} &\leq Q/3 = Q/K \qquad \text{(generally)} \\
\sum \theta_{31}^{k'l'} + \sum \theta_{32}^{k'l'} + \sum \theta_{33}^{k'l'} &\leq Q/3 = Q/K \qquad \text{(generally)} \\
\sum \theta_{11}^{k'l'} + \sum \theta_{21}^{k'l'} + \sum \theta_{31}^{k'l'} &\leq Q/3 = Q/K \qquad \text{(generally)} \\
\sum \theta_{12}^{k'l'} + \sum \theta_{22}^{k'l'} + \sum \theta_{32}^{k'l'} &\leq Q/3 = Q/K \qquad \text{(generally)} \\
\sum \theta_{13}^{k'l'} + \sum \theta_{23}^{k'l'} + \sum \theta_{33}^{k'l'} &\leq Q/3 = Q/K \qquad \text{(generally)}
\end{aligned}$$

- Add Uni-directional assumption
  - Implies

$$\begin{aligned}
\theta_{12}^{11} = \theta_{13}^{11} = \theta_{21}^{11} = \theta_{22}^{11} = \theta_{23}^{11} = \theta_{31}^{11} = \theta_{32}^{11} = \theta_{33}^{11} &= 0 \\
\theta_{13}^{12} = \theta_{21}^{12} = \theta_{22}^{12} = \theta_{23}^{12} = \theta_{31}^{12} = \theta_{32}^{12} = \theta_{33}^{12} &= 0 \\
\theta_{21}^{13} = \theta_{22}^{13} = \theta_{23}^{13} = \theta_{31}^{13} = \theta_{32}^{13} = \theta_{33}^{13} &= 0 \\
\theta_{12}^{21} = \theta_{13}^{21} = \theta_{22}^{21} = \theta_{23}^{21} = \theta_{31}^{21} = \theta_{32}^{21} = \theta_{33}^{21} &= 0 \\
\theta_{13}^{22} = \theta_{23}^{22} = \theta_{31}^{22} = \theta_{32}^{22} = \theta_{33}^{22} &= 0 \\
\theta_{31}^{23} = \theta_{32}^{23} = \theta_{33}^{23} &= 0 \\
\theta_{12}^{31} = \theta_{13}^{31} = \theta_{22}^{31} = \theta_{23}^{31} = \theta_{32}^{31} = \theta_{33}^{31} &= 0 \\
\theta_{13}^{32} = \theta_{23}^{32} = \theta_{33}^{32} &= 0
\end{aligned}$$

  - Now only 27 elements
    - General: # elements $= K^4 + 2K^3 + \sum_k \sum_l [kl - (k+l)(K+1)]$

- Add Temporal Independence assumption

$$
\begin{aligned}
\theta_{kl}^{k'l'} &= \alpha_k^{k'}\beta_l^{l'} \\
\alpha_k^{k'} &= \Pr(y_o \in k', y_o^* \in k) \\
\alpha_k^{k} &= \Pr(y_o \in k, y_o^* \in k) = 1-\alpha_k^{k'} \\
\beta_l^{l'} &= \Pr(y_1 \in l', y_1^* \in l) \\
\beta_l^{l} &= \Pr(y_1 \in l, y_1^* \in l) = 1-\beta_l^{l'}
\end{aligned}
$$

  - Now only 12 elements
    - General: # elements $= 2K(K-1)$
  - Implies

$$
\begin{array}{lllll}
\theta_{11}^{12} = \left(1-\alpha_1^2-\alpha_1^3\right)\beta_1^2 & \theta_{12}^{11} = \left(1-\alpha_1^2-\alpha_1^3\right)\beta_2^1 & \theta_{13}^{11} = \left(1-\alpha_1^2-\alpha_1^3\right)\beta_3^1 & \theta_{21}^{11} = \alpha_2^1\left(1-\beta_1^2-\beta_1^3\right) & \theta_{22}^{11} = \alpha_2^1\beta_2^1 \\
\theta_{11}^{13} = \left(1-\alpha_1^2-\alpha_1^3\right)\beta_1^3 & \theta_{12}^{13} = \left(1-\alpha_1^2-\alpha_1^3\right)\beta_2^3 & \theta_{13}^{12} = \left(1-\alpha_1^2-\alpha_1^3\right)\beta_3^2 & \theta_{21}^{12} = \alpha_2^1\beta_1^2 & \theta_{22}^{12} = \alpha_2^1\left(1-\beta_2^1-\beta_3^1\right) \\
\theta_{11}^{21} = \alpha_1^2\left(1-\beta_1^2-\beta_1^3\right) & \theta_{12}^{21} = \alpha_1^2\beta_2^1 & \theta_{13}^{21} = \alpha_1^2\beta_3^1 & \theta_{21}^{13} = \alpha_2^1\beta_1^3 & \theta_{22}^{13} = \alpha_2^1\beta_2^3 \\
\theta_{11}^{22} = \alpha_1^2\beta_1^2 & \theta_{12}^{22} = \alpha_1^2\left(1-\beta_2^1-\beta_2^3\right) & \theta_{13}^{22} = \alpha_1^2\beta_3^2 & \theta_{21}^{22} = \left(1-\alpha_2^1-\alpha_2^3\right)\beta_1^2 & \theta_{22}^{21} = \left(1-\alpha_2^1-\alpha_2^3\right)\beta_2^1 \\
\theta_{11}^{23} = \alpha_1^2\beta_1^3 & \theta_{12}^{23} = \alpha_1^2\beta_2^3 & \theta_{13}^{23} = \alpha_1^2\left(1-\beta_3^1-\beta_3^3\right) & \theta_{21}^{23} = \left(1-\alpha_2^1-\alpha_2^3\right)\beta_1^3 & \theta_{22}^{23} = \left(1-\alpha_2^1-\alpha_2^3\right)\beta_2^3 \\
\theta_{11}^{31} = \alpha_1^3\left(1-\beta_1^2-\beta_1^3\right) & \theta_{12}^{31} = \alpha_1^3\beta_2^1 & \theta_{13}^{31} = \alpha_1^3\beta_3^1 & \theta_{21}^{31} = \alpha_2^3\left(1-\beta_1^2-\beta_1^3\right) & \theta_{22}^{31} = \alpha_2^3\beta_2^1 \\
\theta_{11}^{32} = \alpha_1^3\beta_1^2 & \theta_{12}^{32} = \alpha_1^3\left(1-\beta_2^1-\beta_2^3\right) & \theta_{13}^{32} = \alpha_1^3\beta_3^2 & \theta_{21}^{32} = \alpha_2^3\beta_1^2 & \theta_{22}^{32} = \alpha_2^3\left(1-\beta_2^1-\beta_2^3\right) \\
\theta_{11}^{33} = \alpha_1^3\beta_1^3 & \theta_{12}^{33} = \alpha_1^3\beta_2^3 & \theta_{13}^{33} = \alpha_1^3\left(1-\beta_3^1-\beta_3^2\right) & \theta_{21}^{33} = \alpha_2^1\beta_1^3 & \theta_{22}^{33} = \alpha_2^3\beta_2^3
\end{array}
$$

$$
\begin{array}{llll}
\theta_{23}^{11} = \alpha_2^1\beta_3^1 & \theta_{31}^{11} = \alpha_3^1\left(1-\beta_1^2-\beta_1^3\right) & \theta_{32}^{11} = \alpha_3^1\beta_2^1 & \theta_{33}^{11} = \alpha_3^1\beta_3^1 \\
\theta_{23}^{12} = \alpha_2^1\beta_3^2 & \theta_{31}^{12} = \alpha_3^1\beta_1^2 & \theta_{32}^{12} = \alpha_3^1\left(1-\beta_2^1-\beta_2^3\right) & \theta_{33}^{12} = \alpha_3^1\beta_3^2 \\
\theta_{23}^{13} = \alpha_2^1\left(1-\beta_3^1-\beta_3^2\right) & \theta_{31}^{13} = \alpha_3^1\beta_1^3 & \theta_{32}^{13} = \alpha_3^1\beta_2^3 & \theta_{33}^{13} = \alpha_3^1\left(1-\beta_3^1-\beta_3^3\right) \\
\theta_{23}^{21} = \left(1-\alpha_2^1-\alpha_2^3\right)\beta_3^1 & \theta_{31}^{21} = \alpha_3^2\left(1-\beta_1^2-\beta_1^3\right) & \theta_{32}^{21} = \alpha_3^2\beta_2^1 & \theta_{33}^{21} = \alpha_3^2\beta_3^1 \\
\theta_{23}^{22} = \left(1-\alpha_2^1-\alpha_2^3\right)\beta_3^2 & \theta_{31}^{22} = \alpha_3^2\beta_1^2 & \theta_{32}^{22} = \alpha_3^2\left(1-\beta_2^1-\beta_2^3\right) & \theta_{33}^{22} = \alpha_3^2\beta_3^2 \\
\theta_{23}^{31} = \alpha_2^3\beta_3^1 & \theta_{31}^{23} = \alpha_3^2\beta_1^3 & \theta_{32}^{23} = \alpha_3^2\beta_2^3 & \theta_{33}^{23} = \alpha_3^2\left(1-\beta_3^1-\beta_3^3\right) \\
\theta_{23}^{32} = \alpha_2^3\beta_3^2 & \theta_{31}^{32} = \left(1-\alpha_3^1-\alpha_3^2\right)\beta_1^2 & \theta_{32}^{31} = \left(1-\alpha_3^1-\alpha_3^2\right)\beta_2^1 & \theta_{33}^{31} = \left(1-\alpha_3^1-\alpha_3^2\right)\beta_3^1 \\
\theta_{23}^{33} = \alpha_2^3\left(1-\beta_3^1-\beta_3^2\right) & \theta_{11}^{33} = \left(1-\alpha_3^1-\alpha_3^2\right)\beta_1^3 & \theta_{32}^{33} = \left(1-\alpha_3^1-\alpha_3^2\right)\beta_2^3 & \theta_{33}^{32} = \left(1-\alpha_3^1-\alpha_3^2\right)\beta_3^2
\end{array}
$$

    - Under Assumption 2(i)

$$
\begin{aligned}
\sum \theta_{kl}^{k'l'} &= \left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_1^2+\beta_1^3\right)+\left(\alpha_1^2+\alpha_1^3\right) \\
&\quad +\left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_2^1+\beta_2^3\right)+\left(\alpha_1^2+\alpha_1^3\right) \\
&\quad +\left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_3^1+\beta_3^2\right)+\left(\alpha_1^2+\alpha_1^3\right) \\
&\quad +\left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_1^2+\beta_1^3\right)+\left(\alpha_2^1+\alpha_2^3\right) \\
&\quad +\left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_2^1+\beta_2^3\right)+\left(\alpha_2^1+\alpha_2^3\right) \\
&\quad +\left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_3^1+\beta_3^2\right)+\left(\alpha_2^1+\alpha_2^3\right) \\
&\quad +\left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_1^2+\beta_1^3\right)+\left(\alpha_3^1+\alpha_3^2\right) \\
&\quad +\left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_2^1+\beta_2^3\right)+\left(\alpha_3^1+\alpha_3^2\right) \\
&\quad +\left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_3^1+\beta_3^2\right)+\left(\alpha_3^1+\alpha_3^2\right) \\
&= \left(3-\alpha_1^2-\alpha_1^3-\alpha_2^1-\alpha_2^3-\alpha_3^1-\alpha_3^2\right)\left(\beta_1^2+\beta_1^3+\beta_2^1+\beta_2^3+\beta_3^1+\beta_3^2\right) \\
&\qquad +3\left(\alpha_1^2+\alpha_1^3+\alpha_2^1+\alpha_2^3+\alpha_3^1+\alpha_3^2\right) \le Q \\
&\Rightarrow \alpha_1^2,\alpha_1^3,\alpha_2^1,\alpha_2^3,\alpha_3^1,\alpha_3^2,\beta_1^2,\beta_1^3,\beta_2^1,\beta_2^3,\beta_3^1,\beta_3^2 \le Q/3 \\
&\Rightarrow \alpha,\beta \le Q/K \qquad \text{(generally)}
\end{aligned}
$$

* Under Assumption 2(ii)

$$
\begin{aligned}
\sum \theta_{11}^{k'l'} &= \left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_1^2+\beta_1^3\right)+\left(\alpha_1^2+\alpha_1^3\right) \\
\sum \theta_{12}^{k'l'} &= \left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_2^1+\beta_2^3\right)+\left(\alpha_1^2+\alpha_1^3\right) \\
\sum \theta_{13}^{k'l'} &= \left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_3^1+\beta_3^2\right)+\left(\alpha_1^2+\alpha_1^3\right) \\
\sum \theta_{21}^{k'l'} &= \left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_1^2+\beta_1^3\right)+\left(\alpha_2^1+\alpha_2^3\right) \\
\sum \theta_{22}^{k'l'} &= \left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_2^1+\beta_2^3\right)+\left(\alpha_2^1+\alpha_2^3\right) \\
\sum \theta_{23}^{k'l'} &= \left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_3^1+\beta_3^2\right)+\left(\alpha_2^1+\alpha_2^3\right) \\
\sum \theta_{31}^{k'l'} &= \left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_1^2+\beta_1^3\right)+\left(\alpha_3^1+\alpha_3^2\right) \\
\sum \theta_{32}^{k'l'} &= \left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_2^1+\beta_2^3\right)+\left(\alpha_3^1+\alpha_3^2\right) \\
\sum \theta_{33}^{k'l'} &= \left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_3^1+\beta_3^2\right)+\left(\alpha_3^1+\alpha_3^2\right)
\end{aligned}
$$

and the following restrictions must hold

$$
\begin{aligned}
\sum \theta_{11}^{k'l'}+\sum \theta_{12}^{k'l'}+\sum \theta_{13}^{k'l'} &= 3\left(\alpha_1^2+\alpha_1^3\right)+\left(1-\alpha_1^2-\alpha_1^3\right)\left(\beta_1^2+\beta_1^3+\beta_2^1+\beta_2^3+\beta_3^1+\beta_3^2\right)\leq Q/3 \\
\sum \theta_{21}^{k'l'}+\sum \theta_{22}^{k'l'}+\sum \theta_{23}^{k'l'} &= 3\left(\alpha_2^1+\alpha_2^3\right)+\left(1-\alpha_2^1-\alpha_2^3\right)\left(\beta_1^2+\beta_1^3+\beta_2^1+\beta_2^3+\beta_3^1+\beta_3^2\right)\leq Q/3 \\
\sum \theta_{31}^{k'l'}+\sum \theta_{32}^{k'l'}+\sum \theta_{33}^{k'l'} &= 3\left(\alpha_3^1+\alpha_3^2\right)+\left(1-\alpha_3^1-\alpha_3^2\right)\left(\beta_1^2+\beta_1^3+\beta_2^1+\beta_2^3+\beta_3^1+\beta_3^2\right)\leq Q/3 \\
\sum \theta_{11}^{k'l'}+\sum \theta_{21}^{k'l'}+\sum \theta_{31}^{k'l'} &= \alpha_1^2+\alpha_1^3+\alpha_2^1+\alpha_2^3+\alpha_3^1+\alpha_3^2+\left(3-\alpha_1^2-\alpha_1^3-\alpha_2^1-\alpha_2^3-\alpha_3^1-\alpha_3^2\right)\left(\beta_1^2+\beta_1^3\right)\leq Q/3 \\
\sum \theta_{12}^{k'l'}+\sum \theta_{22}^{k'l'}+\sum \theta_{32}^{k'l'} &= \alpha_1^2+\alpha_1^3+\alpha_2^1+\alpha_2^3+\alpha_3^1+\alpha_3^2+\left(3-\alpha_1^2-\alpha_1^3-\alpha_2^1-\alpha_2^3-\alpha_3^1-\alpha_3^2\right)\left(\beta_2^1+\beta_2^3\right)\leq Q/3 \\
\sum \theta_{13}^{k'l'}+\sum \theta_{23}^{k'l'}+\sum \theta_{33}^{k'l'} &= \alpha_1^2+\alpha_1^3+\alpha_2^1+\alpha_2^3+\alpha_3^1+\alpha_3^2+\left(3-\alpha_1^2-\alpha_1^3-\alpha_2^1-\alpha_2^3-\alpha_3^1-\alpha_3^2\right)\left(\beta_3^1+\beta_3^2\right)\leq Q/3
\end{aligned}
$$

which imply

$$
\begin{aligned}
&\Rightarrow \quad \alpha_1^2,\alpha_1^3,\alpha_2^1,\alpha_2^3,\alpha_3^1,\alpha_3^2,\beta_1^2,\beta_1^3,\beta_2^1,\beta_2^3,\beta_3^1,\beta_3^2\leq Q/9 \\
&\Rightarrow \quad \left(\left\{\alpha_1^2,\alpha_1^3\right\}+\left\{\alpha_2^1,\alpha_2^3\right\}+\left\{\alpha_3^1,\alpha_3^2\right\}\right),\left(\left\{\beta_1^2,\beta_1^3\right\}+\left\{\beta_2^1,\beta_2^3\right\}+\left\{\beta_3^1,\beta_3^2\right\}\right)\leq Q/3 \\
&\Rightarrow \quad \alpha,\beta\leq Q/K^2 \qquad \text{(generally)} \\
&\Rightarrow \quad \sum\nolimits_k \left\{\alpha_k^{k'}\right\},\sum\nolimits_k \left\{\beta_k^{k'}\right\}\leq Q/K \qquad \text{(generally)}
\end{aligned}
$$

– Add Uni-directional assumption

  * Implies

$$
\alpha_2^1=\beta_2^1=\alpha_3^1=\beta_3^1=\alpha_3^2=\beta_3^2=0
$$

  * Now only 6 elements
  * General: # elements $= K(K-1)$

- Add Temporal Invariance assumption

$$\begin{aligned}
\theta_{kl}^{k'l'} &= \theta_k^{k'}\theta_l^{l'} \\
\theta_k^{k'} &= \Pr(y_o \in k', y_o^* \in k) = \Pr(y_1 \in k', y_1^* \in k) \\
\theta_k^{k} &= \Pr(y_o \in k, y_o^* \in k) = \Pr(y_1 \in k, y_1^* \in k) = 1 - \theta_k^{k'}
\end{aligned}$$

  - Now only 6 elements
    - General: # elements $= K(K-1)$
  - Implies

$$\begin{array}{llll}
\theta_{11}^{12} = \theta_1^2\left(1-\theta_1^2\right) & \theta_{12}^{11} = \theta_2^1\left(1-\theta_1^2\right) & \theta_{21}^{11} = \theta_2^1\left(1-\theta_1^2\right) & \theta_{22}^{11} = \left(\theta_2^1\right)^2 \\
\theta_{11}^{21} = \theta_1^2\left(1-\theta_1^2\right) & \theta_{12}^{21} = \theta_1^2\theta_2^1 & \theta_{21}^{12} = \theta_2^1\theta_1^2 & \theta_{22}^{12} = \theta_2^1\left(1-\theta_2^1\right) \\
\theta_{11}^{22} = \left(\theta_1^2\right)^2 & \theta_{12}^{22} = \theta_1^2\left(1-\theta_2^1\right) & \theta_{21}^{22} = \theta_1^2\left(1-\theta_2^1\right) & \theta_{22}^{21} = \theta_2^1\left(1-\theta_2^1\right)
\end{array}$$

    - Under Assumption 2(i) (solution: set one all θs but one to zero, solve using quadratic formula)

$$\begin{aligned}
\sum \theta_{kl}^{k'l'} &= \left(6-\theta_1^2-\theta_1^3-\theta_2^1-\theta_2^3-\theta_3^1-\theta_3^2\right)\left(\theta_1^2+\theta_1^3+\theta_2^1+\theta_2^3+\theta_3^1+\theta_3^2\right) \leq Q \\
&\Rightarrow \theta_1^2,\theta_1^3,\theta_2^1,\theta_2^3,\theta_3^1,\theta_3^2 \leq 3-\sqrt{9-Q} \\
&\Rightarrow \theta \leq K-\sqrt{K^2-Q} \qquad \text{(generally)}
\end{aligned}$$

    - Under Assumption 2(ii) (solution: set one all θs but one to zero, solve using quadratic formula)

$$\begin{aligned}
\sum\theta_{11}^{k'l'}+\sum\theta_{12}^{k'l'}+\sum\theta_{13}^{k'l'} &= 4\left(\theta_1^2+\theta_1^3\right)+\left(1-\theta_1^2-\theta_1^3\right)\left(\theta_2^1+\theta_2^3+\theta_3^1+\theta_3^2\right)-\left(\theta_1^2+\theta_1^3\right)^2 \leq Q/3 \\
\sum\theta_{21}^{k'l'}+\sum\theta_{22}^{k'l'}+\sum\theta_{23}^{k'l'} &= 4\left(\theta_2^1+\theta_2^3\right)+\left(1-\theta_2^1-\theta_2^3\right)\left(\theta_1^2+\theta_1^3+\theta_3^1+\theta_3^2\right)-\left(\theta_2^1+\theta_2^3\right)^2 \leq Q/3 \\
\sum\theta_{31}^{k'l'}+\sum\theta_{32}^{k'l'}+\sum\theta_{33}^{k'l'} &= 4\left(\theta_3^1+\theta_3^2\right)+\left(1-\theta_3^1-\theta_3^2\right)\left(\theta_1^2+\theta_1^3+\theta_3^1+\theta_3^2\right)-\left(\theta_3^1+\theta_3^2\right)^2 \leq Q/3 \\
\sum\theta_{11}^{k'l'}+\sum\theta_{21}^{k'l'}+\sum\theta_{31}^{k'l'} &= 4\left(\theta_1^2+\theta_1^3\right)+\left(\theta_2^1+\theta_2^3+\theta_3^1+\theta_3^2\right)\left(1-\theta_1^2-\theta_1^3\right)-\left(\theta_1^2+\theta_1^3\right)^2 \leq Q/3 \\
\sum\theta_{12}^{k'l'}+\sum\theta_{22}^{k'l'}+\sum\theta_{32}^{k'l'} &= 4\left(\theta_2^1+\theta_2^3\right)+\left(\theta_1^2+\theta_1^3+\theta_3^1+\theta_3^2\right)\left(1-\theta_2^1-\theta_2^3\right)-\left(\theta_2^1+\theta_2^3\right)^2 \leq Q/3 \\
\sum\theta_{13}^{k'l'}+\sum\theta_{23}^{k'l'}+\sum\theta_{33}^{k'l'} &= 4\left(\theta_3^1+\theta_3^2\right)+\left(\theta_1^2+\theta_1^3+\theta_2^1+\theta_2^3\right)\left(1-\theta_3^1-\theta_3^2\right)-\left(\theta_3^1+\theta_3^2\right)^2 \leq Q/3
\end{aligned}$$

    which imply

$$\begin{aligned}
&\Rightarrow \theta_1^2,\theta_1^3,\theta_2^1,\theta_2^3,\theta_3^1,\theta_3^2 \leq \left(4-\sqrt{16-4Q/3}\right)/2 \\
&\Rightarrow \theta \leq \left(K+1-\sqrt{(K+1)^2-4Q/K}\right)/2 \qquad \text{(generally)}
\end{aligned}$$

  - Add Uni-directional assumption
    - Implies

$$\theta_2^1 = \theta_3^1 = \theta_3^2 = 0$$

    - Now only 3 elements
    - General: # elements $= K(K-1)/2$

A.1.1 p_{11}^*

$$p_{11}^* = \frac{r_{11} + \overbrace{\left[\underset{\bullet}{\theta_{11}^{12}} + \underset{\bullet}{\theta_{11}^{13}} + \theta_{11}^{21} + \theta_{11}^{22} + \theta_{11}^{23} + \theta_{11}^{31} + \theta_{11}^{32} + \theta_{11}^{33}\right]}^{Q_{1,11}} - \overbrace{\left[\underset{\bullet}{\theta_{12}^{11}} + \underset{\bullet}{\theta_{13}^{11}} + \theta_{21}^{11} + \theta_{22}^{11} + \theta_{23}^{11} + \theta_{31}^{11} + \theta_{32}^{11} + \theta_{33}^{11}\right]}^{Q_{2,11}}}{\begin{array}{l} p_1 + \underbrace{\left[\theta_{11}^{21} + \theta_{11}^{22} + \theta_{11}^{23} + \theta_{11}^{31} + \theta_{11}^{32} + \theta_{11}^{33} + \underset{\bullet}{\theta_{12}^{21}} + \underset{\bullet}{\theta_{12}^{22}} + \underset{\bullet}{\theta_{12}^{23}} + \underset{\bullet}{\theta_{12}^{31}} + \underset{\bullet}{\theta_{12}^{32}} + \underset{\bullet}{\theta_{12}^{33}} + \underset{\bullet}{\theta_{13}^{21}} + \underset{\bullet}{\theta_{13}^{22}} + \underset{\bullet}{\theta_{13}^{23}} + \underset{\bullet}{\theta_{13}^{31}} + \underset{\bullet}{\theta_{13}^{32}} + \underset{\bullet}{\theta_{13}^{33}}\right]}_{Q_{3,1}} \\ - \underbrace{\left[\theta_{21}^{11} + \theta_{22}^{11} + \theta_{23}^{11} + \underset{\bullet}{\theta_{21}^{12}} + \underset{\bullet}{\theta_{22}^{12}} + \underset{\bullet}{\theta_{23}^{12}} + \underset{\bullet}{\theta_{21}^{13}} + \underset{\bullet}{\theta_{22}^{13}} + \underset{\bullet}{\theta_{23}^{13}} + \theta_{31}^{11} + \theta_{32}^{11} + \theta_{33}^{11} + \underset{\bullet}{\theta_{31}^{12}} + \underset{\bullet}{\theta_{32}^{12}} + \underset{\bullet}{\theta_{33}^{12}} + \underset{\bullet}{\theta_{31}^{13}} + \underset{\bullet}{\theta_{32}^{13}} + \underset{\bullet}{\theta_{33}^{13}}\right]}_{Q_{4,1}} \end{array}}$$

- $\underset{\bullet}{\theta_{kl}^{k'l'}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$\begin{aligned} LB_{11} &= \frac{r_{11} - \widetilde{Q}}{p_1} \geq 0 & \qquad \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB_{11} &= \frac{r_{11} + \widetilde{Q}}{p_1 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} &= \begin{cases} 0 & \text{AE} \\ \min\{p_1, Q/3\} & \text{UE} \end{cases} \end{aligned}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p_{11}^* = \frac{r_{11} + \overbrace{\left[\underset{\bullet}{\theta_{11}^{12}} + \underset{\bullet}{\theta_{11}^{13}} + \theta_{11}^{21} + \theta_{11}^{22} + \theta_{11}^{23} + \theta_{11}^{31} + \theta_{11}^{32} + \theta_{11}^{33}\right]}^{Q_{1,11}}}{p_1 + \underbrace{\left[\theta_{11}^{21} + \theta_{11}^{22} + \theta_{11}^{23} + \theta_{11}^{31} + \theta_{11}^{32} + \theta_{11}^{33} + \underset{\bullet}{\theta_{12}^{22}} + \underset{\bullet}{\theta_{12}^{23}} + \underset{\bullet}{\theta_{12}^{32}} + \underset{\bullet}{\theta_{12}^{33}} + \underset{\bullet}{\theta_{13}^{23}} + \underset{\bullet}{\theta_{13}^{33}}\right]}_{Q_{3,1}^u}}$$

- Yields

$$\begin{aligned} LB_{11}^u &= \frac{r_{11}}{p_1 + \widetilde{\widetilde{Q}}} \geq 0 & \qquad \widetilde{\widetilde{Q}} &= \min\left\{1 - p_1, \widetilde{Q}\right\},\ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB_{11}^u &= \frac{r_{11} + \widetilde{Q}}{p_1} \leq 1 & \qquad \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \end{aligned}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{11}^{*} = \frac{r_{11} + \overbrace{\left[\alpha_1^1\beta_1^2 + \alpha_1^1\beta_1^3 + \alpha_1^2\beta_1^1 + \alpha_1^2\beta_1^2 + \alpha_1^2\beta_1^3 + \alpha_1^3\beta_1^1 + \alpha_1^3\beta_1^2 + \alpha_1^3\beta_1^3\right]}^{Q_{1,11}} - \overbrace{\left[\alpha_1^1\beta_2^1 + \alpha_1^1\beta_3^1 + \alpha_2^1\beta_1^1 + \alpha_2^1\beta_2^1 + \alpha_2^1\beta_3^1 + \alpha_3^1\beta_1^1 + \alpha_3^1\beta_2^1 + \alpha_3^1\beta_3^1\right]}^{Q_{2,11}}}{\begin{array}{c} p_1 + \underbrace{\left[\alpha_1^2\beta_1^1 + \alpha_1^2\beta_1^2 + \alpha_1^2\beta_1^3 + \alpha_1^3\beta_1^1 + \alpha_1^3\beta_1^2 + \alpha_1^3\beta_1^3 + \alpha_1^2\beta_2^1 + \alpha_1^2\beta_2^2 + \alpha_1^2\beta_2^3 + \alpha_1^3\beta_2^1 + \alpha_1^3\beta_2^2 + \alpha_1^3\beta_2^3 + \alpha_1^2\beta_3^1 + \alpha_1^2\beta_3^2 + \alpha_1^2\beta_3^3 + \alpha_1^3\beta_3^1 + \alpha_1^3\beta_3^2 + \alpha_1^3\beta_3^3\right]}_{Q_{3,1}} - \\ \underbrace{\left[\alpha_2^1\beta_1^1 + \alpha_2^1\beta_2^1 + \alpha_2^1\beta_3^1 + \alpha_2^1\beta_1^2 + \alpha_2^1\beta_2^2 + \alpha_2^1\beta_3^2 + \alpha_2^1\beta_1^3 + \alpha_2^1\beta_2^3 + \alpha_2^1\beta_3^3 + \alpha_3^1\beta_1^1 + \alpha_3^1\beta_2^1 + \alpha_3^1\beta_3^1 + \alpha_3^1\beta_1^2 + \alpha_3^1\beta_2^2 + \alpha_3^1\beta_3^2 + \alpha_3^1\beta_1^3 + \alpha_3^1\beta_2^3 + \alpha_3^1\beta_3^3\right]}_{Q_{4,1}} \end{array}}$$

- Simplifying

$$\begin{aligned}
Q_{1,11} &= \left(\alpha_1^2 + \alpha_1^3\right) + \left(\beta_1^2 + \beta_1^3\right)\left(1 - \alpha_1^2 - \alpha_1^3\right) && \text{(TI)} \\
&= 2\left(\theta_1^2 + \theta_1^3\right) - \left(\theta_1^2 + \theta_1^3\right)^2 && \text{(TIV)} \\
Q_{2,11} &= \left(\alpha_2^1 + \alpha_3^1\right)\left(1 + \beta_2^1 + \beta_3^1 - \beta_1^2 - \beta_1^3\right) + \left(\beta_2^1 + \beta_3^1\right)\left(1 - \alpha_1^2 - \alpha_1^3\right) && \text{(TI)} \\
&= 2\left(\theta_2^1 + \theta_3^1\right)\left(1 - \theta_1^2 - \theta_1^3\right) + \left(\theta_2^1 + \theta_3^1\right)^2 && \text{(TIV)} \\
Q_{3,1} &= 3\left(\alpha_1^2 + \alpha_1^3\right) && \text{(TI)} \\
&= 3\left(\theta_1^2 + \theta_1^3\right) && \text{(TIV)} \\
Q_{4,1} &= 3\left(\alpha_2^1 + \alpha_3^1\right) && \text{(TI)} \\
&= 3\left(\theta_2^1 + \theta_3^1\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{11}^{*}=\frac{r_{11}+\left(\alpha_1^2+\alpha_1^3-\alpha_2^1-\alpha_3^1\right)+\left(\beta_1^2+\beta_1^3-\beta_2^1-\beta_3^1\right)\left(1+\alpha_2^1+\alpha_3^1-\alpha_1^2-\alpha_1^3\right)}{p_1+3\left(\alpha_1^2+\alpha_1^3-\alpha_2^1-\alpha_3^1\right)}$$

  - Yields

$$\begin{aligned}
LB_{11}^{TI} &= \min\left\{\frac{r_{11}-\ddot{Q}}{p_1},\frac{r_{11}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}},\frac{r_{11}-\widehat{Q}}{p_1-3\widehat{Q}}\right\}\geq 0\\
&\qquad \widetilde{\widetilde{Q}}=\min\left\{1-r_{11},(1-p_1)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{11},p_1/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\\
UB_{11}^{TI} &= \max\left\{\frac{r_{11}+\widetilde{Q}}{p_1},\frac{r_{11}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}},\frac{r_{11}-\widehat{Q}}{p_1-3\widehat{Q}}\right\}\leq 1\\
&\qquad \widetilde{\widetilde{Q}}=\min\left\{1-r_{11},(1-p_1)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{11},p_1/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}
\end{aligned}$$

  - Proof:
    1. $\ddot{Q}$ in $\frac{r_{11}-\ddot{Q}}{p_1}$ can be $2Q/9$ as $\beta_2^1,\beta_3^1=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{11}-\widehat{Q}}{p_1-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_2^1,\alpha_3^1=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{11}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}}\right)/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{11}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}}\right)}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(\left(p_1+3\widetilde{\widetilde{Q}}\right)-3\left(r_{11}+\widetilde{\widetilde{Q}}\right)\right)\\
&= sgn\left(p_1-3r_{11}\right)
\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{11}-\widehat{Q}}{p_1-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{11}-\widehat{Q}}{p_1-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_1-3\widehat{Q}\right)+3\left(r_{11}-\widehat{Q}\right)\right)\\
&= sgn\left(3r_{11}-p_1\right)
\end{aligned}$$

  - Adding the uni-directional assumption

$$p_{11}^{*}=\frac{r_{11}+\left(\alpha_1^2+\alpha_1^3\right)+\left(\beta_1^2+\beta_1^3\right)\left(1-\alpha_1^2-\alpha_1^3\right)}{p_1+3\left(\alpha_1^2+\alpha_1^3\right)}$$

    * Yields

$$\begin{aligned}
LB_{11}^{TI,u} &= \frac{r_{11}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}}\geq 0 \qquad \widetilde{\widetilde{Q}}=\begin{cases} 0 & r_{11}<p_1/3\\ \min\left\{1-r_{11},(1-p_1)/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases}\\
UB_{11}^{TI,u} &= \max\left\{\frac{r_{11}+\widetilde{Q}}{p_1},\frac{r_{11}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}}\right\}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{1-r_{11},(1-p_1)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases}
\end{aligned}$$

    * Proof: Same as above.

- Under Temporal Invariance

$$p_{11}^{*} = \frac{r_{11} + 2\left(\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1\right) - \left(\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1\right)^2}{p_1 + 3\left(\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1\right)}$$

  - Yields

$$\begin{aligned}
LB_{11}^{TIV} \quad &= \quad \min\left\{\frac{r_{11} + 2\widehat{Q} - \widehat{Q}^2}{p_1 + 3\widehat{Q}}, \frac{r_{11} - 2\widetilde{\widetilde{Q}} - \widetilde{\widetilde{Q}}^2}{p_1 - 3\widetilde{\widetilde{Q}}}\right\} \geq 0 \\
& \qquad \widehat{Q} = \min\left\{(1-p_1)/3, \widetilde{Q}\right\}, \\
& \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{11} \geq 2p_1/3 \\ \min\left\{\frac{(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}, (-1 + \sqrt{1 + r_{11}}), p_1/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \\
& \qquad \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases} \\
UB_{11}^{TIV} \quad &= \quad \max\left\{\frac{r_{11} + 2\widehat{Q} - \widehat{Q}^2}{p_1 + 3\widehat{Q}}, \frac{r_{11} - 2\widetilde{\widetilde{Q}} - \widetilde{\widetilde{Q}}^2}{p_1 - 3\widetilde{\widetilde{Q}}}\right\} \geq 0 \\
& \qquad \widehat{Q} = \begin{cases} 0 & r_{11} \geq 2p_1/3 \\ \min\left\{\frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}, (1-p_1)/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \\
& \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{11} < 2p_1/3 \\ \min\left\{\frac{(2/3)p_1 - \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}, (-1 + \sqrt{1 + r_{11}}), p_1/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \\
& \qquad \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}
\end{aligned}$$

– Proof:

1. Evaluate $\partial\left(\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(-2-2\tilde{\tilde{Q}}\right)\left(p_1-3\tilde{\tilde{Q}}\right)+3\left(r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2\right)\right) \\
&= sgn\left(-(2/3)p_1\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{11}\right) \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right)\Bigg|_{\tilde{\tilde{Q}}=0} &= sgn\left(-(2/3)p_1+r_{11}\right) \gtrless 0 \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right)\Bigg|_{\tilde{\tilde{Q}}=1} &= sgn\left(-(4/3)p_1+1+r_{11}\right) \gtrless 0
\end{aligned}$$

2. Ensure $r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2 \geq 0$

$$\begin{aligned}
& r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2 \geq 0 \\
\Rightarrow\ & \tilde{\tilde{Q}}^2+2\tilde{\tilde{Q}}-r_{11} \leq 0 \\
\Rightarrow\ & \tilde{\tilde{Q}} \leq \frac{-2+\sqrt{4+4r_{11}}}{2} \\
\Rightarrow\ & \tilde{\tilde{Q}} \leq -1+\sqrt{1+r_{11}}
\end{aligned}$$

3. Minimize $\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}$ s.t. $\tilde{\tilde{Q}}$ being feasible and $r_{11}<2p_1/3$

$$\begin{aligned}
\frac{\partial\left(\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}} &\propto -(2/3)p_1\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{11}=0 \\
\Rightarrow \tilde{\tilde{Q}}^* &= \frac{(2/3)p_1+\sqrt{(4/9)p_1^2+4[(2/3)p_1-r_{11}]}}{2}
\end{aligned}$$

4. Maximize $\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}$ s.t. $\tilde{\tilde{Q}}$ being feasible and $r_{11}>2p_1/3$

$$\begin{aligned}
\frac{\partial\left(\frac{r_{11}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_1-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}} &\propto -(2/3)p_1\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{11}=0 \\
\Rightarrow \tilde{\tilde{Q}}^* &= \frac{(2/3)p_1-\sqrt{(4/9)p_1^2+4[(2/3)p_1-r_{11}]}}{2}
\end{aligned}$$

Note: If $\sqrt{(4/9)p_1^2+4[(2/3)p_1-r_{11}]}=.$, then maximize $\tilde{\tilde{Q}}$.

5. Evaluate $\partial\left(\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(\left(2-2\widehat{Q}\right)\left(p_1+3\widehat{Q}\right)-3\left(r_{11}+2\widehat{Q}-\widehat{Q}^2\right)\right) \\
&= sgn\left((2/3)p_1\left(1-\widehat{Q}\right)-\widehat{Q}^2-r_{11}\right) \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right)\Bigg|_{\widehat{Q}=0} &= sgn\left((2/3)p_1-r_{11}\right) \gtrless 0 \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right)\Bigg|_{\widehat{Q}=1} &= sgn\left(-1-r_{11}\right) < 0
\end{aligned}$$

6. Maximize $\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}}$ s.t. $\widehat{Q}$ being feasible and $r_{11} < 2p_1/3$

$$\begin{aligned}
\frac{\partial\left(\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}}\right)}{\partial\widetilde{\widehat{Q}}} &\propto (2/3)p_1\left(1-\widehat{Q}\right)-\widehat{Q}^2-r_{11}=0 \\
&\Rightarrow \widetilde{\widehat{Q}}^* = \frac{-(2/3)p_1+\sqrt{(4/9)p_1^2+4[(2/3)p_1-r_{11}]}}{2}
\end{aligned}$$

7. Minimize $\frac{r_{11}+2\widehat{Q}-\widehat{Q}^2}{p_1+3\widehat{Q}} \Rightarrow \widehat{Q}=0$ or maximize $\widehat{Q}$. However, if the minimum occurs when $\widehat{Q}=0$, then $\frac{r_{11}-2\widetilde{\widehat{Q}}-\widetilde{\widehat{Q}}^2}{p_1-3\widetilde{\widehat{Q}}} < \frac{r_{11}}{p_1}$ and this will be the binding LB.

– Adding the uni-directional assumption

$$p_{11}^* = \frac{r_{11} + 2\theta_1^2 - \left(\theta_1^2\right)^2}{p_1 + 2\theta_1^2}$$

* Yields

$$\begin{aligned}
LB_{11}^{TIV,u} &= \min\left\{\frac{r_{11}}{p_1}, \frac{r_{11} + 2\widetilde{\widetilde{Q}} - \widetilde{\widetilde{Q}}^2}{p_1 + 3\widetilde{\widetilde{Q}}}\right\} \geq 0 \\
&\qquad \widetilde{\widetilde{Q}} = \min\left\{(1-p_1)/3, \widetilde{Q}\right\}, \\
&\qquad \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases} \\
UB_{11}^{TIV,u} &= \frac{r_{11} + 2\widetilde{\widetilde{Q}} - \widetilde{\widetilde{Q}}^2}{p_1 + 3\widetilde{\widetilde{Q}}} \leq 1 \\
&\qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{11} \geq 2p_1/3 \\ \min\left\{\frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}, (1-p_1)/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \\
&\qquad \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}
\end{aligned}$$

* Proof: Same as above.

A.1.2 p_{12}^*

$$
p_{12}^* = \frac{r_{12} + \overbrace{\left[\underset{\bullet}{\theta_{12}^{11}} + \underset{\bullet}{\theta_{12}^{13}} + \theta_{12}^{21} + \theta_{12}^{22} + \theta_{12}^{23} + \theta_{12}^{31} + \theta_{12}^{32} + \theta_{12}^{33}\right]}^{Q_{1,12}} - \overbrace{\left[\underset{\bullet}{\theta_{11}^{12}} + \underset{\bullet}{\theta_{13}^{12}} + \theta_{21}^{12} + \theta_{22}^{12} + \theta_{23}^{12} + \theta_{31}^{12} + \theta_{32}^{12} + \theta_{33}^{12}\right]}^{Q_{2,12}}}{\begin{array}{l} p_1 + \underbrace{\left[\theta_{11}^{21} + \theta_{11}^{22} + \theta_{11}^{23} + \theta_{11}^{31} + \theta_{11}^{32} + \theta_{11}^{33} + \underset{\bullet}{\theta_{12}^{21}} + \underset{\bullet}{\theta_{12}^{22}} + \underset{\bullet}{\theta_{12}^{23}} + \underset{\bullet}{\theta_{12}^{31}} + \underset{\bullet}{\theta_{12}^{32}} + \underset{\bullet}{\theta_{12}^{33}} + \underset{\bullet}{\theta_{13}^{21}} + \underset{\bullet}{\theta_{13}^{22}} + \underset{\bullet}{\theta_{13}^{23}} + \underset{\bullet}{\theta_{13}^{31}} + \underset{\bullet}{\theta_{13}^{32}} + \underset{\bullet}{\theta_{13}^{33}}\right]}_{Q_{3,1}} \\ \quad - \underbrace{\left[\theta_{21}^{11} + \theta_{22}^{11} + \theta_{23}^{11} + \underset{\bullet}{\theta_{21}^{12}} + \underset{\bullet}{\theta_{22}^{12}} + \underset{\bullet}{\theta_{23}^{12}} + \underset{\bullet}{\theta_{21}^{13}} + \underset{\bullet}{\theta_{22}^{13}} + \underset{\bullet}{\theta_{23}^{13}} + \theta_{31}^{11} + \theta_{32}^{11} + \theta_{33}^{11} + \underset{\bullet}{\theta_{31}^{12}} + \underset{\bullet}{\theta_{32}^{12}} + \underset{\bullet}{\theta_{33}^{12}} + \underset{\bullet}{\theta_{31}^{13}} + \underset{\bullet}{\theta_{32}^{13}} + \underset{\bullet}{\theta_{33}^{13}}\right]}_{Q_{4,1}} \end{array}}
$$

- $\underset{\bullet}{\theta_{kl}^{k'l'}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$
\begin{aligned}
LB_{12} &= \frac{r_{12} - \widetilde{Q}}{p_1} \geq 0 \qquad & \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\
UB_{12} &= \frac{r_{12} + \widetilde{Q}}{p_1 - \widetilde{\widetilde{Q}}} \leq 1 \qquad & \widetilde{\widetilde{Q}} = \begin{cases} 0 & \text{AE} \\ \min\{p_1, Q/3\} & \text{UE} \end{cases}
\end{aligned}
$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$
p_{12}^* = \frac{r_{12} + \overbrace{\left[\underset{\bullet}{\theta_{12}^{13}} + \theta_{12}^{22} + \theta_{12}^{23} + \theta_{12}^{32} + \theta_{12}^{33}\right]}^{Q_{1,12}^u} - \overbrace{\left[\underset{\bullet}{\theta_{11}^{12}}\right]}^{Q_{2,12}^u}}{p_1 + \underbrace{\left[\underset{\bullet}{\theta_{11}^{21}} + \underset{\bullet}{\theta_{11}^{22}} + \underset{\bullet}{\theta_{11}^{23}} + \underset{\bullet}{\theta_{11}^{31}} + \underset{\bullet}{\theta_{11}^{32}} + \underset{\bullet}{\theta_{11}^{33}} + \theta_{12}^{22} + \theta_{12}^{23} + \theta_{12}^{32} + \theta_{12}^{33} + \underset{\bullet}{\theta_{13}^{23}} + \underset{\bullet}{\theta_{13}^{33}}\right]}_{Q_{3,1}^u}}
$$

- Yields

$$
\begin{aligned}
LB_{12}^u &= \min\left\{\frac{r_{12} - \widetilde{Q}}{p_1}, \frac{r_{12}}{p_1 + \widetilde{\widetilde{Q}}}\right\} \geq 0 \qquad & \widetilde{\widetilde{Q}} = \min\{1 - p_1, \widetilde{Q}\},\ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\
UB_{12}^u &= \frac{r_{12} + \widetilde{Q}}{p_1} \leq 1
\end{aligned}
$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{12}^{*} = \frac{r_{12} + \overbrace{\left[\alpha_1^1\beta_2^1 + \alpha_1^1\beta_2^3 + \alpha_1^2\beta_2^1 + \alpha_1^2\beta_2^2 + \alpha_1^2\beta_2^3 + \alpha_1^3\beta_2^1 + \alpha_1^3\beta_2^2 + \alpha_1^3\beta_2^3\right]}^{Q_{1,12}} - \overbrace{\left[\alpha_1^1\beta_1^2 + \alpha_1^1\beta_3^2 + \alpha_2^1\beta_1^2 + \alpha_2^1\beta_2^2 + \alpha_2^1\beta_3^2 + \alpha_3^1\beta_1^2 + \alpha_3^1\beta_2^2 + \alpha_3^1\beta_3^2\right]}^{Q_{2,12}}}{p_1 + \underbrace{\left[\alpha_1^2\beta_1^1 + \alpha_1^2\beta_1^2 + \alpha_1^2\beta_1^3 + \alpha_1^3\beta_1^1 + \alpha_1^3\beta_1^2 + \alpha_1^3\beta_1^3 + \alpha_1^2\beta_2^1 + \alpha_1^2\beta_2^2 + \alpha_1^2\beta_2^3 + \alpha_1^3\beta_2^1 + \alpha_1^3\beta_2^2 + \alpha_1^3\beta_2^3 + \alpha_1^2\beta_3^1 + \alpha_1^2\beta_3^2 + \alpha_1^2\beta_3^3 + \alpha_1^3\beta_3^1 + \alpha_1^3\beta_3^2 + \alpha_1^3\beta_3^3\right]}_{Q_{3,1}} - \underbrace{\left[\alpha_2^1\beta_1^1 + \alpha_2^1\beta_2^1 + \alpha_2^1\beta_3^1 + \alpha_2^1\beta_1^2 + \alpha_2^1\beta_2^2 + \alpha_2^1\beta_3^2 + \alpha_2^1\beta_1^3 + \alpha_2^1\beta_2^3 + \alpha_2^1\beta_3^3 + \alpha_3^1\beta_1^1 + \alpha_3^1\beta_2^1 + \alpha_3^1\beta_3^1 + \alpha_3^1\beta_1^2 + \alpha_3^1\beta_2^2 + \alpha_3^1\beta_3^2 + \alpha_3^1\beta_1^3 + \alpha_3^1\beta_2^3 + \alpha_3^1\beta_3^3\right]}_{Q_{4,1}}}$$

- Simplifying

$$\begin{aligned}
Q_{1,12} &= \left(\alpha_1^2 + \alpha_1^3\right) + \left(\beta_2^1 + \beta_2^3\right)\left(1 - \alpha_1^2 - \alpha_1^3\right) && \text{(TI)} \\
&= \left(\theta_1^2 + \theta_1^3\right) + \left(\theta_2^1 + \theta_2^3\right)\left(1 - \theta_1^2 - \theta_1^3\right) && \text{(TIV)} \\
Q_{2,12} &= \left(\alpha_2^1 + \alpha_3^1\right)\left(1 + \beta_1^2 + \beta_3^2 - \beta_2^1 - \beta_2^3\right) + \left(\beta_1^2 + \beta_3^2\right)\left(1 - \alpha_1^2 - \alpha_1^3\right) && \text{(TI)} \\
&= \left(\theta_2^1 + \theta_3^1\right)\left(1 + \theta_1^2 + \theta_3^2 - \theta_2^1 - \theta_2^3\right) + \left(\theta_1^2 + \theta_3^2\right)\left(1 - \theta_1^2 - \theta_1^3\right) && \text{(TIV)} \\
Q_{3,1} &= 3\left(\alpha_1^2 + \alpha_1^3\right) && \text{(TI)} \\
&= 3\left(\theta_1^2 + \theta_1^3\right) && \text{(TIV)} \\
Q_{4,1} &= 3\left(\alpha_2^1 + \alpha_3^1\right) && \text{(TI)} \\
&= 3\left(\theta_2^1 + \theta_3^1\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{12}^{*}=\frac{r_{12}+\left(\beta_2^1+\beta_2^3-\beta_1^2-\beta_3^2\right)+\left(\alpha_1^2+\alpha_1^3-\alpha_2^1-\alpha_3^1\right)\left(1+\beta_1^2+\beta_3^2-\beta_2^1-\beta_2^3\right)}{p_1+3\left(\alpha_1^2+\alpha_1^3-\alpha_2^1-\alpha_3^1\right)}$$

  - Yields

$$\begin{aligned}
LB_{12}^{TI} &= \min\left\{\frac{r_{12}-\ddot{Q}}{p_1},\frac{r_{12}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}},\frac{r_{12}-\widehat{Q}}{p_1-3\widehat{Q}}\right\}\geq 0\\
&\quad \tilde{\tilde{Q}}<\min\left\{1-r_{12},(1-p_1)/3,\widetilde{Q}\right\},\ \widehat{Q}<\min\left\{r_{12},p_1/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\\
UB_{12}^{TI} &= \max\left\{\frac{r_{12}+\widetilde{Q}}{p_1},\frac{r_{12}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}},\frac{r_{12}-\widehat{Q}}{p_1-3\widehat{Q}}\right\}\leq 1\\
&\quad \tilde{\tilde{Q}}<\min\left\{1-r_{12},(1-p_1)/3,\widetilde{Q}\right\},\ \widehat{Q}<\min\left\{r_{12},p_1/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases}Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}
\end{aligned}$$

  - Proof:
    1. $\ddot{Q}$ in $\frac{r_{12}-\ddot{Q}}{p_1}$ can be $2Q/9$ as $\beta_1^2,\beta_3^2=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{12}-\widehat{Q}}{p_1-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_2^1,\alpha_3^1=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{12}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{12}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(p_1+3\tilde{\tilde{Q}}\right)-3\left(r_{12}+\tilde{\tilde{Q}}\right)\right)\\
&= sgn\left(p_1-3r_{12}\right)
\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{12}-\widehat{Q}}{p_1-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{12}-\widehat{Q}}{p_1-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_1-3\widehat{Q}\right)+3\left(r_{12}-\widehat{Q}\right)\right)\\
&= sgn\left(3r_{12}-p_1\right)
\end{aligned}$$

    Since both derivatives can take either sign, it is possible either could be the LB, UB.

  - Adding the uni-directional assumption

$$p_{12}^{*}=\frac{r_{12}+\left(\beta_2^3-\beta_1^2\right)+\left(\alpha_1^2+\alpha_1^3\right)\left(1+\beta_1^2-\beta_2^3\right)}{p_1+3\left(\alpha_1^2+\alpha_1^3\right)}$$

    * Yields

$$\begin{aligned}
LB_{12}^{TI,u} &= \min\left\{\frac{r_{12}-\widetilde{Q}}{p_1},\frac{r_{12}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right\}\geq 0 \qquad \tilde{\tilde{Q}}<\min\left\{1-r_{12},(1-p_1)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases}\\
UB_{12}^{TI,u} &= \max\left\{\frac{r_{12}+\widetilde{Q}}{p_1},\frac{r_{12}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right\}\leq 1 \qquad \tilde{\tilde{Q}}<\min\left\{1-r_{12},(1-p_1)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases}
\end{aligned}$$

    * Proof: Same as above except now $\ddot{Q}$ is not feasible.

- Under Temporal Invariance

$$p_{12}^{*}=\frac{r_{12}+\left(\theta_1^3+\theta_2^3-\theta_3^1-\theta_3^2\right)+\left(\theta_1^2+\theta_1^3-\theta_2^1-\theta_3^1\right)\left(\theta_1^2+\theta_3^2-\theta_2^1-\theta_2^3\right)}{p_1+3\left(\theta_1^2+\theta_1^3-\theta_2^1-\theta_3^1\right)}$$

  – Yields

$$\begin{aligned}
LB_{12}^{TIV} &= \min\left\{\frac{r_{12}-\widetilde{Q}}{p_1},\frac{r_{12}+\widetilde{\widetilde{Q}}^2}{p_1+3\widetilde{\widetilde{Q}}}\right\}\geq 0\\
&\qquad \widetilde{\widetilde{Q}}=\min\left\{\frac{-(2/3)p_1+\sqrt{(4/9)p_1^2+4r_{12}}}{2},\sqrt{1-r_{12}},(1-p_1)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\left\{\begin{array}{ll}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{array}\right.\\
UB_{12}^{TIV} &= \max\left\{\frac{r_{12}+\widetilde{Q}}{p_1},\frac{r_{12}+\widetilde{\widetilde{Q}}^2}{p_1-3\widetilde{\widetilde{Q}}}\right\}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{\sqrt{1-r_{12}},p_1/3,\widetilde{Q}\right\},\ \widetilde{Q}=\left\{\begin{array}{ll}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{array}\right.
\end{aligned}$$

  – Proof:

    1. Evaluate $\partial LB_{12}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned}
sgn\left(\frac{\partial LB_{12}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(2\widetilde{\widetilde{Q}}\left(p_1+3\widetilde{\widetilde{Q}}\right)-3\left(r_{12}+\widetilde{\widetilde{Q}}^2\right)\right)\\
&= sgn\left(\widetilde{\widetilde{Q}}\left((2/3)p_1+\widetilde{\widetilde{Q}}\right)-r_{12}\right)\\
\Rightarrow sgn\left(\frac{\partial LB_{12}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right)\Bigg|_{\widetilde{\widetilde{Q}}=0} &= sgn\left(-r_{12}\right)<0\\
\Rightarrow\widetilde{\widetilde{Q}} &> 0
\end{aligned}$$

    2. Minimize LB_{12}^{TIV} s.t. $\widetilde{\widetilde{Q}}$ being feasible

$$\begin{aligned}
\frac{\partial LB_{12}^{TIV}}{\partial\widetilde{\widetilde{Q}}} &\propto \widetilde{\widetilde{Q}}\left((2/3)p_1+\widetilde{\widetilde{Q}}\right)-r_{12}=0\\
\Rightarrow\ \widetilde{\widetilde{Q}}^{*} &= \frac{-(2/3)p_1+\sqrt{(4/9)p_1^2+4r_{12}}}{2}
\end{aligned}$$

    So, derivative starts off negative and then reaches zero at $\widetilde{\widetilde{Q}}^{*}$. Thus, $\frac{r_{12}+\widetilde{\widetilde{Q}}^2}{p_1+3\widetilde{\widetilde{Q}}}$ is minimized at $\widetilde{\widetilde{Q}}^{*}$.

  – Adding the uni-directional assumption

$$p_{12}^{*}=\frac{r_{12}+\left(\theta_1^3+\theta_2^3\right)+\left(\theta_1^2+\theta_1^3\right)\left(\theta_1^2-\theta_2^3\right)}{p_1+3\left(\theta_1^2+\theta_1^3\right)}$$

    * Yields

$$\begin{aligned}
LB_{12}^{TIV,u} &= \frac{r_{12}+\widetilde{\widetilde{Q}}^2}{p_1+3\widetilde{\widetilde{Q}}}\geq 0\\
&\qquad \widetilde{\widetilde{Q}}=\min\left\{\frac{-(2/3)p_1+\sqrt{(4/9)p_1^2+4r_{12}}}{2},\sqrt{1-r_{12}},(1-p_1)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\left\{\begin{array}{ll}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{array}\right.\\
UB_{12}^{TIV,u} &= \frac{r_{12}+\widetilde{Q}}{p_1}\leq 1 \qquad \widetilde{Q}=\left\{\begin{array}{ll}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{array}\right.
\end{aligned}$$

    * Proof: Same as above.

A.1.3 p^*_{13}

$$
p^*_{13} = \frac{r_{13} + \overbrace{\left[\underset{\bullet}{\theta^{11}_{13}} + \underset{\bullet}{\theta^{12}_{13}} + \theta^{21}_{13} + \theta^{22}_{13} + \theta^{23}_{13} + \theta^{31}_{13} + \theta^{32}_{13} + \theta^{33}_{13}\right]}^{Q_{1,13}} - \overbrace{\left[\underset{\bullet}{\theta^{13}_{11}} + \underset{\bullet}{\theta^{13}_{12}} + \theta^{13}_{21} + \theta^{13}_{22} + \theta^{13}_{23} + \theta^{13}_{31} + \theta^{13}_{32} + \theta^{13}_{33}\right]}^{Q_{2,13}}}{\begin{array}{l} p_1 + \underbrace{\left[\theta^{21}_{11} + \theta^{22}_{11} + \theta^{23}_{11} + \theta^{31}_{11} + \theta^{32}_{11} + \theta^{33}_{11} + \underset{\bullet}{\theta^{21}_{12}} + \underset{\bullet}{\theta^{22}_{12}} + \underset{\bullet}{\theta^{23}_{12}} + \underset{\bullet}{\theta^{31}_{12}} + \underset{\bullet}{\theta^{32}_{12}} + \underset{\bullet}{\theta^{33}_{12}} + \underset{\bullet}{\theta^{21}_{13}} + \underset{\bullet}{\theta^{22}_{13}} + \underset{\bullet}{\theta^{23}_{13}} + \underset{\bullet}{\theta^{31}_{13}} + \underset{\bullet}{\theta^{32}_{13}} + \underset{\bullet}{\theta^{33}_{13}}\right]}_{Q_{3,1}} \\ \quad - \underbrace{\left[\theta^{11}_{21} + \theta^{11}_{22} + \theta^{11}_{23} + \underset{\bullet}{\theta^{12}_{21}} + \underset{\bullet}{\theta^{12}_{22}} + \underset{\bullet}{\theta^{12}_{23}} + \underset{\bullet}{\theta^{13}_{21}} + \underset{\bullet}{\theta^{13}_{22}} + \underset{\bullet}{\theta^{13}_{23}} + \theta^{11}_{31} + \theta^{11}_{32} + \theta^{11}_{33} + \underset{\bullet}{\theta^{12}_{31}} + \underset{\bullet}{\theta^{12}_{32}} + \underset{\bullet}{\theta^{12}_{33}} + \underset{\bullet}{\theta^{13}_{31}} + \underset{\bullet}{\theta^{13}_{32}} + \underset{\bullet}{\theta^{13}_{33}}\right]}_{Q_{4,1}} \end{array}}
$$

- $\underset{\bullet}{\theta^{k'l'}_{kl}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$
\begin{aligned}
LB_{13} &= \frac{r_{13} - \widetilde{Q}}{p_1} \geq 0 & \qquad \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\
UB_{13} &= \frac{r_{13} + \widetilde{Q}}{p_1 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} &= \begin{cases} 0 & \text{AE} \\ \min\{p_1, Q/3\} & \text{UE} \end{cases}
\end{aligned}
$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$
p^*_{13} = \frac{r_{13} + \overbrace{\left[\theta^{23}_{13} + \theta^{33}_{13}\right]}^{Q^u_{1,12}} - \overbrace{\left[\underset{\bullet}{\theta^{13}_{11}} + \underset{\bullet}{\theta^{13}_{12}}\right]}^{Q^u_{2,12}}}{p_1 + \underbrace{\left[\underset{\bullet}{\theta^{21}_{11}} + \underset{\bullet}{\theta^{22}_{11}} + \underset{\bullet}{\theta^{23}_{11}} + \underset{\bullet}{\theta^{31}_{11}} + \underset{\bullet}{\theta^{32}_{11}} + \underset{\bullet}{\theta^{33}_{11}} + \underset{\bullet}{\theta^{22}_{12}} + \underset{\bullet}{\theta^{23}_{12}} + \underset{\bullet}{\theta^{32}_{12}} + \underset{\bullet}{\theta^{33}_{12}} + \theta^{23}_{13} + \theta^{33}_{13}\right]}_{Q^u_{3,1}}}
$$

- Yields

$$
\begin{aligned}
LB^u_{13} &= \min\left\{\frac{r_{13} - \widetilde{Q}}{p_1}, \frac{r_{13}}{p_1 + \widetilde{\widetilde{Q}}}\right\} \geq 0 & \qquad \widetilde{\widetilde{Q}} &= \min\{1 - p_1, \widetilde{Q}\},\ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\
UB^u_{13} &= \frac{r_{13} + \widetilde{\widetilde{Q}}}{p_1 + \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} &= \min\left\{1 - p_1, \widetilde{Q}\right\},\ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}
\end{aligned}
$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{13}^{*}=\frac{r_{13}+\overbrace{\left[\alpha_1^1\beta_3^1+\alpha_1^1\beta_3^2+\alpha_1^2\beta_3^1+\alpha_1^2\beta_3^2+\alpha_1^2\beta_3^3+\alpha_1^3\beta_3^1+\alpha_1^3\beta_3^2+\alpha_1^3\beta_3^3\right]}^{Q_{1,13}}-\overbrace{\left[\alpha_1^1\beta_1^3+\alpha_1^1\beta_2^3+\alpha_2^1\beta_1^3+\alpha_2^1\beta_2^3+\alpha_2^1\beta_3^3+\alpha_3^1\beta_1^3+\alpha_3^1\beta_2^3+\alpha_3^1\beta_3^3\right]}^{Q_{2,13}}}{\begin{array}{c}p_1+\underbrace{\left[\alpha_1^2\beta_1^1+\alpha_1^2\beta_1^2+\alpha_1^2\beta_1^3+\alpha_1^3\beta_1^1+\alpha_1^3\beta_1^2+\alpha_1^3\beta_1^3+\alpha_1^2\beta_2^1+\alpha_1^2\beta_2^2+\alpha_1^2\beta_2^3+\alpha_1^3\beta_2^1+\alpha_1^3\beta_2^2+\alpha_1^3\beta_2^3+\alpha_1^2\beta_3^1+\alpha_1^2\beta_3^2+\alpha_1^2\beta_3^3+\alpha_1^3\beta_3^1+\alpha_1^3\beta_3^2+\alpha_1^3\beta_3^3\right]}_{Q_{3,1}}-\\ \underbrace{\left[\alpha_2^1\beta_1^1+\alpha_2^1\beta_2^1+\alpha_2^1\beta_3^1+\alpha_2^1\beta_1^2+\alpha_2^1\beta_2^2+\alpha_2^1\beta_3^2+\alpha_2^1\beta_1^3+\alpha_2^1\beta_2^3+\alpha_2^1\beta_3^3+\alpha_3^1\beta_1^1+\alpha_3^1\beta_2^1+\alpha_3^1\beta_3^1+\alpha_3^1\beta_1^2+\alpha_3^1\beta_2^2+\alpha_3^1\beta_3^2+\alpha_3^1\beta_1^3+\alpha_3^1\beta_2^3+\alpha_3^1\beta_3^3\right]}_{Q_{4,1}}\end{array}}$$

- Simplifying

$$\begin{aligned}
Q_{1,13} &= \left(\alpha_1^2+\alpha_1^3\right)+\left(\beta_3^1+\beta_3^2\right)\left(1-\alpha_1^2-\alpha_1^3\right) && \text{(TI)}\\
&= \left(\theta_1^2+\theta_1^3\right)+\left(\theta_3^1+\theta_3^2\right)\left(1-\theta_1^2-\theta_1^3\right) && \text{(TIV)}\\
Q_{2,13} &= \left(\alpha_2^1+\alpha_3^1\right)\left(1+\beta_1^3+\beta_2^3-\beta_3^1-\beta_3^2\right)+\left(\beta_1^3+\beta_2^3\right)\left(1-\alpha_1^2-\alpha_1^3\right) && \text{(TI)}\\
&= \left(\theta_2^1+\theta_3^1\right)\left(1+\theta_1^3+\theta_2^3-\theta_3^1-\theta_3^2\right)+\left(\theta_1^3+\theta_2^3\right)\left(1-\theta_1^2-\theta_1^3\right) && \text{(TIV)}\\
Q_{3,1} &= 3\left(\alpha_1^2+\alpha_1^3\right) && \text{(TI)}\\
&= 3\left(\theta_1^2+\theta_1^3\right) && \text{(TIV)}\\
Q_{4,1} &= 3\left(\alpha_2^1+\alpha_3^1\right) && \text{(TI)}\\
&= 3\left(\theta_2^1+\theta_3^1\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{13}^{*}=\frac{r_{13}+\left(\beta_3^1+\beta_3^2-\beta_1^3-\beta_2^3\right)+\left(\alpha_1^2+\alpha_1^3-\alpha_2^1-\alpha_3^1\right)\left(1+\beta_1^3+\beta_2^3-\beta_3^1-\beta_3^2\right)}{p_1+3\left(\alpha_1^2+\alpha_1^3-\alpha_2^1-\alpha_3^1\right)}$$

  - Yields

$$\begin{aligned}
LB_{13}^{TI} &= \min\left\{\frac{r_{13}-\ddot{Q}}{p_1},\frac{r_{13}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}},\frac{r_{13}-\widehat{Q}}{p_1-3\widehat{Q}}\right\}\geq 0 \\
& \qquad \tilde{\tilde{Q}}=\min\left\{1-r_{13},(1-p_1)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{13},p_1/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE} \\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE} \\ 2Q/9 & \text{UE}\end{cases} \\
UB_{13}^{TI} &= \max\left\{\frac{r_{13}+\widetilde{Q}}{p_1},\frac{r_{13}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}},\frac{r_{13}-\widehat{Q}}{p_1-3\widehat{Q}}\right\}\leq 1 \\
& \qquad \tilde{\tilde{Q}}=\min\left\{1-r_{13},(1-p_1)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{13},p_1/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE} \\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE} \\ 2Q/9 & \text{UE}\end{cases}
\end{aligned}$$

  - Proof:
    1. $\ddot{Q}$ in $\frac{r_{13}-\ddot{Q}}{p_1}$ can be $2Q/9$ as $\beta_1^3,\beta_3^3=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{13}-\widehat{Q}}{p_1-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_2^1,\alpha_3^1=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{13}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{13}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(p_1+3\tilde{\tilde{Q}}\right)-3\left(r_{13}+\tilde{\tilde{Q}}\right)\right) \\
&= sgn\left(p_1-3r_{13}\right)
\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{13}-\widehat{Q}}{p_1-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{13}-\widehat{Q}}{p_1-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_1-3\widehat{Q}\right)+3\left(r_{13}-\widehat{Q}\right)\right) \\
&= sgn\left(3r_{13}-p_1\right)
\end{aligned}$$

  - Adding the uni-directional assumption

$$p_{13}^{*}=\frac{r_{13}-\left(\beta_1^3+\beta_2^3\right)+\left(\alpha_1^2+\alpha_1^3\right)\left(1+\beta_1^3+\beta_2^3\right)}{p_1+3\left(\alpha_1^2+\alpha_1^3\right)}$$

    * Yields

$$\begin{aligned}
LB_{13}^{TI,u} &= \min\left\{\frac{r_{13}-\ddot{Q}}{p_1},\frac{r_{13}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\right\}\geq 0 \\
& \qquad \tilde{\tilde{Q}}=\min\left\{1-r_{13},(1-p_1)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE} \\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE} \\ 2Q/9 & \text{UE}\end{cases} \\
UB_{13}^{TI,u} &= \frac{r_{13}+\tilde{\tilde{Q}}}{p_1+3\tilde{\tilde{Q}}}\leq 1 \qquad \tilde{\tilde{Q}}=\begin{cases} 0 & r_{13}\geq p_1/3 \\ \min\left\{(1-p_1)/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE} \\ Q/9 & \text{UE}\end{cases}
\end{aligned}$$

    * Proof: Same as above.

- Under Temporal Invariance

$$p_{13}^{*}=\frac{r_{13}+\left(\theta_3^2+\theta_1^2-\theta_2^1-\theta_2^3\right)+\left(\theta_1^2+\theta_1^3-\theta_2^1-\theta_3^1\right)\left(\theta_1^3+\theta_2^3-\theta_3^1-\theta_3^2\right)}{p_1+3\left(\theta_1^2+\theta_1^3-\theta_2^1-\theta_3^1\right)}$$

  – Yields

$$LB_{13}^{TIV} = \min\left\{\frac{r_{13}-\widetilde{Q}}{p_1},\frac{r_{13}+\widetilde{\widetilde{Q}}^2}{p_1+3\widetilde{\widetilde{Q}}}\right\}\geq 0$$

$$\widetilde{\widetilde{Q}}=\min\left\{\frac{-(2/3)p_1+\sqrt{(4/9)p_1^2+4r_{13}}}{2},\sqrt{1-r_{13}},(1-p_1)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

$$UB_{13}^{TIV} = \max\left\{\frac{r_{13}+\widetilde{Q}}{p_1},\frac{r_{13}+\widetilde{\widetilde{Q}}^2}{p_1-3\widetilde{\widetilde{Q}}}\right\}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{\sqrt{1-r_{13}},p_1/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}2-\sqrt{4-Q} & \text{AE}\\ \left(3-\sqrt{9-2Q}\right)/2 & \text{UE}\end{cases}$$

  – Proof:

    1. Evaluate $\partial LB_{13}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned}sgn\left(\frac{\partial LB_{13}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(2\widetilde{\widetilde{Q}}\left(p_1+3\widetilde{\widetilde{Q}}\right)-3\left(r_{13}+\widetilde{\widetilde{Q}}^2\right)\right)\\ &= sgn\left(\widetilde{\widetilde{Q}}\left((2/3)p_1+\widetilde{\widetilde{Q}}\right)-r_{13}\right)\\ \Rightarrow sgn\left(\frac{\partial LB_{13}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right)\Bigg|_{\widetilde{\widetilde{Q}}=0} &= sgn\left(-r_{13}\right)<0\\ \Rightarrow\widetilde{\widetilde{Q}} &> 0\end{aligned}$$

    2. Minimize LB_{13}^{TIV} s.t. $\widetilde{\widetilde{Q}}$ being feasible

$$\begin{aligned}\frac{\partial LB_{13}^{TIV}}{\partial\widetilde{\widetilde{Q}}} &\propto \widetilde{\widetilde{Q}}\left((2/3)p_1+\widetilde{\widetilde{Q}}\right)-r_{13}=0\\ \Rightarrow\ \widetilde{\widetilde{Q}}^{*} &= \frac{-(2/3)p_1+\sqrt{(4/9)p_1^2+4r_{13}}}{2}\end{aligned}$$

    So, derivative starts off negative and then reaches zero at $\widetilde{\widetilde{Q}}^{*}$. Thus, $\frac{r_{13}+\widetilde{\widetilde{Q}}^2}{p_1+3\widetilde{\widetilde{Q}}}$ is minimized at $\widetilde{\widetilde{Q}}^{*}$.

– Adding the uni-directional assumption

$$p^*_{13} = \frac{r_{13} + \left(\theta_1^2 - \theta_2^3\right) + \left(\theta_1^2 + \theta_1^3\right)\left(\theta_1^3 + \theta_2^3\right)}{p_1 + 3\left(\theta_1^2 + \theta_1^3\right)}$$

* Yields

$$LB_{13}^{TIV,u} = \min\left\{\frac{r_{13} - \widetilde{Q}}{p_1}, \frac{r_{13} + \widetilde{\widetilde{Q}}^2}{p_1 + 3\widetilde{\widetilde{Q}}}\right\} \geq 0$$

$$\widetilde{\widetilde{Q}} = \min\left\{\frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4r_{13}}}{2}, \sqrt{1 - r_{13}}, (1 - p_1)/3, \widetilde{Q}\right\}, \; \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

$$UB_{13}^{TIV,u} = \frac{r_{13} + \widetilde{\widetilde{Q}}}{p_1 + 3\widetilde{\widetilde{Q}}} \leq 1 \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{13} \geq p_1/3 \\ \min\left\{(1 - p_1)/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \; \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

* Proof: Evaluate $\partial\left(\frac{r_{13}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}}\right)/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} sgn\left(\frac{\partial\left(\frac{r_{13}+\widetilde{\widetilde{Q}}}{p_1+3\widetilde{\widetilde{Q}}}\right)}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(\left(p_1 + 3\widetilde{\widetilde{Q}}\right) - 3\left(r_{13} + \widetilde{\widetilde{Q}}\right)\right) \\ &= sgn\left(p_1 - 3r_{13}\right) \end{aligned}$$

A.1.4 p^*_{21}

$$p^*_{21} = \frac{r_{21} + \overbrace{\left[\theta^{11}_{21} + \theta^{12}_{21} + \theta^{13}_{21} + \underset{\bullet}{\theta^{22}_{21}} + \underset{\bullet}{\theta^{23}_{21}} + \theta^{31}_{21} + \theta^{32}_{21} + \theta^{33}_{21}\right]}^{Q_{1,21}} - \overbrace{\left[\theta^{21}_{11} + \theta^{21}_{12} + \theta^{21}_{13} + \underset{\bullet}{\theta^{21}_{22}} + \underset{\bullet}{\theta^{21}_{23}} + \theta^{21}_{31} + \theta^{21}_{32} + \theta^{21}_{33}\right]}^{Q_{2,21}}}{\begin{array}{c} p_2 + \underbrace{\left[\theta^{11}_{21} + \theta^{12}_{21} + \theta^{13}_{21} + \theta^{31}_{21} + \theta^{32}_{21} + \theta^{33}_{21} + \underset{\bullet}{\theta^{11}_{22}} + \underset{\bullet}{\theta^{12}_{22}} + \underset{\bullet}{\theta^{13}_{22}} + \underset{\bullet}{\theta^{31}_{22}} + \underset{\bullet}{\theta^{32}_{22}} + \underset{\bullet}{\theta^{33}_{22}} + \underset{\bullet}{\theta^{11}_{23}} + \underset{\bullet}{\theta^{12}_{23}} + \underset{\bullet}{\theta^{13}_{23}} + \underset{\bullet}{\theta^{31}_{23}} + \underset{\bullet}{\theta^{32}_{23}} + \underset{\bullet}{\theta^{33}_{23}}\right]}_{Q_{3,2}} \\ - \underbrace{\left[\theta^{21}_{11} + \theta^{21}_{12} + \theta^{21}_{13} + \underset{\bullet}{\theta^{22}_{11}} + \underset{\bullet}{\theta^{22}_{12}} + \underset{\bullet}{\theta^{22}_{13}} + \underset{\bullet}{\theta^{23}_{11}} + \underset{\bullet}{\theta^{23}_{12}} + \underset{\bullet}{\theta^{23}_{13}} + \theta^{21}_{31} + \theta^{21}_{32} + \theta^{21}_{33} + \underset{\bullet}{\theta^{22}_{31}} + \underset{\bullet}{\theta^{22}_{32}} + \underset{\bullet}{\theta^{22}_{33}} + \underset{\bullet}{\theta^{23}_{31}} + \underset{\bullet}{\theta^{23}_{32}} + \underset{\bullet}{\theta^{23}_{33}}\right]}_{Q_{4,2}} \end{array}}$$

- $\underset{\bullet}{\theta^{k'l'}_{kl}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$\begin{aligned} LB_{21} &= \frac{r_{21} - \widetilde{Q}}{p_2} \geq 0 & \qquad \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB_{21} &= \frac{r_{21} + \widetilde{Q}}{p_2 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} &= \begin{cases} 0 & \text{AE} \\ \min\{p_2, Q/3\} & \text{UE} \end{cases} \end{aligned}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p^*_{21} = \frac{r_{21} + \overbrace{\left[\underset{\bullet}{\theta^{22}_{21}} + \underset{\bullet}{\theta^{23}_{21}} + \theta^{31}_{21} + \theta^{32}_{21} + \theta^{33}_{21}\right]}^{Q^u_{1,21}} - \overbrace{\left[\theta^{21}_{11}\right]}^{Q^u_{2,21}}}{p_2 + \underbrace{\left[\theta^{31}_{21} + \theta^{32}_{21} + \theta^{33}_{21} + \underset{\bullet}{\theta^{32}_{22}} + \underset{\bullet}{\theta^{33}_{22}} + \underset{\bullet}{\theta^{33}_{23}}\right]}_{Q^u_{3,2}} - \underbrace{\left[\theta^{21}_{11} + \underset{\bullet}{\theta^{22}_{11}} + \underset{\bullet}{\theta^{22}_{12}} + \underset{\bullet}{\theta^{23}_{11}} + \underset{\bullet}{\theta^{23}_{12}} + \underset{\bullet}{\theta^{23}_{13}}\right]}_{Q^u_{4,2}}}$$

- Yields

$$\begin{aligned} LB^u_{21} &= \min\left\{\frac{r_{21} - \widetilde{\widetilde{Q}}}{p_2 - \widetilde{\widetilde{Q}}}, \frac{r_{21}}{p_2 + \widehat{Q}}\right\}, \geq 0 & \qquad & \widetilde{\widetilde{Q}} = \min\{r_{21}, p_2, \widetilde{Q}\},\ \widehat{Q} = \min\{1 - p_2, \widetilde{Q}\},\ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB_{21} &= \frac{r_{21} + \widetilde{Q}}{p_2 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad & \widetilde{\widetilde{Q}} = \begin{cases} 0 & \text{AE} \\ \min\{p_2, \widetilde{Q}\} & \text{UE} \end{cases},\ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \end{aligned}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{21}^{*}=\frac{r_{21}+\overbrace{\left[\alpha_2^1\beta_1^1+\alpha_2^1\beta_1^2+\alpha_2^1\beta_1^3+\alpha_2^2\beta_1^2+\alpha_2^2\beta_1^3+\alpha_2^3\beta_1^1+\alpha_2^3\beta_1^2+\alpha_2^3\beta_1^3\right]}^{Q_{1,21}}-\overbrace{\left[\alpha_1^2\beta_1^1+\alpha_1^2\beta_2^1+\alpha_1^2\beta_3^1+\alpha_2^2\beta_2^1+\alpha_2^2\beta_3^1+\alpha_3^2\beta_1^1+\alpha_3^2\beta_2^1+\alpha_3^2\beta_3^1\right]}^{Q_{2,21}}}{\begin{array}{c}p_2+\underbrace{\left[\alpha_2^1\beta_1^1+\alpha_2^1\beta_1^2+\alpha_2^1\beta_1^3+\alpha_2^3\beta_1^1+\alpha_2^3\beta_1^2+\alpha_2^3\beta_1^3+\alpha_2^1\beta_2^1+\alpha_2^1\beta_2^2+\alpha_2^1\beta_2^3+\alpha_2^3\beta_2^1+\alpha_2^3\beta_2^2+\alpha_2^3\beta_2^3+\alpha_2^1\beta_3^1+\alpha_2^1\beta_3^2+\alpha_2^1\beta_3^3+\alpha_2^3\beta_3^1+\alpha_2^3\beta_3^2+\alpha_2^3\beta_3^3\right]}_{Q_{3,2}}\\-\underbrace{\left[\alpha_1^2\beta_1^1+\alpha_1^2\beta_2^1+\alpha_1^2\beta_3^1+\alpha_1^2\beta_1^2+\alpha_1^2\beta_2^2+\alpha_1^2\beta_3^2+\alpha_1^2\beta_1^3+\alpha_1^2\beta_2^3+\alpha_1^2\beta_3^3+\alpha_3^2\beta_1^1+\alpha_3^2\beta_2^1+\alpha_3^2\beta_3^1+\alpha_3^2\beta_1^2+\alpha_3^2\beta_2^2+\alpha_3^2\beta_3^2+\alpha_3^2\beta_1^3+\alpha_3^2\beta_2^3+\alpha_3^2\beta_3^3\right]}_{Q_{4,2}}\end{array}}$$

- Simplifying

$$\begin{aligned}
Q_{1,21} &= \left(\alpha_2^1+\alpha_2^3\right)+\left(\beta_1^2+\beta_1^3\right)\left(1-\alpha_2^1-\alpha_2^3\right) && \text{(TI)}\\
&= \left(\theta_2^1+\theta_2^3\right)+\left(\theta_1^2+\theta_1^3\right)\left(1-\theta_2^1-\theta_2^3\right) && \text{(TIV)}\\
Q_{2,21} &= \left(\alpha_1^2+\alpha_3^2\right)\left(1+\beta_2^1+\beta_3^1-\beta_1^2-\beta_1^3\right)+\left(\beta_2^1+\beta_3^1\right)\left(1-\alpha_2^1-\alpha_2^3\right) && \text{(TI)}\\
&= \left(\theta_1^2+\theta_3^2\right)\left(1+\theta_2^1+\theta_3^1-\theta_1^2-\theta_1^3\right)+\left(\theta_2^1+\theta_3^1\right)\left(1-\theta_2^1-\theta_2^3\right) && \text{(TIV)}\\
Q_{3,2} &= 3\left(\alpha_2^1+\alpha_2^3\right) && \text{(TI)}\\
&= 3\left(\theta_2^1+\theta_2^3\right) && \text{(TIV)}\\
Q_{4,2} &= 3\left(\alpha_1^2+\alpha_3^2\right) && \text{(TI)}\\
&= 3\left(\theta_1^2+\theta_3^2\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{21}^{*}=\frac{r_{21}+\left(\beta_1^2+\beta_1^3-\beta_2^1-\beta_3^1\right)+\left(\alpha_2^1+\alpha_2^3-\alpha_1^2-\alpha_3^2\right)\left(1+\beta_2^1+\beta_3^1-\beta_1^2-\beta_1^3\right)}{p_2+3\left(\alpha_2^1+\alpha_2^3-\alpha_1^2-\alpha_3^2\right)}$$

  - Yields

$$\begin{aligned}
LB_{21}^{TI} &= \min\left\{\frac{r_{21}-\ddot{Q}}{p_2},\frac{r_{21}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}},\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\geq 0\\
&\tilde{\tilde{Q}}=\min\left\{1-r_{21},(1-p_2)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{21},p_2/3,\ddot{Q}\right\},\ \widetilde{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{array}\right.,\ \ddot{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{array}\right.\\
UB_{21}^{TI} &= \max\left\{\frac{r_{21}+\widetilde{Q}}{p_2},\frac{r_{21}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}},\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\leq 1\\
&\tilde{\tilde{Q}}=\min\left\{1-r_{21},(1-p_2)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{21},p_2/3,\ddot{Q}\right\},\ \widetilde{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{array}\right.,\ \ddot{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{array}\right.
\end{aligned}$$

  - Proof:
    1. $\ddot{Q}$ in $\frac{r_{21}-\ddot{Q}}{p_2}$ can be $2Q/9$ as $\beta_2^1,\beta_3^1=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_1^2,\alpha_3^2=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{21}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{21}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(p_2+3\tilde{\tilde{Q}}\right)-3\left(r_{21}+\tilde{\tilde{Q}}\right)\right)\\
&= sgn\left(p_2-3r_{21}\right)
\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_2-3\widehat{Q}\right)+3\left(r_{21}-\widehat{Q}\right)\right)\\
&= sgn\left(3r_{21}-p_2\right)
\end{aligned}$$

– Adding the uni-directional assumption

$$p_{21}^{*} = \frac{r_{21} + \left(\beta_1^2 + \beta_1^3\right) + \left(\alpha_2^3 - \alpha_1^2\right)\left(1 - \beta_1^2 - \beta_1^3\right)}{p_2 + 3\left(\alpha_2^3 - \alpha_1^2\right)}$$

∗ Yields

$$LB_{21}^{TI,u} = \min\left\{\frac{r_{21} + \widetilde{\widetilde{Q}}}{p_2 + 3\widetilde{\widetilde{Q}}}, \frac{r_{21} - \widehat{Q}}{p_2 - 3\widehat{Q}}\right\} \geq 0$$

$$\widetilde{\widetilde{Q}} = \min\left\{1 - r_{21}, (1 - p_2)/3, \widetilde{Q}\right\},\ \widehat{Q} = \min\left\{r_{21}, p_2/3, \widetilde{Q}\right\},\ \widetilde{Q} = \left\{\begin{array}{ll} Q/3 & \text{AE} \\ Q/9 & \text{UE}\end{array}\right.$$

$$UB_{21}^{TI,u} = \max\left\{\frac{r_{21} + \widetilde{Q}}{p_2}, \frac{r_{21} + \widetilde{\widetilde{Q}}}{p_2 + 3\widetilde{\widetilde{Q}}}, \frac{r_{21} - \widehat{Q}}{p_2 - 3\widehat{Q}}\right\} \leq 1$$

$$\widetilde{\widetilde{Q}} = \min\left\{1 - r_{21}, (1 - p_2)/3, \widetilde{Q}\right\},\ \widehat{Q} = \min\left\{r_{21}, p_2/3, \widetilde{Q}\right\},\ \widetilde{Q} = \left\{\begin{array}{ll} Q/3 & \text{AE} \\ Q/9 & \text{UE}\end{array}\right.$$

∗ Proof: UB is same as above except now $\ddot{Q}$ is not feasible. The LB is not r_{21}/p_2 as the derivative of one of the terms in $\min\{\cdot\}$ wrt Q must be negative.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{21}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}}\right)}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(p_2 + 3\widetilde{\widetilde{Q}} - 3\left(r_{21} + \widetilde{\widetilde{Q}}\right)\right) \\
&= sgn\left(p_2 - 3r_{21}\right) \\
sgn\left(\frac{\partial\left(\frac{r_{21}-\widehat{Q}}{p_2-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(p_2 + 3\widehat{Q} - 3\left(r_{21} + \widehat{Q}\right)\right) \\
&= sgn\left(3r_{21} - p_2\right)
\end{aligned}$$

- Under Temporal Invariance

$$p_{21}^{*}=\frac{r_{21}+\left(\theta_1^3+\theta_2^3-\theta_3^1-\theta_3^2\right)+\left(\theta_2^1+\theta_2^3-\theta_1^2-\theta_3^2\right)\left(\theta_2^1+\theta_3^1-\theta_1^2-\theta_1^3\right)}{p_2+3\left(\theta_2^1+\theta_2^3-\theta_1^2-\theta_3^2\right)}$$

  - Yields

$$LB_{21}^{TIV} = \min\left\{\frac{r_{21}-\widetilde{Q}}{p_2},\frac{r_{21}+\widetilde{\widetilde{Q}}^2}{p_1+3\widetilde{\widetilde{Q}}}\right\}\geq 0$$

$$\widetilde{\widetilde{Q}}=\min\left\{\frac{-(2/3)p_2+\sqrt{(4/9)p_2^2+4r_{21}}}{2},\sqrt{1-r_{21}},(1-p_2)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

$$UB_{21}^{TIV} = \max\left\{\frac{r_{21}+\widetilde{Q}}{p_2},\frac{r_{21}+\widetilde{\widetilde{Q}}^2}{p_2-3\widetilde{\widetilde{Q}}}\right\}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{\sqrt{1-r_{21}},p_2/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

  - Proof:
    1. Evaluate $\partial LB_{21}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned} sgn\left(\frac{\partial LB_{21}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(2\widetilde{\widetilde{Q}}\left(p_2+3\widetilde{\widetilde{Q}}\right)-3\left(r_{21}+\widetilde{\widetilde{Q}}^2\right)\right)\\ &= sgn\left(\widetilde{\widetilde{Q}}\left((2/3)p_2+\widetilde{\widetilde{Q}}\right)-r_{21}\right)\\ \Rightarrow sgn\left(\frac{\partial LB_{21}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right)\Bigg|_{\widetilde{\widetilde{Q}}=0} &= sgn\left(-r_{21}\right)<0\\ \Rightarrow\widetilde{\widetilde{Q}} &> 0 \end{aligned}$$

    2. Minimize LB_{21}^{TIV} s.t. $\widetilde{\widetilde{Q}}$ being feasible

$$\begin{aligned} \frac{\partial LB_{21}^{TIV}}{\partial\widetilde{\widetilde{Q}}} &\propto \widetilde{\widetilde{Q}}\left((2/3)p_2+\widetilde{\widetilde{Q}}\right)-r_{21}=0\\ \Rightarrow \widetilde{\widetilde{Q}}^{*} &= \frac{-(2/3)p_2+\sqrt{(4/9)p_2^2+4r_{21}}}{2} \end{aligned}$$

    So, derivative starts off negative and then reaches zero at $\widetilde{\widetilde{Q}}^{*}$. Thus, $\frac{r_{21}+\widetilde{\widetilde{Q}}^2}{p_2+3\widetilde{\widetilde{Q}}}$ is minimized at $\widetilde{\widetilde{Q}}^{*}$.

– Adding the uni-directional assumption

$$p_{21}^{*}=\frac{r_{21}+\left(\theta_1^3+\theta_2^3\right)-\left(\theta_2^3-\theta_1^2\right)\left(\theta_1^2+\theta_1^3\right)}{p_2+3\left(\theta_2^3-\theta_1^2\right)}$$

* Yields

$$LB_{21}^{TIV,u} = \frac{r_{21}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}}\geq 0$$

$$\widetilde{\widetilde{Q}}=\begin{cases}0 & r_{21}<p_2/3\\ \min\left\{1-r_{21},(1-p_2)/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

$$UB_{21}^{TIV,u} = \max\left\{\frac{r_{21}+\widetilde{Q}}{p_2},\frac{r_{21}+\widetilde{\widetilde{Q}}^2}{p_2-3\widetilde{\widetilde{Q}}}\right\}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{\sqrt{1-r_{21}},p_2/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

* Proof: Evaluate $\partial LB_{21}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned} sgn\left(\frac{\partial LB_{21}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(p_2+3\widetilde{\widetilde{Q}}-3\left(r_{21}+\widetilde{\widetilde{Q}}\right)\right)\\ &= sgn\left(p_2-3r_{21}\right)\end{aligned}$$

A.1.5 p^*_{22}

$$p^*_{22} = \frac{r_{22} + \overbrace{\left[\theta^{11}_{22} + \theta^{12}_{22} + \theta^{13}_{22} + \underset{\bullet}{\theta^{21}_{22}} + \underset{\bullet}{\theta^{23}_{22}} + \theta^{31}_{22} + \theta^{32}_{22} + \theta^{33}_{22}\right]}^{Q_{1,22}} - \overbrace{\left[\theta^{22}_{11} + \theta^{22}_{12} + \theta^{22}_{13} + \underset{\bullet}{\theta^{22}_{21}} + \underset{\bullet}{\theta^{22}_{23}} + \theta^{22}_{31} + \theta^{22}_{32} + \theta^{22}_{33}\right]}^{Q_{2,22}}}{\begin{array}{l} p_2 + \underbrace{\left[\underset{\bullet}{\theta^{11}_{21}} + \underset{\bullet}{\theta^{12}_{21}} + \underset{\bullet}{\theta^{13}_{21}} + \underset{\bullet}{\theta^{31}_{21}} + \underset{\bullet}{\theta^{32}_{21}} + \underset{\bullet}{\theta^{33}_{21}} + \theta^{11}_{22} + \theta^{12}_{22} + \theta^{13}_{22} + \theta^{31}_{22} + \theta^{32}_{22} + \theta^{33}_{22} + \underset{\bullet}{\theta^{11}_{23}} + \underset{\bullet}{\theta^{12}_{23}} + \underset{\bullet}{\theta^{13}_{23}} + \underset{\bullet}{\theta^{31}_{23}} + \underset{\bullet}{\theta^{32}_{23}} + \underset{\bullet}{\theta^{33}_{23}}\right]}_{Q_{3,2}} \\ \quad - \underbrace{\left[\underset{\bullet}{\theta^{21}_{11}} + \underset{\bullet}{\theta^{21}_{12}} + \underset{\bullet}{\theta^{21}_{13}} + \theta^{22}_{11} + \theta^{22}_{12} + \theta^{22}_{13} + \underset{\bullet}{\theta^{23}_{11}} + \underset{\bullet}{\theta^{23}_{12}} + \underset{\bullet}{\theta^{23}_{13}} + \underset{\bullet}{\theta^{21}_{31}} + \underset{\bullet}{\theta^{21}_{32}} + \underset{\bullet}{\theta^{21}_{33}} + \theta^{22}_{31} + \theta^{22}_{32} + \theta^{22}_{33} + \underset{\bullet}{\theta^{23}_{31}} + \underset{\bullet}{\theta^{23}_{32}} + \underset{\bullet}{\theta^{23}_{33}}\right]}_{Q_{4,2}} \end{array}}$$

- $\underset{\bullet}{\theta^{k'l'}_{kl}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$\begin{aligned} LB_{22} &= \frac{r_{22} - \widetilde{Q}}{p_2} \geq 0 & \qquad \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB_{22} &= \frac{r_{22} + \widetilde{Q}}{p_2 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} &= \begin{cases} 0 & \text{AE} \\ \min\{p_2, Q/3\} & \text{UE} \end{cases} \end{aligned}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p^*_{22} = \frac{r_{22} + \overbrace{\left[\underset{\bullet}{\theta^{23}_{22}} + \theta^{32}_{22} + \theta^{33}_{22}\right]}^{Q^u_{1,22}} - \overbrace{\left[\theta^{22}_{11} + \theta^{22}_{12} + \underset{\bullet}{\theta^{22}_{21}}\right]}^{Q^u_{2,22}}}{p_2 + \underbrace{\left[\underset{\bullet}{\theta^{31}_{21}} + \underset{\bullet}{\theta^{32}_{21}} + \underset{\bullet}{\theta^{33}_{21}} + \theta^{32}_{22} + \theta^{33}_{22} + \underset{\bullet}{\theta^{33}_{23}}\right]}_{Q^u_{3,2}} - \underbrace{\left[\underset{\bullet}{\theta^{21}_{11}} + \theta^{22}_{11} + \theta^{22}_{12} + \underset{\bullet}{\theta^{23}_{11}} + \underset{\bullet}{\theta^{23}_{12}} + \underset{\bullet}{\theta^{23}_{13}}\right]}_{Q^u_{4,2}}}$$

- Yields

$$\begin{aligned} LB^u_{22} &= \frac{r_{22} - \widetilde{Q}}{p_2} \geq 0 & \qquad \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB^u_{22} &= \frac{r_{22} + \widetilde{Q}}{p_2 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} &= \begin{cases} 0 & \text{AE} \\ \min\{p_2, Q/3\} & \text{UE} \end{cases}, \; \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \end{aligned}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{22}^{*}=\frac{r_{22}+\overbrace{\left[\alpha_2^1\beta_2^1+\alpha_2^1\beta_2^2+\alpha_2^1\beta_2^3+\alpha_2^2\beta_2^1+\alpha_2^2\beta_2^3+\alpha_2^3\beta_2^1+\alpha_2^3\beta_2^2+\alpha_2^3\beta_2^3\right]}^{Q_{1,22}}-\overbrace{\left[\alpha_1^2\beta_1^2+\alpha_1^2\beta_2^2+\alpha_1^2\beta_3^2+\alpha_2^2\beta_1^2+\alpha_2^2\beta_3^2+\alpha_3^2\beta_1^2+\alpha_3^2\beta_2^2+\alpha_3^2\beta_3^2\right]}^{Q_{2,22}}}{p_2+\underbrace{\left[\alpha_2^1\beta_1^1+\alpha_2^1\beta_1^2+\alpha_2^1\beta_1^3+\alpha_2^3\beta_1^1+\alpha_2^3\beta_1^2+\alpha_2^3\beta_1^3+\alpha_2^1\beta_2^1+\alpha_2^1\beta_2^2+\alpha_2^1\beta_2^3+\alpha_2^3\beta_2^1+\alpha_2^3\beta_2^2+\alpha_2^3\beta_2^3+\alpha_2^1\beta_3^1+\alpha_2^1\beta_3^2+\alpha_2^1\beta_3^3+\alpha_2^3\beta_3^1+\alpha_2^3\beta_3^2+\alpha_2^3\beta_3^3\right]}_{Q_{3,2}}-\underbrace{\left[\alpha_1^2\beta_1^1+\alpha_1^2\beta_2^1+\alpha_1^2\beta_3^1+\alpha_1^2\beta_1^2+\alpha_1^2\beta_2^2+\alpha_1^2\beta_3^2+\alpha_1^2\beta_1^3+\alpha_1^2\beta_2^3+\alpha_1^2\beta_3^3+\alpha_3^2\beta_1^1+\alpha_3^2\beta_2^1+\alpha_3^2\beta_3^1+\alpha_3^2\beta_1^2+\alpha_3^2\beta_2^2+\alpha_3^2\beta_3^2+\alpha_3^2\beta_1^3+\alpha_3^2\beta_2^3+\alpha_3^2\beta_3^3\right]}_{Q_{4,2}}}$$

- Simplifying

$$\begin{aligned}
Q_{1,22} &= \left(\alpha_2^1+\alpha_2^3\right)+\left(\beta_2^1+\beta_2^3\right)\left(1-\alpha_2^1-\alpha_2^3\right) && \text{(TI)}\\
&= 2\left(\theta_2^1+\theta_2^3\right)-\left(\theta_2^1+\theta_2^3\right)^2 && \text{(TIV)}\\
Q_{2,22} &= \left(\alpha_1^2+\alpha_3^2\right)\left(1+\beta_1^2+\beta_3^2-\beta_2^1-\beta_2^3\right)+\left(\beta_1^2+\beta_3^2\right)\left(1-\alpha_2^1-\alpha_2^3\right) && \text{(TI)}\\
&= 2\left(\theta_1^2+\theta_3^2\right)\left(1-\theta_2^1-\theta_2^3\right)+\left(\theta_1^2+\theta_3^2\right)^2 && \text{(TIV)}\\
Q_{3,2} &= 3\left(\alpha_2^1+\alpha_2^3\right) && \text{(TI)}\\
&= 3\left(\theta_2^1+\theta_2^3\right) && \text{(TIV)}\\
Q_{4,2} &= 3\left(\alpha_1^2+\alpha_3^2\right) && \text{(TI)}\\
&= 3\left(\theta_1^2+\theta_3^2\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{22}^{*}=\frac{r_{22}+\left(\alpha_2^1+\alpha_2^3-\alpha_1^2-\alpha_3^2\right)+\left(\beta_2^1+\beta_2^3-\beta_1^2-\beta_3^2\right)\left(1+\alpha_1^2+\alpha_3^2-\alpha_2^1-\alpha_2^3\right)}{p_2+3\left(\alpha_2^1+\alpha_2^3-\alpha_1^2-\alpha_3^2\right)}$$

  - Yields

$$\begin{aligned}
LB_{22}^{TI} &= \min\left\{\frac{r_{22}-\ddot{Q}}{p_2},\frac{r_{22}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}},\frac{r_{22}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\geq 0\\
&\widetilde{\widetilde{Q}}=\min\left\{1-r_{22},(1-p_2)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{22},p_2/3,\ddot{Q}\right\},\ \widetilde{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{array}\right.,\ \ddot{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{array}\right.\\
UB_{22}^{TI} &= \max\left\{\frac{r_{22}+\widetilde{Q}}{p_2},\frac{r_{22}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}},\frac{r_{22}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\leq 1\\
&\widetilde{\widetilde{Q}}=\min\left\{1-r_{22},(1-p_2)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{22},p_2/3,\ddot{Q}\right\},\ \widetilde{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{array}\right.,\ \ddot{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{array}\right.
\end{aligned}$$

  - Proof:
    1. $\ddot{Q}$ in $\frac{r_{22}-\ddot{Q}}{p_2}$ can be $2Q/9$ as $\beta_1^2,\beta_3^2=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{22}-\widehat{Q}}{p_2-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_1^2,\alpha_3^2=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{22}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}}\right)/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{22}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}}\right)}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(\left(p_2+3\widetilde{\widetilde{Q}}\right)-3\left(r_{22}+\widetilde{\widetilde{Q}}\right)\right)\\
&= sgn\left(p_2-3r_{22}\right)
\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{22}-\widehat{Q}}{p_2-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{22}-\widehat{Q}}{p_2-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_2-3\widehat{Q}\right)+3\left(r_{22}-\widehat{Q}\right)\right)\\
&= sgn\left(3r_{22}-p_2\right)
\end{aligned}$$

  - Adding the uni-directional assumption

$$p_{22}^{*}=\frac{r_{22}+\left(\alpha_2^3-\alpha_1^2\right)+\left(\beta_2^3-\beta_1^2\right)\left(1+\alpha_1^2-\alpha_2^3\right)}{p_2+3\left(\alpha_2^3-\alpha_1^2\right)}$$

    * Yields

$$\begin{aligned}
LB_{22}^{TI,u} &= \min\left\{\frac{r_{22}-\widetilde{Q}}{p_2},\frac{r_{22}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}},\frac{r_{22}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\geq 0\\
&\widetilde{\widetilde{Q}}=\min\left\{1-r_{22},(1-p_2)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{22},p_2/3,\widetilde{Q}\right\},\ \widetilde{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{array}\right.\\
UB_{22}^{TI,u} &= \max\left\{\frac{r_{22}+\widetilde{Q}}{p_2},\frac{r_{22}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}},\frac{r_{22}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\leq 1\\
&\widetilde{\widetilde{Q}}=\min\left\{1-r_{22},(1-p_2)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{22},p_2/3,\widetilde{Q}\right\},\ \widetilde{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{array}\right.
\end{aligned}$$

    * Proof: Same as above except now $\ddot{Q}$ is not feasible.

- Under Temporal Invariance

$$p_{22}^{*}=\frac{r_{22}+2\left(\theta_2^1+\theta_2^3-\theta_1^2-\theta_3^2\right)-\left(\theta_2^1+\theta_2^3-\theta_1^2-\theta_3^2\right)^2}{p_2+3\left(\theta_2^1+\theta_2^3-\theta_1^2-\theta_3^2\right)}$$

  - Yields

$$\begin{aligned}
LB_{22}^{TIV} &= \min\left\{\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}},\frac{r_{22}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^2}{p_2-3\widetilde{\widetilde{Q}}}\right\}\geq 0\\
&\quad \widehat{Q}=\min\left\{(1-p_2)/3,\widetilde{Q}\right\},\\
&\quad \widetilde{\widetilde{Q}}=\begin{cases}0 & r_{22}\geq 2p_2/3\\ \min\left\{\frac{(2/3)p_2+\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]}}{2},(-1+\sqrt{1+r_{22}}),p_2/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}\\
UB_{22}^{TIV} &= \min\left\{\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}},\frac{r_{22}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^2}{p_2-3\widetilde{\widetilde{Q}}}\right\}\geq 0\\
&\quad \widehat{Q}=\begin{cases}0 & r_{22}\geq 2p_2/3\\ \min\left\{\frac{-(2/3)p_2+\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]}}{2},(1-p_2)/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{\widetilde{Q}}=\begin{cases}0 & r_{22}< 2p_2/3\\ \min\left\{\frac{(2/3)p_2-\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]}}{2},(-1+\sqrt{1+r_{22}}),p_2/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}
\end{aligned}$$

– Proof:

1. Evaluate $\partial\left(\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(-2-2\tilde{\tilde{Q}}\right)\left(p_2-3\tilde{\tilde{Q}}\right)+3\left(r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2\right)\right) \\
&= sgn\left(-(2/3)p_2\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{22}\right) \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right)\Bigg|_{\tilde{\tilde{Q}}=0} &= sgn\left(-(2/3)p_2+r_{22}\right) \gtrless 0 \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right)\Bigg|_{\tilde{\tilde{Q}}=1} &= sgn\left(-(4/3)p_2+1+r_{22}\right) \gtrless 0
\end{aligned}$$

2. Ensure $r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2 \geq 0$

$$\begin{aligned}
& r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2 \geq 0 \\
\Rightarrow\ & \tilde{\tilde{Q}}^2+2\tilde{\tilde{Q}}-r_{22} \leq 0 \\
\Rightarrow\ & \tilde{\tilde{Q}} \leq \frac{-2+\sqrt{4+4r_{22}}}{2} \\
\Rightarrow\ & \tilde{\tilde{Q}} \leq -1+\sqrt{1+r_{22}}
\end{aligned}$$

3. Minimize $\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}$ s.t. $\tilde{\tilde{Q}}$ being feasible and $r_{22} < 2p_2/3$

$$\begin{aligned}
\frac{\partial\left(\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}} &\propto -(2/3)p_2\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{22}=0 \\
\Rightarrow\ \tilde{\tilde{Q}}^* &= \frac{(2/3)p_2+\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]}}{2}
\end{aligned}$$

4. Maximize $\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}$ s.t. $\tilde{\tilde{Q}}$ being feasible and $r_{22} \geq 2p_2/3$

$$\begin{aligned}
\frac{\partial\left(\frac{r_{22}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_2-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}} &\propto -(2/3)p_2\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{22}=0 \\
\Rightarrow\ \tilde{\tilde{Q}}^* &= \frac{(2/3)p_2-\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]}}{2}
\end{aligned}$$

Note: If $\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]} = .$, then maximize $\tilde{\tilde{Q}}$.

5. Evaluate $\partial\left(\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(\left(2-2\widehat{Q}\right)\left(p_2+3\widehat{Q}\right)-3\left(r_{22}+2\widehat{Q}-\widehat{Q}^2\right)\right)\\
&= sgn\left((2/3)p_2\left(1-\widehat{Q}\right)-\widehat{Q}^2-r_{22}\right)\\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right)\Bigg|_{\widehat{Q}=0} &= sgn\left((2/3)p_2-r_{22}\right)\gtrless 0\\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right)\Bigg|_{\widehat{Q}=1} &= sgn\left(-1-r_{22}\right)<0
\end{aligned}$$

6. Maximize $\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}}$ s.t. $\widehat{Q}$ being feasible and $r_{22}<2p_2/3$

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}}\right)}{\partial\widetilde{\widetilde{Q}}}\right) &\propto (2/3)p_2\left(1-\widehat{Q}\right)-\widehat{Q}^2-r_{22}=0\\
\Rightarrow \widetilde{\widetilde{Q}}^* &= \frac{-(2/3)p_2+\sqrt{(4/9)p_2^2+4[(2/3)p_2-r_{22}]}}{2}
\end{aligned}$$

7. Minimize $\frac{r_{22}+2\widehat{Q}-\widehat{Q}^2}{p_2+3\widehat{Q}}\Rightarrow\widehat{Q}=0$ or maximize $\widehat{Q}$. However, if the minimum occurs when $\widehat{Q}=0$, then $\frac{r_{221}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^2}{p_2-3\widetilde{\widetilde{Q}}}<\frac{r_{22}}{p_2}$ and this will be the binding LB.

– Adding the uni-directional assumption

$$p_{22}^{*}=\frac{r_{22}+2\left(\theta_{2}^{3}-\theta_{1}^{2}\right)-\left(\theta_{2}^{3}-\theta_{1}^{2}\right)^{2}}{p_{2}+3\left(\theta_{2}^{3}-\theta_{1}^{2}\right)}$$

* Yields

$$\begin{aligned}
LB_{22}^{TIV,u} &= \min\left\{\frac{r_{22}+2\widehat{Q}-\widehat{Q}^{2}}{p_{2}+3\widehat{Q}},\frac{r_{22}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^{2}}{p_{2}-3\widetilde{\widetilde{Q}}}\right\}\geq 0\\
&\quad \widehat{Q}=\min\left\{(1-p_{2})/3,\widetilde{Q}\right\},\\
&\quad \widetilde{\widetilde{Q}}=\begin{cases}0 & r_{22}\geq 2p_{2}/3\\ \min\left\{\frac{(2/3)p_{2}+\sqrt{(4/9)p_{2}^{2}+4[(2/3)p_{2}-r_{22}]}}{2},(-1+\sqrt{1+r_{22}}),p_{2}/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}\\
UB_{22}^{TIV,u} &= \min\left\{\frac{r_{22}+2\widehat{Q}-\widehat{Q}^{2}}{p_{2}+3\widehat{Q}},\frac{r_{22}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^{2}}{p_{2}-3\widetilde{\widetilde{Q}}}\right\}\geq 0\\
&\quad \widehat{Q}=\begin{cases}0 & r_{22}\geq 2p_{2}/3\\ \min\left\{\frac{-(2/3)p_{2}+\sqrt{(4/9)p_{2}^{2}+4[(2/3)p_{2}-r_{22}]}}{2},(1-p_{2})/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{\widetilde{Q}}=\begin{cases}0 & r_{22}<2p_{2}/3\\ \min\left\{(-1+\sqrt{1+r_{22}}),p_{2}/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}
\end{aligned}$$

* Proof: Same as above.

A.1.6 p^*_{23}

$$p^*_{23} = \frac{r_{23} + \overbrace{\left[\theta^{11}_{23} + \theta^{12}_{23} + \theta^{13}_{23} + \underset{\bullet}{\theta^{21}_{23}} + \underset{\bullet}{\theta^{22}_{23}} + \theta^{31}_{23} + \theta^{32}_{23} + \theta^{33}_{23}\right]}^{Q_{1,23}} - \overbrace{\left[\theta^{23}_{11} + \theta^{23}_{12} + \theta^{23}_{13} + \underset{\bullet}{\theta^{23}_{21}} + \underset{\bullet}{\theta^{23}_{22}} + \theta^{23}_{31} + \theta^{23}_{32} + \theta^{23}_{33}\right]}^{Q_{2,23}}}{\begin{array}{l} p_2 + \underbrace{\left[\underset{\bullet}{\theta^{11}_{21}} + \underset{\bullet}{\theta^{12}_{21}} + \underset{\bullet}{\theta^{13}_{21}} + \underset{\bullet}{\theta^{31}_{21}} + \underset{\bullet}{\theta^{32}_{21}} + \underset{\bullet}{\theta^{33}_{21}} + \underset{\bullet}{\theta^{11}_{22}} + \underset{\bullet}{\theta^{12}_{22}} + \underset{\bullet}{\theta^{13}_{22}} + \underset{\bullet}{\theta^{31}_{22}} + \underset{\bullet}{\theta^{32}_{22}} + \underset{\bullet}{\theta^{33}_{22}} + \theta^{11}_{23} + \theta^{12}_{23} + \theta^{13}_{23} + \theta^{31}_{23} + \theta^{32}_{23} + \theta^{33}_{23}\right]}_{Q_{3,2}} \\ \quad - \underbrace{\left[\underset{\bullet}{\theta^{21}_{11}} + \underset{\bullet}{\theta^{21}_{12}} + \underset{\bullet}{\theta^{21}_{13}} + \underset{\bullet}{\theta^{22}_{11}} + \underset{\bullet}{\theta^{22}_{12}} + \underset{\bullet}{\theta^{22}_{13}} + \theta^{23}_{11} + \theta^{23}_{12} + \theta^{23}_{13} + \underset{\bullet}{\theta^{21}_{31}} + \underset{\bullet}{\theta^{21}_{32}} + \underset{\bullet}{\theta^{21}_{33}} + \underset{\bullet}{\theta^{22}_{31}} + \underset{\bullet}{\theta^{22}_{32}} + \underset{\bullet}{\theta^{22}_{33}} + \theta^{23}_{31} + \theta^{23}_{32} + \theta^{23}_{33}\right]}_{Q_{4,2}} \end{array}}$$

- $\underset{\bullet}{\theta^{k'l'}_{kl}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$\begin{aligned} LB_{23} &= \frac{r_{23} - \widetilde{Q}}{p_2} \geq 0 & \qquad \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB_{23} &= \frac{r_{23} + \widetilde{Q}}{p_2 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} &= \begin{cases} 0 & \text{AE} \\ \min\{p_2, Q/3\} & \text{UE} \end{cases} \end{aligned}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p^*_{23} = \frac{r_{23} + \overbrace{\left[\theta^{33}_{23}\right]}^{Q^u_{1,23}} - \overbrace{\left[\theta^{23}_{11} + \theta^{23}_{12} + \theta^{23}_{13} + \underset{\bullet}{\theta^{23}_{21}} + \underset{\bullet}{\theta^{23}_{22}}\right]}^{Q^u_{2,23}}}{p_2 + \underbrace{\left[\underset{\bullet}{\theta^{31}_{21}} + \underset{\bullet}{\theta^{32}_{21}} + \underset{\bullet}{\theta^{33}_{21}} + \underset{\bullet}{\theta^{32}_{22}} + \underset{\bullet}{\theta^{33}_{22}} + \theta^{33}_{23}\right]}_{Q^u_{3,2}} - \underbrace{\left[\underset{\bullet}{\theta^{21}_{11}} + \underset{\bullet}{\theta^{22}_{11}} + \underset{\bullet}{\theta^{22}_{12}} + \theta^{23}_{11} + \theta^{23}_{12} + \theta^{23}_{13}\right]}_{Q^u_{4,2}}}$$

- Yields

$$\begin{aligned} LB^u_{23} &= \frac{r_{23} - \widetilde{Q}}{p_2} \geq 0 & \qquad \widetilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB^u_{23} &= \max\left\{\frac{r_{23}}{p_2 - \widehat{Q}}, \frac{r_{23} + \widetilde{\widetilde{Q}}}{p_2 + \widetilde{\widetilde{Q}}}\right\} \leq 1 & \qquad \widehat{Q} &= \min\left\{p_2, \widetilde{Q}\right\},\ \widetilde{\widetilde{Q}} = \min\left\{1 - p_2, \widetilde{Q}\right\},\ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \end{aligned}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{23}^{*}=\frac{r_{23}+\overbrace{\left[\alpha_2^1\beta_3^1+\alpha_2^1\beta_3^2+\alpha_2^1\beta_3^3+\alpha_2^2\beta_3^1+\alpha_2^2\beta_3^2+\alpha_2^3\beta_3^1+\alpha_2^3\beta_3^2+\alpha_2^3\beta_3^3\right]}^{Q_{1,23}}-\overbrace{\left[\alpha_1^2\beta_1^3+\alpha_1^2\beta_2^3+\alpha_1^2\beta_3^3+\alpha_2^2\beta_1^3+\alpha_2^2\beta_2^3+\alpha_3^2\beta_1^3+\alpha_3^2\beta_2^3+\alpha_3^2\beta_3^3\right]}^{Q_{2,23}}}{\begin{array}{c}p_2+\underbrace{\left[\alpha_2^1\beta_1^1+\alpha_2^1\beta_1^2+\alpha_2^1\beta_1^3+\alpha_2^3\beta_1^1+\alpha_2^3\beta_1^2+\alpha_2^3\beta_1^3+\alpha_2^1\beta_2^1+\alpha_2^1\beta_2^2+\alpha_2^1\beta_2^3+\alpha_2^3\beta_2^1+\alpha_2^3\beta_2^2+\alpha_2^3\beta_2^3+\alpha_2^1\beta_3^1+\alpha_2^1\beta_3^2+\alpha_2^1\beta_3^3+\alpha_2^3\beta_3^1+\alpha_2^3\beta_3^2+\alpha_2^3\beta_3^3\right]}_{Q_{3,2}}\\-\underbrace{\left[\alpha_1^2\beta_1^1+\alpha_1^2\beta_2^1+\alpha_1^2\beta_3^1+\alpha_1^2\beta_1^2+\alpha_1^2\beta_2^2+\alpha_1^2\beta_3^2+\alpha_1^2\beta_1^3+\alpha_1^2\beta_2^3+\alpha_1^2\beta_3^3+\alpha_3^2\beta_1^1+\alpha_3^2\beta_2^1+\alpha_3^2\beta_3^1+\alpha_3^2\beta_1^2+\alpha_3^2\beta_2^2+\alpha_3^2\beta_3^2+\alpha_3^2\beta_1^3+\alpha_3^2\beta_2^3+\alpha_3^2\beta_3^3\right]}_{Q_{4,2}}\end{array}}$$

- Simplifying

$$\begin{aligned}
Q_{1,23} &= \left(\alpha_2^1+\alpha_2^3\right)+\left(\beta_3^1+\beta_3^2\right)\left(1-\alpha_2^1-\alpha_2^3\right) && \text{(TI)}\\
&= \left(\theta_2^1+\theta_2^3\right)+\left(\theta_3^1+\theta_3^2\right)\left(1-\theta_2^1-\theta_2^3\right) && \text{(TIV)}\\
Q_{2,23} &= \left(\alpha_1^2+\alpha_3^2\right)\left(1+\beta_1^3+\beta_2^3-\beta_3^1-\beta_3^2\right)+\left(\beta_1^3+\beta_2^3\right)\left(1-\alpha_2^1-\alpha_2^3\right) && \text{(TI)}\\
&= \left(\theta_1^2+\theta_3^2\right)\left(1+\theta_1^3+\theta_2^3-\theta_3^1-\theta_3^2\right)+\left(\theta_1^3+\theta_2^3\right)\left(1-\theta_2^1-\theta_2^3\right) && \text{(TIV)}\\
Q_{3,2} &= 3\left(\alpha_2^1+\alpha_2^3\right) && \text{(TI)}\\
&= 3\left(\theta_2^1+\theta_2^3\right) && \text{(TIV)}\\
Q_{4,2} &= 3\left(\alpha_1^2+\alpha_3^2\right) && \text{(TI)}\\
&= 3\left(\theta_1^2+\theta_3^2\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{23}^{*}=\frac{r_{23}+\left(\beta_3^1+\beta_3^2-\beta_1^3-\beta_2^3\right)+\left(\alpha_2^1+\alpha_2^3-\alpha_1^2-\alpha_3^2\right)\left(1+\beta_1^3+\beta_2^3-\beta_3^1-\beta_3^2\right)}{p_2+3\left(\alpha_2^1+\alpha_2^3-\alpha_1^2-\alpha_3^2\right)}$$

- Yields

$$\begin{aligned}
LB_{23}^{TI} &= \min\left\{\frac{r_{23}-\ddot{Q}}{p_1},\frac{r_{23}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}},\frac{r_{23}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\geq 0\\
&\quad \widetilde{\widetilde{Q}}=\min\left\{1-r_{23},(1-p_2)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{23},p_2/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\\
UB_{23}^{TI} &= \max\left\{\frac{r_{23}+\widetilde{Q}}{p_2},\frac{r_{23}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}},\frac{r_{23}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\leq 1\\
&\quad \widetilde{\widetilde{Q}}=\min\left\{1-r_{23},(1-p_2)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{23},p_2/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}
\end{aligned}$$

- Proof:
  1. $\ddot{Q}$ in $\frac{r_{23}-\ddot{Q}}{p_2}$ can be $2Q/9$ as $\beta_2^3,\beta_1^3=Q/9$ under UE.
  2. $\widehat{Q}$ in $\frac{r_{23}-\widehat{Q}}{p_2-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_1^2,\alpha_3^2=Q/9$ under UE.
  3. Evaluate $\partial\left(\frac{r_{23}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}}\right)/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{23}+\widetilde{\widetilde{Q}}}{p_2+3\widetilde{\widetilde{Q}}}\right)}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(\left(p_2+3\widetilde{\widetilde{Q}}\right)-3\left(r_{23}+\widetilde{\widetilde{Q}}\right)\right)\\
&= sgn\left(p_2-3r_{23}\right)
\end{aligned}$$

  4. Evaluate $\partial\left(\frac{r_{23}-\widehat{Q}}{p_2-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{23}-\widehat{Q}}{p_2-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_2-3\widehat{Q}\right)+3\left(r_{23}-\widehat{Q}\right)\right)\\
&= sgn\left(3r_{23}-p_2\right)
\end{aligned}$$

– Adding the uni-directional assumption

$$p_{23}^{*}=\frac{r_{23}-\left(\beta_{1}^{3}+\beta_{2}^{3}\right)+\left(\alpha_{2}^{3}-\alpha_{1}^{2}\right)\left(1+\beta_{1}^{3}+\beta_{2}^{3}\right)}{p_{2}+3\left(\alpha_{2}^{3}-\alpha_{1}^{2}\right)}$$

* Yields

$$\begin{aligned}
LB_{23}^{TI,u} &= \min\left\{\frac{r_{23}-\ddot{Q}}{p_2},\frac{r_{23}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}},\frac{r_{23}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\geq 0\\
&\quad \tilde{\tilde{Q}}=\min\left\{1-r_{23},(1-p_2)/3,\tilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{23},p_2/3,\tilde{Q}\right\},\ \tilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\\
UB_{23}^{TI,u} &= \max\left\{\frac{r_{23}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}},\frac{r_{23}-\widehat{Q}}{p_2-3\widehat{Q}}\right\}\\
&\quad \tilde{\tilde{Q}}=\min\left\{1-r_{23},(1-p_2)/3,\tilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{23},p_2/3,\tilde{Q}\right\},\ \tilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases}
\end{aligned}$$

* Proof: LB is the same except that $\ddot{Q}$ is no longer feasible in $\widehat{Q}$. The UB is not r_{23}/p_2 as the derivative of one of the terms in $\max\{\cdot\}$ wrt Q must be positive.

$$\begin{aligned}
sgn\left(\frac{\partial\frac{r_{23}+\tilde{\tilde{Q}}}{p_2+3\tilde{\tilde{Q}}}}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(p_2+3\tilde{\tilde{Q}}-3\left(r_{23}+\tilde{\tilde{Q}}\right)\right)\\
&= sgn\left(p_2-3r_{23}\right)\\
sgn\left(\frac{\partial\frac{r_{23}-\widehat{Q}}{p_2-3\widehat{Q}}}{\partial\widehat{Q}}\right) &= sgn\left(p_2+3\widehat{Q}-3\left(r_{23}+\widehat{Q}\right)\right)\\
&= sgn\left(3r_{23}-p_2\right)
\end{aligned}$$

- Under Temporal Invariance

$$p_{23}^{*}=\frac{r_{23}+\left(\theta_2^1+\theta_3^1-\theta_1^2-\theta_1^3\right)+\left(\theta_2^1+\theta_2^3-\theta_1^2-\theta_3^2\right)\left(\theta_1^3+\theta_2^3-\theta_3^1-\theta_3^2\right)}{p_2+3\left(\theta_2^1+\theta_2^3-\theta_1^2-\theta_3^2\right)}$$

  - Yields

$$LB_{23}^{TIV} = \min\left\{\frac{r_{23}-\widetilde{Q}}{p_2},\frac{r_{23}+\widetilde{\widetilde{Q}}^2}{p_2+3\widetilde{\widetilde{Q}}}\right\}\geq 0$$

$$\widetilde{\widetilde{Q}}=\min\left\{\frac{-(2/3)p_2+\sqrt{(4/9)p_2^2+4r_{23}}}{2},\sqrt{1-r_{23}},(1-p_2)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

$$UB_{23}^{TIV} = \max\left\{\frac{r_{23}+\widetilde{Q}}{p_2},\frac{r_{23}+\widetilde{\widetilde{Q}}^2}{p_2-3\widetilde{\widetilde{Q}}}\right\}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{\sqrt{1-r_{23}},p_2/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

  - Proof:

    1. Evaluate $\partial LB_{23}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned}
sgn\left(\frac{\partial LB_{23}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(2\widetilde{\widetilde{Q}}\left(p_2+3\widetilde{\widetilde{Q}}\right)-3\left(r_{23}+\widetilde{\widetilde{Q}}^2\right)\right)\\
&= sgn\left(\widetilde{\widetilde{Q}}\left((2/3)p_2+\widetilde{\widetilde{Q}}\right)-r_{23}\right)\\
\Rightarrow sgn\left(\frac{\partial LB_{23}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right)\Bigg|_{\widetilde{\widetilde{Q}}=0} &= sgn\left(-r_{23}\right)<0\\
\Rightarrow\widetilde{\widetilde{Q}} &> 0
\end{aligned}$$

    2. Minimize LB_{23}^{TIV} s.t. $\widetilde{\widetilde{Q}}$ being feasible

$$\begin{aligned}
\frac{\partial LB_{23}^{TIV}}{\partial\widetilde{\widetilde{Q}}} &\propto \widetilde{\widetilde{Q}}\left((2/3)p_2+\widetilde{\widetilde{Q}}\right)-r_{23}=0\\
\Rightarrow & \widetilde{\widetilde{Q}}^* = \frac{-(2/3)p_2+\sqrt{(4/9)p_2^2+4r_{23}}}{2}
\end{aligned}$$

So, derivative starts off negative and then reaches zero at $\widetilde{\widetilde{Q}}^*$. Thus, $\frac{r_{23}+\widetilde{\widetilde{Q}}^2}{p_2+3\widetilde{\widetilde{Q}}}$ is minimized at $\widetilde{\widetilde{Q}}^*$.

– Adding the uni-directional assumption

$$p_{23}^{*} = \frac{r_{23} - \left(\theta_1^2 + \theta_1^3\right) + \left(\theta_2^3 - \theta_1^2\right)\left(\theta_1^3 + \theta_2^3\right)}{p_2 + 3\left(\theta_2^3 - \theta_1^2\right)}$$

* Yields

$$LB_{23}^{TIV,u} = \min\left\{\frac{r_{23} - \widetilde{Q}}{p_2}, \frac{r_{23} + \widetilde{\widetilde{Q}}^2}{p_2 + 3\widetilde{\widetilde{Q}}}\right\} \geq 0$$

$$\widetilde{\widetilde{Q}} = \min\left\{\frac{-(2/3)p_2 + \sqrt{(4/9)p_2^2 + 4r_{23}}}{2}, \sqrt{1 - r_{23}}, (1 - p_2)/3, \widetilde{Q}\right\}, \ \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

$$UB_{23}^{TIV,u} = \frac{r_{23} - \widetilde{\widetilde{Q}}}{p_2 - 3\widetilde{\widetilde{Q}}} \leq 1$$

$$\widetilde{\widetilde{Q}} = \begin{cases} 0 & r_{23} < p_2/3 \\ \min\left\{r_{23}, p_2/3, \widetilde{Q}\right\} & \text{otherwise} \end{cases}, \ \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

* Proof: Evaluate $\partial\left(\frac{r_{23} - \widetilde{\widetilde{Q}}}{p_2 - 3\widetilde{\widetilde{Q}}}\right)/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned} sgn\left(\frac{\partial\left(\frac{r_{23} - \widetilde{\widetilde{Q}}}{p_2 - 3\widetilde{\widetilde{Q}}}\right)}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(-\left(p_2 - 3\widetilde{\widetilde{Q}}\right) + 3\left(r_{23} - \widetilde{\widetilde{Q}}\right)\right) \\ &= sgn\left(3r_{23} - p_2\right) \end{aligned}$$

A.1.7 p_{31}^{*}

$$p_{31}^{*}=\frac{r_{31}+\overbrace{\left[\theta_{31}^{11}+\theta_{31}^{12}+\theta_{31}^{13}+\theta_{31}^{21}+\theta_{31}^{22}+\theta_{31}^{23}+\underset{\bullet}{\theta_{31}^{32}}+\underset{\bullet}{\theta_{31}^{33}}\right]}^{Q_{1,31}}-\overbrace{\left[\theta_{11}^{31}+\theta_{12}^{31}+\theta_{13}^{31}+\theta_{21}^{31}+\theta_{22}^{31}+\theta_{23}^{31}+\underset{\bullet}{\theta_{32}^{31}}+\underset{\bullet}{\theta_{33}^{31}}\right]}^{Q_{2,31}}}{\begin{array}{l}p_{3}+\underbrace{\left[\theta_{31}^{11}+\theta_{31}^{12}+\theta_{31}^{13}+\theta_{31}^{21}+\theta_{31}^{22}+\theta_{31}^{23}+\underset{\bullet}{\theta_{32}^{11}}+\underset{\bullet}{\theta_{32}^{12}}+\underset{\bullet}{\theta_{32}^{13}}+\underset{\bullet}{\theta_{32}^{21}}+\underset{\bullet}{\theta_{32}^{22}}+\underset{\bullet}{\theta_{32}^{23}}+\underset{\bullet}{\theta_{33}^{11}}+\underset{\bullet}{\theta_{33}^{12}}+\underset{\bullet}{\theta_{33}^{13}}+\underset{\bullet}{\theta_{33}^{21}}+\underset{\bullet}{\theta_{33}^{22}}+\underset{\bullet}{\theta_{33}^{23}}\right]}_{Q_{3,3}}\\ \quad-\underbrace{\left[\theta_{11}^{31}+\theta_{12}^{31}+\theta_{13}^{31}+\underset{\bullet}{\theta_{11}^{32}}+\underset{\bullet}{\theta_{12}^{32}}+\underset{\bullet}{\theta_{13}^{32}}+\underset{\bullet}{\theta_{11}^{33}}+\underset{\bullet}{\theta_{12}^{33}}+\underset{\bullet}{\theta_{13}^{33}}+\theta_{21}^{31}+\theta_{22}^{31}+\theta_{23}^{31}+\underset{\bullet}{\theta_{21}^{32}}+\underset{\bullet}{\theta_{22}^{32}}+\underset{\bullet}{\theta_{23}^{32}}+\underset{\bullet}{\theta_{21}^{33}}+\underset{\bullet}{\theta_{22}^{33}}+\underset{\bullet}{\theta_{23}^{33}}\right]}_{Q_{4,3}}\end{array}}$$

- $\underset{\bullet}{\theta_{kl}^{k'l'}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$\begin{aligned} LB_{31} &= \frac{r_{31}-\widetilde{Q}}{p_3}\geq 0 & \qquad \widetilde{Q}&=\begin{cases} Q & \text{AE}\\ Q/3 & \text{UE}\end{cases}\\ UB_{31} &= \frac{r_{31}+\widetilde{Q}}{p_3-\widetilde{\widetilde{Q}}}\leq 1 & \qquad \widetilde{\widetilde{Q}}&=\begin{cases} 0 & \text{AE}\\ \min\{p_3,Q/3\} & \text{UE}\end{cases}\end{aligned}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p_{31}^{*}=\frac{r_{31}+\overbrace{\left[\underset{\bullet}{\theta_{31}^{32}}+\underset{\bullet}{\theta_{31}^{33}}\right]}^{Q_{1,31}^{u}}-\overbrace{\left[\theta_{11}^{31}+\theta_{21}^{31}\right]}^{Q_{2,31}^{u}}}{p_{3}-\underbrace{\left[\theta_{11}^{31}+\underset{\bullet}{\theta_{11}^{32}}+\underset{\bullet}{\theta_{12}^{32}}+\underset{\bullet}{\theta_{11}^{33}}+\underset{\bullet}{\theta_{12}^{33}}+\underset{\bullet}{\theta_{13}^{33}}+\theta_{21}^{31}+\underset{\bullet}{\theta_{21}^{32}}+\underset{\bullet}{\theta_{22}^{32}}+\underset{\bullet}{\theta_{21}^{33}}+\underset{\bullet}{\theta_{22}^{33}}+\underset{\bullet}{\theta_{23}^{33}}\right]}_{Q_{4,3}^{u}}}$$

- Yields

$$\begin{aligned} LB_{31}^{u} &= \frac{r_{31}-\widetilde{\widetilde{Q}}}{p_3-\widetilde{\widetilde{Q}}}\geq 0 & \qquad \widetilde{\widetilde{Q}}&=\min\left\{r_{31},\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases} Q & \text{AE}\\ Q/3 & \text{UE}\end{cases}\\ UB_{31} &= \frac{r_{31}+\widetilde{Q}}{p_3-\widetilde{\widetilde{Q}}}\leq 1 & \qquad \widetilde{\widetilde{Q}}&=\begin{cases} 0 & \text{AE}\\ \min\{p_3,Q/3\} & \text{UE}\end{cases},\ \widetilde{Q}=\begin{cases} Q & \text{AE}\\ Q/3 & \text{UE}\end{cases}\end{aligned}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{31}^{*}=\frac{r_{31}+\overbrace{\left[\alpha_3^1\beta_1^1+\alpha_3^1\beta_1^2+\alpha_3^1\beta_1^3+\alpha_3^2\beta_1^1+\alpha_3^2\beta_1^2+\alpha_3^2\beta_1^3+\alpha_3^3\beta_1^2+\alpha_3^3\beta_1^3\right]}^{Q_{1,31}}-\overbrace{\left[\alpha_1^3\beta_1^1+\alpha_1^3\beta_2^1+\alpha_1^3\beta_3^1+\alpha_2^3\beta_1^1+\alpha_2^3\beta_2^1+\alpha_2^3\beta_3^1+\alpha_3^3\beta_2^1+\alpha_3^3\beta_3^1\right]}^{Q_{2,31}}}{\begin{array}{l}p_3+\underbrace{\left[\alpha_3^1\beta_1^1+\alpha_3^1\beta_1^2+\alpha_3^1\beta_1^3+\alpha_3^2\beta_1^1+\alpha_3^2\beta_1^2+\alpha_3^2\beta_1^3+\alpha_3^1\beta_2^1+\alpha_3^1\beta_2^2+\alpha_3^1\beta_2^3+\alpha_3^2\beta_2^1+\alpha_3^2\beta_2^2+\alpha_3^2\beta_2^3+\alpha_3^1\beta_3^1+\alpha_3^1\beta_3^2+\alpha_3^1\beta_3^3+\alpha_3^2\beta_3^1+\alpha_3^2\beta_3^2+\alpha_3^2\beta_3^3\right]}_{Q_{3,3}}\\ -\underbrace{\left[\alpha_1^3\beta_1^1+\alpha_1^3\beta_2^1+\alpha_1^3\beta_3^1+\alpha_1^3\beta_1^2+\alpha_1^3\beta_2^2+\alpha_1^3\beta_3^2+\alpha_1^3\beta_1^3+\alpha_1^3\beta_2^3+\alpha_1^3\beta_3^3+\alpha_2^3\beta_1^1+\alpha_2^3\beta_2^1+\alpha_2^3\beta_3^1+\alpha_2^3\beta_1^2+\alpha_2^3\beta_2^2+\alpha_2^3\beta_3^2+\alpha_2^3\beta_1^3+\alpha_2^3\beta_2^3+\alpha_2^3\beta_3^3\right]}_{Q_{4,3}}\end{array}}$$

- Simplifying

$$\begin{aligned}
Q_{1,31} &= \left(\alpha_3^1+\alpha_3^2\right)+\left(\beta_1^2+\beta_1^3\right)\left(1-\alpha_3^1-\alpha_3^2\right) && \text{(TI)}\\
&= \left(\theta_3^1+\theta_3^2\right)+\left(\theta_1^2+\theta_1^3\right)\left(1-\theta_3^1-\theta_3^2\right) && \text{(TIV)}\\
Q_{2,31} &= \left(\alpha_1^3+\alpha_2^3\right)\left(1+\beta_2^1+\beta_3^1-\beta_1^2-\beta_1^3\right)+\left(\beta_2^1+\beta_3^1\right)\left(1-\alpha_3^1-\alpha_3^2\right) && \text{(TI)}\\
&= \left(\theta_1^3+\theta_2^3\right)\left(1+\theta_2^1+\theta_3^1-\theta_1^2-\theta_1^3\right)+\left(\theta_2^1+\theta_3^1\right)\left(1-\theta_3^1-\theta_3^2\right) && \text{(TIV)}\\
Q_{3,3} &= 3\left(\alpha_3^1+\alpha_3^2\right) && \text{(TI)}\\
&= 3\left(\theta_3^1+\theta_3^2\right) && \text{(TIV)}\\
Q_{4,3} &= 3\left(\alpha_1^3+\alpha_2^3\right) && \text{(TI)}\\
&= 3\left(\theta_1^3+\theta_2^3\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{31}^{*}=\frac{r_{31}+\left(\beta_1^2+\beta_1^3-\beta_2^1-\beta_3^1\right)+\left(\alpha_3^1+\alpha_3^2-\alpha_1^3-\alpha_2^3\right)\left(1+\beta_2^1+\beta_3^1-\beta_1^2-\beta_1^3\right)}{p_3+3\left(\alpha_3^1+\alpha_3^2-\alpha_1^3-\alpha_2^3\right)}$$

  - Yields

$$LB_{31}^{TI} = \min\left\{\frac{r_{31}-\ddot{Q}}{p_3},\frac{r_{31}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}},\frac{r_{31}-\widehat{Q}}{p_3-3\widehat{Q}}\right\}\geq 0$$

$$\tilde{\tilde{Q}}=\min\left\{1-r_{31},(1-p_3)/3,\tilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{31},p_3/3,\ddot{Q}\right\},\ \tilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

$$UB_{31}^{TI} = \max\left\{\frac{r_{31}+\tilde{Q}}{p_3},\frac{r_{31}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}},\frac{r_{31}-\widehat{Q}}{p_3-3\widehat{Q}}\right\}\leq 1$$

$$\tilde{\tilde{Q}}=\min\left\{1-r_{31},(1-p_3)/3,\tilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{31},p_3/3,\ddot{Q}\right\},\ \tilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

  - Proof:

    1. $\ddot{Q}$ in $\frac{r_{31}-\ddot{Q}}{p_3}$ can be $2Q/9$ as $\beta_2^1,\beta_3^1=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{31}-\widehat{Q}}{p_3-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_1^3,\alpha_2^3=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{31}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} sgn\left(\frac{\partial\left(\frac{r_{31}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(p_3+3\tilde{\tilde{Q}}\right)-3\left(r_{31}+\tilde{\tilde{Q}}\right)\right)\\ &= sgn\left(p_3-3r_{31}\right)\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{31}-\widehat{Q}}{p_3-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} sgn\left(\frac{\partial\left(\frac{r_{31}-\widehat{Q}}{p_3-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_3-3\widehat{Q}\right)+3\left(r_{31}-\widehat{Q}\right)\right)\\ &= sgn\left(3r_{31}-p_3\right)\end{aligned}$$

  - Adding the uni-directional assumption

$$p_{31}^{*}=\frac{r_{31}+\left(\beta_1^2+\beta_1^3\right)-\left(\alpha_1^3+\alpha_2^3\right)\left(1-\beta_1^2-\beta_1^3\right)}{p_3-3\left(\alpha_1^3+\alpha_2^3\right)}$$

    * Yields

$$LB_{31}^{TI} = \frac{r_{31}-\tilde{\tilde{Q}}}{p_3-3\tilde{\tilde{Q}}}\geq 0 \qquad \tilde{\tilde{Q}}=\begin{cases} 0 & r_{31}\geq p_3/3\\ \min\left\{r_{31},p_3/3,\ddot{Q}\right\} & \text{otherwise}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

$$UB_{31}^{TI} = \max\left\{\frac{r_{31}+\tilde{Q}}{p_3},\frac{r_{31}-\tilde{\tilde{Q}}}{p_3-3\tilde{\tilde{Q}}}\right\}\leq 1$$

$$\tilde{\tilde{Q}}=\min\left\{r_{31},p_3/3,\ddot{Q}\right\},\ \tilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}$$

    * Proof: Same as above.

- Under Temporal Invariance

$$p_{31}^* = \frac{r_{31} + \left(\theta_1^2 + \theta_3^2 - \theta_2^1 - \theta_2^3\right) + \left(\theta_3^1 + \theta_3^2 - \theta_1^3 - \theta_2^3\right)\left(\theta_2^1 + \theta_3^1 - \theta_1^2 - \theta_1^3\right)}{p_3 + 3\left(\theta_3^1 + \theta_3^2 - \theta_1^3 - \theta_2^3\right)}$$

– Yields

$$LB_{31}^{TIV} = \min\left\{\frac{r_{31} - \widetilde{Q}}{p_3}, \frac{r_{31} + \widetilde{\widetilde{Q}}^2}{p_3 + 3\widetilde{\widetilde{Q}}}\right\} \geq 0$$

$$\widetilde{\widetilde{Q}} = \min\left\{\frac{-(2/3)p_3 + \sqrt{(4/9)p_3^2 + 4r_{31}}}{2}, \sqrt{1 - r_{31}}, (1 - p_3)/3, \widetilde{Q}\right\}, \ \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

$$UB_{31}^{TIV} = \max\left\{\frac{r_{31} + \widetilde{Q}}{p_3}, \frac{r_{31} + \widetilde{\widetilde{Q}}^2}{p_3 - 3\widetilde{\widetilde{Q}}}\right\} \leq 1 \qquad \widetilde{\widetilde{Q}} = \min\left\{\sqrt{1 - r_{31}}, p_3/3, \widetilde{Q}\right\}, \ \widetilde{Q} = \begin{cases} 3 - \sqrt{9 - Q} & \text{AE} \\ \left(4 - \sqrt{16 - 4Q/3}\right)/2 & \text{UE} \end{cases}$$

– Proof:

1. Evaluate $\partial LB_{31}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned}
sgn\left(\frac{\partial LB_{31}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(2\widetilde{\widetilde{Q}}\left(p_3 + 3\widetilde{\widetilde{Q}}\right) - 3\left(r_{31} + \widetilde{\widetilde{Q}}^2\right)\right) \\
&= sgn\left(\widetilde{\widetilde{Q}}\left((2/3)\,p_3 + \widetilde{\widetilde{Q}}\right) - r_{31}\right) \\
\Rightarrow sgn\left(\frac{\partial LB_{31}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right)\Bigg|_{\widetilde{\widetilde{Q}}=0} &= sgn\left(-r_{31}\right) < 0 \\
\Rightarrow \widetilde{\widetilde{Q}} &> 0
\end{aligned}$$

2. Minimize LB_{31}^{TIV} s.t. $\widetilde{\widetilde{Q}}$ being feasible

$$\begin{aligned}
\frac{\partial LB_{31}^{TIV}}{\partial\widetilde{\widetilde{Q}}} &\propto \widetilde{\widetilde{Q}}\left((2/3)\,p_3 + \widetilde{\widetilde{Q}}\right) - r_{31} = 0 \\
\Rightarrow \quad \widetilde{\widetilde{Q}}^* &= \frac{-(2/3)p_3 + \sqrt{(4/9)p_3^2 + 4r_{31}}}{2}
\end{aligned}$$

So, derivative starts off negative and then reaches zero at $\widetilde{\widetilde{Q}}^*$. Thus, $\frac{r_{31}+\widetilde{\widetilde{Q}}^2}{p_3+3\widetilde{\widetilde{Q}}}$ is minimized at $\widetilde{\widetilde{Q}}^*$.

– Adding the uni-directional assumption

$$p_{31}^{*}=\frac{r_{31}+\left(\theta_1^2-\theta_2^3\right)+\left(\theta_1^3+\theta_2^3\right)\left(\theta_1^2+\theta_1^3\right)}{p_3-3\left(\theta_1^3+\theta_2^3\right)}$$

* Yields

$$LB_{31}^{TIV,u} = \frac{r_{31}-\tilde{\tilde{Q}}}{p_3-3\tilde{\tilde{Q}}}\geq 0$$

$$\tilde{\tilde{Q}}=\begin{cases}0 & r_{31}\geq p_3/3\\ \min\left\{r_{31},p_3/3,\tilde{Q}\right\} & \text{otherwise}\end{cases},\ \tilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

$$UB_{31}^{TIV,u} = \max\left\{\frac{r_{31}+\tilde{Q}}{p_3},\frac{r_{31}+\tilde{\tilde{Q}}^2}{p_3-3\tilde{\tilde{Q}}}\right\}\leq 1 \qquad \tilde{\tilde{Q}}=\min\left\{\sqrt{1-r_{31}},p_3/3,\tilde{Q}\right\},\ \tilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

* Proof: Evaluate $\partial\left(LB_{31}^{TIV,u}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned} sgn\left(\frac{\partial LB_{31}^{TIV,u}}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(-\left(p_3-3\tilde{\tilde{Q}}\right)+3\left(r_{31}-\tilde{\tilde{Q}}\right)\right)\\ &= sgn\left(3r_{31}-p_3\right)\end{aligned}$$

A.1.8 p^*_{32}

$$
\begin{aligned}
p^*_{32} = \frac{r_{32} + \overbrace{\left[\theta^{11}_{32} + \theta^{12}_{32} + \theta^{13}_{32} + \theta^{21}_{32} + \theta^{22}_{32} + \theta^{23}_{32} + \underset{\bullet}{\theta}^{31}_{32} + \underset{\bullet}{\theta}^{33}_{32}\right]}^{Q_{1,32}} - \overbrace{\left[\theta^{32}_{11} + \theta^{32}_{12} + \theta^{32}_{13} + \theta^{32}_{21} + \theta^{32}_{22} + \theta^{32}_{23} + \underset{\bullet}{\theta}^{32}_{31} + \underset{\bullet}{\theta}^{32}_{33}\right]}^{Q_{2,32}}}{\begin{aligned} & p_3 + \underbrace{\left[\underset{\bullet}{\theta}^{11}_{31} + \underset{\bullet}{\theta}^{12}_{31} + \underset{\bullet}{\theta}^{13}_{31} + \underset{\bullet}{\theta}^{21}_{31} + \underset{\bullet}{\theta}^{22}_{31} + \underset{\bullet}{\theta}^{23}_{31} + \underset{\bullet}{\theta}^{11}_{32} + \underset{\bullet}{\theta}^{12}_{32} + \underset{\bullet}{\theta}^{13}_{32} + \underset{\bullet}{\theta}^{21}_{32} + \underset{\bullet}{\theta}^{22}_{32} + \underset{\bullet}{\theta}^{23}_{32} + \theta^{11}_{33} + \theta^{12}_{33} + \theta^{13}_{33} + \theta^{21}_{33} + \theta^{22}_{33} + \theta^{23}_{33}\right]}_{Q_{3,3}} \\ & - \underbrace{\left[\underset{\bullet}{\theta}^{31}_{11} + \underset{\bullet}{\theta}^{31}_{12} + \underset{\bullet}{\theta}^{31}_{13} + \underset{\bullet}{\theta}^{32}_{11} + \underset{\bullet}{\theta}^{32}_{12} + \underset{\bullet}{\theta}^{32}_{13} + \theta^{33}_{11} + \theta^{33}_{12} + \theta^{33}_{13} + \underset{\bullet}{\theta}^{31}_{21} + \underset{\bullet}{\theta}^{31}_{22} + \underset{\bullet}{\theta}^{31}_{23} + \underset{\bullet}{\theta}^{32}_{21} + \underset{\bullet}{\theta}^{32}_{22} + \underset{\bullet}{\theta}^{32}_{23} + \theta^{33}_{21} + \theta^{33}_{22} + \theta^{33}_{23}\right]}_{Q_{4,3}} \end{aligned}}
\end{aligned}
$$

- $\underset{\bullet}{\theta}^{k'l'}_{kl}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$
\begin{aligned}
LB_{32} &= \frac{r_{32} - \widetilde{Q}}{p_3} \geq 0 & \qquad \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\
UB_{32} &= \frac{r_{32} + \widetilde{Q}}{p_3 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & \text{AE} \\ \min\{p_3, Q/3\} & \text{UE} \end{cases}
\end{aligned}
$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$
p^*_{32} = \frac{r_{32} + \overbrace{\left[\underset{\bullet}{\theta}^{33}_{32}\right]}^{Q^u_{1,32}} - \overbrace{\left[\theta^{32}_{11} + \theta^{32}_{12} + \theta^{32}_{21} + \theta^{32}_{22} + \underset{\bullet}{\theta}^{32}_{31}\right]}^{Q^u_{2,32}}}{p_3 - \underbrace{\left[\theta^{31}_{11} + \underset{\bullet}{\theta}^{32}_{11} + \underset{\bullet}{\theta}^{32}_{12} + \underset{\bullet}{\theta}^{33}_{11} + \underset{\bullet}{\theta}^{33}_{12} + \underset{\bullet}{\theta}^{33}_{13} + \theta^{31}_{21} + \underset{\bullet}{\theta}^{32}_{21} + \underset{\bullet}{\theta}^{32}_{22} + \underset{\bullet}{\theta}^{33}_{21} + \underset{\bullet}{\theta}^{33}_{22} + \underset{\bullet}{\theta}^{33}_{23}\right]}_{Q^u_{4,3}}}
$$

- Yields

$$
\begin{aligned}
LB^u_{32} &= \frac{r_{32} - \widetilde{Q}}{p_3} \geq 0 & \qquad \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\
UB^u_{32} &= \frac{r_{32} + \widetilde{Q}}{p_3 - \widetilde{\widetilde{Q}}} \leq 1 & \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & \text{AE} \\ \min\{p_3, Q/3\} & \text{UE} \end{cases}, \; \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}
\end{aligned}
$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{32}^{*}=\frac{r_{32}+\overbrace{\left[\alpha_3^1\beta_2^1+\alpha_3^1\beta_2^2+\alpha_3^1\beta_2^3+\alpha_3^2\beta_2^1+\alpha_3^2\beta_2^2+\alpha_3^2\beta_2^3+\alpha_3^3\beta_2^1+\alpha_3^3\beta_2^3\right]}^{Q_{1,32}}-\overbrace{\left[\alpha_1^3\beta_1^2+\alpha_1^3\beta_2^2+\alpha_1^3\beta_3^2+\alpha_2^3\beta_1^2+\alpha_2^3\beta_2^2+\alpha_2^3\beta_3^2+\alpha_3^3\beta_1^2+\alpha_3^3\beta_3^2\right]}^{Q_{2,32}}}{\begin{array}{c} p_3+\underbrace{\left[\alpha_3^1\beta_1^1+\alpha_3^1\beta_1^2+\alpha_3^1\beta_1^3+\alpha_3^2\beta_1^1+\alpha_3^2\beta_1^2+\alpha_3^2\beta_1^3+\alpha_3^1\beta_2^1+\alpha_3^1\beta_2^2+\alpha_3^1\beta_2^3+\alpha_3^2\beta_2^1+\alpha_3^2\beta_2^2+\alpha_3^2\beta_2^3+\alpha_3^1\beta_3^1+\alpha_3^1\beta_3^2+\alpha_3^1\beta_3^3+\alpha_3^2\beta_3^1+\alpha_3^2\beta_3^2+\alpha_3^2\beta_3^3\right]}_{Q_{3,3}} \\ -\underbrace{\left[\alpha_1^3\beta_1^1+\alpha_1^3\beta_2^1+\alpha_1^3\beta_3^1+\alpha_1^3\beta_1^2+\alpha_1^3\beta_2^2+\alpha_1^3\beta_3^2+\alpha_1^3\beta_1^3+\alpha_1^3\beta_2^3+\alpha_1^3\beta_3^3+\alpha_2^3\beta_1^1+\alpha_2^3\beta_2^1+\alpha_2^3\beta_3^1+\alpha_2^3\beta_1^2+\alpha_2^3\beta_2^2+\alpha_2^3\beta_3^2+\alpha_2^3\beta_1^3+\alpha_2^3\beta_2^3+\alpha_2^3\beta_3^3\right]}_{Q_{4,3}}\end{array}}$$

- Simplifying

$$\begin{aligned}
Q_{1,32} &= \left(\alpha_3^1+\alpha_3^2\right)+\left(\beta_2^1+\beta_2^3\right)\left(1-\alpha_3^1-\alpha_3^2\right) && \text{(TI)}\\
&= \left(\theta_3^1+\theta_3^2\right)+\left(\theta_2^2+\theta_2^3\right)\left(1-\theta_3^1-\theta_3^2\right) && \text{(TIV)}\\
Q_{2,32} &= \left(\alpha_1^3+\alpha_2^3\right)\left(1+\beta_1^2+\beta_3^2-\beta_2^1-\beta_2^3\right)+\left(\beta_1^2+\beta_3^2\right)\left(1-\alpha_3^1-\alpha_3^2\right) && \text{(TI)}\\
&= \left(\theta_1^3+\theta_2^3\right)\left(1+\theta_1^2+\theta_3^2-\theta_2^1-\theta_2^3\right)+\left(\theta_1^2+\theta_3^2\right)\left(1-\theta_3^1-\theta_3^2\right) && \text{(TIV)}\\
Q_{3,3} &= 3\left(\alpha_3^1+\alpha_3^2\right) && \text{(TI)}\\
&= 3\left(\theta_3^1+\theta_3^2\right) && \text{(TIV)}\\
Q_{4,3} &= 3\left(\alpha_1^3+\alpha_2^3\right) && \text{(TI)}\\
&= 3\left(\theta_1^3+\theta_2^3\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{32}^{*}=\frac{r_{32}+\left(\beta_2^1+\beta_2^3-\beta_1^2-\beta_3^2\right)+\left(\alpha_3^1+\alpha_3^2-\alpha_1^3-\alpha_2^3\right)\left(1+\beta_1^2+\beta_3^2-\beta_2^1-\beta_2^3\right)}{p_3+3\left(\alpha_3^1+\alpha_3^2-\alpha_1^3-\alpha_2^3\right)}$$

  – Yields

$$\begin{aligned}
LB_{32}^{TI} &= \min\left\{\frac{r_{32}-\ddot{Q}}{p_3},\frac{r_{32}+\widetilde{\widetilde{Q}}}{p_3+3\widetilde{\widetilde{Q}}},\frac{r_{32}-\widehat{Q}}{p_3-3\widehat{Q}}\right\}\geq 0\\
&\quad \widetilde{\widetilde{Q}}=\min\left\{1-r_{32},(1-p_3)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{32},p_3/3,\ddot{Q}\right\},\ \widetilde{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{array}\right.,\ \ddot{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{array}\right.\\
UB_{32}^{TI} &= \max\left\{\frac{r_{32}+\widetilde{Q}}{p_3},\frac{r_{32}+\widetilde{\widetilde{Q}}}{p_3+3\widetilde{\widetilde{Q}}},\frac{r_{32}-\widehat{Q}}{p_3-3\widehat{Q}}\right\}\leq 1\\
&\quad \widetilde{\widetilde{Q}}=\min\left\{1-r_{32},(1-p_3)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{32},p_3/3,\ddot{Q}\right\},\ \widetilde{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{array}\right.,\ \ddot{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{array}\right.
\end{aligned}$$

  – Proof:

    1. $\ddot{Q}$ in $\frac{r_{32}-\ddot{Q}}{p_3}$ can be $2Q/9$ as $\beta_1^2,\beta_3^2=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{32}-\widehat{Q}}{p_3-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_1^3,\alpha_2^3=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{32}+\widetilde{\widetilde{Q}}}{p_3+3\widetilde{\widetilde{Q}}}\right)/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{32}+\widetilde{\widetilde{Q}}}{p_3+3\widetilde{\widetilde{Q}}}\right)}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(\left(p_3+3\widetilde{\widetilde{Q}}\right)-3\left(r_{32}+\widetilde{\widetilde{Q}}\right)\right)\\
&= sgn\left(p_3-3r_2\right)
\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{32}-\widehat{Q}}{p_3-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{32}-\widehat{Q}}{p_3-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_3-3\widehat{Q}\right)+3\left(r_{32}-\widehat{Q}\right)\right)\\
&= sgn\left(3r_{32}-p_3\right)
\end{aligned}$$

  – Adding the uni-directional assumption

$$p_{32}^{*}=\frac{r_{32}+\left(\beta_2^3-\beta_1^2\right)-\left(\alpha_1^3+\alpha_2^3\right)\left(1+\beta_1^2-\beta_2^3\right)}{p_3-3\left(\alpha_1^3+\alpha_2^3\right)}$$

    * Yields

$$\begin{aligned}
LB_{32}^{TI,u} &= \min\left\{\frac{r_{32}-\widetilde{Q}}{p_3},\frac{r_{32}-\widetilde{\widetilde{Q}}}{p_3-3\widetilde{\widetilde{Q}}}\right\}\geq 0\\
&\quad \widetilde{\widetilde{Q}}<\min\left\{r_{32},p_3/3,\ddot{Q}\right\},\ \widetilde{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{array}\right.,\ \ddot{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{array}\right.\\
UB_{32}^{TI,u} &= \max\left\{\frac{r_{32}+\widetilde{Q}}{p_3},\frac{r_{32}-\widetilde{\widetilde{Q}}}{p_3-3\widetilde{\widetilde{Q}}}\right\}\leq 1\\
&\quad \widetilde{\widetilde{Q}}<\min\left\{r_{32},p_3/3,\ddot{Q}\right\},\ \widetilde{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{array}\right.,\ \ddot{Q}=\left\{\begin{array}{ll} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{array}\right.
\end{aligned}$$

    * Proof: Same as above.

- Under Temporal Invariance

$$p_{32}^{*}=\frac{r_{32}+\left(\theta_2^1+\theta_3^1-\theta_1^2-\theta_1^3\right)+\left(\theta_3^1+\theta_3^2-\theta_1^3-\theta_2^3\right)\left(\theta_1^2+\theta_3^2-\theta_2^1-\theta_2^3\right)}{p_3+3\left(\theta_3^1+\theta_3^2-\theta_1^3-\theta_2^3\right)}$$

  – Yields

$$LB_{32}^{TIV} = \min\left\{\frac{r_{32}-\widetilde{Q}}{p_3},\frac{r_{32}+\widetilde{\widetilde{Q}}^2}{p_3+3\widetilde{\widetilde{Q}}}\right\}\geq 0$$

$$\widetilde{\widetilde{Q}}=\min\left\{\frac{-(2/3)p_3+\sqrt{(4/9)p_3^2+4r_{32}}}{2},\sqrt{1-r_{32}},(1-p_3)/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

$$UB_{32}^{TIV} = \max\left\{\frac{r_{32}+\widetilde{Q}}{p_3},\frac{r_{32}+\widetilde{\widetilde{Q}}^2}{p_3-3\widetilde{\widetilde{Q}}}\right\}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{\sqrt{1-r_{32}},p_3/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

  – Proof:

    1. Evaluate $\partial LB_{32}^{TIV}/\partial\widetilde{\widetilde{Q}}$ and see when the sign is positive/negative.

$$\begin{aligned} sgn\left(\frac{\partial LB_{32}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right) &= sgn\left(2\widetilde{\widetilde{Q}}\left(p_3+3\widetilde{\widetilde{Q}}\right)-3\left(r_{32}+\widetilde{\widetilde{Q}}^2\right)\right)\\ &= sgn\left(\widetilde{\widetilde{Q}}\left((2/3)\,p_3+\widetilde{\widetilde{Q}}\right)-r_{32}\right)\\ \Rightarrow sgn\left(\frac{\partial LB_{32}^{TIV}}{\partial\widetilde{\widetilde{Q}}}\right)\Bigg|_{\widetilde{\widetilde{Q}}=0} &= sgn\left(-r_{32}\right)<0\\ \Rightarrow\widetilde{\widetilde{Q}} &> 0\end{aligned}$$

    2. Minimize LB_{32}^{TIV} s.t. $\widetilde{\widetilde{Q}}$ being feasible

$$\begin{aligned}\frac{\partial LB_{32}^{TIV}}{\partial\widetilde{\widetilde{Q}}} &\propto \widetilde{\widetilde{Q}}\left((2/3)\,p_3+\widetilde{\widetilde{Q}}\right)-r_{32}=0\\ \Rightarrow\quad \widetilde{\widetilde{Q}}^{*} &= \frac{-(2/3)p_3+\sqrt{(4/9)p_3^2+4r_{32}}}{2}\end{aligned}$$

    So, derivative starts off negative and then reaches zero at $\widetilde{\widetilde{Q}}^{*}$. Thus, $\frac{r_{32}+\widetilde{\widetilde{Q}}^2}{p_3+3\widetilde{\widetilde{Q}}}$ is minimized at $\widetilde{\widetilde{Q}}^{*}$.

  – Adding the uni-directional assumption

$$p_{32}^{*}=\frac{r_{32}-\left(\theta_1^2+\theta_1^3\right)-\left(\theta_1^3+\theta_2^3\right)\left(\theta_1^2-\theta_2^3\right)}{p_3-3\left(\theta_1^3+\theta_2^3\right)}$$

    * Yields

$$LB_{32}^{TIV,u} = \frac{r_{32}-\widetilde{Q}}{p_3} \qquad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

$$UB_{32}^{TIV,u} = \frac{r_{32}+\widetilde{\widetilde{Q}}^2}{p_3-3\widetilde{\widetilde{Q}}}\leq 1 \qquad \widetilde{\widetilde{Q}}=\min\left\{\sqrt{1-r_{32}},p_3/3,\widetilde{Q}\right\},\ \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}$$

    * Proof: Same as above.

A.1.9 p^*_{33}

$$p^*_{33} = \frac{r_{33} + \overbrace{\left[\theta^{11}_{33} + \theta^{12}_{33} + \theta^{13}_{33} + \theta^{21}_{33} + \theta^{22}_{33} + \theta^{23}_{33} + \theta^{31}_{\underset{\bullet}{33}} + \theta^{32}_{\underset{\bullet}{33}}\right]}^{Q_{1,33}} - \overbrace{\left[\theta^{33}_{11} + \theta^{33}_{12} + \theta^{33}_{13} + \theta^{33}_{21} + \theta^{33}_{22} + \theta^{33}_{23} + \theta^{33}_{\underset{\bullet}{31}} + \theta^{33}_{\underset{\bullet}{32}}\right]}^{Q_{2,33}}}{\begin{array}{l} p_3 + \underbrace{\left[\theta^{11}_{\underset{\bullet}{31}} + \theta^{12}_{\underset{\bullet}{31}} + \theta^{13}_{\underset{\bullet}{31}} + \theta^{21}_{\underset{\bullet}{31}} + \theta^{22}_{\underset{\bullet}{31}} + \theta^{23}_{\underset{\bullet}{31}} + \theta^{11}_{\underset{\bullet}{32}} + \theta^{12}_{\underset{\bullet}{32}} + \theta^{13}_{\underset{\bullet}{32}} + \theta^{21}_{\underset{\bullet}{32}} + \theta^{22}_{\underset{\bullet}{32}} + \theta^{23}_{\underset{\bullet}{32}} + \theta^{11}_{33} + \theta^{12}_{33} + \theta^{13}_{33} + \theta^{21}_{33} + \theta^{22}_{33} + \theta^{23}_{33}\right]}_{Q_{3,3}} \\ \quad - \underbrace{\left[\theta^{31}_{\underset{\bullet}{11}} + \theta^{31}_{\underset{\bullet}{12}} + \theta^{31}_{\underset{\bullet}{13}} + \theta^{32}_{\underset{\bullet}{11}} + \theta^{32}_{\underset{\bullet}{12}} + \theta^{32}_{\underset{\bullet}{13}} + \theta^{33}_{11} + \theta^{33}_{12} + \theta^{33}_{13} + \theta^{31}_{\underset{\bullet}{21}} + \theta^{31}_{\underset{\bullet}{22}} + \theta^{31}_{\underset{\bullet}{23}} + \theta^{32}_{\underset{\bullet}{21}} + \theta^{32}_{\underset{\bullet}{22}} + \theta^{32}_{\underset{\bullet}{23}} + \theta^{33}_{21} + \theta^{33}_{22} + \theta^{33}_{23}\right]}_{Q_{4,3}} \end{array}}$$

- $\theta^{k'l'}_{\underset{\bullet}{kl}}$ = unique element

Arbitrary, Uniform Errors: Assumptions 2(i), 2(ii)

$$LB_{33} = \frac{r_{33} - \widetilde{Q}}{p_3} \geq 0 \qquad \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

$$UB_{33} = \frac{r_{33} + \widetilde{Q}}{p_3 - \widetilde{\widetilde{Q}}} \leq 1 \qquad \widetilde{\widetilde{Q}} = \begin{cases} 0 & \text{AE} \\ \min\{p_3, Q/3\} & \text{UE} \end{cases}$$

Uni-Directional Errors: Assumption 3

- Simplifying

$$p^*_{33} = \frac{r_{33} - \overbrace{\left[\theta^{33}_{11} + \theta^{33}_{12} + \theta^{33}_{13} + \theta^{33}_{21} + \theta^{33}_{22} + \theta^{33}_{23} + \theta^{33}_{\underset{\bullet}{31}} + \theta^{33}_{\underset{\bullet}{32}}\right]}^{Q^u_{2,33}}}{p_3 - \underbrace{\left[\theta^{31}_{\underset{\bullet}{11}} + \theta^{32}_{\underset{\bullet}{11}} + \theta^{32}_{\underset{\bullet}{12}} + \theta^{33}_{11} + \theta^{33}_{12} + \theta^{33}_{13} + \theta^{31}_{\underset{\bullet}{21}} + \theta^{32}_{\underset{\bullet}{21}} + \theta^{32}_{\underset{\bullet}{22}} + \theta^{33}_{21} + \theta^{33}_{22} + \theta^{33}_{23}\right]}_{Q^u_{4,3}}}$$

- Yields

$$LB^u_{33} = \frac{r_{33} - \widetilde{Q}}{p_3} \geq 0 \qquad \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

$$UB^u_{33} = \frac{r_{33}}{p_3 - \widetilde{\widetilde{Q}}} \leq 1 \qquad \widetilde{\widetilde{Q}} = \min\{p_3, \widetilde{Q}\},\ \widetilde{Q} = \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases}$$

Temporal Independence, Temporal Invariance

- Implies

$$p_{33}^{*} = \frac{r_{33} + \overbrace{\left[\alpha_3^1\beta_3^1 + \alpha_3^1\beta_3^2 + \alpha_3^1\beta_3^3 + \alpha_3^2\beta_3^1 + \alpha_3^2\beta_3^2 + \alpha_3^2\beta_3^3 + \alpha_3^3\beta_3^1 + \alpha_3^3\beta_3^2\right]}^{Q_{1,33}} - \overbrace{\left[\alpha_1^3\beta_1^3 + \alpha_1^3\beta_2^3 + \alpha_1^3\beta_3^3 + \alpha_2^3\beta_1^3 + \alpha_2^3\beta_2^3 + \alpha_2^3\beta_3^3 + \alpha_3^3\beta_1^3 + \alpha_3^3\beta_2^3\right]}^{Q_{2,33}}}{\begin{array}{c} p_3 + \underbrace{\left[\alpha_3^1\beta_1^1 + \alpha_3^1\beta_1^2 + \alpha_3^1\beta_1^3 + \alpha_3^2\beta_1^1 + \alpha_3^2\beta_1^2 + \alpha_3^2\beta_1^3 + \alpha_3^1\beta_2^1 + \alpha_3^1\beta_2^2 + \alpha_3^1\beta_2^3 + \alpha_3^2\beta_2^1 + \alpha_3^2\beta_2^2 + \alpha_3^2\beta_2^3 + \alpha_3^1\beta_3^1 + \alpha_3^1\beta_3^2 + \alpha_3^1\beta_3^3 + \alpha_3^2\beta_3^1 + \alpha_3^2\beta_3^2 + \alpha_3^2\beta_3^3\right]}_{Q_{3,3}} \\ - \underbrace{\left[\alpha_1^3\beta_1^1 + \alpha_1^3\beta_2^1 + \alpha_1^3\beta_3^1 + \alpha_1^3\beta_1^2 + \alpha_1^3\beta_2^2 + \alpha_1^3\beta_3^2 + \alpha_1^3\beta_1^3 + \alpha_1^3\beta_2^3 + \alpha_1^3\beta_3^3 + \alpha_2^3\beta_1^1 + \alpha_2^3\beta_2^1 + \alpha_2^3\beta_3^1 + \alpha_2^3\beta_1^2 + \alpha_2^3\beta_2^2 + \alpha_2^3\beta_3^2 + \alpha_2^3\beta_1^3 + \alpha_2^3\beta_2^3 + \alpha_2^3\beta_3^3\right]}_{Q_{4,3}} \end{array}}$$

- Simplifying

$$\begin{aligned}
Q_{1,33} &= \left(\alpha_3^1 + \alpha_3^2\right) + \left(\beta_3^1 + \beta_3^2\right)\left(1 - \alpha_3^1 - \alpha_3^2\right) && \text{(TI)} \\
&= 2\left(\theta_3^1 + \theta_3^2\right) - \left(\theta_3^1 + \theta_3^2\right)^2 && \text{(TIV)} \\
Q_{2,33} &= \left(\alpha_1^3 + \alpha_2^3\right)\left(1 + \beta_1^3 + \beta_2^3 - \beta_3^1 - \beta_3^2\right) + \left(\beta_1^3 + \beta_2^3\right)\left(1 - \alpha_3^1 - \alpha_3^2\right) && \text{(TI)} \\
&= 2\left(\theta_1^3 + \theta_2^3\right)\left(1 - \theta_3^1 - \theta_3^2\right) + \left(\theta_1^3 + \theta_2^3\right)^2 && \text{(TIV)} \\
Q_{3,3} &= 3\left(\alpha_3^1 + \alpha_3^2\right) && \text{(TI)} \\
&= 3\left(\theta_3^1 + \theta_3^2\right) && \text{(TIV)} \\
Q_{4,3} &= 3\left(\alpha_1^3 + \alpha_2^3\right) && \text{(TI)} \\
&= 3\left(\theta_1^3 + \theta_2^3\right) && \text{(TIV)}
\end{aligned}$$

- Under Temporal Independence

$$p_{33}^{*}=\frac{r_{33}+\left(\alpha_3^1+\alpha_3^2-\alpha_1^3-\alpha_2^3\right)+\left(\beta_3^1+\beta_3^2-\beta_1^3-\beta_2^3\right)\left(1+\alpha_1^3+\alpha_2^3-\alpha_3^1-\alpha_3^2\right)}{p_2+3\left(\alpha_3^1+\alpha_3^2-\alpha_1^3-\alpha_2^3\right)}$$

  – Yields

$$\begin{aligned}
LB_{33}^{TI} &= \min\left\{\frac{r_{33}-\ddot{Q}}{p_3},\frac{r_{33}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}},\frac{r_{33}-\widehat{Q}}{p_3-3\widehat{Q}}\right\}\geq 0\\
&\quad \tilde{\tilde{Q}}=\min\left\{1-r_{33},(1-p_3)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{33},p_3/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\\
UB_{33}^{TI} &= \max\left\{\frac{r_{33}+\widetilde{Q}}{p_3},\frac{r_{33}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}},\frac{r_{33}-\widehat{Q}}{p_3-3\widehat{Q}}\right\}\leq 1\\
&\quad \tilde{\tilde{Q}}=\min\left\{1-r_{33},(1-p_3)/3,\widetilde{Q}\right\},\ \widehat{Q}=\min\left\{r_{33},p_3/3,\ddot{Q}\right\},\ \widetilde{Q}=\begin{cases} Q/3 & \text{AE}\\ Q/9 & \text{UE}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}
\end{aligned}$$

  – Proof:

    1. $\ddot{Q}$ in $\frac{r_{33}-\ddot{Q}}{p_3}$ can be $2Q/9$ as $\beta_1^3,\beta_2^3=Q/9$ under UE.
    2. $\widehat{Q}$ in $\frac{r_{33}-\widehat{Q}}{p_3-3\widehat{Q}}$ can be $2Q/9$ as $\alpha_1^2,\alpha_2^3=Q/9$ under UE.
    3. Evaluate $\partial\left(\frac{r_{33}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{33}+\tilde{\tilde{Q}}}{p_3+3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(p_3+3\tilde{\tilde{Q}}\right)-3\left(r_{33}+\tilde{\tilde{Q}}\right)\right)\\
&= sgn\left(p_3-3r_{33}\right)
\end{aligned}$$

    4. Evaluate $\partial\left(\frac{r_{33}-\widehat{Q}}{p_3-3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{33}-\widehat{Q}}{p_3-3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(-\left(p_3-3\widehat{Q}\right)+3\left(r_{33}-\widehat{Q}\right)\right)\\
&= sgn\left(3r_{33}-p_3\right)
\end{aligned}$$

  – Adding the uni-directional assumption

$$p_{33}^{*}=\frac{r_{33}-\left(\alpha_1^3+\alpha_2^3\right)-\left(\beta_1^3+\beta_2^3\right)\left(1+\alpha_1^3+\alpha_2^3\right)}{p_2-3\left(\alpha_1^3+\alpha_2^3\right)}$$

    * Yields

$$\begin{aligned}
LB_{33}^{TI,u} &= \min\left\{\frac{r_{33}-\ddot{Q}}{p_3},\frac{r_{33}-\tilde{\tilde{Q}}}{p_3-3\tilde{\tilde{Q}}}\right\}\geq 0 \qquad \tilde{\tilde{Q}}=\min\left\{r_{33},p_3/3,\ddot{Q}\right\},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}\\
UB_{33}^{TI,u} &= \frac{r_{33}-\tilde{\tilde{Q}}}{p_3-3\tilde{\tilde{Q}}}\leq 1 \qquad \tilde{\tilde{Q}}=\begin{cases} 0 & r_{33}<p_3/3\\ \min\left\{r_{33},p_3/3,\ddot{Q}\right\} & \text{otherwise}\end{cases},\ \ddot{Q}=\begin{cases} Q/3 & \text{AE}\\ 2Q/9 & \text{UE}\end{cases}
\end{aligned}$$

    * Proof: Same as above.

- Under Temporal Invariance

$$p_{33}^{*}=\frac{r_{33}+2\left(\theta_3^1+\theta_3^2-\theta_1^3-\theta_2^3\right)-\left(\theta_3^1+\theta_3^2-\theta_1^3-\theta_2^3\right)^2}{p_2+3\left(\theta_3^1+\theta_3^2-\theta_1^3-\theta_2^3\right)}$$

  – Yields

$$\begin{aligned}
LB_{33}^{TIV} &= \min\left\{\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}},\frac{r_{33}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^2}{p_3-3\widetilde{\widetilde{Q}}}\right\}\geq 0\\
&\quad \widehat{Q}=\min\left\{(1-p_3)/3,\widetilde{Q}\right\},\\
&\quad \widetilde{\widetilde{Q}}=\begin{cases}0 & r_{33}\geq 2p_3/3\\ \min\left\{\frac{(2/3)p_3+\sqrt{(4/9)p_3^2+4[(2/3)p_3-r_{33}]}}{2},(-1+\sqrt{1+r_{33}}),p_3/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}\\
UB_{33}^{TIV} &= \min\left\{\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}},\frac{r_{33}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^2}{p_3-3\widetilde{\widetilde{Q}}}\right\}\geq 0\\
&\quad \widehat{Q}=\begin{cases}0 & r_{33}\geq 2p_3/3\\ \min\left\{\frac{-(2/3)p_3+\sqrt{(4/9)p_3^2+4[(2/3)p_3-r_{33}]}}{2},(1-p_3)/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{\widetilde{Q}}=\begin{cases}0 & r_{33}< 2p_3/3\\ \min\left\{\frac{(2/3)p_3-\sqrt{(4/9)p_3^2+4[(2/3)p_3-r_{33}]}}{2},(-1+\sqrt{1+r_{33}}),p_3/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\quad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}
\end{aligned}$$

– Proof:

1. Evaluate $\partial\left(\frac{r_{33}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_3-3\tilde{\tilde{Q}}}\right)/\partial\tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{33}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_3-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right) &= sgn\left(\left(-2-2\tilde{\tilde{Q}}\right)\left(p_3-3\tilde{\tilde{Q}}\right)+3\left(r_{33}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2\right)\right)\\
&= sgn\left(-(2/3)p_3\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{33}\right)\\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{33}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_3-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right)\Bigg|_{\tilde{\tilde{Q}}=0} &= sgn\left(-(2/3)p_3+r_{33}\right)\gtreqless 0\\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{33}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_3-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}}\right)\Bigg|_{\tilde{\tilde{Q}}=1} &= sgn\left(-(4/3)p_3+1+r_{33}\right)\gtreqless 0
\end{aligned}$$

2. Ensure $r_{33}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2\geq 0$

$$\begin{aligned}
& r_{33}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2\geq 0\\
\Rightarrow\ & \tilde{\tilde{Q}}^2+2\tilde{\tilde{Q}}-r_{33}\leq 0\\
\Rightarrow\ & \tilde{\tilde{Q}}\leq\frac{-2+\sqrt{4+4r_{33}}}{2}\\
\Rightarrow\ & \tilde{\tilde{Q}}\leq -1+\sqrt{1+r_{33}}
\end{aligned}$$

3. Minimize $\frac{r_{33}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_3-3\tilde{\tilde{Q}}}$ s.t. $\tilde{\tilde{Q}}$ being feasible and $r_{33}<2p_3/3$

$$\begin{aligned}
\frac{\partial\left(\frac{r_{33}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_3-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}} &\propto -(2/3)p_3\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{33}=0\\
\Rightarrow\ \tilde{\tilde{Q}}^* &= \frac{(2/3)p_3+\sqrt{(4/9)p_3^2+4[(2/3)p_3-r_{33}]}}{2}
\end{aligned}$$

4. Maximize $\frac{r_{33}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_3-3\tilde{\tilde{Q}}}$ s.t. $\tilde{\tilde{Q}}$ being feasible and $r_{33}>2p_3/3$

$$\begin{aligned}
\frac{\partial\left(\frac{r_{33}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_3-3\tilde{\tilde{Q}}}\right)}{\partial\tilde{\tilde{Q}}} &\propto -(2/3)p_3\left(1+\tilde{\tilde{Q}}\right)+\tilde{\tilde{Q}}^2+r_{33}=0\\
\Rightarrow\ \tilde{\tilde{Q}}^* &= \frac{(2/3)p_3-\sqrt{(4/9)p_3^2+4[(2/3)p_3-r_{33}]}}{2}
\end{aligned}$$

Note: If $\sqrt{(4/9)p_3^2+4[(2/3)p_3-r_{33}]}=.$, then maximize $\tilde{\tilde{Q}}$.

5. Evaluate $\partial\left(\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}\right)/\partial\widehat{Q}$ and see when the sign is positive/negative. Both are possible.

$$
\begin{aligned}
sgn\left(\frac{\partial\left(\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right) &= sgn\left(\left(2-2\widehat{Q}\right)\left(p_3+3\widehat{Q}\right)-3\left(r_{33}+2\widehat{Q}-\widehat{Q}^2\right)\right) \\
&= sgn\left((2/3)p_3\left(1-\widehat{Q}\right)-\widehat{Q}^2-r_{33}\right) \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right)\Bigg|_{\widehat{Q}=0} &= sgn\left((2/3)p_3-r_{33}\right)\gtrless 0 \\
\Rightarrow sgn\left(\frac{\partial\left(\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}\right)}{\partial\widehat{Q}}\right)\Bigg|_{\widehat{Q}=1} &= sgn\left(-1-r_{33}\right)<0
\end{aligned}
$$

6. Maximize $\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}$ s.t. $\widehat{Q}$ being feasible and $r_{33}<2p_3/3$

$$
\begin{aligned}
\frac{\partial\left(\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}\right)}{\partial\widehat{Q}} &\propto (2/3)p_3\left(1-\widehat{Q}\right)-\widehat{Q}^2-r_{33}=0 \\
&\Rightarrow \widehat{Q}^*=\frac{-(2/3)p_3+\sqrt{(4/9)p_3^2+4[(2/3)p_3-r_{33}]}}{2}
\end{aligned}
$$

7. Minimize $\frac{r_{33}+2\widehat{Q}-\widehat{Q}^2}{p_3+3\widehat{Q}}\Rightarrow\widehat{Q}=0$ or maximize $\widehat{Q}$. However, if the minimum occurs when $\widehat{Q}=0$, then $\frac{r_{33}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_3-3\tilde{\tilde{Q}}}<\frac{r_{33}}{p_3}$ and this will be the binding LB.

– Adding the uni-directional assumption

$$p_{33}^{*}=\frac{r_{33}-2\left(\theta_{1}^{3}+\theta_{2}^{3}\right)-\left(\theta_{1}^{3}+\theta_{2}^{3}\right)^{2}}{p_{2}-3\left(\theta_{1}^{3}+\theta_{2}^{3}\right)}$$

* Yields

$$\begin{aligned}
LB_{33}^{TIV} &= \frac{r_{33}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^{2}}{p_{3}-3\widetilde{\widetilde{Q}}}\geq 0\\
&\qquad \widetilde{\widetilde{Q}}=\begin{cases}0 & r_{33}\geq 2p_{3}/3\\ \min\left\{\frac{(2/3)p_{3}+\sqrt{(4/9)p_{3}^{2}+4[(2/3)p_{3}-r_{33}]}}{2},(-1+\sqrt{1+r_{33}}),p_{3}/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\qquad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}\\
UB_{33}^{TIV} &= \frac{r_{33}-2\widetilde{\widetilde{Q}}-\widetilde{\widetilde{Q}}^{2}}{p_{3}-3\widetilde{\widetilde{Q}}}\geq 0\\
&\qquad \widetilde{\widetilde{Q}}=\begin{cases}0 & r_{33}< 2p_{3}/3\\ \min\left\{\frac{(2/3)p_{3}-\sqrt{(4/9)p_{3}^{2}+4[(2/3)p_{3}-r_{33}]}}{2},(-1+\sqrt{1+r_{33}}),p_{3}/3,\widetilde{Q}\right\} & \text{otherwise}\end{cases},\\
&\qquad \widetilde{Q}=\begin{cases}3-\sqrt{9-Q} & \text{AE}\\ \left(4-\sqrt{16-4Q/3}\right)/2 & \text{UE}\end{cases}
\end{aligned}$$

* Proof: Same as above.

A.2 Tightening the Bounds

A.2.1 Shape Restrictions

$p^*_{11}, p^*_{22}, p^*_{33}$

$$\begin{aligned} LB^S_{kk} &= \max\left\{\sup_{k' \neq k} LB_{kk'}, \sup_{k' \neq k} LB_{k'k}\right\} \\ UB^S_{kk} &= UB_{kk} \end{aligned}$$

$p^*_{12}, p^*_{13}, p^*_{23}$

$$\begin{aligned} LB^S_{kk} &= LB_{kk} \\ UB^S_{kk} &= \min\{UB_{11}, UB_{12}, UB_{22}\} \end{aligned}$$

$p^*_{21}, p^*_{31}, p^*_{32}$

$$\begin{aligned} LB^S_{kk} &= LB_{kk} \\ UB^S_{kk} &= \min\{UB_{11}, UB_{21}, UB_{22}\} \end{aligned}$$

A.2.2 Level Set Restrictions

$$p^*_{kl}(x) = \frac{r_{kl}(x) + Q_{1,kl}(x) - Q_{2,kl}(x)}{p_k(x) + Q_{3,k}(x) - Q_{4,k}(x)}$$

- Let $Q(x)$ be probability of misclassification conditional on $X = x$. Then

$$\sum\nolimits_x p_x Q(x) \leq Q$$

- Implies

$$Q(x) = \begin{cases} Q/p_x & \text{No Independence} \\ Q & \text{Independence} \end{cases}$$

- Bounds
  - Bounds on $p^*_{kl}(x)$ are identical to baseline with Q replaced by $Q(x)$
  - After bounding $P^*_{01}(x)$, impose shape if desired
  - Derive bounds on P^*_{01}
  - Impose shape if desired

A.2.3 Monotonicity Restrictions

$$p^*_{kl}(u) = \frac{r_{kl}(u) + Q_{1,kl}(u) - Q_{2,kl}(u)}{p_k(u) + Q_{3,k}(u) - Q_{4,k}(u)}$$

- Let $Q(u)$ be probability of misclassification conditional on $U = u$. Then

$$\sum\nolimits_u p_u Q(u) \leq Q$$

- Implies

$$Q(u) = \begin{cases} Q/p_u & \text{No Independence} \\ Q & \text{Independence} \end{cases}$$

- Bounds
  - Bounds on $p^*_{kl}(u)$ are identical to baseline with Q replaced by $Q(u)$
  - After bounding $P^*_{01}(u)$, impose shape if desired
  - Derive bounds on P^*_{01}
  - Impose shape if desired

- Adding level set restrictions

$$p^*_{kl}(x,u) = \frac{r_{kl}(x,u) + Q_{1,kl}(x,u) - Q_{2,kl}(x,u)}{p_k(x,u) + Q_{3,k}(x,u) - Q_{4,k}(x,u)}$$

- Let $Q(x,u)$ be probability of misclassification conditional on $X = x, U = u$. Then

$$\sum\nolimits_x p_{xu} Q(x,u) \leq Q(u)$$

- Implies

$$Q(x,u) = \begin{cases} Q/\left(p_{xu} p_u\right) & \text{No Independence} \\ Q & \text{Independence} \end{cases}$$

  where $p_{xu} = \Pr(X = x|U = u)$

- Bounds
  - Bounds on $p^*_{kl}(x,u)$ are identical to baseline with Q replaced by $Q(x,u)$
  - After bounding $P^*_{01}(x,u)$, impose shape if desired
  - Derive bounds on $P^*_{01}(u)$
  - Impose shape if desired
  - Derive bounds on P^*_{01}
  - Impose shape if desired

B Supplemental Tables

Table B1. State specific Poverty Lines (Tendulkar Committee estimates): Monthly Per Capita Expenditure (Rs.)

Id	State/Union Territories	2005		2012	
		Rural	Urban	Rural	Urban
1	Jammu & Kashmir	522	603	891	988
2	Himachal Pradesh	520	606	913	1,064
3	Punjab	544	643	1,054	1,155
4	Chandigarh	643	643	1,155	1,155
5	Uttarkhand/Uttaranchal	486	602	880	1,082
6	Haryana	529	626	1,015	1,169
7	Delhi	541	642	1,145	1,134
8	Rajasthan	478	568	905	1,002
9	Uttar Pradesh	435	532	768	941
10	Bihar	433	526	778	923
11	Sikkim	532	742	930	1,226
12	Arunachal Pradesh	547	618	930	1,060
13	Nagaland	687	783	1,270	1,302
14	Manipur	578	641	1,118	1,170
15	Mizoram	639	700	1,066	1,155
16	Tripura	450	556	798	920
17	Meghalaya	503	746	888	1,154
18	Assam	478	600	828	1,008
19	West Bengal	445	573	783	981
20	Jharkhand	405	531	748	974
21	Orissa	408	497	695	861
22	Chhattisgarh	399	514	738	849
23	Madhya Pradesh	408	532	771	897
24	Gujarat	502	659	932	1,152
25	Daman & Diu	609	671	1,090	1,134
26	Dadra & Nagar Haveli	485	632	967	1,126
27	Maharashtra	485	632	967	1,126
28	Andhra Pradesh	433	563	860	1,009
29	Karnataka	418	588	902	1,089
30	Goa	609	671	1,090	1,134
32	Kerala	537	585	1,018	987
33	Tamil Nadu	442	560	880	937
34	Puducherry	385	506	1,301	1,309

Source: Planning Commission (Available at http://niti.gov.in/state-statistics)

C Supplemental Figures

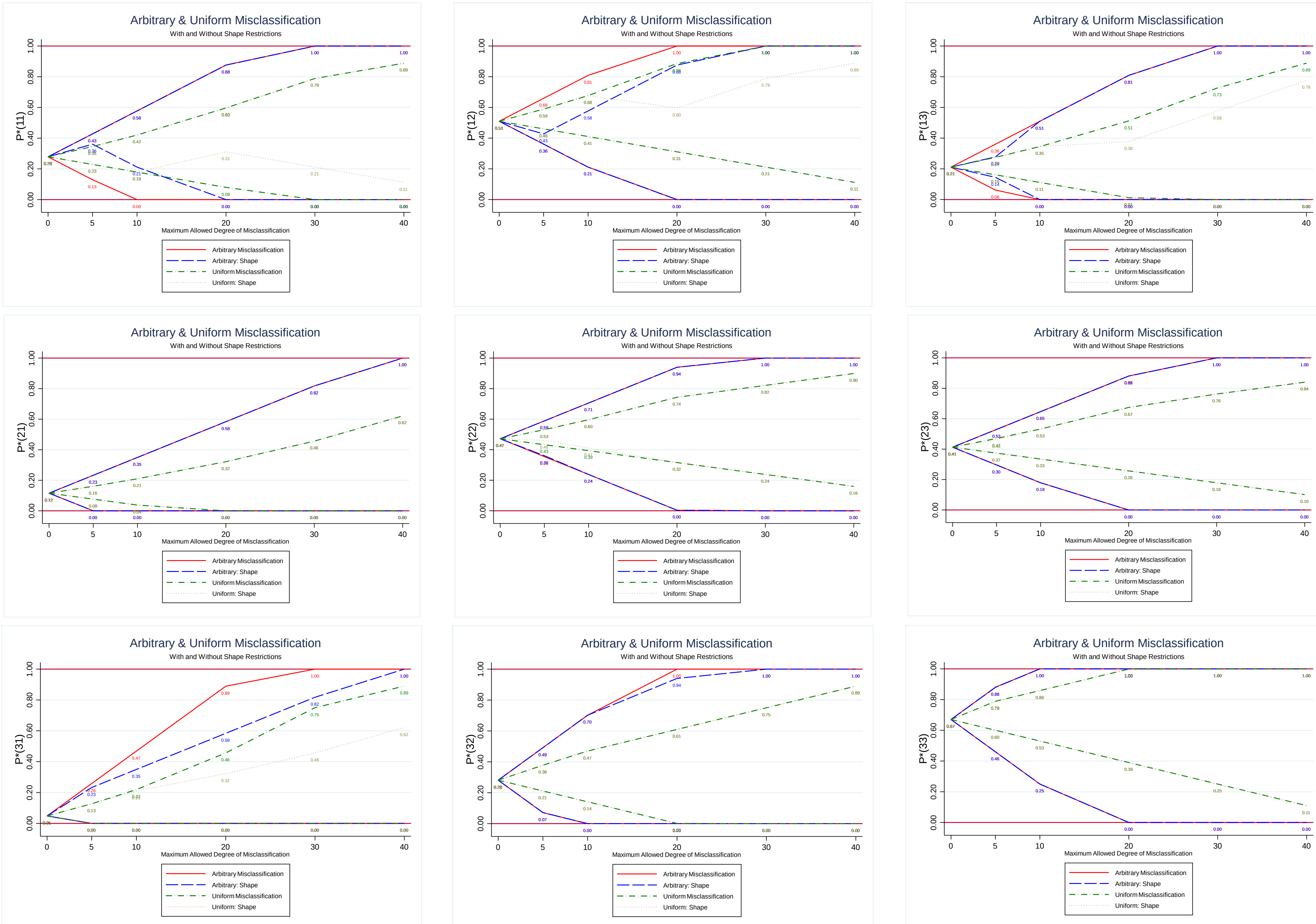

Figure C1. Transition Matrices Bounds (Arbitrary & Uniform Misclassification (with & without Shape)): Sensitivity to *Q*

Notes: Sample based on IHDS panel data. Point estimates for bounds obtained using 100 subsamples of size N/2 for bias correction. See text for further details.

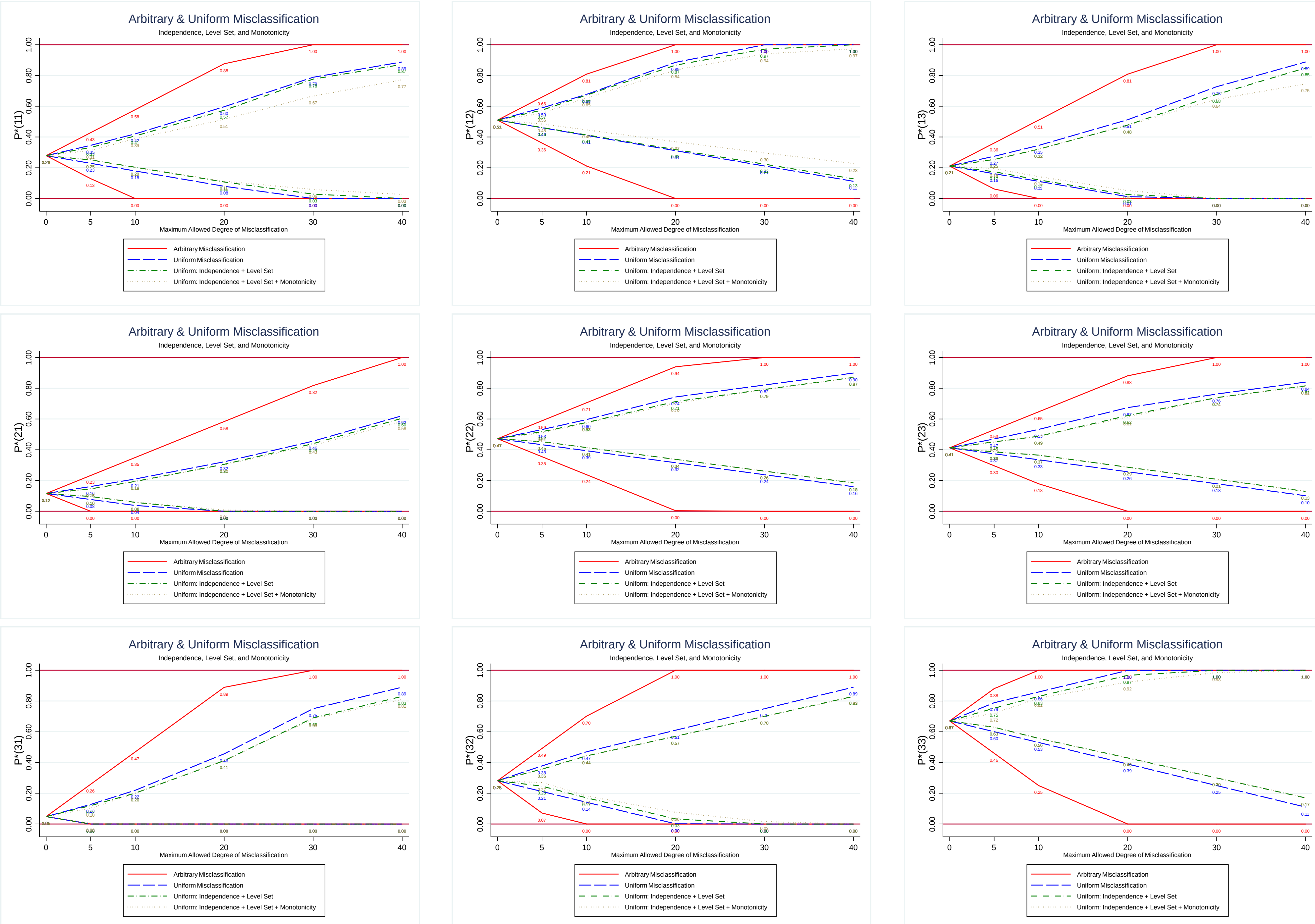

Figure C2. Transition Matrices Bounds (Arbitrary & Uniform Misclassification (with Independence, Level set & Monotonicity)): Sensitivity to *Q*

Notes: Sample based on IHDS panel data. Point estimates for bounds obtained using 100 subsamples of size N/2 for bias correction. See text for further details.

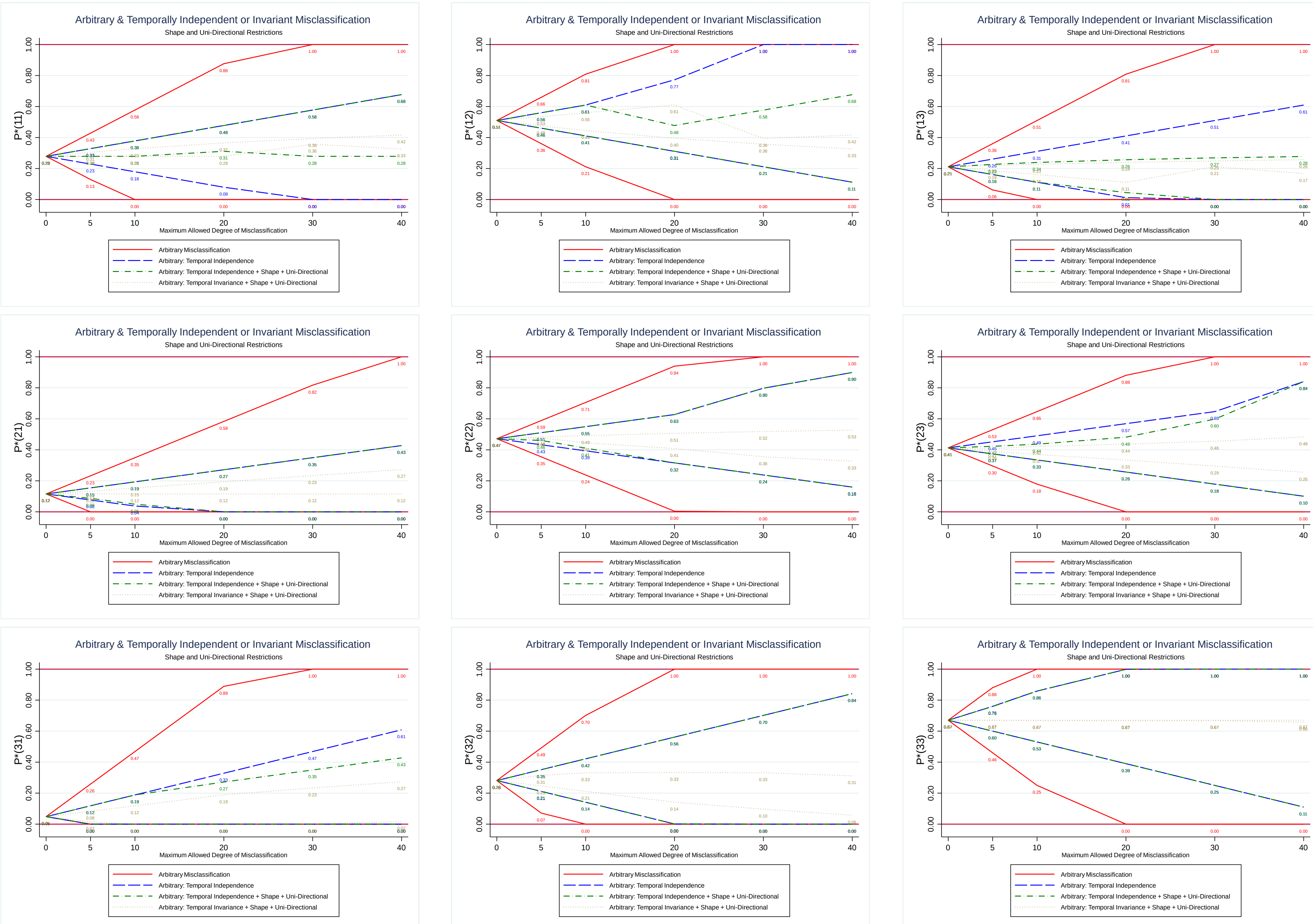

Figure C3. Transition Matrices Bounds (Arbitrary and Temporally Independent/Invariant Misclassification (with Shape & Unidirectional)): Sensitivity to *Q*

Notes: Sample based on IHDS panel data. Point estimates for bounds obtained using 100 subsamples of size N/2 for bias correction. See text for further details.

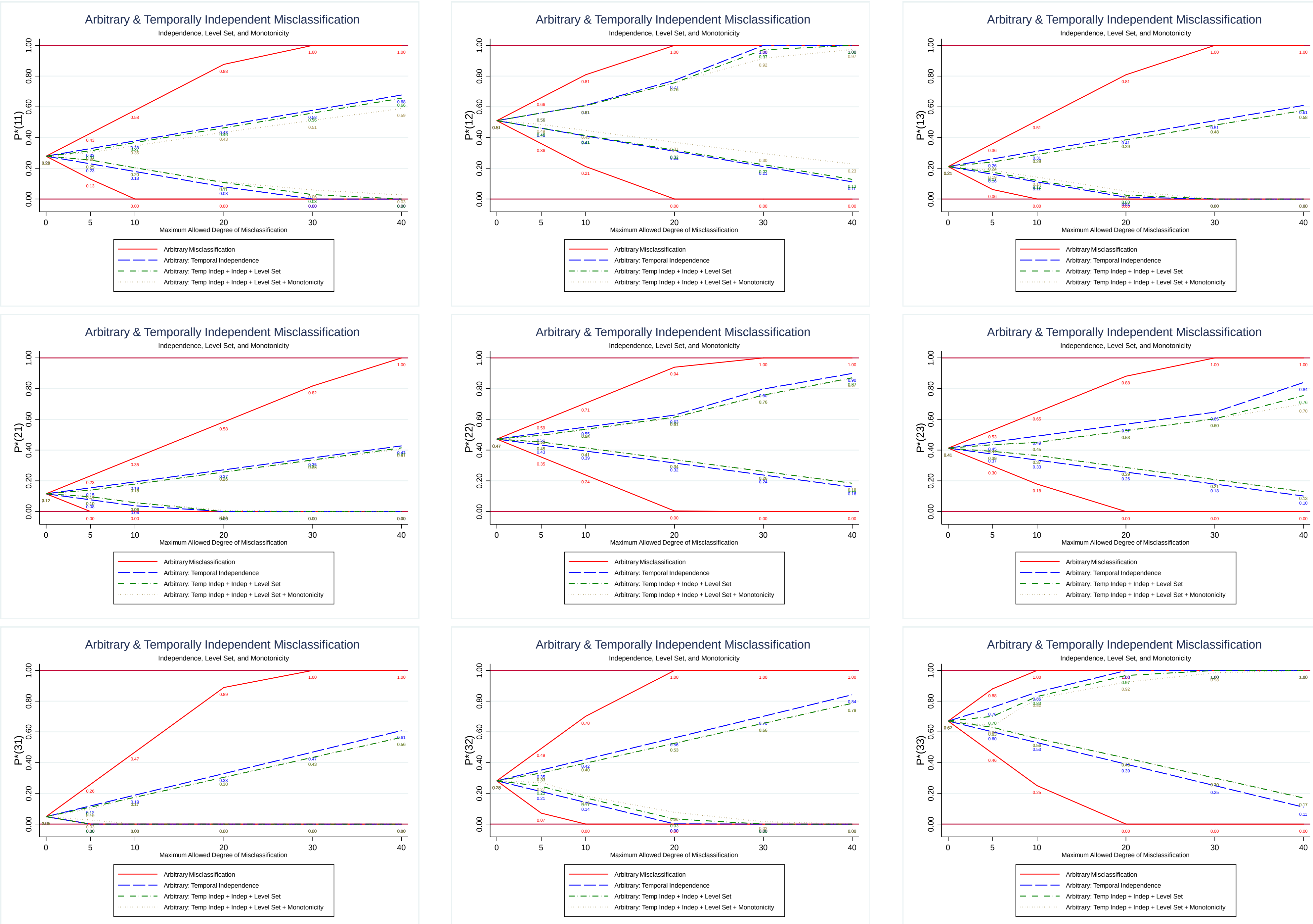

Figure C4. Transition Matrices Bounds (Arbitrary and Temporally Independent Misclassification (with Level set and Monotonicity)): Sensitivity to *Q*

Notes: Sample based on IHDS panel data. Point estimates for bounds obtained using 100 subsamples of size N/2 for bias correction. See text for further details.

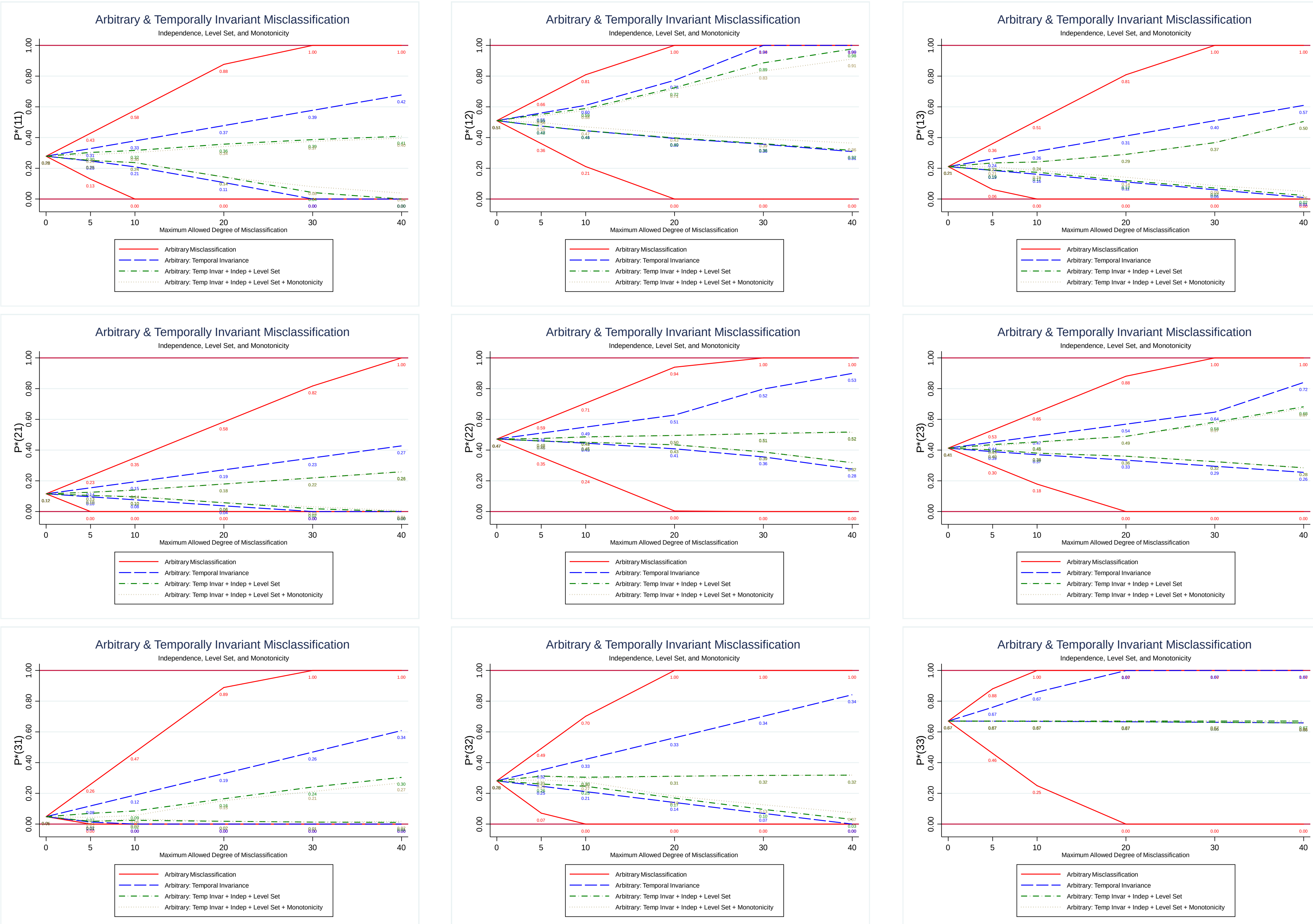

Figure C5. Transition Matrices Bounds (Arbitrary and Temporally Invariant Misclassification (with Level set and Monotonicity)): Sensitivity to *Q*

Notes: Sample based on IHDS panel data. Point estimates for bounds obtained using 100 subsamples of size N/2 for bias correction. See text for further details.